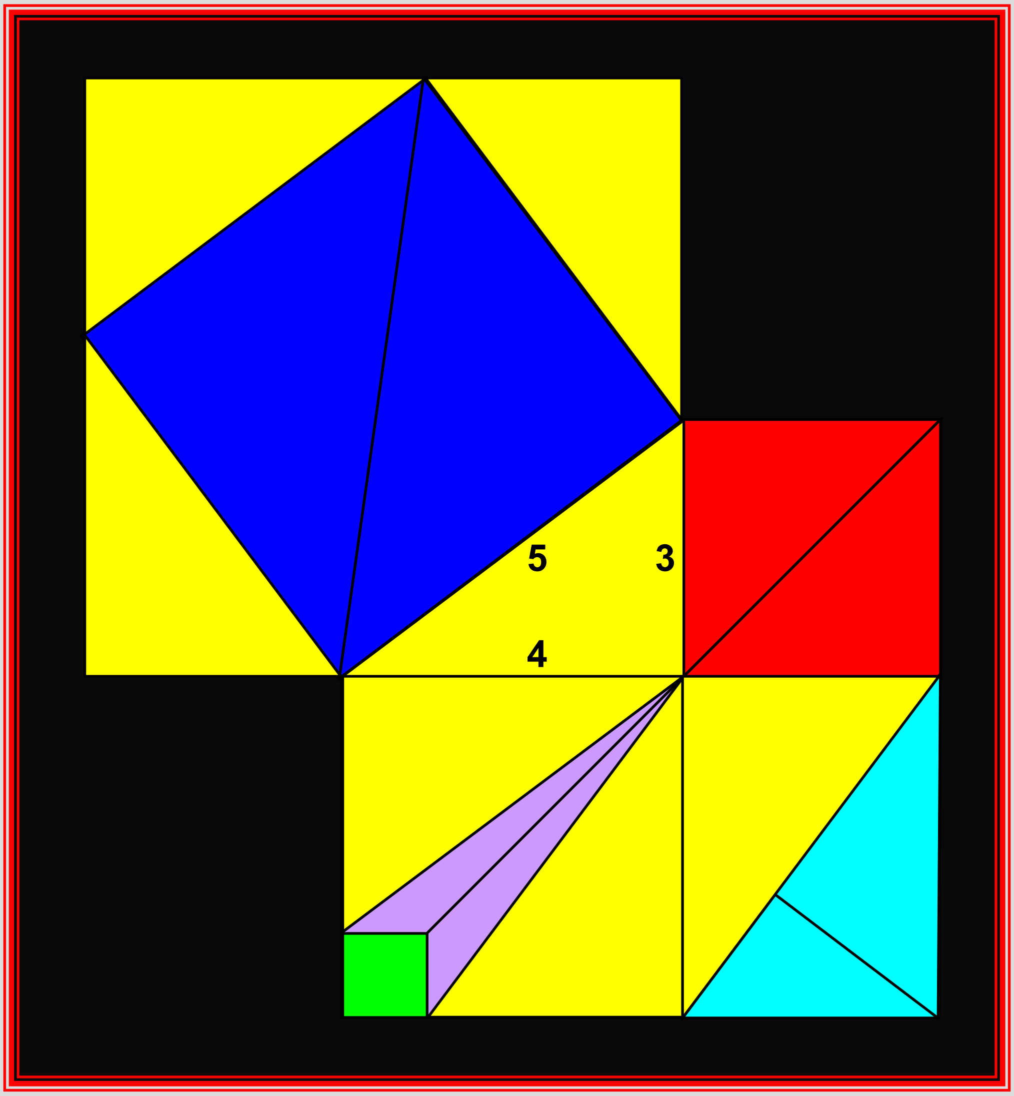
The Pythagorean Theorem

Crown Jewel of Mathematics



John C. Sparks

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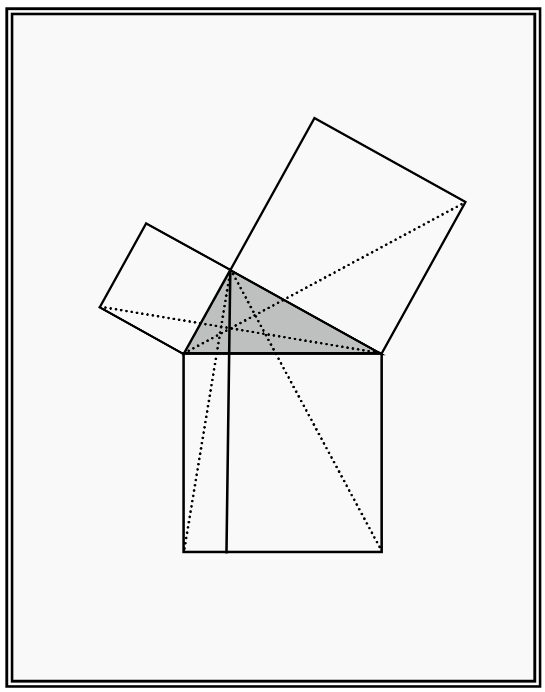


By John C. Sparks

The Pythagorean Theorem
Crown Jewel of Mathematics

 Front cover, Pythagorean Dreams, a composite mosaic of historical Pythagorean proofs. Back cover photo by Curtis Sparks

Produced by *Sparrow-Hawke †reasures*
Xenia, Ohio 45385

Dedication

I would like to dedicate The Pythagorean Theorem to:
Carolyn Sparks, my wife, best friend, and life partner for
40 years; our two grown sons, Robert and Curtis;
My father, Roscoe C. Sparks (1910-1994).

From Earth with Love

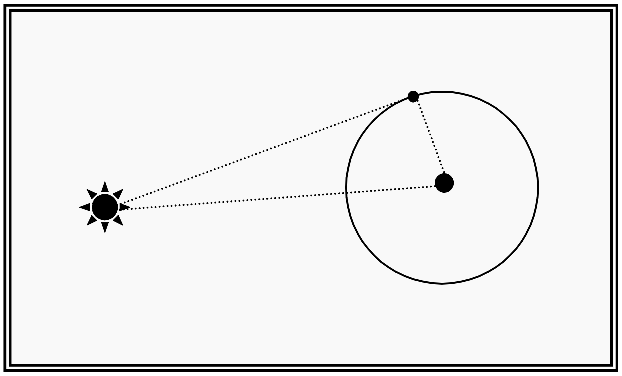
Do you remember, as do I,
When Neil walked, as so did we,
On a calm and sun-lit sea
One July, Tranquillity,
Filled with dreams and futures?

For in that month of long ago,
Lofty visions raptured all
Moonstruck with that starry call
From life beyond this earthen ball...
Not wedded to its surface.

But marriage is of dust to dust
Where seasoned limbs reclaim the ground
Though passing thoughts still fly around
Supernal realms never found
On the planet of our birth.

And I, a man, love you true,
Love as God had made it so,
Not angel rust when then aglow,
But coupled here, now rib to soul,
Dear Carolyn of mine.

July 2002: 33rd Wedding Anniversary



Conceptual Use of the Pythagorean Theorem by Ancient Greeks to Estimate the Distance From the Earth to the Sun

Significance

The wisp in my glass on a clear winter's night
Is home for a billion wee glimmers of light,
Each crystal itself one faraway dream
With faraway worlds surrounding its gleam.

And locked in the realm of each tiny sphere
Is all that is met through an eye or an ear;
Too, all that is felt by a hand or our love,
For we are but whits in the sea seen above.

Such scales immense make wonder abound
And make a lone knee touch the cold ground.
For what is this man that he should be made
To sing to The One Whose breath heavens laid?

July 1999

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* Section 2.10 requires knowledge of college-level calculus and be omitted without loss of continuity.

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List of Tables and Figures

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Section and Topic **Page**

* These are actual distinct proofs of the Pythagorean Theorem. This book has 20 such proofs in total.

Preface

The Pythagorean Theorem has been with us for over 4000 years and has never ceased to yield its bounty to mathematicians, scientists, and engineers. Amateurs love it in that most new proofs are discovered by amateurs. Without the Pythagorean Theorem, none of the following is possible: radio, cell phone, television, internet, flight, pistons, cyclic motion of all sorts, surveying and associated infrastructure development, and interstellar measurement. **The Pythagorean Theorem, Crown Jewel of Mathematics** chronologically traces the Pythagorean Theorem from a conjectured beginning, **Consider the Squares** (Chapter 1), through 4000 years of Pythagorean proofs, **Four Thousand Years of Discovery** (Chapter 2), from all major proof categories, 20 proofs in total. Chapter 3, **Diamonds of the Same Mind**, presents several mathematical results closely allied to the Pythagorean Theorem along with some major Pythagorean "spin-offs" such as Trigonometry. Chapter 4, **Pearls of Fun and Wonder**, is a potpourri of classic puzzles, amusements, and applications. An Epilogue, The **Crown and the Jewels**, summarizes the importance of the Pythagorean Theorem and suggests paths for further exploration. Four appendices service the reader: A] **Greek Alphabet**, B] **Mathematical Symbols**, C] **Geometric Foundations**, and D] **References**. For the reader who may need a review of elementary geometric concepts before engaging this book, Appendix C is highly recommended. A **Topical Index** completes the book.

A Word on Formats and Use of Symbols

One of my interests is poetry, having written and studied poetry for several years now. If you pick up a textbook on poetry and thumb the pages, you will see poems interspersed between explanations, explanations that English professors will call prose. Prose differs from poetry in that it is a major subcategory of how language is used. Even to the casual eye, prose and poetry each have a distinct look and feel.

So what does poetry have to do with mathematics? Any mathematics text can be likened to a poetry text. In it, the author is interspersing two languages: a language of qualification (English in the case of this book) and a language of quantification (the universal language of algebra). The way these two languages are interspersed is very similar to that of the poetry text. When we are describing, we use English prose interspersed with an illustrative phrase or two of algebra. When it is time to do an extensive derivation or problem-solving activity—using the concise algebraic language—then the whole page (or two or three pages!) may consist of nothing but algebra. Algebra then becomes the alternate language of choice used to unfold the idea or solution. **The Pythagorean Theorem** follows this general pattern, which is illustrated below by a discussion of the well-known quadratic formula.

☺

Let $ax^2 + bx + c = 0$ be a quadratic equation written in the standard form as shown with $a \neq 0$. Then $ax^2 + bx + c = 0$ has two solutions (including complex and multiple) given by the formula below, called the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a quadratic equation, using the quadratic formula, one needs to apply the following four steps considered to be a *solution process.*

1. Rewrite the quadratic equation in standard form.
2. Identify the two coefficients and constant term $a, b, \& c$.
3. Apply the formula and solve for the two x values.
4. Check your two answers in the original equation.

To illustrate this four-step process, we will solve the quadratic equation $2x^2 = 13x + 7$.

$\overset{1}{\mapsto}: 2x^2 = 13x + 7 \Rightarrow$

$2x^2 - 13x - 7 = 0$

****

$\overset{2}{\mapsto}: a = 2, b = -13, c = -7$

****

$$\overset{3}{\mapsto}: x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(2)(-7)}}{2(2)} \Rightarrow$$

$$x = \frac{13 \pm \sqrt{169 + 56}}{4} \Rightarrow$$

$$x = \frac{13 \pm \sqrt{225}}{4} = \frac{13 \pm 15}{4} \Rightarrow$$

$x \in \{-\frac{1}{2}, 7\}$

****

$\overset{4}{\mapsto}$: This step is left to the reader.

☺

Taking a look at the text between the two happy-face symbols ☺ ☺, we first see the usual mixture of algebra and prose common to math texts. The quadratic formula itself, being a major algebraic result, is presented first as a stand-alone result. If an associated process, such as solving a quadratic equation, is best described by a sequence of enumerated steps, the steps will be presented in indented, enumerated fashion as shown. Appendix B provides a detailed list of all mathematical symbols used in this book along with explanations.

Regarding other formats, italicized 9-font text is used throughout the book to convey special cautionary notes to the reader, items of historical or personal interest, etc. Rather than footnote these items, I have chosen to place them within the text exactly at the place where they augment the overall discussion.

Lastly, throughout the book, the reader will notice a three-squared triangular figure at the bottom of the page. One such figure signifies a section end; two, a chapter end; and three, the book end.

Credits

No book such as this is an individual effort. Many people have inspired it: from concept to completion. Likewise, many people have *made it so* from drafting to publishing. I shall list just a few and their contributions.

Elisha Loomis, I never knew you except through your words in The Pythagorean Proposition; but thank you for propelling me to fashion an every-person's update suitable for a new millennium. To those great Americans of my youth—President John F. Kennedy, John Glenn, Neil Armstrong, and the like—thank you all for inspiring an entire generation to think and dream of bigger things than themselves.

To my two editors, Curtis and Stephanie Sparks, thank you for helping the raw material achieve full publication. This has truly been a family affair.

To my wife Carolyn, the Heart of it All, what can I say. You have been my constant and loving partner for some 40 years now. You gave me the space to complete this project and rejoiced with me in its completion. As always, we are a proud team!

John C. Sparks
October 2008
Xenia, Ohio

1) Consider the Squares

"If it was good enough for old Pythagoras,
It is good enough for me." Unknown

How did the Pythagorean Theorem come to be a theorem? Having not been trained as mathematical historian, I shall leave the answer to that question to those who have. What I do offer in Chapter 1 is a speculative, logical sequence of how the Pythagorean Theorem might have been originally discovered and then extended to its present form. Mind you, the following idealized account describes a discovery process much too smooth to have actually occurred through time. Human inventiveness in reality always has entailed plenty of dead ends and false starts. Nevertheless, in this chapter, I will play the role of the proverbial Monday-morning quarterback and execute a perfect play sequence as one modern-day teacher sees it.

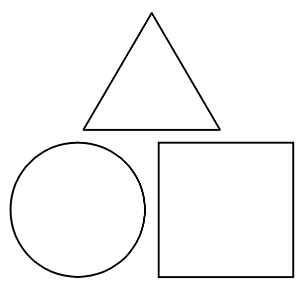


Figure 1.1: The Circle, Square, and Equilateral Triangle

Of all regular, planar geometric figures, the square ranks in the top three for elegant simplicity, the other two being the circle and equilateral triangle, **Figure 1.1**. All three figures would be relatively easy to draw by our distant ancestors: either freehand or, more precisely, with a stake and fixed length of rope.

For this reason, I would think that the square would be one of the earliest geometrics objects examined.

Note: Even in my own early-sixties high-school days, string, chalk, and chalk-studded compasses were used to draw 'precise' geometric figures on the blackboard. Whether or not this ranks me with the ancients is a matter for the reader to decide.

So, how might an ancient mathematician study a simple square? Four things immediately come to my modern mind: translate it (move the position in planar space), rotate it, duplicate it, and partition it into two triangles by insertion of a diagonal as shown in **Figure 1.2**.

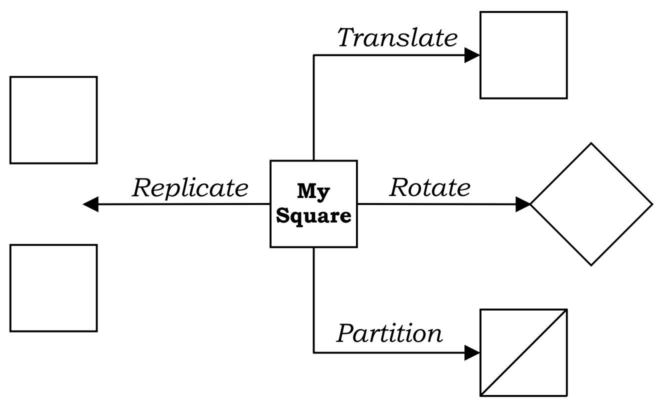


Figure 1.2: Four Ways to Contemplate a Square

I personally would consider the partitioning of the square to be the most interesting operation of the four in that I have generated two triangles, two new geometric objects, from one square. The two right-isosceles triangles so generated are congruent—perfect copies of each other—as shown on the next page in **Figure 1.3** with annotated side lengths s and angle measurements in degrees.

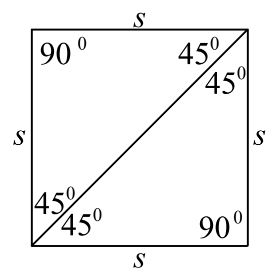


Figure 1.3: One Square to Two Triangles

For this explorer, the partitioning of the square into perfect triangular replicates would be a fascination starting point for further exploration. Continuing with our speculative journey, one could imagine the replication of a partitioned square with perhaps a little decorative shading as shown in **Figure 1.4**. Moreover, let us not replicate just once, but four times.

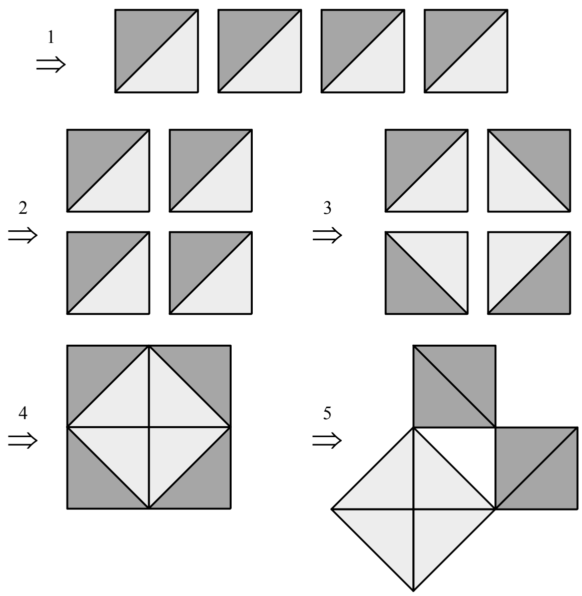


Figure 1.4: One Possible Path to Discovery

Now, continue to translate and rotate the four replicated, shaded playing piece pieces as if working a jigsaw puzzle. After spending some trial-and-error time—perhaps a few hours, perhaps several years—we stop to ponder a fascinating composite pattern when it finally meanders into view, Step 5 in **Figure 1.4**.

Note: I have always found it very amusing to see a concise and logical textbook sequence [e.g. the five steps shown on the previous page] presented in such a way that a student is left to believe that this is how the sequence actually happened in a historical context. Recall that Thomas Edison had four-thousand failures before finally succeeding with the light bulb. Mathematicians are no less prone to dead ends and frustrations!

Since the sum of any two acute angles in any one of the right triangles is again 90^0, the lighter-shaded figure bounded by the four darker triangles (resulting from Step 4) is a square with area double that of the original square. Further rearrangement in Step 5 reveals the fundamental Pythagorean sum-of-squares pattern when the three squares are used to enclose an empty triangular area congruent to each of the eight original right-isosceles triangles.

Of the two triangle properties for each little triangle—*the fact that each was right or the fact that each was isosceles*—which was the key for the sum of the two smaller areas to be equal to the one larger area? Or, were both properties needed? To explore this question, we will start by eliminating one of the properties, isosceles; in order to see if this magical sum-of-squares pattern still holds. **Figure 1.5** is a general right triangle where the three interior angles and side lengths are labeled.

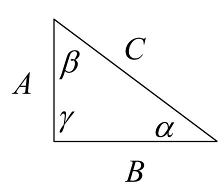


Figure 1.5: General Right Triangle

Notice that the right-triangle property implies that the sum of the two acute interior angles equals the right angle as proved below.

$$\alpha + \beta + \gamma = 180^0 \ \&$$
$$\gamma = 90^0 \Rightarrow$$
$$\alpha + \beta = 90^0 \Rightarrow$$
$$\alpha + \beta = \gamma \therefore$$

Thus, for any right triangle, the sum of the two acute angles equals the remaining right angle or 90^0; eloquently stated in terms of **Figure 1.5** as $\alpha + \beta = \gamma$.

Continuing our exploration, let's replicate the general right triangle in **Figure 1.5** eight times, dropping all algebraic annotations. Two triangles will then be fused together in order to form a rectangle, which is shaded via the same shading scheme in **Figure 1.4**. **Figure 1.6** shows the result, four bi-shaded rectangles mimicking the four bi-shaded squares in **Figure 1.4**.

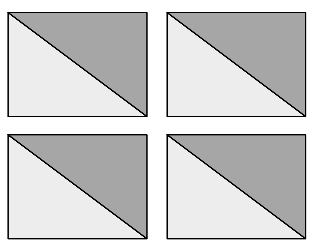


Figure 1.6: Four Bi-Shaded Congruent Rectangles

With our new playing pieces, we rotate as before, finally arriving at the pattern shown in **Figure 1.7**.

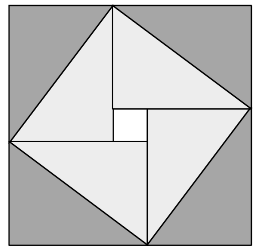


Figure 1.7: A Square Donut within a Square

That the rotated interior quadrilateral—the 'square donut'—is indeed a square is easily shown. Each interior corner angle associated with the interior quadrilateral is part of a three-angle group that totals 180^0. The two acute angles flanking the interior corner angle sum to 90^0 since these are the two different acute angles associated with the right triangle. Thus, simple subtraction gives the measure of any one of the four interior corners as 90^0. The four sides of the quadrilateral are equal in length since they are simply four replicates of the hypotenuse of our basic right triangle. Therefore, the interior quadrilateral is indeed a square *generated from our basic triangle and its hypotenuse.*

Suppose we remove the four lightly shaded playing pieces and lay them aside as shown in **Figure 1.8**.

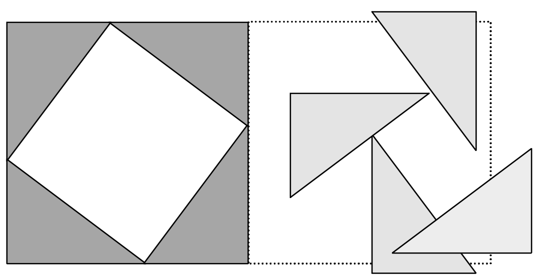


Figure 1.8: The Square within the Square is Still There

The middle square (minus the donut hole) is still plainly visible and nothing has changed with respect to size or orientation. Moreover, in doing so, we have freed up four playing pieces, which can be used for further explorations.

If we use the four lighter pieces to experiment with different ways of filling the outline generated by the four darker pieces, an amazing discover will eventually manifest itself—again, perhaps after a few hours of fiddling and twiddling or, perhaps after several years—**Figure 1.9**.

Note: To reiterate, Thomas Edison tried 4000 different light-bulb filaments before discovering the right material for such an application.

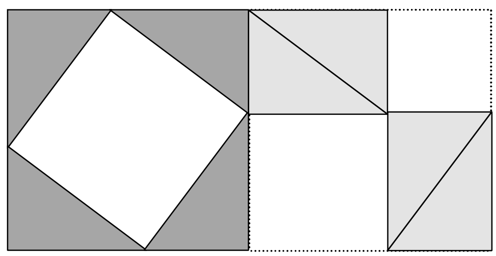


Figure 1.9: A Discovery Comes into View

That the ancient discovery is undeniable is plain from **Figure 1.10** on the next page, which includes yet another pattern and, for comparison, the original square shown in **Figure 1.7** comprised of all eight playing pieces. The 12th century Indian mathematician Bhaskara was alleged to have simply said, “Behold!” when showing these diagrams to students. Decoding Bhaskara’s terseness, one can create four different, equivalent-area square patterns using eight congruent playing pieces. Three of the patterns use half of the playing pieces and one uses the full set. Of the three patterns using half the pieces, the sum of the areas for the two smaller squares equals the area of the rotated square in the middle as shown in the final pattern with the three outlined squares.

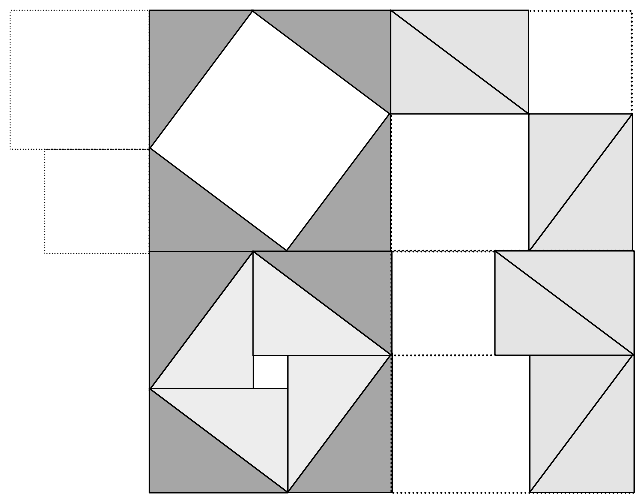
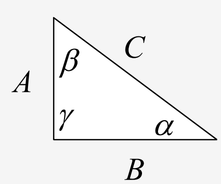


Figure 1.10: Behold!

Phrasing Bhaskara's "proclamation" in modern algebraic terms, we would state the following:

> **The Pythagorean Theorem**
>
> Suppose we have a right triangle with side lengths and angles labeled as shown below.
>
> 
>
>
> Then $\alpha + \beta = \gamma$ and
>
> $$\alpha + \beta = \gamma \Rightarrow A^2 + B^2 = C^2$$

Our proof in Chapter 1 has been by visual inspection and consideration of various arrangements of eight triangular playing pieces. I can imagine our mathematically minded ancestors doing much the same thing some three to four thousand years ago when this theorem was first discovered and utilized in a mostly pre-algebraic world.

To conclude this chapter, we need to address one loose end. Suppose we have a non-right triangle. Does the Pythagorean Theorem still hold? The answer is a resounding no, but we will hold off proving what is known as the converse of the Pythagorean Theorem, $A^2 + B^2 = C^2 \Rightarrow \alpha + \beta = \gamma$, until Chapter 2. However, we will close Chapter 1 by visually exploring two extreme cases where non-right angles definitely imply that $A^2 + B^2 \neq C^2$.

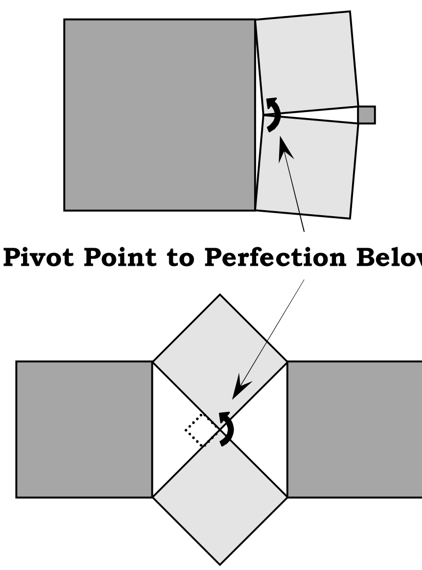


Figure 1.11: Extreme Differences Versus Pythagorean Perfection

In **Figure 1.11**, the lightly shaded squares in the upper diagram form two equal sides for two radically different isosceles triangles. One isosceles triangle has a large central obtuse angle and the other isosceles triangle has a small central acute angle. For both triangles, the darker shaded square is formed from the remaining side. It is obvious to the eye that two light areas do not sum to a dark area no matter which triangle is under consideration. By way of contrast, compare the upper diagram to the lower diagram where an additional rotation of the lightly shaded squares creates two central right angles and the associated Pythagorean perfection.

Euclid Alone Has Looked on Beauty Bare

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.

O blinding hour, O holy, terrible day,
When first the shaft into his vision shown
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once and then but far away,
Have heard her massive sandal set on stone.

Edna St. Vincent Millay

2) Four Thousand Years of Discovery

Consider old Pythagoras,
A Greek of long ago,
And all that he did give to us,
Three sides whose squares now show

In houses, fields and highways straight;
In buildings standing tall;
In mighty planes that leave the gate;
And, micro-systems small.

Yes, all because he got it right
When angles equal ninety—
One geek (BC), his plain delight—
One world changed aplenty! January 2002

2.1) Pythagoras and the First Proof

Pythagoras was not the first in antiquity to know about the remarkable theorem that bears his name, but he was the first to formally prove it using deductive geometry and the first to actively 'market' it (using today's terms) throughout the ancient world. One of the earliest indicators showing knowledge of the relationship between right triangles and side lengths is a hieroglyphic-style picture, **Figure 2.1**, of a knotted rope having twelve equally-spaced knots.

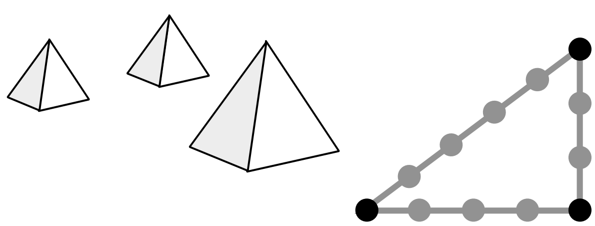


Figure 2.1: Egyptian Knotted Rope, Circa 2000 BCE

The rope was shown in a context suggesting its use as a workman's tool for creating right angles, done via the fashioning of a 3-4-5 right triangle. Thus, the Egyptians had a mechanical device for demonstrating the converse of the Pythagorean Theorem for the 3-4-5 special case:

$$3^2 + 4^2 = 5^2 \Rightarrow \gamma = 90^0 .$$

Not only did the Egyptians know of specific instances of the Pythagorean Theorem, but also the Babylonians and Chinese some 1000 years before Pythagoras definitively institutionalized the general result circa 500 BCE. And to be fair to the Egyptians, Pythagoras himself, who was born on the island of Samos in 572 BCE, traveled to Egypt at the age of 23 and spent 21 years there as a student before returning to Greece. While in Egypt, Pythagoras studied a number of things under the guidance of Egyptian priests, including geometry. **Table 2.1** briefly summarizes what is known about the Pythagorean Theorem before Pythagoras.

Date	Culture	Person	Evidence
2000 BCE	Egyptian	Unknown	Workman's rope for fashioning a 3-4-5 triangle
1500 BCE	Babylonian & Chaldean	Unknown	Rules for right triangles written on clay tablets along with geometric diagrams
1100 BCE	Chinese	Tschou-Gun	Written geometric characterizations of right angles
520 BCE	Greek	Pythagoras	Generalized result and deductively proved

Table 2.1: Prior to Pythagoras

The proof Pythagoras is thought to have actually used is shown in **Figure 2.2**. It is a visual proof in that no algebraic language is used to support numerically the deductive argument. In the top diagram, the ancient observer would note that removing the eight congruent right triangles, four from each identical master square, brings the magnificent sum-of-squares equality into immediate view.

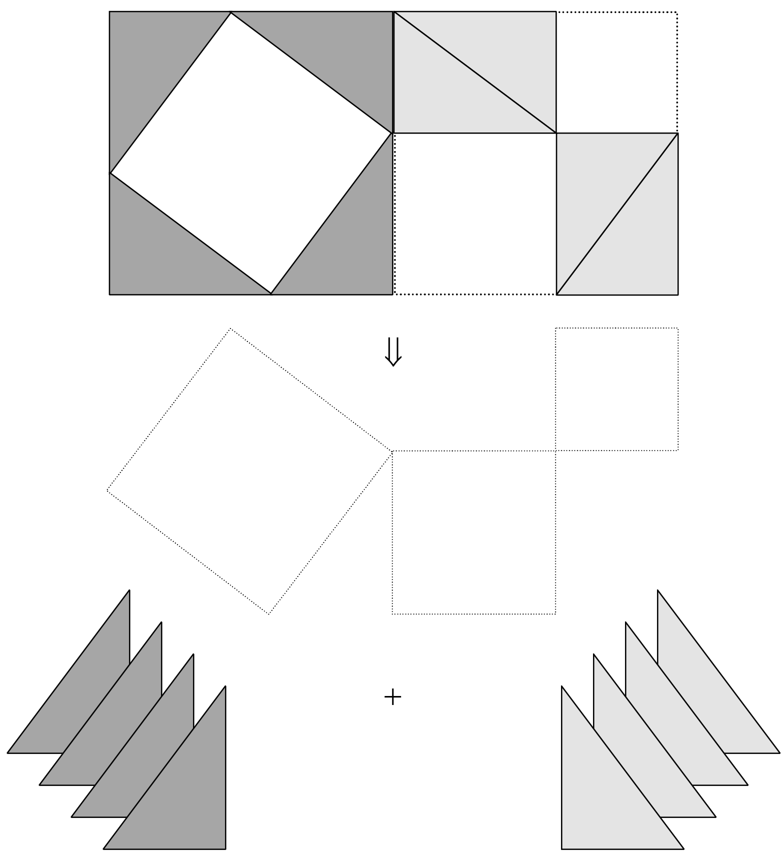


Figure 2.2: The First Proof by Pythagoras

Figure 2.3 is another original, visual proof attributed to Pythagoras. Modern mathematicians would say that this proof is more 'elegant' in that the same deductive message is conveyed using one less triangle. Even today, 'elegance' in proof is measured in terms of logical conciseness coupled with the amount insight provided by the conciseness. Without any further explanation on my part, the reader is invited to engage in the mental deductive gymnastics needed to derive the sum-of-squares equality from the diagram below.

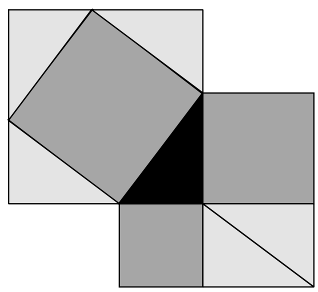


Figure 2.3: An Alternate Visual Proof by Pythagoras

Neither of Pythagoras' two visual proofs requires the use of an algebraic language as we know it. Algebra in its modern form as a precise language of numerical quantification wasn't fully developed until the Renaissance. The branch of mathematics that utilizes algebra to facilitate the understanding and development of geometric concepts is known as *analytic geometry*. Analytic geometry allows for a deductive elegance unobtainable by the use of visual geometry alone. **Figure 2.4** is the square-within-the-square (as first fashioned by Pythagoras) where the length of each triangular side is algebraically annotated just one time.

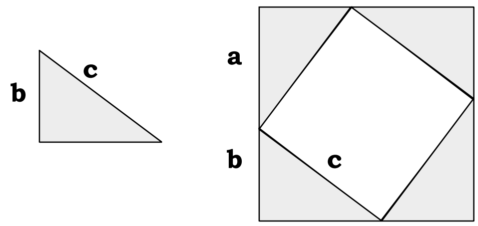


Figure 2.4: Annotated Square within a Square

The proof to be shown is called a *dissection* proof due to the fact that the larger square has been dissected into five smaller pieces. In all dissection proofs, our arbitrary right triangle, shown on the left, is at least one of the pieces. One of the two keys leading to a successful dissection proof is the writing of the total area in two different algebraic ways: as a singular unit and as the sum of the areas associated with the individual pieces. The other key is the need to utilize each critical right triangle dimension—**a**, **b**, **c**—at least once in writing the two expressions for area. Once the two expressions are written, algebraic simplification will lead (hopefully) to the Pythagorean Theorem. Let us start our proof. The first step is to form the two expressions for area.

$$\overset{1}{\mapsto}: A_{bigsquare} = (a+b)^2 \ \&$$
$$A_{bigsquare} = A_{littlesquare} + 4 \cdot (A_{onetriangle}) \Rightarrow$$
$$A_{bigsquare} = c^2 + 4(\tfrac{1}{2}ab)$$

The second step is to equate these expressions and algebraically simplify.

$$\overset{2}{\mapsto}: (a+b)^2 \overset{set}{=} c^2 + 4(\tfrac{1}{2}ab) \Rightarrow$$
$$a^2 + 2ab + b^2 = c^2 + 2ab \Rightarrow$$
$$a^2 + b^2 = c^2 \therefore$$

Notice how quickly and easily our result is obtained once algebraic is used to augment the geometric picture. Simply put, algebra coupled with geometry is superior to geometry alone in quantifying and tracking the diverse and subtle relationships between geometric whole and the assorted pieces. Hence, throughout the remainder of the book, analytic geometry will be used to help prove and develop results as much as possible.

Since the larger square in **Figure 2.4** is dissected into five smaller pieces, we will say that this is a Dissection Order V (**DRV**) proof. It is a good proof in that all three critical dimensions—**a**, **b**, **c**—and *only these dimensions* are used to verify the result. This proof is the proof most commonly used when the Pythagorean Theorem is first introduced. As we have seen, the origins of this proof can be traced to Pythagoras himself.

We can convey the proof in simpler fashion by simply showing the square-within-the-square diagram (**Figure 2.5**) and the associated algebraic development below unencumbered by commentary. Here forward, this will be our standard way of presenting smaller and more obvious proofs and/or developments.

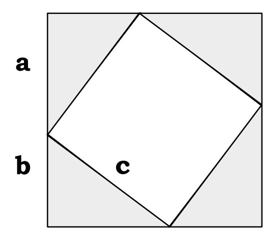


Figure 2.5: Algebraic Form of the First Proof

$$\overset{1}{\mapsto}: A = (a+b)^2 \;\&\; A = c^2 + 4(\tfrac{1}{2}ab)$$

$$\overset{2}{\mapsto}: (a+b)^2 \overset{set}{=} c^2 + 4(\tfrac{1}{2}ab)$$

$$\Rightarrow a^2 + 2ab + b^2 = c^2 + 2ab \Rightarrow a^2 + b^2 = c^2 \;\therefore$$

Figure 2.6 is the diagram for a second not-so-obvious dissection proof where a rectangle encloses the basic right triangle as shown. The three triangles comprising the rectangle are similar (left to reader to show), allowing the unknown dimensions **x, y**, **z** to be solved via similarity principles in terms of **a**, **b**, and **c**. Once we have **x, y**, and **z** in hand**,** the proof proceeds as a normal dissection.

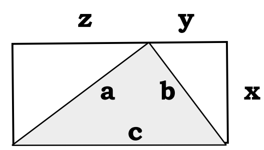


Figure 2.6: A Rectangular Dissection Proof

$$\overset{1}{\mapsto}: \frac{x}{b} = \frac{a}{c} \Rightarrow x = \frac{ab}{c}, \frac{y}{b} = \frac{b}{c} \Rightarrow y = \frac{b^2}{c}, \&$$

$$\frac{z}{a} = \frac{a}{c} \Rightarrow z = \frac{a^2}{c}$$

$$\overset{2}{\mapsto}: A = c\left\{\frac{ab}{c}\right\} = ab$$

$$\overset{3}{\mapsto}: A = \tfrac{1}{2}\left\{\frac{ab}{c} \cdot \frac{b^2}{c} + ab + \frac{ab}{c} \cdot \frac{a^2}{c}\right\} = \frac{ab}{2}\left\{\frac{b^2}{c^2} + 1 + \frac{a^2}{c^2}\right\}$$

$$\overset{4}{\mapsto}: ab \overset{set}{=} \frac{ab}{2}\left\{\frac{b^2}{c^2} + 1 + \frac{a^2}{c^2}\right\} \Rightarrow$$

$$2ab = ab\left\{\frac{b^2}{c^2} + 1 + \frac{a^2}{c^2}\right\} \Rightarrow 2 = \left\{\frac{b^2}{c^2} + 1 + \frac{a^2}{c^2}\right\} \Rightarrow$$

$$1 = \frac{b^2}{c^2} + \frac{a^2}{c^2} \Rightarrow \frac{b^2}{c^2} + \frac{a^2}{c^2} = 1 \Rightarrow$$

$$a^2 + b^2 = c^2 \therefore$$

As the reader can immediately discern, this proof, a **DRIII**, is not visually apparent. Algebra must be used along with the diagram in order to quantify the needed relationships and carry the proof to completion.

Note: This is not a good proof for beginning students—say your average eighth or ninth grader—for two reasons. One, the algebra is somewhat extensive. Two, derived and not intuitively obvious quantities representing various lengths are utilized to formulate the various areas. Thus, the original Pythagorean proof remains superior for introductory purposes.

Our last proof in this section is a four-step **DRV** developed by a college student, Michelle Watkins (1997), which also requires similarity principles to carry the proof to completion. **Figure 2.7** shows our two fundamental, congruent right triangles where a heavy dashed line outlines the second triangle. The lighter dashed line completes a master triangle ΔABC for which we will compute the area two using different methods. *The reader is to verify that each right triangle created by the merger of the congruent right triangles is similar to the original right triangle.*

Step 1 is to compute the length of line segment **x** using similarity principles. The two distinct area calculations in Steps 2 and 3 result from viewing the master triangle as either ΔABC or ΔCBA. Step 4 sets the equality and completes the proof.

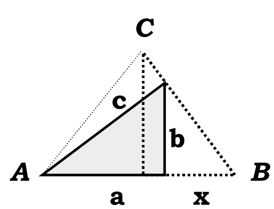


Figure 2.7: Twin Triangle Proof

$$\overset{1}{\mapsto}: \frac{x}{b} = \frac{b}{a} \Rightarrow x = \frac{b^2}{a}$$

$$\overset{2}{\mapsto}: Area(\Delta CBA) = \tfrac{1}{2}(c)(c) = \tfrac{1}{2}c^2$$

$$\overset{3}{\mapsto}: Area(\Delta ABC) = \tfrac{1}{2}(a)\left\{a + \frac{b^2}{a}\right\} \Rightarrow$$

$$Area(\Delta ABC) = \tfrac{1}{2}(a)\left\{\frac{a^2 + b^2}{a}\right\} = \tfrac{1}{2}\{a^2 + b^2\}$$

$$\overset{4}{\mapsto}: \tfrac{1}{2}c^2 \overset{set}{=} \tfrac{1}{2}\{a^2 + b^2\} \Rightarrow$$

$$c^2 = a^2 + b^2 \Rightarrow a^2 + b^2 = c^2 \therefore$$

One of the interesting features of this proof is that even though it is a **DRV**, the five individual areas were not all needed in order to compute the area associated with triangle **ABC** in two different ways. However, some areas were critical in a construction sense in that they allowed for the determination of the critical parameter **x**. Other areas traveled along as excess baggage so to speak. Hence, we could characterize this proof as elegant but a tad inefficient. However, our Twin Triangle Proof did allow for the introduction of the construction principle, a principle that Euclid exploited fully in his great Windmill Proof, the subject of our next section.

2.2) Euclid's Wonderful Windmill

Euclid, along with Archimedes and Apollonius, is considered one of the three great mathematicians of antiquity. All three men were Greeks, and Euclid was the earliest, having lived from approximately 330BCE to 275BCE. Euclid was the first master mathematics teacher and pedagogist. He wrote down in logical systematic fashion all that was known about plane geometry, solid geometry, and number theory during his time. The result is a treatise known as The Elements, a work that consists of 13 books and 465 propositions. Euclid's The Elements is one of most widely read books of all times. Great minds throughout twenty-three centuries (e.g. Bertrand Russell in the 20^{th} century) have been initiated into the power of critical thinking by its wondrous pages.

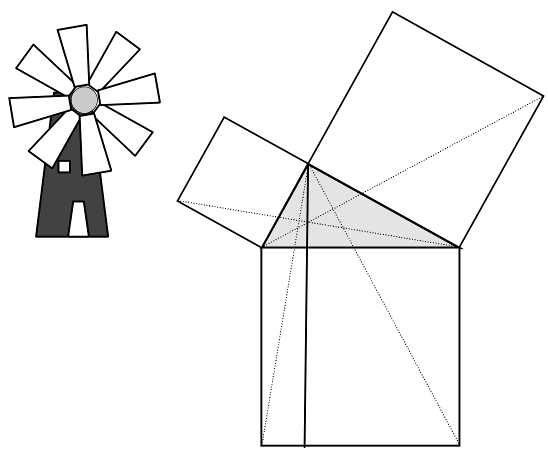


Figure 2.8: Euclid's Windmill without Annotation

Euclid's proof of the Pythagorean Theorem (Book 1, Proposition 47) is commonly known as the Windmill Proof due to the stylized windmill appearance of the associated intricate geometric diagram, **Figure 2.8**.

Note: I think of it as more Art Deco.

There is some uncertainty whether or not Euclid was the actual originator of the Windmill Proof, but that is really of secondary importance. The important thing is that Euclid captured it in all of elegant step-by-step logical elegance via The Elements. The Windmill Proof is best characterized as a construction proof as apposed to a dissection proof. In **Figure 2.8**, the six 'extra' lines—five dashed and one solid—are inserted to generate additional *key* geometric objects within the diagram needed to prove the result. *Not all geometric objects generated by the intersecting lines are needed to actualize the proof*. Hence, to characterize the associated proof as a **DRXX** (the reader is invited to verify this last statement) is a bit unfair.

How and when the Windmill Proof first came into being is a topic for historical speculation. **Figure 2.9** reflects my personal view on how this might have happened.

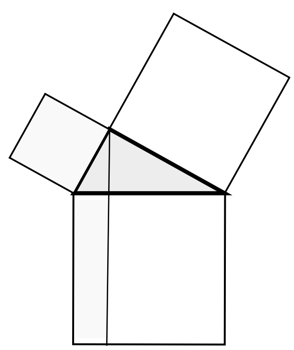


Figure 2.9: Pondering Squares and Rectangles

First, three squares were constructed, perhaps by the old compass and straightedge, from the three sides of our standard right triangle as shown in **Figure 2.9**. The resulting structure was then leveled on the hypotenuse square in a horizontal position. The stroke of intuitive genius was the creation of the additional line emanating from the vertex angle and parallel to the vertical sides of the square. So, with this in view, what exactly was the beholder suppose to behold?

My own intuition tells me that two complimentary observations were made: 1) the area of the lightly-shaded square and rectangle are identical and 2) the area of the non-shaded square and rectangle are identical. Perhaps both observations started out as nothing more than a curious conjecture. However, subsequent measurements for specific cases turned conjecture into conviction and initiated the quest for a general proof. Ancient Greek genius finally inserted (period of time unknown) two additional dashed lines and annotated the resulting diagram as shown in **Figure 2.10**. Euclid's proof follows on the next page.

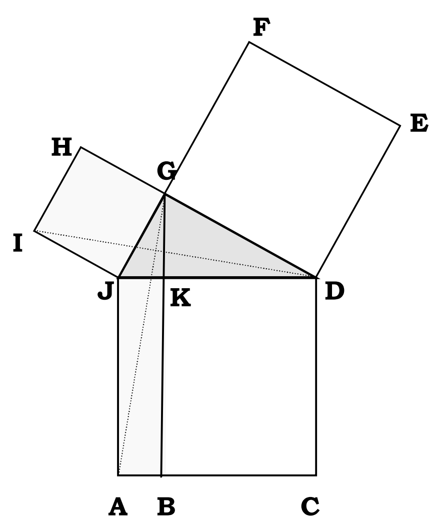


Figure 2.10: Annotated Windmill

First we establish that the two triangles **IJD** and **GJA** are congruent.

$$\overset{1}{\mapsto}: IJ = JG, JD = JA, \& \angle IJD = \angle GJA \Rightarrow$$
$$\Delta IJD \cong \Delta GJA$$

$$\overset{2}{\mapsto}: \Delta IJD \cong \Delta GJA \Rightarrow$$
$$Area(\Delta IJD) = Area(\Delta GJA) \Rightarrow$$
$$2Area(\Delta IJD) = 2Area(\Delta GJA)$$

The next step is to establish that the area of the square **IJGH** is double the area of ΔIJD. This is done by carefully observing the length of the base and associated altitude for each. Equivalently, we do a similar procedure for rectangle **JABK** and ΔGJA.
Thus:

$$\overset{3}{\mapsto}: Area(IJGH) = 2Area(\Delta IJD)$$
$$\overset{4}{\mapsto}: Area(JABK) = 2Area(\Delta GJA).$$
$$\overset{5}{\mapsto}: Area(IJGH) = Area(JABK)$$

The equivalency of the two areas associated with the square **GDEF** and rectangle **BCDK** is established in like fashion (necessitating the drawing of two more dashed lines as previously shown in **Figure 2.8**). With this last result, we have enough information to bring to completion Euclid's magnificent proof.

$$\overset{6}{\mapsto}: Area(IJGH) + Area(GDEF) =$$
$$Area(JABK) + Area(BCDK) \Rightarrow$$
$$Area(IJGH) + Area(GDEF) = Area(ACDJ) \therefore$$

Modern analytic geometry greatly facilitates Euclid's central argument. **Figure 2.11** is a much-simplified windmill with only key dimensional lengths annotated.

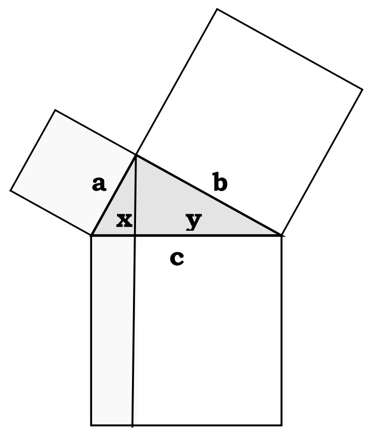


Figure 2.11: Windmill Light

The analytic-geometry proof below rests on the central fact that the two right triangles created by the insertion of the perpendicular bisector are both similar to the original right triangle. First, we establish the equality of the two areas associated with the lightly shaded square and rectangle via the following logic sequence:

$$\overset{1}{\mapsto}: \frac{x}{a} = \frac{a}{c} \Rightarrow x = \frac{a^2}{c}$$

$$\overset{2}{\mapsto}: A_{shadedsquare} = a^2$$

$$\overset{3}{\mapsto}: A_{shadedrect} = \frac{a^2}{c}\{c\} = a^2$$

$$\overset{4}{\mapsto}: A_{shadedsquare} = A_{shadedrect}$$

Likewise, for the non-shaded square and rectangle:

$$\overset{1}{\mapsto}: \frac{y}{b} = \frac{b}{c} \Rightarrow y = \frac{b^2}{c}$$

$$\overset{2}{\mapsto}: A_{unshadedsquare} = b^2$$

$$\overset{3}{\mapsto}: A_{unshadedrect} = \frac{b^2}{c}\{c\} = b^2$$

$$\overset{4}{\mapsto}: A_{unshadedsquare} = A_{unshadedrect}$$

Putting the two pieces together (quite literally), we have:

$$\overset{1}{\mapsto}: A_{bigsquare} = c^2 \Rightarrow$$
$$c^2 = a^2 + b^2 \Rightarrow a^2 + b^2 = c^2 \therefore$$

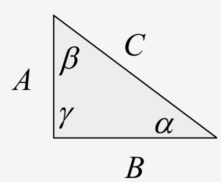
The reader probably has discerned by now that similarity arguments play a key role in many proofs of the Pythagorean Theorem. This is indeed true. In fact, proof by *similarity* can be thought of a major subcategory just like proof by *dissection* or proof by *construction*. Proof by *visualization* is also a major subcategory requiring crystal-clear, additive dissections in order to make the Pythagorean Theorem visually obvious without the help of analytic geometry. Similarity proofs were first exploited in wholesale fashion by Legendre, a Frenchman that had the full power of analytic geometry at his disposal. In Section 2.7, we will further reduce Euclid's Windmill to its primal bare-bones form via similarity as first exploited by Legendre.

Note: More complicated proofs of the Pythagorean Theorem usually are a hybrid of several approaches. The proof just given can be thought of as a combination of construction, dissection, and similarity. Since similarity was the driving element in forming the argument, I would primarily characterize it as a similarity proof. Others may characteristic it as a construction proof since no argument is possible without the insertion of the perpendicular bisector. Nonetheless, creation of a perpendicular bisection creates a dissection essential to the final addition of squares and rectangles! Bottom line: all things act together in concert.

We close this section with a complete restatement of the Pythagorean Theorem as found in Chapter 2, but now with the inclusion of the converse relationship $A^2 + B^2 = C^2 \Rightarrow \gamma = 90^0$. Euclid's subtle proof of the Pythagorean Converse follows (Book 1 of The Elements, Proposition 48).

The Pythagorean Theorem and
Pythagorean Converse

Suppose we have a triangle with side lengths and angles labeled as shown below.



Then: $\alpha + \beta = \gamma \Leftrightarrow A^2 + B^2 = C^2$

Figure 2.12 on the next page shows Euclid's original construction used to prove the Pythagorean Converse. The shaded triangle conforms to the hypothesis where $A^2 + B^2 = C^2$ by design. From the intentional design, one is to show or deduce that $\gamma = 90^0 \Rightarrow \beta + \alpha = \gamma$ in order to prove the converse.

Note: the ancients and even some of my former public-school teachers would have said 'by construction' instead of 'by design.' However, the year is 2008, not 1958, and the word design seems to be a superior conveyor of the intended meaning.

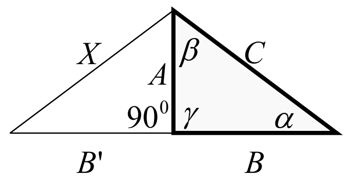


Figure 2.12: Euclid's Converse Diagram

Euclid's first step was to construct a line segment of length B' where $B' = B$. Then, this line segment was joined as shown to the shaded triangle in such a fashion that the corner angle that mirrors γ was indeed a right angle of 90^0 measure—again by design! Euclid then added a second line of unknown length X in order to complete a companion triangle with common vertical side as shown in **Figure 2.12**. Euclid finally used a formal verbally-descriptive logic stream quite similar to the annotated algebraic logic stream below in order to complete his proof.

Algebraic Logic	Verbal Annotation
$\overset{1}{\mapsto}: A^2 + B^2 = C^2$	1] By hypothesis
$\overset{2}{\mapsto}: B = B'$	2] By design
$\overset{3}{\mapsto}: \angle B'A = 90^0$	3] By design
$\overset{4}{\mapsto}: A^2 + (B')^2 = X^2$	4] Pythagorean Theorem
$\overset{5}{\mapsto}: X^2 = A^2 + (B')^2 \Rightarrow$ $X^2 = A^2 + B^2 \Rightarrow$ $X^2 = C^2 \Rightarrow$ $X = C$	5] Properties of algebraic equality

$\overset{6}{\mapsto}: \Delta ABC \cong \Delta AB'X$ — 6] The three corresponding sides are equal in length (**SSS**)

$\overset{7}{\mapsto}: \gamma = 90^0$ — 7] The triangles ΔABC and $\Delta AB'X$ are congruent

$\overset{8}{\mapsto}: \alpha + \beta + \gamma = 180^0 \Rightarrow$

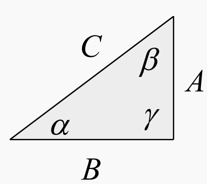
$\alpha + \beta + 90^0 = 180^0 \Rightarrow$

$\alpha + \beta = 90^0 \Rightarrow$

$\alpha + \beta = \gamma \therefore$

8] Properties of algebraic equality

We close this section by simply admiring the simple and profound algebraic symmetry of the Pythagorean Theorem and its converse as 'chiseled' below



$$\alpha + \beta = \gamma \Leftrightarrow A^2 + B^2 = C^2$$

2.3) Liu Hui Packs the Squares

Liu Hui was a Chinese philosopher and mathematician that lived in the third century ACE. By that time, the great mathematical ideas of the Greeks would have traveled the Silk Road to China and visa-versa, with the cross-fertilization of two magnificent cultures enhancing the further global development of mathematics. As just described, two pieces of evidence strongly suggest that indeed this was the case.

Figure 2.13 is Liu Hui's exquisite diagram associated with his *visual proof* of the Pythagorean Theorem.

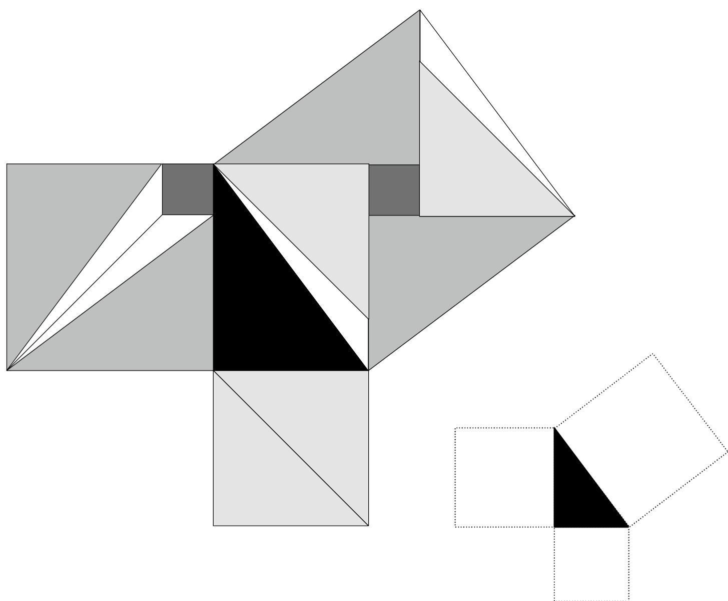


Figure 2.13: Liu Hui's Diagram with Template

In it, one clearly sees the Greek influence of Pythagoras and Euclid. However, one also sees much, much more: a more intricate and clever visual demonstration of the 'Pythagorean Proposition' than those previously accomplished.

Note: Elisha Loomis, a fellow Ohioan, whom we shall meet in Section 2.10, first used the expression 'Pythagorean Proposition' over a century ago.

Is Liu Hui's diagram best characterized as a dissection (a humongous **DRXIIII** not counting the black right triangle) or a construction?

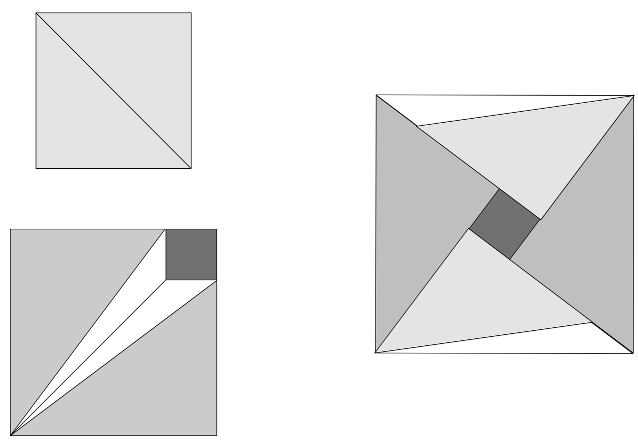


Figure 2.14: Packing Two Squares into One

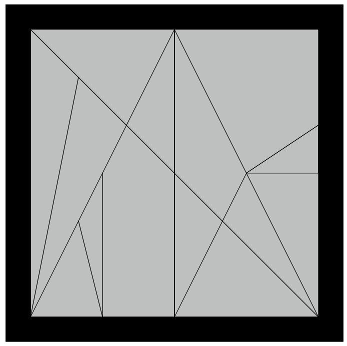
We will say neither although the diagram has elements of both. Liu Hui's proof is best characterized as a *packing proof* in that the two smaller squares have been dissected in such a fashion as to allow them to pack themselves into the larger square, **Figure 2.14**. Some years ago when our youngest son was still living at home, we bought a Game Boy™ and gave it to him as a Christmas present. In time, I took a liking to it due to the nifty puzzle games.

One of my favorites was Boxel™, a game where the player had to pack boxes into a variety of convoluted warehouse configurations. In a sense, I believe this is precisely what Liu Hui did: he perceived the Pythagorean Proposition as a packing problem and succeeded to solve the problem by the masterful dismemberment and reassembly as shown above. *In the spirit of Liu Hui, actual step-by-step confirmation of the 'packing of the pieces' is left to the reader as a challenging visual exercise.*

Note: One could say that Euclid succeeded in packing two squares into two rectangles, the sum of which equaled the square formed on the hypotenuse.

So what might have been the origin of Liu Hui's packing idea? Why did Liu Hui use such odd-shaped pieces, especially the two obtuse, scalene triangles? Finally, why did Liu Hui dissect the three squares into exactly fourteen pieces as opposed to twenty? Archimedes (287BCE-212BCE), a Greek and one of the three greatest mathematicians of all time—Isaac Newton and Karl Gauss being the other two—may provide some possible answers.

Archimedes is commonly credited (rightly or wrongly) with a puzzle known by two names, the Archimedes' Square or the Stomachion, **Figure 2.15** on the next page. In the Stomachion, a 12 by 12 square grid is expertly dissected into 14 polygonal playing pieces where each piece has an integral area. Each of the fourteen pieces is labeled with two numbers. The first is the number of the piece and the second is the associated area. Two views of the Stomachion are provided in **Figure 2.15**, an 'artist's concept' followed by an 'engineering drawing'. I would like to think that the Stomachion somehow played a key role in Liu Hui's development of his magnificent packing solution to the Pythagorean Proposition. Archimedes' puzzle could have traveled the Silk Road to China and eventually found its way into the hands of another ancient and great out-of-the-box thinker!



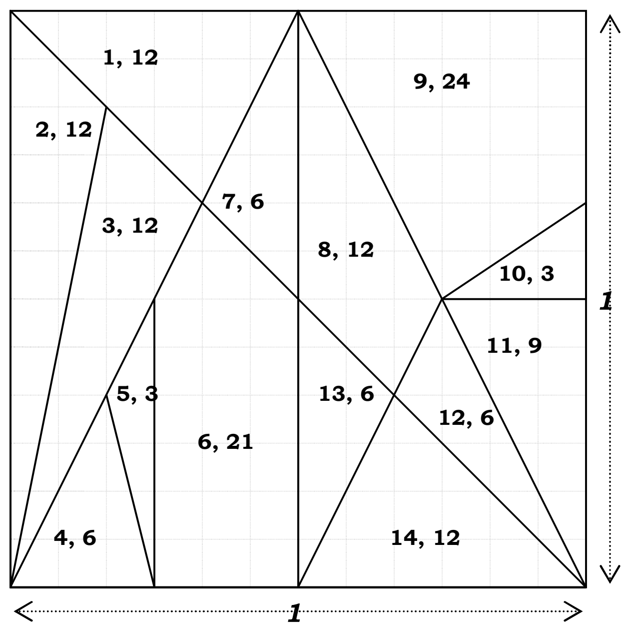


Figure 2.15: The Stomachion Created by Archimedes

2.4) Kurrah Transforms the Bride's Chair

Our youngest son loved Transformers™ as a child, and our oldest son was somewhat fond of them too. For those of you who may not remember, a transformer is a mechanical toy that can take on a variety of shapes—e.g. from truck to robot to plane to boat—depending how you twist and turn the various appendages. The idea of transforming shapes into shapes is not new, even though the 1970s brought renewed interest in the form of highly marketable toys for the children of Baby Boomers. Even today, a new league of Generation X parents are digging in their pockets and shelling out some hefty prices for those irresistible Transformers™.

Thabit ibn Kurrah (836-901), a Turkish-born mathematician and astronomer, lived in Baghdad during Islam's Era of Enlightenment paralleling Europe's Dark Ages. Kurrah (also Qurra or Qorra) developed a clever and original proof of the Pythagorean Theorem along with a non-right triangle extension of the same (Section 3.6) Kurrah's proof has been traditionally classified as a dissection proof. Then again, Kurrah's proof can be equally classified as a *transformer proof*. Let's have a look.

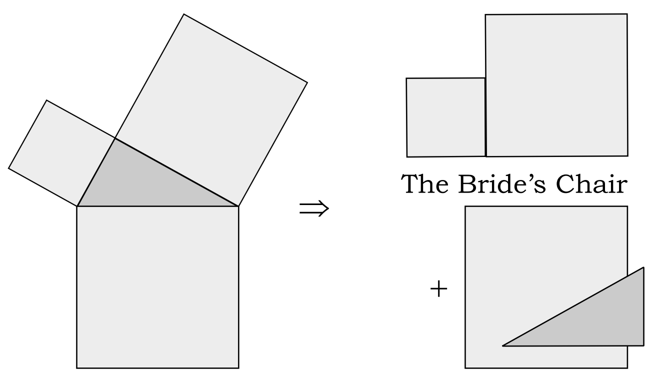


Figure 2.16: Kurrah Creates the Bride's Chair

Figure 2.16 shows Kurrah's creation of the Bride's Chair. The process is rather simple, but shows Kurrah's intimate familiarity with our fundamental Pythagorean geometric structure on the left. Four pieces comprise the basic structure and these are pulled apart and rearranged as depicted. The key rearrangement is the one on the top right that reassembles the two smaller squares into a new configuration known as the bride's chair. Where the name 'Bride's Chair' originated is a matter for speculation; personally, I think the chair-like structure looks more like a Lazy Boy™.

Now what? Kurrah had a packing problem—two little squares to be packed into one big square—which he cleverly solved by the following dissection and subsequent transformation. **Figure 2.17** pictorially captures Kurrah's dilemma, and his key dissection that allowed the transformation to proceed.

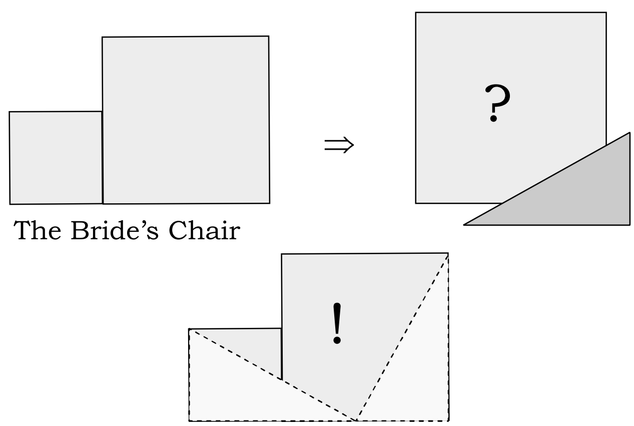


Figure 2.17: Packing the Bride's Chair into The Big Square

What Kurrah did was to replicate the shaded triangle and use it to frame two cutouts on the Bride's Chair as shown. **Figure 2.18** is 'Operation Transformation' showing Kurrah's rotational sequence that leads to a successful packing of the large square.

Note: as is the occasional custom in this volume, the reader is asked to supply all dimensional details knowing that the diagram is dimensionally correct. I am convinced that Kurrah himself would have demanded the same.

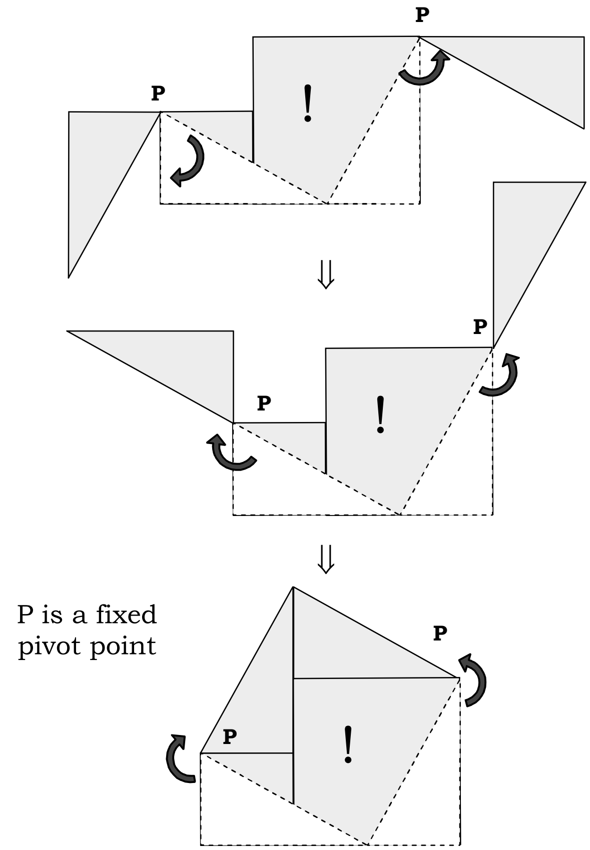


Figure 2.18: Kurrah's Operation Transformation

As **Figure 2.18** clearly illustrates, Kurrah took a cleverly dissected Bride's Chair and masterfully packed it into the big square though a sequence of rotations akin to those employed by the toy Transformers™ of today—a demonstration of pure genius!

The Bride's Chair and Kurrah's subsequent dissection has long been the source for a little puzzle that has found its way into American stores for at least forty years. I personally dub this puzzle 'The Devil's Teeth', **Figure 2.19**. As one can see, it nothing more than the Bride's Chair cut into four pieces, two of which are identical right triangles. The two remaining pieces are arbitrarily cut from the residual of the Bride's Chair. **Figure 2.19** depicts two distinct versions of 'The Devil's Teeth'.

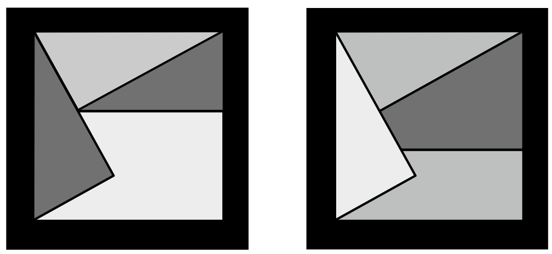


Figure 2.19: 'The Devil's Teeth'

The name 'Devil's Teeth' is obvious: the puzzle is a devilish one to reassemble. If one adds a little mysticism about the significance of the number four, you probably got a winner on your hands. *In closing, I can imagine Paul Harvey doing a radio spot focusing on Kurrah, the Bride's Chair, and the thousand-year-old Transformers™ proof. After the commercial break, he describes 'The Devil's Teeth' and the successful marketer who started this business out of a garage. "And that is the rest of the story. Good day."*

Note: I personally remember this puzzle from the early 1960s.

2.5) Bhaskara Unleashes the Power of Algebra

We first met Bhaskara in Chapter 2. He was the 12th century (circa 1115 to 1185) Indian mathematician who drew the top diagram shown in **Figure 2.20** and simply said, "Behold!", completing his proof of the Pythagorean Theorem. However, legend has a way of altering details and fish stories often times get bigger. Today, what is commonly ascribed to Bhaskara's "Behold!" is nothing more than the non-annotated square donut in the lower diagram. *I, for one, have a very hard time beholding exactly what I am suppose to behold when viewing the non-annotated square donut*. Appealing to Paul Harvey's famous radio format a second time, perhaps there is more to this story. There is. Bhaskara had at his disposal a well-developed algebraic language, a language that allowed him to capture precisely via analytic geometry those truths that descriptive geometry alone could not easily convey.

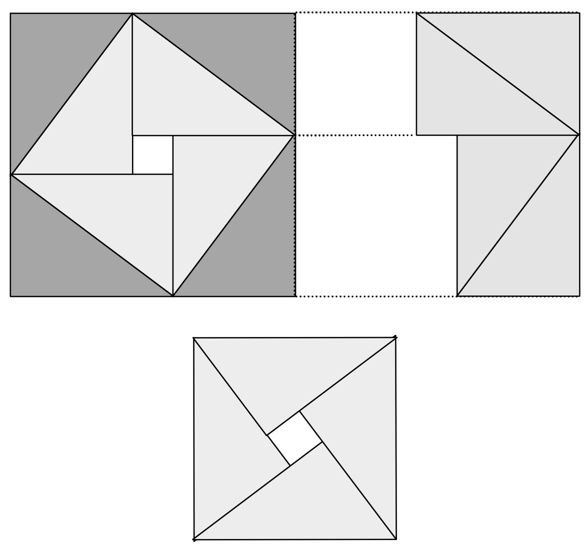


Figure 2.20: Truth Versus Legend

What Bhaskara most likely did as an accomplished algebraist was to annotate the lower figure as shown again in **Figure 2.21**. The former proof easily follows in a few steps using analytic geometry. Finally, we are ready for the famous "Behold! as Bhaskara's magnificent **DRV** Pythagorean proof unfolds before our eyes.

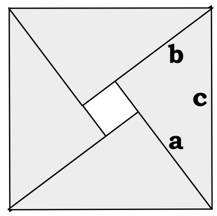
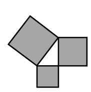


Figure 2.21: Bhaskara's Real Power

$$\overset{1}{\mapsto}: A_{bigsquare} = c^2 \ \&$$
$$A_{bigsquare} = A_{littlesquare} + 4 \cdot (A_{onetriangle}) \Rightarrow$$
$$A_{bigsquare} = (a-b)^2 + 4(\tfrac{1}{2}ab)$$
$$\overset{2}{\mapsto}: c^2 \overset{set}{=} (a-b)^2 + 4(\tfrac{1}{2}ab) \Rightarrow$$
$$c^2 = a^2 - 2ab + b^2 + 2ab \Rightarrow$$
$$c^2 = a^2 + b^2 \Rightarrow a^2 + b^2 = c^2 \ \therefore$$

Bhaskara's proof is minimal in that the large square has the smallest possible linear dimension, namely **c**. It also utilizes the three fundamental dimensions—**a**, **b**, & **c**—as they naturally occur with no scaling or proportioning. The tricky part is size of the donut hole, which Bhaskara's use of analytic geometry easily surmounts. Thus, only one word remains to describe this historic first—behold!



2.6) Leonardo da Vinci's Magnificent Symmetry

Leonardo da Vinci (1452-1519) was born in Anchiano, Italy. In his 67 years, Leonardo became an accomplished painter, architect, designer, engineer, and mathematician. If alive today, the whole world would recognize Leonardo as 'world class' in all the aforementioned fields. It would be as if Stephen Hawking and Stephen Spielberg were both joined into one person. For this reason, Leonardo da Vinci is properly characterized as the first and greatest Renaissance man. The world has not seen his broad-ranging intellectual equivalent since! Thus, it should come as no surprise that Leonardo da Vinci, the eclectic master of many disciplines, would have thoroughly studied and concocted an independent proof of the Pythagorean Theorem.

Figure 2.22 is the diagram that Leonardo used to demonstrate his proof of the Pythagorean Proposition. The added dotted lines are used to show that the right angle of the fundamental right triangle is bisected by the solid line joining the two opposite corners of the large dotted square enclosing the lower half of the diagram. Alternately, the two dotted circles can also be used to show the same (reader exercise).

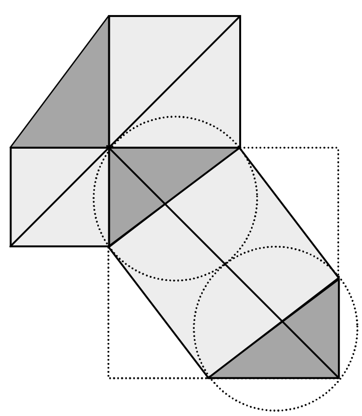


Figure 2.22: Leonardo da Vinci's Symmetry Diagram

Figure 2.23 is the six-step sequence that visually demonstrates Leonardo's proof. The critical step is Step 5 where the two figures are acknowledged by the observer to be equivalent in area. Step 6 immediately follows.

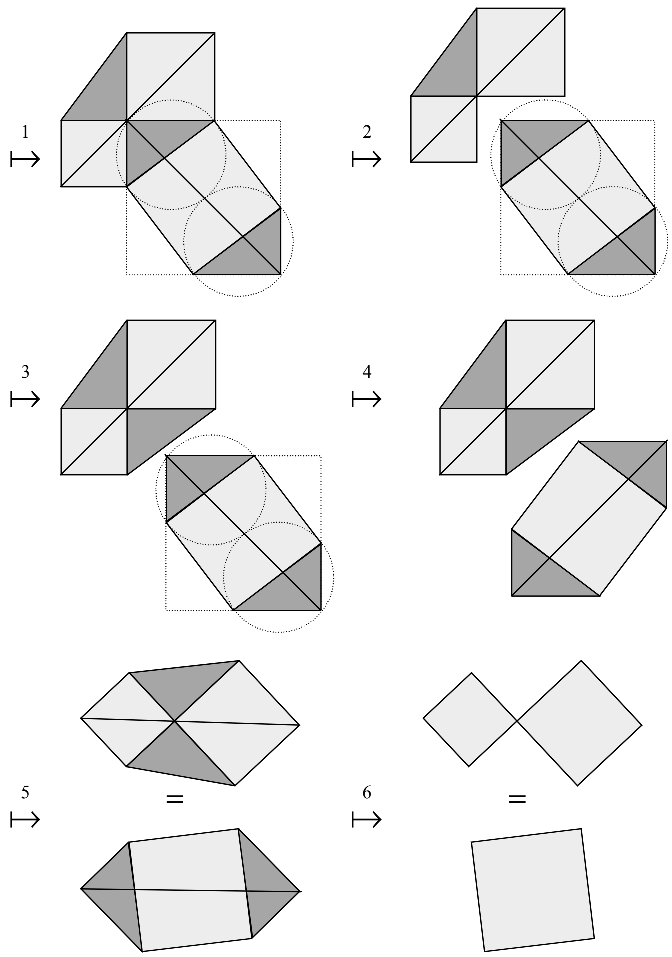


Figure 2.23: Da Vinci's Proof in Sequence

In **Figure 2.24**, we enlarge Step 5 and annotate the critical internal equalities. **Figure 2.24** also depicts the subtle rotational symmetry between the two figures by labeling the pivot point **P** for an out-of-plane rotation where the lower half of the top diagram is rotated 180^0 in order to match the bottom diagram. The reader is to supply the supporting rationale. While doing so, take time to reflect on the subtle and brilliant genius of the Renaissance master—Leonardo da Vinci!

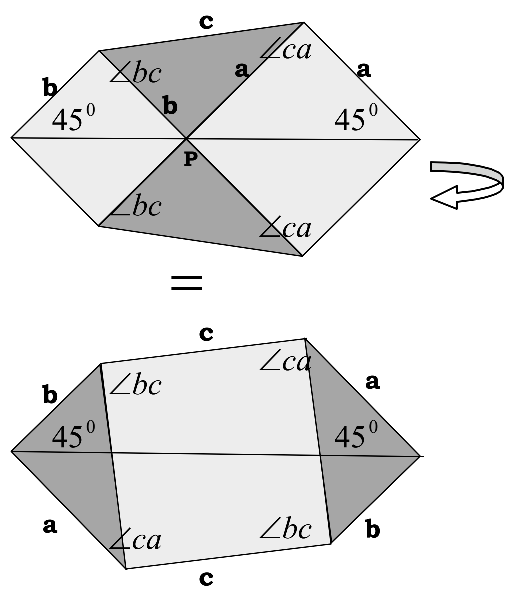


Figure 2.24: Subtle Rotational Symmetry

2.7) Legendre Exploits Embedded Similarity

Adrian Marie Legendre was a well-known French mathematician born at Toulouse in 1752. He died at Paris in 1833. Along with Lagrange and Laplace, Legendre can be considered on of the three fathers of modern analytic geometry, a geometry that incorporates all the inherent power of both algebra and calculus. With much of his life's work devoted to the new analytic geometry, it should come as no surprise that Legendre should be credited with a powerful, simple and thoroughly modern—for the time—new proof of the Pythagorean Proposition. Legendre's proof starts with the Windmill Light (**Figure 2.11**). Legendre then pared it down to the diagram shown in **Figure 2.25**.

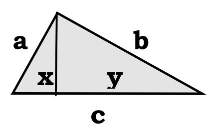


Figure 2.25: Legendre's Diagram

He then demonstrated that the two *right* triangles formed by dropping a perpendicular from the vertex angle to the hypotenuse are both similar to the master triangle. Armed with this knowledge, a little algebra finished the job.

$$\overset{1}{\mapsto}: \frac{x}{a} = \frac{a}{c} \Rightarrow x = \frac{a^2}{c}$$

$$\overset{2}{\mapsto}: \frac{y}{b} = \frac{b}{c} \Rightarrow y = \frac{b^2}{c}$$

$$\overset{3}{\mapsto}: x + y = c \Rightarrow$$

$$\frac{a^2}{c} + \frac{b^2}{c} = c \Rightarrow a^2 + b^2 = c^2 \therefore$$

Notice that this is the first proof in our historical sequence lacking an obvious visual component.

But, this is precisely the nature of algebra and analytic geometry where abstract ideas are more precisely (and abstractly) conveyed than by descriptive (visual) geometry alone. The downside is that visual intuition plays a minimal role as similarity arguments produce the result via a few algebraic pen strokes. Thus, this is not a suitable beginner's proof.

Similarity proofs have been presented throughout this chapter, but Legendre's is historically the absolute minimum in terms of both geometric augmentation (the drawing of additional construction lines, etc.) and algebraic terseness. Thus, it is included as a major milestone in our survey of Pythagorean proofs. To summarize, Legendre's proof can be characterized as an embedded similarity proof where two smaller triangles are created by the dropping of just one perpendicular from the vertex of the master triangle. All three triangles—master and the two created—are mutually similar. Algebra and similarity principles complete the argument in a masterful and modern way.

Note: as a dissection proof, Legendre's proof could be characterized as a ***DRII****, but the visual dissection is useless without the powerful help of algebra, essential to the completion of the argument.*

Not all similarity proofs rely on complicated ratios such as $x = a^2 / c$ to evaluate constructed linear dimensions in terms of the three primary quantities **a**, **b**, & **c**. **Figure 2.26** is the diagram recently used (2002) by J. Barry Sutton to prove the Pythagorean Proposition using similarity principles with minimally altered primary quantities.

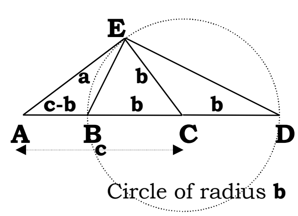


Figure 2.26 Barry Sutton's Diagram

We end Section 2.7 by presenting Barry's proof in step-by-step fashion so that the reader will get a sense of what formal geometric logic streams look like, as they are found in modern geometry textbooks at the high school or college level.

$\overset{1}{\mapsto}$: Construct right triangle ΔAEC with sides **a**, **b**, **c**.	
$\overset{2}{\mapsto}$: Construct circle **Cb** centered at **C** with radius **b**.	
$\overset{3}{\mapsto}$: Construct triangle ΔBED with hypotenuse **2b**.	
$\overset{4}{\mapsto}: \Delta BED = right\Delta$	By inscribed triangle theorem since the hypotenuse for ΔBED equals and exactly overlays the diameter for **Cb**
$\overset{5}{\mapsto}: \angle AEB = \angle CED$	The same common angle $\angle BEC$ is subtracted from the right angles $\angle AEC$ and $\angle BED$
$\overset{6}{\mapsto}: \angle CED = \angle CDE$	The triangle ΔCED is isosceles.
$\overset{7}{\mapsto}: \angle AEB = \angle CDE$	Transitivity of equality
$\overset{8}{\mapsto}: \Delta AEB \approx \Delta AED$	The angle $\angle DAE$ is common to both triangles and $\angle AEB = \angle CDE$. Hence the third angle is equal and similarity is assured by **AAA**.

With the critical geometric similarity firmly established by traditional logic, Barry finishes his proof with an algebraic coup-de-grace that is typical of the modern approach!

$$\overset{9}{\mapsto}: \frac{AE}{AD} = \frac{AB}{AE} \Rightarrow \frac{a}{c+b} = \frac{c-b}{a} \Rightarrow$$

$$a^2 = (c+b)(c-b) \Rightarrow$$ Equality of similar ratios

$$a^2 = c^2 - b^2 \Rightarrow a^2 + b^2 = c^2 \therefore$$

2.8) Henry Perigal's Tombstone

Henry Perigal was an amateur mathematician and astronomer who spent most of his long life (1801-1898) near London, England. Perigal was an accountant by trade, but stargazing and mathematics was his passion. He was a Fellow of the Royal Astronomical Society and treasurer of the Royal Meteorological Society. Found of geometric dissections, Perigal developed a novel proof of the Pythagorean Theorem in 1830 based on a rather intricate dissection, one not as transparent to the casual observer when compared to some of the proofs from antiquity. Henry must have considered his proof of the Pythagorean Theorem to be the crowning achievement of his life, for the diagram is chiseled on his tombstone, **Figure 2.27**. Notice the clever use of key letters found in his name: **H**, **P**, **R**, **G**, and **L**.

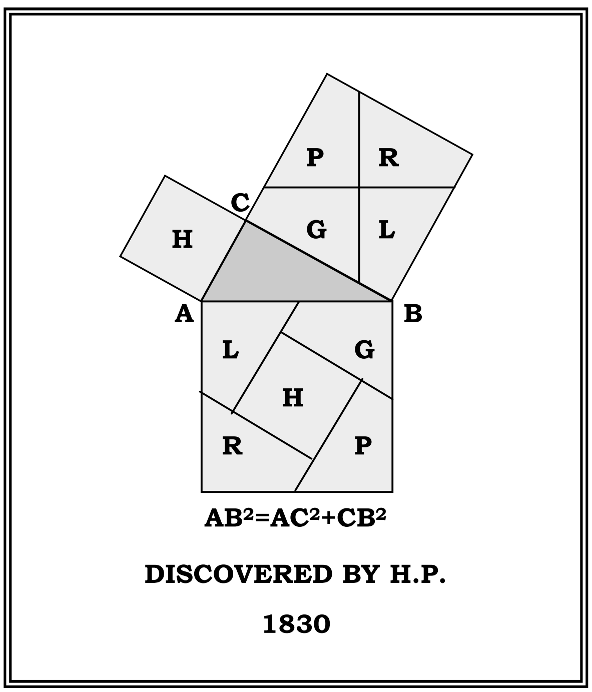


Figure 2.27: Diagram on Henry Perigal's Tombstone

In **Figure 2.28,** we update the annotations used by Henry and provide some key geometric information on his overall construction.

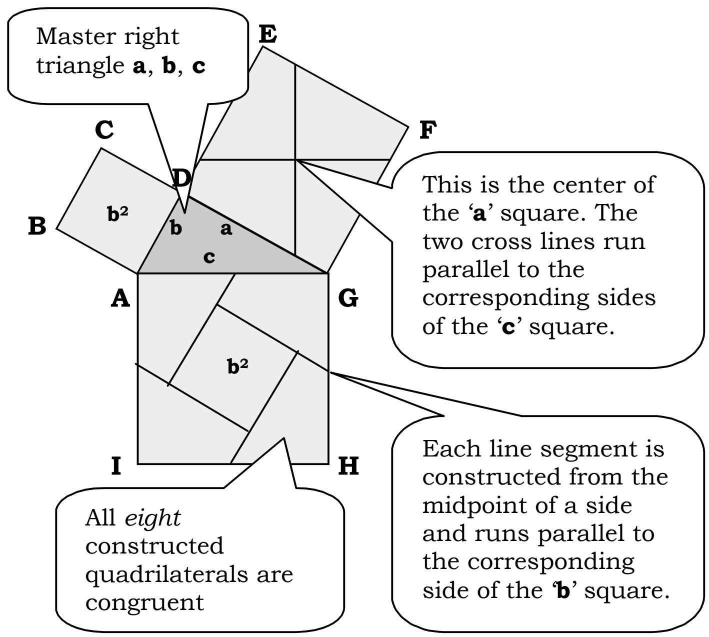


Figure 2.28: Annotated Perigal Diagram

We are going to leave the proof to the reader as a challenge. Central to the Perigal argument is the fact that all eight of the constructed quadrilaterals are congruent. This immediately leads to fact that the middle square embedded in the '**c**' square is identical to the '**a**' square from which $a^2 + b^2 = c^2$ can be established.

Perigal's proof has since been cited as one of the most ingenious examples of a proof associated with a phenomenon that modern mathematicians call a Pythagorean Tiling.

In the century following Perigal, both Pythagorean Tiling and tiling phenomena in general were extensively studied by mathematicians resulting in two fascinating discoveries:

1] Pythagorean Tiling guaranteed that the existence of countless dissection proofs of the Pythagorean Theorem.
2] Many previous dissection proofs were in actuality simple variants of each, inescapably linked by Pythagorean Tiling.

Gone forever was the keeping count of the number of proofs of the Pythagorean Theorem! For classical dissections, the continuing quest for new proofs became akin to writing the numbers from $1{,}234{,}567$ to $1{,}334{,}567$. People started to ask, what is the point other than garnering a potential entry in the Guinness Book of World Records? As we continue our Pythagorean journey, keep in mind Henry Perigal, for it was he (albeit unknowingly) that opened the door to this more general way of thinking.

Note: Elisha Loomis whom we shall meet in Section 2.10, published a book in 1927 entitled The Pythagorean Proposition, in which he details over 350 original proofs of the Pythagorean Theorem.

We are now going to examine Perigal's novel proof and quadrilateral filling using the modern methods associated with Pythagorean Tiling, an example of which is shown in **Figure 2.29** on the next page. From **Figure 2.29**, we see that three items comprise a Pythagorean Tiling where each item is generated, either directly or indirectly, from the master right triangle.

1) The Bride's Chair, which serves as a basic tessellation unit when repeatedly drawn.
2) The master right triangle itself, which serves as an 'anchor-point' somewhere within the tessellation pattern.
3) A square cutting grid, aligned as shown with the triangular anchor point. The length of each line segment within the grid equals the length of the hypotenuse for the master triangle. Therefore, the area of each square hole equals the area of the square formed on the hypotenuse.

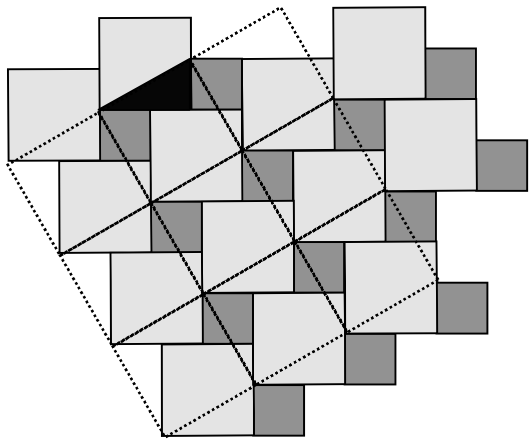


Figure 2.29: An Example of Pythagorean Tiling

As **Figure 2.29** illustrates, the square cutting grid immediately visualizes the cuts needed in order to dissect and pack the two smaller squares into the hypotenuse square, given a particular placement of the black triangle. **Figure 2.30** shows four different, arbitrary placements of the anchor point that ultimately lead to four dissections and four proofs once the cutting grid is properly placed. Bottom line: a different placement means a different proof!

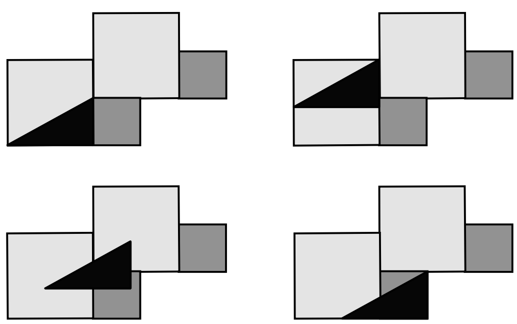


Figure 2.30: Four Arbitrary Placements of the Anchor Point

Returning to Henry Perigal, **Figure 2.31** shows the anchor placement and associated Pythagorean Tiling needed in order to verify his 1830 dissection. Notice how the viability of Henry Perigal's proof and novel quadrilateral is rendered immediately apparent by the grid placement. As we say in 2008, slick!

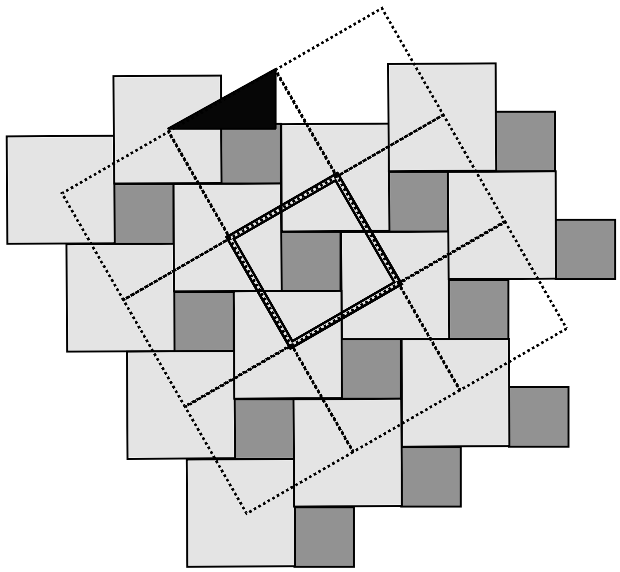


Figure 2.31: Exposing Henry's Quadrilaterals

With Pythagorean Tiling, we can have a thousand different placements leading to a thousand different proofs. Should we try for a million? Not a problem! Even old Pythagoras and Euclid might have been impressed.

2.9) President Garfield's Ingenious Trapezoid

The Ohioan James A. Garfield (1831-1881) was the 20th president of the United States. Tragically, Garfield's first term in office was cut short by an assassin's bullet: inaugurated on 4 March 1881, shot on 2 July 1881, and died of complications on 19 Sept 1881. Garfield came from modest Midwestern roots. However, per hard work he was able to save enough extra money in order to attend William's College in Massachusetts. He graduated with honors in 1856 with a degree in classical studies. After a meteoric stint as a classics professor and (within two years) President of Hiram College in Ohio, Garfield was elected to the Ohio Senate in 1859 as a Republican. He fought in the early years of the Civil War and in 1862 obtained the rank of Brigadier General at age 31 (achieving a final rank of Major General in 1864). However, Lincoln had other plans for the bright young Garfield and urged him to run for the U.S. Congress. Garfield did just that and served from 1862 to 1880 as a Republican Congressman from Ohio, eventually rising to leading House Republican.

While serving in the U.S. Congress, Garfield fabricated one of the most amazing and simplistic proofs of the Pythagorean Theorem ever devised—a dissection proof that looks back to the original diagram attributed to Pythagoras himself yet reduces the number of playing pieces from five to three.

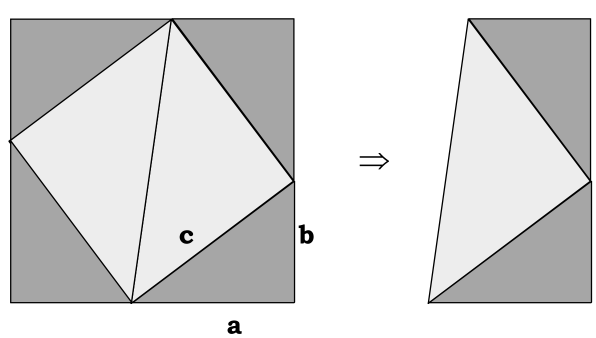


Figure 2.32: President Garfield's Trapezoid

Figure 2.32 is President Garfield's Trapezoid diagram in upright position with its origin clearly linked to **Figure 2.3**. Recall that the area of a trapezoid, in particular the area of the trapezoid in **Figure 2.32**, is given by the formula:

$$A_{Trap} = \left\{\tfrac{1}{2}(a+b)\right\}\left\{a+b\right\}.$$

Armed with this information, Garfield completes his proof with a minimum of algebraic pen strokes as follows.

$$\overset{1}{\mapsto}: A_{Trap} = \left\{\tfrac{1}{2}(a+b)\right\}\left\{a+b\right\}$$

$$\overset{2}{\mapsto}: A_{Trap} = \tfrac{1}{2}ab + \tfrac{1}{2}c^2 + \tfrac{1}{2}ab$$

$$\begin{array}{l} \overset{3}{\mapsto}: \left\{\tfrac{1}{2}(a+b)\right\}\left\{a+b\right\} \overset{set}{=} \tfrac{1}{2}ab + \tfrac{1}{2}c^2 + \tfrac{1}{2}ab \Rightarrow \\ \tfrac{1}{2}\left\{a^2 + 2ab + b^2\right\} = \tfrac{1}{2}ab + \tfrac{1}{2}c^2 + \tfrac{1}{2}ab \Rightarrow \\ \tfrac{1}{2}\left\{a^2 + 2ab + b^2\right\} = \tfrac{1}{2}ab + \tfrac{1}{2}c^2 + \tfrac{1}{2}ab \Rightarrow \\ \tfrac{1}{2}a^2 + ab + \tfrac{1}{2}b^2 = ab + \tfrac{1}{2}c^2 \Rightarrow \\ \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 = \tfrac{1}{2}c^2 \Rightarrow a^2 + b^2 = c^2 \therefore \end{array}.$$

Note: President Garfield actually published his proof in the 1876 edition of the Journal of Education, Volume 3, Issue 161, where the trapezoid is shown lying on its right side.

It does not get any simpler than this! Garfield's proof is a magnificent **DRIII** where all three fundamental quantities **a**, **b**, **c** are used in their natural and fundamental sense. An extraordinary thing is that the proof was not discovered sooner considering the ancient origins of Garfield's trapezoid. Isaac Newton, the co-inventor of calculus, once said. "If I have seen further, it has been by standing on the shoulders of giants." I am sure President Garfield, a giant in his own right, would concur. Lastly, speaking of agreement, Garfield did have this to say about his extraordinary and simple proof of the Pythagorean Proposition, "This is one thing upon which Republicans and Democrats should both agree."

2.10] Ohio and the Elusive Calculus Proof

Elisha Loomis (1852-1940), was a Professor of Mathematics, active Mason, and contemporary of President Garfield. Loomis taught at a number of Ohio colleges and high schools, finally retiring as mathematics department head for Cleveland West High School in 1923. In 1927, Loomis published a still-actively-cited book entitled The Pythagorean Proposition, a compendium of over 250 proofs of the Pythagorean Theorem—increased to 365 proofs in later editions. The Pythagorean Proposition was reissued in 1940 and finally reprinted by the National Council of Teachers of Mathematics in 1968, 2nd printing 1972, as part of its "Classics in Mathematics Education" Series.

Per the Pythagorean Proposition, Loomis is credited with the following statement; *there can be no proof of the Pythagorean Theorem using either the methods of trigonometry or calculus*. Even today, this statement remains largely unchallenged as it is still found with source citation on at least two academic-style websites[1]. For example, Jim Loy states on his website, "The book The Pythagorean Proposition, by Elisha Scott Loomis, is a fairly amazing book. It contains 256 proofs of the Pythagorean Theorem. It shows that you can devise an infinite number of algebraic proofs, like the first proof above. It shows that you can devise an infinite number of geometric proofs, like Euclid's proof. And it shows that there can be no proof using trigonometry, analytic geometry, or calculus. The book is out of print, by the way."

That the Pythagorean Theorem is not provable using the methods of trigonometry is obvious since trigonometric relationships have their origin in a presupposed Pythagorean right-triangle condition. Hence, any proof by trigonometry would be a circular proof and logically invalid. However, calculus is a different matter.

[1] *See Math Forum@Drexel, http://mathforum.org/library/drmath/view/6259.html ; Jim Loy website, http://www.jimloy.com/geometry/pythag.htm .*

Even though the Cartesian coordinate finds its way into many calculus problems, this backdrop is not necessary in order for calculus to function since the primary purpose of a Cartesian coordinate system is to enhance our visualization capability with respect to functional and other algebraic relationships. In the same regard, calculus most definitely does not require a metric of distance—as defined by the Distance Formula, another Pythagorean derivate—in order to function. There are many ways for one to metricize Euclidean n-space that will lead to the establishment of rigorous limit and continuity theorems. **Table 2.2** lists the Pythagorean metric and two alternatives. Reference 19 presents a complete and rigorous development of the differential calculus for one and two independent variables using the rectangular metric depicted in **Table 2.2**.

METRIC	**SET DEFINITION**	**SHAPE**
Pythagorean	$\sqrt{(x-x_0)^2+(y-y_0)^2} < \varepsilon$	Circle
Taxi Cab	$\lvert x-x_0 \rvert + \lvert y-y_0 \rvert < \varepsilon$	Diamond
Rectangular	$\lvert x-x_0 \rvert < \varepsilon$ and $\lvert y-y_0 \rvert < \varepsilon$	Square

Table 2.2 Three Euclidean Metrics

Lastly, the derivative concept—albeit enhanced via the geometric concept of slope introduced with a touch of metrics—is actually a much broader notion than instantaneous "rise over run". So what mathematical principle may have prompted Elisha Loomis, our early 20th century Ohioan, to discount the methods of calculus as a viable means for proving the Pythagorean Theorem? Only that calculus requires geometry as a substrate. The implicit and untrue assumption is that all reality-based geometry is Pythagorean. For a realty-based geometric counterexample, the reader is encouraged to examine Taxicab Geometry: an Adventure in Non-Euclidean Geometry by Eugene Krauss, (Reference 20).

Whatever the original intent or implication, the Pythagorean Proposition has most definitely discouraged the quest for calculus-based proofs of the Pythagorean Theorem, for they are rarely found or even mentioned on the worldwide web. This perplexing and fundamental void in elementary mathematics quickly became a personal challenge to search for a new calculus-based proof of the Pythagorean Theorem. Calculus excels in its power to analyze changing processes incorporating one or more independent variables. Thus, one would think that there ought to be something of value in Isaac Newton and Gottfried Leibniz's brainchild—hailed by many as the greatest achievement of Western science and certainly equal to the Pythagorean brainchild—that would allow for an independent metrics-free investigation of the Pythagorean Proposition.

Note: I personally remember a copy of the Pythagorean Proposition—no doubt, the 1940 edition—sitting on my father's bookshelf while yet a high-school student, Class of 1965.

Initial thoughts/questions were twofold. Could calculus be used to analyze a general triangle as it dynamically changed into a right triangle? Furthermore, could calculus be used to analyze the relationship amongst the squares of the three sides A^2, B^2, C^2 throughout the process and establish the sweet spot of equality $A^2 + B^2 = C^2$—the Pythagorean Theorem? Being a lifelong Ohioan from the Greater Dayton area personally historicized this quest in that I was well aware of the significant contributions Ohioans have made to technical progress in a variety of fields. By the end of 2004, a viable approach seemed to be in hand, as the inherent power of calculus was unleashed on several ancient geometric structures dating back to the time of Pythagoras himself.

Figure 2.33, Carolyn's Cauliflower (so named in honor of my wife who suggested that the geometric structure looked like a head of cauliflower) is the geometric anchor point for a calculus-based proof of the Pythagorean Theorem.

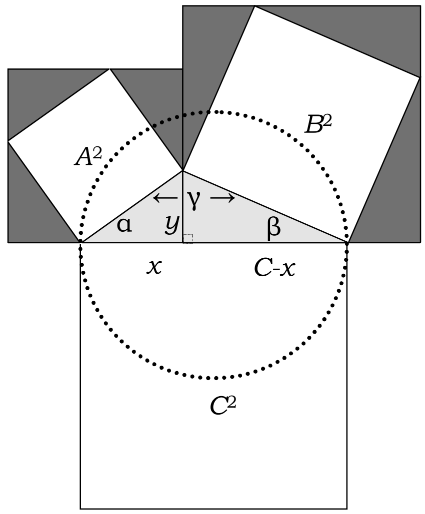


Figure 2.33: Carolyn's Cauliflower

The goal is to use the optimization techniques of multivariable differential calculus to show that the three squares A^2, B^2, and C^2 constructed on the three sides of the general triangle shown in **Figure 2.33**, with angles are α, β, and γ, satisfy the Pythagorean condition if and only if $\alpha+\beta=\gamma$. Since $\alpha+\beta+\gamma=180^0$ for any triangle, the rightmost equality $\alpha+\beta=\gamma$ is equivalent to the condition $\gamma=90^0$, which in turn implies that triangle (α,β,γ) is a right triangle.

$\overset{1}{\mapsto}$:To start the proof , let $C>0$ be the fixed length of an arbitrary line segment placed in a horizontal position. Let x be an arbitrary point on the line segment which cuts the line segment into two sub-lengths: x and $C-x$. Let $0<y\le C$ be an arbitrary length of a perpendicular line segment erected at the point x. Since x and y are both arbitrary, they are both independent variables in the classic sense.

In addition, y serves as the altitude for the *arbitrary triangle* (α, β, γ) defined by the construction shown in **Figure 2.33**. The sum of the two square areas A^2 and B^2 in **Figure 2.33** can be determined in terms of x and y as follows:

$$A^2 + B^2 = (x + y)^2 + ([C - x] + y)^2 - 2Cy .$$

The terms $(x + y)^2$ and $([C - x] + y)^2$ are the areas of the left and right outer enclosing squares, and the term $2Cy$ is the combined area of the eight shaded triangles expressed as an equivalent rectangle. Define $F(x, y)$ as follows:

$$F(x, y) = \{A^2 + B^2 - C^2\}^2$$

Then, substituting the expression for $A^2 + B^2$, we have

$$F(x, y) = \{(x + y)^2 + ([C - x] + y)^2 - 2Cy - C^2\}^2 \Rightarrow$$
$$F(x, y) = \{2x^2 + 2y^2 - 2Cx\}^2 \Rightarrow$$
$$F(x, y) = 4\{x[C - x] - y^2\}^2$$

We now restrict the function F to the compact, square domain $D = \{(x, y) \mid 0 \le x \le C, 0 \le y \le C\}$ shown in **Figure 2.34** where the symbols $BndD$ and $IntD$ denote the boundary of D and interior of D respectively.

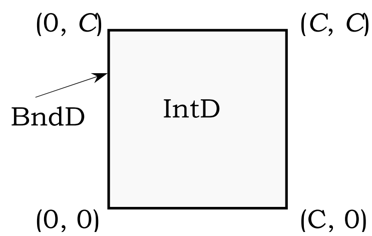


Figure 2.34: Domain D of F

Being polynomial in form, the function F is both continuous and differentiable on D. Continuity implies that F achieves both an absolute maximum and absolute minimum on D, which occur either on $BndD$ or $IntD$. Additionally, $F(x, y) \geq 0$ for all points (x, y) in D due to the presence of the outermost square in

$$F(x, y) = 4\{x[C - x] - y^2\}^2.$$

This implies in turn that $F(x_{\min}, y_{\min}) \geq 0$ for point(s) $(x_{\min}, y_{\min})$ corresponding to absolute minimum(s) for F on D. Equality to zero will be achieved if and only if

$$x_{\min}[C - x_{\min}] - y_{\min}{}^2 = 0.$$

Returning to the definition for F, one can immediately see that the following four expressions are mutually equivalent

$$F(x_{\min}, y_{\min}) = 0 \Leftrightarrow$$
$$x_{\min}[C - x_{\min}] - y_{\min}{}^2 = 0 \Leftrightarrow$$
$$A^2 + B^2 - C^2 = 0 \Leftrightarrow$$
$$A^2 + B^2 = C^2$$

$\overset{2}{\mapsto}$: We now employ the optimization methods of multivariable differential calculus to search for those points $(x_{\min}, y_{\min})$ where $F(x_{\min}, y_{\min}) = 0$ (if such points exist) and study the implications. First we examine $F(x, y)$ for points (x, y) restricted to the four line segments comprising $BndD$.

1. $F(x,0) = 4\{x[C - x]\}^2$. This implies $F(x,0) = 0$ only when $x = 0$ or $x = C$ on the lower segment of $BndD$. Both of these x values lead to degenerate cases.

2. $F(x,C) = 4\{x[C-x]-C^2\}^2 > 0$ for all points on the upper segment of $BndD$ since the smallest value that $|x[C-x]-C^2|$ achieves is $3C^2/4$, as determined via the techniques of single-variable differential calculus.

3. $F(0,y) = F(C,y) = 4y^4 > 0$ for all points other than $y=0$ (a degenerate case) on the two vertical segments of $BndD$.

To examine F on $IntD$, first take the partial derivatives of F with respect to x and y. This gives after simplification:

$$\partial F(x,y)/\partial x = 8[C-2x]\{x[C-x]-y^2\}$$
$$\partial F(x,y)/\partial y = -16y\{x[C-x]-y^2\}$$

Next, set the two partial derivatives equal to z

$$\partial F(x,y)/\partial x = \partial F(x,y)/\partial y = 0.$$

Solving for the associated critical points (x_{cp}, y_{cp}) yields one specific critical point and an entire locus of critical points as follows:

$(C/2, 0)$, a specific critical point

$x_{cp}[C-x_{cp}]-y_{cp}{}^2 = 0$,

an entire locus of critical points.

The specific critical point $(C/2, 0)$ is the midpoint of the lower segment for $BndD$. We have that

$$F(C/2, 0) = C^4/4 > 0.$$

Thus, the critical point $(C/2, 0)$ is removed from further consideration since $F(C/2, 0) > 0$, which in turn implies $A^2 + B^2 \neq C^2$. As a geometric digression, any point $(x,0)$ on the lower segment of $BndD$ represents a degenerate case associated with **Figure 2.33** since a viable triangle cannot be generated if the y value (the altitude) is zero as depicted in **Figure 2.35**

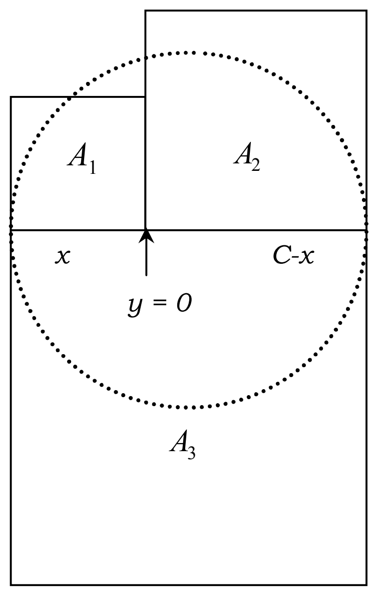


Figure 2.35: Carolyn's Cauliflower for y = 0

To examine the locus of critical points $x_{cp}[C - x_{cp}] - y_{cp}^2 = 0$, we first need to ask the following question: Given a critical point (x_{cp}, y_{cp}) on the locus with $0 < x_{cp} < C$, does the critical point necessarily lie in $IntD$? Again, we turn to the techniques of single-variable differential calculus for an answer.

Let x_{cp} be the x component of an arbitrary critical point (x_{cp}, y_{cp}). Now x_{cp} must be such that $0 < x_{cp} < C$ 0 in order for the quantity $x_{cp}[C - x_{cp}] > 0$. This in turn allows two real-number values for y_{cp}. In light of **Figure 2.33**, only those y_{cp} values where $y_{cp} > 0$ are of interest.

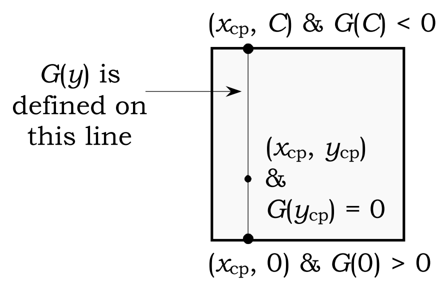


Figure 2.36: Behavior of G on IntD

Define the continuous quadratic function $G(y) = x_{cp}[C - x_{cp}] - y^2$ on the vertical line segment connecting the two points $(x_{cp}, 0)$ and (x_{cp}, C) on $BndD$ as shown in **Figure 2.36**. On the lower segment, we have $G(0) = x_{cp}[C - x_{cp}] > 0$ for all x_{cp} in $0 < x_{cp} < C$. On the upper segment, we have that $G(C) = x_{cp}[C - x_{cp}] - C^2 < 0$ for all x_{cp} in $0 < x_{cp} < C$. This is since the maximum that $G(x, C)$ can achieve for any x_{cp} in $0 < x_{cp} < C$ is $-3C^2/4$ per the optimization techniques of single-variable calculus. Now, by the intermediate value theorem, there must be a value $0 < y^* < C$ where $G(y^*) = x_{cp}[C - x_{cp}] - (y^*)^2 = 0$. By inspection, the associated point (x_{cp}, y^*) is in $IntD$, and, thus, by definition is part of the locus of critical points with $(x_{cp}, y^*) = (x_{cp}, y_{cp})$.

Since x_{cp} was chosen on an arbitrary basis, all critical points (x_{cp}, y_{cp}) defined by $x_{cp}[C - x_{cp}] - y_{cp}{}^2 = 0$ lie in $IntD$.

$\overset{3}{\mapsto}$: Thus far, we have used the techniques of differential calculus (both single and multi-variable) to establish a locus of critical points $x_{cp}[C - x_{cp}] - y_{cp}{}^2 = 0$ lying entirely within $IntD$. What is the relationship of each point (x_{cp}, y_{cp}) to the Pythagorean Theorem? As observed at the end of Step 1, one can immediately state the following:

$$\begin{aligned} &F(x_{\min}, y_{\min}) = 0 \Leftrightarrow \\ &x_{\min}[C - x_{\min}] - y_{\min}{}^2 = 0 \Leftrightarrow \\ &A^2 + B^2 - C^2 = 0 \Leftrightarrow \\ &A^2 + B^2 = C^2 \end{aligned}.$$

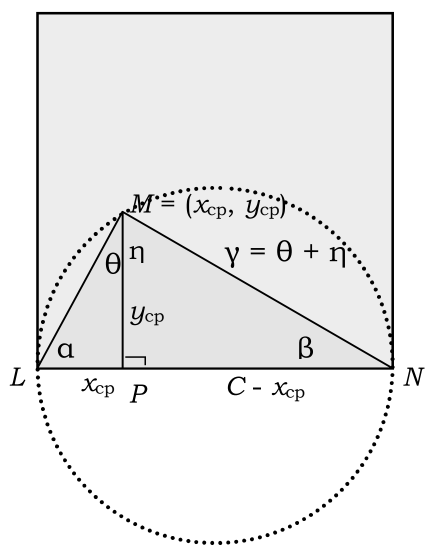


Figure 2.37: D and Locus of Critical Points

However, the last four-part equivalency is not enough. We need information about the three angles in our arbitrary triangle generated via the critical point (x_{cp}, y_{cp}) as shown in **Figure 2.37**. In particular, is $\gamma = \eta + \theta$ a right angle? If so, then the condition $A^2 + B^2 = C^2$ corresponds to the fact that ΔLMN is a right triangle and we are done! To proceed, first rewrite $x_{cp}[C - x_{cp}] - {y_{cp}}^2 = 0$ as

$$\frac{x_{cp}}{y_{cp}} = \frac{y_{cp}}{C - x_{cp}}.$$

Now study this proportional equality in light of **Figure 2.37**, where one sees that it establishes direct proportionality of non-hypotenuse sides for the two triangles ΔLPM and ΔMPN. From **Figure 2.37**, we see that both triangles have interior right angles, establishing that

$$\Delta LPM \approx \Delta MPN.$$

Thus $\alpha = \eta$ and $\theta = \beta$. Since the sum of the remaining two angles in a right triangle is 90^0 both $\alpha + \theta = 90^0$ and $\eta + \beta = 90^0$. Combining $\eta + \beta = 90^0$ with the equality $\theta = \beta$ and the definition for γ immediately leads to $\gamma = \eta + \theta = 90^0$, establishing the key fact that ΔLMN is a right triangle and the subsequent simultaneity of the two conditions

$$A^2 + B^2 = C^2 \Leftrightarrow$$
$$\alpha + \beta = \gamma \therefore$$

Figure 2.38 on the next page summarizes the various logic paths applicable to the now established *Cauliflower Proof of the Pythagorean Theorem with Converse*.

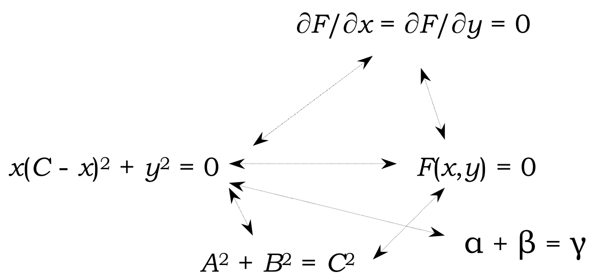


Figure 2.38: Logically Equivalent Starting Points

Starting with any one of the five statements in **Figure 2.38**, any of the remaining four statements can be deduced via the Cauliflower Proof by following a permissible path as indicated by the dashed lines. Each line is double-arrowed indicating total reversibility along that particular line.

Once the Pythagorean Theorem is established, one can show that the locus of points

$$x_{cp}[C - x_{cp}] - y_{cp}{}^2 = 0$$

describes a circle centered at $(C/2, 0)$ with radius $C/2$ by rewriting the equation as

$$(x_{cp} - C/2)^2 + y_{cp}{}^2 = [C/2]^2.$$

This was not possible prior to the establishment of the Pythagorean Theorem since the analytic equation for a circle is derived using the distance formula, a corollary of the Pythagorean Theorem. The dashed circle described by

$$(x_{cp} - C/2)^2 + y_{cp}{}^2 = [C/2]^2$$

in **Figure 2.37** nicely reinforces the fact that ΔLMN is a right triangle by the Inscribed Triangle Theorem of basic geometry.

Our final topic in this Section leads to a more general statement of the Pythagorean Theorem.

Let (x_{cp}, y_{cp}) be an arbitrary interior critical point as established by the Cauliflower Proof. Except for the single boundary point $(\frac{c}{2},0)$ on the lower boundary of $D(f)$, the interior critical points (x_{cp}, y_{cp}) are the only possible critical points. Create a vertical line segment passing through (x_{cp}, y_{cp}) and joining the two points $(x_{cp},0)$ and (x_{cp},c) as shown in **Figure 2.39** below. As created, this vertical line segment will pass through one and only one critical point.

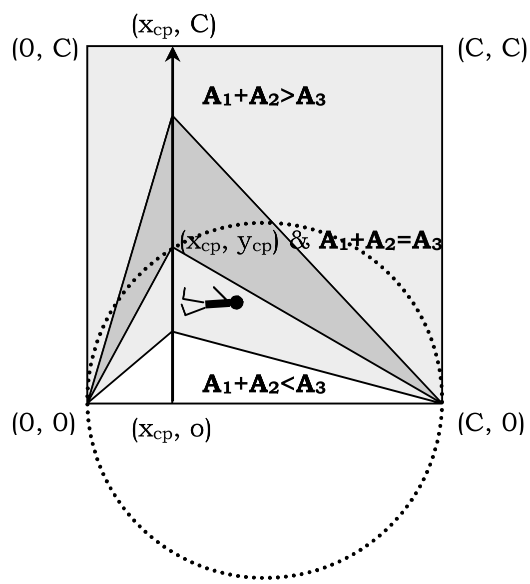


Figure 2.39: Walking From Inequality to Equality to Inequality

Define a new function

$$g(x_{cp}, y) = A_1 + A_2 - A_3 = 2x_{cp}{}^2 - 2cx_{cp} + 2y^2$$

for all points y on the vertical line segment where the stick figure is walking. The function $g(x_{cp}, y)$ is clearly continuous on the closed line segment $\{y : y \in [0, c]\}$ where $y_{cp} \in (0, c)$. We note the following three results:

1. $g(x_{cp}, 0) = 2x_{cp}(x_{cp} - c) < 0$

2. $g(x_{cp}, y_{cp}) = -2\left(x_{cp}[c - x_{cp}] - y_{cp}{}^2\right) = 0$

3. $g(x_{cp}, c) = 2x_{cp}{}^2 + 2c(c - x_{cp}) > 0$.

By continuity, we can immediately deduce that our function $g(x_{cp}, y)$ has the following behavior directly associated with the three results as given above:

1. $g(x_{cp}, y) < 0$ for all $y \in [0, y_{cp})$

   This in turn implies that $A_1 + A_2 < A_3$.

2. $g(x_{cp}, y_{cp}) = 0$

   This in turn implies that $A_1 + A_2 = A_3$.

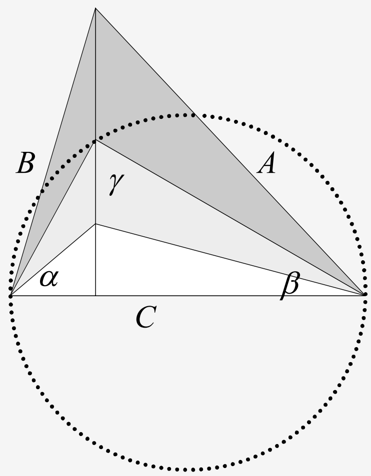
3. $g(x_{cp}, y) > 0$ for all $y \in (y_{cp}, c]$

   This in turn implies that $A_1 + A_2 > A_3$.

Tying these results to the obvious measure of the associated interior vertex angle γ in **Figure 2.37** leads to our final revised statement of the Pythagorean Theorem and converse on the next page.

The Pythagorean Theorem and Pythagorean Converse

Suppose we have a family of triangles built from a common hypotenuse C with side lengths and angles labeled in general fashion as shown below.



Then the following three cases apply

1. $\alpha + \beta < \gamma \Leftrightarrow A^2 + B^2 < C^2$
2. $\alpha + \beta = \gamma \Leftrightarrow A^2 + B^2 = C^2$
3. $\alpha + \beta > \gamma \Leftrightarrow A^2 + B^2 > C^2$

with the Pythagorean Theorem and Converse being Case 2.

2.11) Shear, Shape, and Area

Our last major category of proof for the Pythagorean Proposition is that of a shearing proof. Shearing proofs have been around for at least one thousand years, but they have increased in popularity with the advent of the modern computer and associated computer graphics.

The heart of a shearing proof is a rectangle that changes to a parallelogram preserving area as shown in **Figure 2.40**. Area is preserved as long as the length and altitude remain the same. In a sense, one could say that a shearing force **F** is needed to alter the shape of the rectangle into the associated parallelogram, hence the name *shearing proof*. Shearing proofs distinguish themselves from transformer proofs in that playing pieces will undergo both shape changes and position changes. In transformer proofs, the playing pieces only undergo position changes.

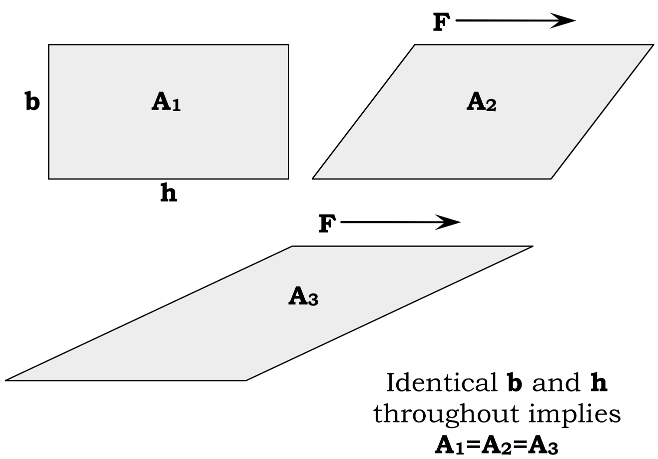


Figure 2.40: Shearing a Rectangle

Shearing proofs are primarily visual in nature, making them fantastic to watch when animated on a computer. As such, they are of special interest to mathematical hobbyists, who collectively maintain their delightful pursuit of new ones.

The one shearing proof that we will illustrate, without the benefit of modern technology, starts with a variant of Euclid's Windmill, **Figure 2.41**. Four basic steps are needed to move the total area of the big square into the two smaller squares.

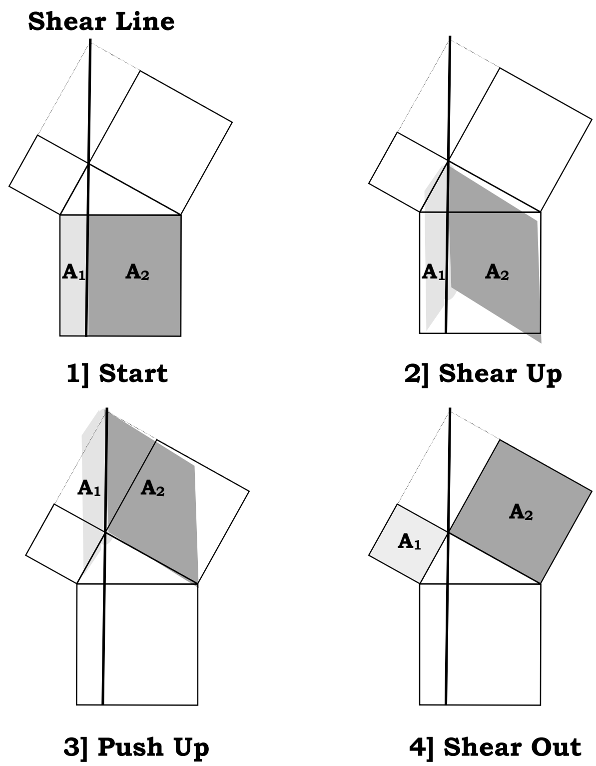


Figure 2.41: A Four Step Shearing Proof

Step descriptors are as follows:

Step 1: Cut the big square into two pieces using Euclid's perpendicular as a cutting guide.

Step 2: Shear a first time, transforming the two rectangles into associated equal-area parallelograms whose slanted sides run parallel to the two doglegs of the master triangle.

Step 3: Push the two parallelograms are pushed vertically upward to the top of the shear line, which is a projection of the altitude for each square.

Step 4: Shear a second time to squeeze both parallelograms into the associated little squares.

To summarize the shear proof: The hypotenuse square is cut into two rectangles per the Euclidean shear line. The rectangles are sheared into equivalent-area parallelograms, which are then pushed up the Euclidean shear line into a position allowing each parallelogram to be sheared back to the associated little square.

2.12) A Challenge for All Ages

As shown in this chapter, proving the Pythagorean Theorem has provided many opportunities for mathematical discovery for nearly 4000 years. Moreover, the Pythagorean Theorem does not cease in its ability to attract new generations of amateurs and professionals who want to add yet another proof to the long list of existing proofs. Proofs can be of many types and at many levels. Some are suitable for children in elementary school such as the visual proof attributed to Pythagoras himself—super effective if made into a plastic or wood hand-manipulative set. Other proofs only require background in formal geometry such as the shearing proof in Section 2.11. Still others require background in both algebra and geometry. The Cauliflower Proof requires a background in calculus. Thus, one can say *there is a proof for all ages as, indeed, there have been proofs throughout the ages*. The Pythagorean Crown Jewel never ceases to awe and inspire!

Type	Example	Comment
Visual Dissection	Pythagoras	Elementary School
Advanced Visual Dissection	Liu Hui	Middle School
Algebraic Dissection	Bhaskara	Middle School
Construction	Euclid	Early High School
Transformer	Kurrah	Middle School
Similarity	Legendre	High School
Tiling	Perigal	Early High School
Calculus	'Cauliflower'	Early College
Shearing	Section 2.11	Early High School

Table 2.3: Categories of Pythagorean Proofs

In **Table 2.3**, we briefly summarize the categories of existing proofs with an example for each. Not all proofs neatly tuck into one category or another, but rather combine various elements of several categories. A prime example is Legendre's similarity proof. In Chapter 3, **Diamonds of the Same Mind**, we will explore a sampling of major mathematical 'spin offs' attributed to the Pythagorean Theorem. In doing so, we will get a glimpse of how our Crown Jewel has permeated many aspects of today's mathematics and, by doing so, underpinned many of our modern technological marvels and discoveries.

Euclid's Beauty Revisited

Never did Euclid, as Newton ', discern
Areas between an edge and a curve
Where ancient precisions of straight defer
To infinitesimal addends of turn,
Precisely tallied in order to learn
Those planes that Euclid could only observe
As beauty…then barren of quadrature
And numbers for which Fair Order did yearn.

Thus beauty of worth meant beauty in square,
Or, those simple forms that covered the same.
And, though, Archimedes reckoned with care
Arcs of exhaustion no purist would claim,
Yet, his were the means for Newton to bare
True Beauty posed…in Principia's frame.

October 2006

3) Diamonds of the Same Mind

"Philosophy is written in this grand book—I mean the universe—which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth." Galileo Galilei

3.1) Extension to Similar Areas

All squares are geometrically similar. When we 'fit' length—the primary dimension associated with a square—to all three sides of a right triangle, we have that $a^2 + b^2 = c^2$ or $A_1 + A_2 = A_3$ by the Pythagorean Theorem. The equation $A_1 + A_2 = A_3$ not only applies to areas of squares fitted to the sides of a right triangle, but also to any geometrically similar planar figure fitted to the sides of a right triangle such as the three similar crosses shown in **Figure 3.1**.

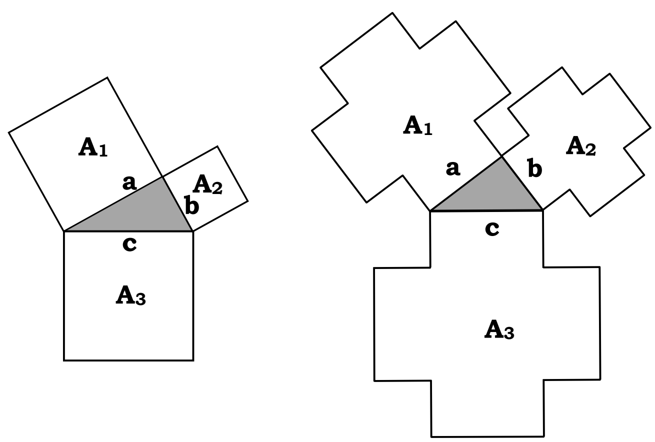


Figure 3.1: Three Squares, Three Crosses

We can formally state this similarity principle as a theorem, one first espoused by Euclid for the special case of similar rectangles (Book 6, Proposition 31).

Similar-Figure Theorem: Suppose three similar geometric figures are fitted to the three sides of a right triangle in such a fashion that $A_1 = ka^2$, $A_2 = kb^2$, and $A_3 = kc^2$ where the constant of proportionality k is identical for all three figures. Then we have that $A_1 + A_2 = A_3$.
Proof: From the Pythagorean Theorem

$$a^2 + b^2 = c^2 \Rightarrow ka^2 + kb^2 = kc^2 \Rightarrow A_1 + A_2 = A_3 \therefore$$

Table 3.1 provides several area formulas of the form $A = kc^2$ for a sampling of geometric figures fitted to a hypotenuse of length c. In order to apply the similar-figure theorem to any given set of three geometric figures, the fitting constant k must remain the same for the remaining two sides a and b.

SIMILAR FIGURE	**DIMENSION**	**AREA FORMULA**
Square	Side length	$A = 1 \cdot c^2$
Rectangle	Length or height	$A = \frac{h}{c} \cdot c^2$
Semicircle	Diameter	$A = \frac{\pi}{8} \cdot c^2$
Equilateral Triangle	Side length	$A = \frac{\sqrt{3}}{4} c^2$
Cross (**Figure 3.1**)	Side length	$A = 3c^2$
Pentagon	Side length	$A = 1.72048c^2$
Hexagon	Side length	$A = \frac{3\sqrt{3}}{2} c^2$

Table 3.1: A Sampling of Similar Areas

3.2) Pythagorean Triples and Triangles

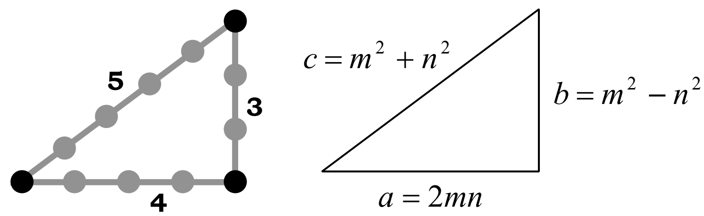


Figure 3.2: Pythagorean Triples

A *Pythagorean Triple* is a set of three positive integers (a, b, c) satisfying the classical Pythagorean relationship

$$a^2 + b^2 = c^2.$$

Furthermore, a *Pythagorean Triangle* is simply a right triangle having side lengths corresponding to the integers in a Pythagorean Triple. **Figure 3.2** shows the earliest and smallest such triple, $(3,4,5)$, known to the Egyptians some 4000 years ago in the context of a construction device for laying out right angles (Section 2.1). The triple $(6,8,10)$ was also used by early builders to lay out right triangles. Pythagorean triples were studied in their own right by the Babylonians and the Greeks. Even today, Pythagorean Triples are a continuing and wonderful treasure chest for professionals and amateurs alike as they explore various numerical relationships and oddities using the methods of number theory, in particular Diophantine analysis (the study of algebraic equations whose answers can only be positive integers). We will not even attempt to open the treasure chest in this volume, but simply point the reader to the great and thorough references by Beiler and Sierpinski where the treasures are revealed in full glory.

What we will do in Section 3.2 is present the formulas for a well-known method for generating Pythagorean Triples. This method provides one of the traditional starting points for further explorations of Pythagorean Triangles and Triples.

Theorem: Let $m > n > 0$ be two positive integers. Then a set of Pythagorean Triples is given by the formula:

$$(a,b,c) = (2mn, m^2 - n^2, m^2 + n^2)$$

Proof:

$$a^2 + b^2 = (2mn)^2 + (m^2 - n^2)^2 \Rightarrow$$
$$a^2 + b^2 = 4m^2n^2 + m^4 - 2m^2n^2 + n^4 \Rightarrow$$
$$a^2 + b^2 = m^4 + 2m^2n^2 + n^4 \Rightarrow$$
$$a^2 + b^2 = (m^2 + n^2)^2 = c^2 \therefore$$

From the formulas for the three Pythagorean Triples, we can immediately develop expressions for both the perimeter P and area A of the associated Pythagorean Triangle.

$$A = \tfrac{1}{2}(2mn)(m^2 - n^2) = mn(m^2 - n^2)$$
$$P = 2mn + m^2 - n^2 + m^2 + n^2 = 2m(m+n)$$

One example of an elementary exploration is to find all Pythagorean Triangles whose area numerically equals the perimeter. To do so, first set

$$A = P \Rightarrow$$
$$mn(m^2 - n^2) = 2m(m+n) \Rightarrow$$
$$n(m-n) = 2 \Rightarrow$$
$$m = 3, n = 1 \;\&\; m = 3, n = 2$$

For $m = 3, n = 1$, we have that $A = P = 24$.

For $m = 3, n = 2$, we have that $A = P = 30$.

Table 3.2 lists all possible Pythagorean Triples with long side $c < 100$ generated via the $m \& n$ formulas on the previous page. Shaded are the two solutions where $A = P$.

M	N	A=2MN	B=M²-N²	C=M²+N²	T	P	A
2	1	4	3	5	PT	12	6
3	1	6	8	10	C	24	24
3	2	12	5	13	PT	30	30
4	1	8	15	17	P	40	60
4	2	16	12	20	C	48	96
4	3	24	7	25	PT	56	84
5	1	10	24	26	C	60	120
5	2	20	21	29	P	70	210*
5	3	30	16	34	C	80	240
5	4	40	9	41	PT	90	180
6	1	12	35	37	P	84	210*
6	2	24	32	40	C	96	384
6	3	36	27	45	C	108	486
6	4	48	20	52	C	120	480
6	5	60	11	61	PT	132	330
7	1	14	48	50	C	112	336
7	2	28	45	53	P	126	630
7	3	42	40	58	C	140	840*
7	4	56	33	65	P	154	924
7	5	70	24	74	C	168	840*
7	6	84	13	85	PT	182	546
8	1	16	63	65	P	144	504
8	2	32	60	68	C	160	960
8	3	48	55	73	P	176	1320
8	4	64	48	80	C	192	1536
8	5	80	39	89	P	208	1560
9	1	18	80	82	C	180	720
9	2	36	77	85	P	198	1386
9	3	54	72	90	C	216	1944
9	4	72	65	97	P	234	2340

Table 3.2: Pythagorean Triples with c<100

Examining the table, one will first notice a column headed by the symbol **T**, which doesn't match with any known common abbreviations such as **A** (area) or **P** (perimeter). The symbol **T** stands for *type* or *triple type*.

There are three Pythagorean-triple types: *primitive* (P), *primitive twin* (PT), and *composite* (C). The definitions for each are as follows.

1. *Primitive*: A Pythagorean Triple (a,b,c) where there is no common factor for all three positive integers $a,b,\&c$.

2. *Primitive twin*: A Pythagorean Triple (a,b,c) where the longest leg differs from the hypotenuse by one.

3. *Composite*: A Pythagorean Triple (a,b,c) where there is a common factor for all three positive integers $a,b,\&c$.

The definition for primitive twin can help us find an associated $m\&n$ condition for identifying the same. If $a=2mn$ is the longest leg, then

$$
\begin{aligned}
&c=a+1\Rightarrow\\
&m^2+n^2=2mn+1\Rightarrow\\
&m^2-2mn+n^2=1\Rightarrow\\
&(m-n)^2=1\Rightarrow m=n+1
\end{aligned}
$$

Examining **Table 3.2** confirms the last equality; primitive twins occur whenever $m=n+1$. As a quick exercise, we invite the reader to confirm that the two cases $m,n=8,7$ and $m,n=9,8$ also produce Pythagorean twins with $c>100$.

If $b=m^2-n^2$ is the longest leg [such as the case for $m=7\&n=2$], then

$$
\begin{aligned}
&c=b+1\Rightarrow\\
&m^2+n^2=m^2-n^2+1\Rightarrow.\\
&2n^2=1
\end{aligned}
$$

The last equality has no solutions that are positive integers. Therefore, we can end our quest for Pythagorean Triples where $c = b + 1$.

Table 3.2 reveals two ways that Composite Pythagorean Triples are formed. The first way is when the two generators $m \& n$ have factors in common, examples of which are $m = 6 \& n = 2$, $m = 6 \& n = 3$, $m = 6 \& n = 4$ or $m = 8 \& n = 4$. Suppose the two generators $m \& n$ have a factor in common which simply means $m = kp \& m = kq$. Then the following is true:

$$a = 2mn = 2kpkq = 2k^2 pq$$
$$b = m^2 - n^2 = (kp)^2 - (kq)^2 = k^2(p^2 - q^2).$$
$$c = m^2 + n^2 = (kp)^2 + (kq)^2 = k^2(p^2 + q^2)$$

The last expressions show that the three positive integers $a, b \& c$ have the same factor k in common, but now to the second power. The second way is when the two generators $m \& n$ differ by a factor of two: $m = n + 2k, k = 1,2,3....$ Exploring this further, we have

$$a = 2(n + 2k)n = 2n^2 + 2kn$$
$$b = (n + 2k)^2 - n^2 = 2k^n + 4k^2 \quad .$$
$$c = (n + 2k)^2 + n^2 = 2n^2 + 4nk + 4n^2$$

Each of the integers $a, b \& c$ is divisible by two. Thus, all three integers share, as a minimum, the common factor two.

In General: If $k > 0$ is a positive integer and (a, b, c) is a Pythagorean triple, then (ka, kb, kc) is also a Pythagorean Triple. *Proof: left as a challenge to the reader.*

Thus, if we multiply any given primitive Pythagorean Triple, say $(4,3,5)$, by successive integers $k = 2,3...$; one can create an unlimited number composite Pythagorean Triples and associated Triangles $(8,6,10)$, $(12,9,15)$, etc.

We close this section by commenting on the asterisked * values in **Table 3.2**. These correspond to cases where two or more Pythagorean Triangles have identical planar areas. These equal-area Pythagorean Triangles are somewhat rare and provide plenty of opportunity for amateurs to discover new pairs. **Table 3.3** is a small table of selected equal-area Pythagorean Triangles.

A	B	C	AREA
20	21	29	210
12	35	37	210
42	40	58	840
70	24	74	840
112	15	113	840
208	105	233	10920
182	120	218	10920
390	56	392	10920

Table 3.3: Equal-Area Pythagorean Triangles

Rarer yet are equal-perimeter Pythagorean Triangles. **Table 3.4** shows one set of four equal-perimeter Pythagorean Triangles where the perimeter $P < 1{,}000{,}000$. Rumor has it that there are six other sets of four where $P < 1{,}000{,}000$!

A	B	C	PERIM
153868	9435	154157	317460
99660	86099	131701	317460
43660	133419	140381	317460
13260	151811	152389	317460

Table 3.4: Equal-Perimeter Pythagorean Triangles

3.3) Inscribed Circle Theorem

The Inscribed Circle Theorem states that the radius of a circle inscribed within a Pythagorean Triangle is a positive integer given that the three sides have been generated by the $m \,\&\, n$ process described in Section 3.2. **Figure 3.3** shows the layout for the Inscribed Circle Theorem.

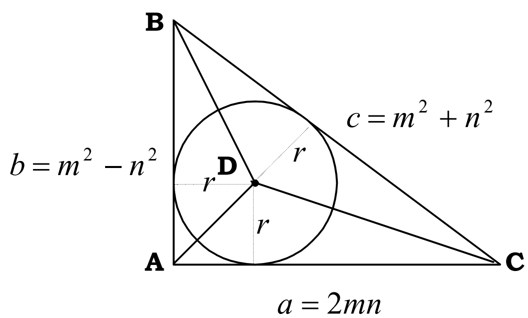


Figure 3.3: Inscribed Circle Theorem

The proof is simple once you see the needed dissection. The key is to equate the area of the big triangle ΔABC to the sum of the areas for the three smaller triangles ΔADB, ΔBDC, and ΔADC. Analytic geometry deftly yields the result.

$$\overset{1}{\rightarrow}: Area(\Delta ABC) =$$
$$Area(\Delta ADB) + Area(\Delta BDC) + Area(\Delta ADC)$$
$$\overset{2}{\rightarrow}: (\tfrac{1}{2})2mn(m^2 - n^2) =$$
$$(\tfrac{1}{2})(r)2mn + (\tfrac{1}{2})(r)(m^2 - n^2) + (\tfrac{1}{2})(r)(m^2 + n^2) \Rightarrow$$
$$2rmn + (m^2 - n^2)r + (m^2 + n^2)r = 2mn(m^2 - n^2) \Rightarrow$$
$$2rmn + 2m^2 r = 2mn(m^2 - n^2) \Rightarrow$$
$$m(n + m)r = mn(m + n)(m - n) \Rightarrow$$
$$r = n(m - n) \therefore$$

The last equality shows that the inscribed radius r is a product of two positive integral quantities $n(m-n)$. Thus the inscribed radius itself, called a *Pythagorean Radius*, is a positive integral quantity, which is what we set out to prove. The proof also gives us as a bonus the actual formula for finding the inscribed radius $r = n(m-n)$. **Table 3.5** gives r values for all possible m & n values where $m \leq 7$.

M	**N**	**A=2MN**	**$B=M^2-N^2$**	**$C=M^2+N^2$**	**R=N(M-N)**
2	1	4	3	5	1
3	1	6	8	10	2
3	2	12	5	13	2
4	1	8	15	17	3
4	2	16	12	20	4
4	3	24	7	25	3
5	1	10	24	26	4
5	2	20	21	29	6
5	3	30	16	34	6
5	4	40	9	41	4
6	1	12	35	37	5
6	2	24	32	40	8
6	3	36	27	45	9
6	4	48	20	52	8
6	5	60	11	61	5
7	1	14	48	50	6
7	2	28	45	53	10
7	3	42	40	58	12
7	4	56	33	65	12
7	5	70	24	74	10
7	6	84	13	85	6

Table 3.5: Select Pythagorean Radii

3.4) Adding a Dimension

Inscribe a right triangle in cattycorner fashion and within a rectangular solid having the three side lengths **A**, **B**, and **C** as shown in **Figure 3.4**. Once done, the Pythagorean Theorem can be easily extended to three dimensions as follows.

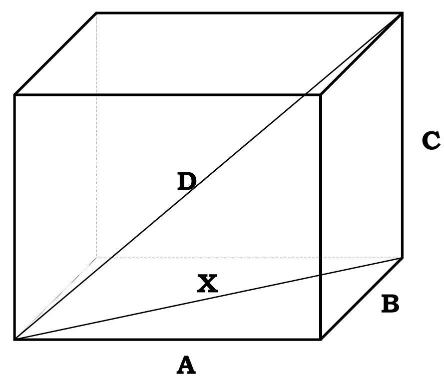


Figure 3.4: Three Dimensional Pythagorean Theorem

Extension Theorem: Let three mutually-perpendicular line segments have lengths A, B and C respectively, and let these line segments be joined in unbroken end-to-end fashion as shown in **Figure 3.4**. Then the square of the straight-line distance from the beginning of the first segment to the end of the last line segment is given by

$$A^2 + B^2 + C^2 = D^2$$

Proof: From the Pythagorean Theorem, we have $X^2 + C^2 = D^2$. From the Pythagorean Theorem a second time, we have $A^2 + B^2 = X^2$. Thus

$$X^2 + C^2 = D^2 \ \&$$

$$A^2 + B^2 = X^2 \Rightarrow$$

$$A^2 + B^2 + C^2 = D^2 \ \therefore$$

A Pythagorean Quartet is a set of four positive integers (a, b, c, d) satisfying the three-dimensional Pythagorean relationship $a^2 + b^2 + c^2 = d^2$.

Pythagorean quartets can be generated using the $m \& n$ formulas below.

$$a = m^2$$
$$b = 2mn$$
$$c = 2n^2$$
$$d = m^2 + 2n^2$$

To verify, square the expressions for $a, b \& c$, add, and simplify.

$$(m^2)^2 + (2mn)^2 + (2n^2)^2 =$$
$$m^4 + 4mn + 4n^4 =$$
$$(m_2 + 2n^2)^2 = d^2 \therefore$$

Table 3.6 lists the first twelve Pythagorean quartets so generated.

M	N	A=M²	B=2MN	C=2N²	D=M²+2N²
2	1	4	4	2	6
3	1	9	6	2	11
3	2	9	10	8	17
4	1	16	8	2	18
4	2	16	16	8	24
4	3	16	24	18	34
5	1	25	10	2	27
5	2	25	20	8	33
5	3	25	30	18	43
5	4	25	40	32	57
6	1	36	12	2	38
6	2	36	24	8	44

Table 3.6: Select Pythagorean Quartets

Our last topic in Section 3.4 addresses the famous Distance Formula, a far-reaching result that is a direct consequence of the Pythagorean Theorem. The Distance Formula—an alternate formulation of the Pythagorean Theorem in analytic-geometry form—permeates all of basic and advanced mathematics due to its fundamental utility.

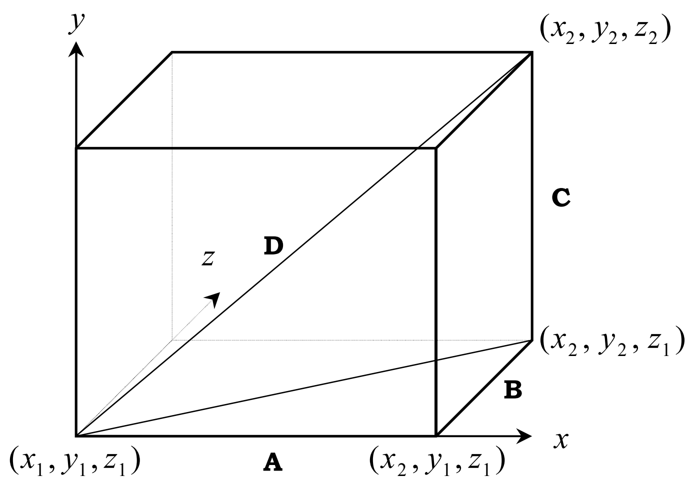


Figure 3.5: Three-Dimensional Distance Formula

Theorem: Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points in three-dimensional Cartesian coordinate space as shown in **Figure 3.5**. Let D be the straight-line distance between the two points. Then the distance D is given by

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Proof:

$$D^2 = A^2 + B^2 + C^2 \Rightarrow$$

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \Rightarrow$$

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \;\therefore$$

3.5) Pythagoras and the Three Means

The Pythagorean Theorem can be used to visually display three different statistical means or averages: *arithmetic mean, geometric mean*, and *harmonic mean.*

Suppose $a \& b$ are two numbers. From these two numbers, we can create three different means defined by

Arithmetic Mean: $M_A = \dfrac{a+b}{2}$

Geometric Mean: $M_G = \sqrt{ab}$

Harmonic Mean: $M_H = \dfrac{2}{\frac{1}{a}+\frac{1}{b}} = \dfrac{2ab}{a+b}$.

Collectively, these three means are called the Pythagorean Means due to their interlocking relationships with respect to *six right-triangles* as shown in **Figure 3.6**.

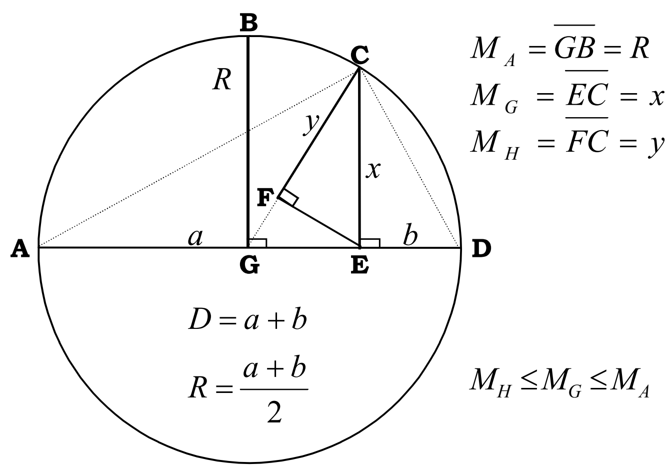


Figure 3.6: The Three Pythagorean Means

In **Figure 3.6**, the line $\overline{AD}$ has length $a+b$ and is the diameter for a circle with center $[a+b]/2$. Triangle ΔACD utilizes $\overline{AD}$ as its hypotenuse and the circle rim as its vertex making ΔACD a right triangle by construction. Since $\overline{EC}$ is perpendicular to $\overline{AD}$, ΔACE and ΔECD are both similar to ΔACD. Likewise, the three triangles ΔGCE, ΔGFE, and ΔEFC form a second similarity group by construction.

Next, we show that the three line segments $\overline{GB}$, $\overline{EC}$, and $\overline{FC}$ are the three Pythagorean Means M_A, M_G, and M_H as defined.

Arithmetic Mean: The line segment $\overline{GB}$ is a radius and has length $[a+b]/2$ by definition. It immediately follows that:

$$M_A = \overline{GB} = \frac{a+b}{2}.$$

Geometric Mean: The similarity relationship $\Delta ACE \approx \Delta ECD$ leads to the following proportion:

$$\frac{x}{a} = \frac{b}{x} \Rightarrow$$
$$x^2 = ab \Rightarrow x = \sqrt{ab} \Rightarrow .$$
$$M_G = \overline{EC} = x = \sqrt{ab}$$

Harmonic Mean: By construction, the line segment $\overline{GC}$ is a radius with length $[a+b]/2$. Line segment $\overline{EF}$ is constructed perpendicular to $\overline{GC}$. This in turn makes $\Delta GCE \approx \Delta EFC$, which leads to the following proportion and expression for $\overline{FC}$.

$$\frac{y}{x}=\frac{x}{\frac{a+b}{2}} \Rightarrow y=\frac{x^2}{\frac{a+b}{2}}$$

$$y=\frac{ab}{\frac{a+b}{2}} \Rightarrow y=\frac{2ab}{a+b} \Rightarrow$$

$$M_H=\overline{FC}=y=\frac{2}{\frac{1}{a}+\frac{1}{b}}$$

Examining the relative lengths of the three line segments $\overline{GB}$, $\overline{EC}$, and $\overline{FC}$ in **Figure 3.6** immediately establishes the inequality $M_H \le M_G \le M_A$ for the three Pythagorean Means. The reader is invited to verify that $M_H = M_G = M_A$ when $a=b$.

As is usually done in mathematics, our two-number Pythagorean Mean definitions can easily be generalized to multi-number definitions. Let $a_i, i=1,n$ be n numbers. Then the following definitions apply:

General Arithmetic Mean: $M_A=\frac{\sum_{i=1}^{n} a_i}{n}$

General Geometric Mean: $M_G=\sqrt[n]{\prod_{i=1}^{n} a_i}$

General Harmonic Mean: $M_H=\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}$.

To close, various sorts of means are some of the most widely used quantities in modern statistics. Three of these have geometric origins and can be interpreted within a Pythagorean context.

3.6) The Theorems of Heron, Pappus, Kurrah, and Stewart

The Pythagorean Theorem uses as its starting point a right triangle. Suppose a triangle is not a right triangle. What can we say about relationships amongst the various parts of a triangle's anatomy: sides, areas, and altitudes? Most students of mathematics can recite the two basic formulas as presented in **Figure 3.7**. *Are there other facts that one can glean*?

$$Area = \tfrac{1}{2}ch$$

$$Perimeter = a + b + c$$

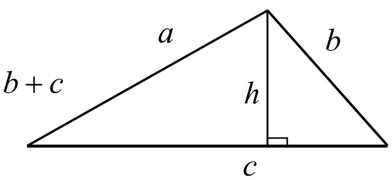


Figure 3.7: The Two Basic Triangle Formulas

In this section, we will look at four investigations into this very question spanning a period of approximately 1700 years. Three of the associated scientists and mathematicians—Heron, Pappus, and Stewart—are new to our study. We have previously introduced Kurrah in Section 2.4.

Heron, also known as Hero of Alexandria, was a physicist, mathematician, and engineer who lived in the 1st century ACE.

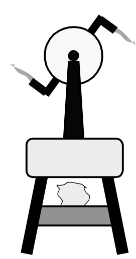


Figure 3.8: Schematic of Hero's Steam Engine

Hero was the founder of the Higher Technical School at the then world-renown center of learning at Alexandria, Egypt. He invented the first steam engine, schematically depicted in **Figure 3.8**, a device that rotated due to the thrust generated by steam exiting through two nozzles diametrically placed on opposite sides of the axis of rotation. Working models of Hero's steam engine are still sold today as tabletop scientific collectables.

In his role as an accomplished mathematician, Heron (as he is better known in mathematical circles) discovered one of the great results of classical mathematics, a general formula for triangular area known as Heron's Theorem. Heron did not originally use the Pythagorean Theorem to prove his result. Both the Pythagorean Theorem and Heron's Theorem are independent and co-equal results in that each can be derived without the use of the other and each can be used to prove the other. In this section, we will show how the Pythagorean Theorem, coupled with 'heavy' analytic geometry, is used to render the truth of Heron's Theorem. First, we state Heron's historic result.

Heron's Theorem: Suppose a general triangle (a triangle having no special angles such as right angles) has three sides whose lengths are a, b, and c respectively. Let $s = [a+b+c]/2$ be the semi-perimeter for the same. Then the internal area A enclosed by the general triangle, **Figure 3.9**, is given by the formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

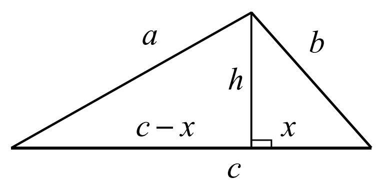


Figure 3.9: Diagram for Heron's Theorem

Proof of Heron's Theorem:

$\overset{1}{\mapsto}$: Using the Pythagorean Theorem, create two equations E_1 & E_2 in the two unknowns h and x.

$$E_1 : h^2 + x^2 = b^2$$
$$E_2 : h^2 + (c-x)^2 = a^2$$

$\overset{2}{\mapsto}$: Subtract E_2 from E_1 and then solve for x.

$$x^2 - (c-x)^2 = b^2 - a^2 \Rightarrow$$
$$2cx - c^2 = b^2 - a^2 \Rightarrow$$
$$x = \frac{c^2 + b^2 - a^2}{2c}$$

$\overset{3}{\mapsto}$: Substitute the last value for x into E_1.

$$h^2 + \left[\frac{c^2 + b^2 - a^2}{2c}\right]^2 = b^2$$

$\overset{4}{\mapsto}$: Solve for h.

$$h = \sqrt{\frac{4c^2b^2 - \left[c^2 + b^2 - a^2\right]^2}{4c^2}} \Rightarrow$$
$$h = \sqrt{\frac{4c^2b^2 - \left[c^2 + b^2 - a^2\right]^2}{4c^2}} \Rightarrow$$
$$h = \sqrt{\frac{\left\{2cb - \left[c^2 + b^2 - a^2\right]\right\}\left\{2cb + \left[c^2 + b^2 - a^2\right]\right\}}{4c^2}} \Rightarrow$$
$$h = \sqrt{\frac{\left\{a^2 - [c-b]^2\right\}\left\{[c+b]^2 - a^2\right\}}{4c^2}} \Rightarrow$$
$$h = \sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{4c^2}}$$

$\overset{5}{\mapsto}$: Solve for area using the formula $A = \frac{1}{2}ch$.

$$A = \tfrac{1}{2}c\sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{4c^2}} \Rightarrow$$
$$A = \sqrt{\frac{\{a+b-c\}\{a+c-b\}\{c+b-a\}\{c+b+a\}}{16}} \Rightarrow$$
$$A = \sqrt{\frac{\{a+b+c-2c\}\{a+b+c-2b\}\{a+c+b-2a\}\{c+b+a\}}{2\cdot 2\cdot 2\cdot 2}}$$

$\overset{6}{\mapsto}$: Substitute $s = [a+b+c]/2$ and simplify to obtain the final result.

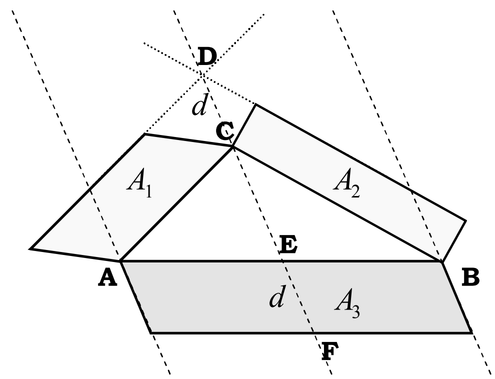
$$A = \sqrt{\left\{s - \frac{2c}{2}\right\}\left\{s - \frac{2b}{2}\right\}\left\{s - \frac{2a}{2}\right\}\{s\}} \Rightarrow$$
$$A = \sqrt{(s-c)(s-b)(s-a)s} \Rightarrow$$
$$A = \sqrt{s(s-a)(s-b)(s-c)} \therefore$$

***

Pappus, like Heron, was a prominent mathematician of the Alexandrian School and contributed much to the field of mathematics. There are several 'Pappus Theorems' still in use today. Notable is the Pappus Theorem utilized in multi-variable calculus to determine the volume of a solid of revolution via the circular distance traveled by a centroid. In his masterwork, The Mathematical Collection (circa 300 ACE), he published the following generalization of the Pythagorean Theorem which goes by his name.

Pappus' Theorem: Let ΔABC be a general triangle with sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Suppose arbitrary parallelograms having areas A_1 and A_2 are fitted to sides $\overline{AC}$ and $\overline{BC}$ as shown in **Figure 3.10**.

If a parallelogram of area A_3 is fitted to side $\overline{AB}$ via the construction method in **Figure 3.10**, then $A_1 + A_2 = A_3$.



$$A_1 + A_2 = A_3$$

Figure 3.10: Diagram for Pappus' Theorem

Proof of Pappus' Theorem:

$\overset{1}{\mapsto}$: Let ΔABC be a general triangle. Fit an arbitrary parallelogram of area A_1 to the side $\overline{AC}$; likewise, fit the arbitrary parallelogram of area A_2 to the side $\overline{BC}$.

$\overset{2}{\mapsto}$: Linearly extend the outer sides—dotted lines—of the two fitted parallelograms until they meet at the point **D**.

$\overset{3}{\mapsto}$: Construct a dashed shear line through the points **D** and **C** as shown in **Figure 3.10**. Let $\overline{DC} = \overline{EF} = d$. Construct two additional shear lines through points **A** and **B** with both lines parallel to the line passing through points **D** and **C**.

$\overset{4}{\mapsto}$: Construct the parallelogram with area A_3 as shown with two sides of length $\overline{AB}$ and two sides of length $\overline{DC}$. We claim that if the third parallelogram is constructed in this fashion, then $A_1 + A_2 = A_3$.

$\overset{5}{\mapsto}$: **Figure 3.11** depicts the two-step shearing proof for Pappus' Theorem. The proof is rather easy once the restrictions for the rectangle to be fitted on side $\overline{AB}$ are known. **Figure 3.11** shows the areas of the two original parallelograms being preserved through a two-step transformation that preserves equality of altitudes. This is done by keeping each area contained within *two pairs of parallel guidelines—like railroad tracks*. In this fashion, A_1 is morphed to the left side of the constructed parallelogram having area A_3. Likewise, A_2 is morphed to the right side of the same. Summing the two areas gives the sought after result:

$$A_1 + A_2 = A_3 \therefore$$

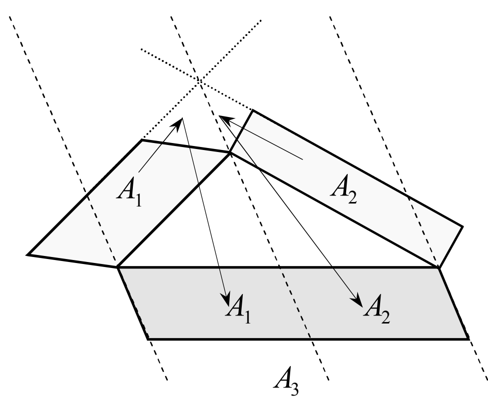


Figure 3.11: Pappus Triple Shear Line Proof

Note[1]: The reader is invited to compare the shearing proof of Pappus to the one presented in Section 2.11 and notice the sequence reversal.

Note[2]: Pappus' Theorem is restrictive in that one must construct the third parallelogram according to the process laid out in the proof. Perhaps the theorem evolved after many trials with the squares of Pythagoras as Pappus tried to relate size of squares to the sides of a general triangle. In doing so, squares became rectangles and parallelograms as the investigation broadened. I suspect, as is usually the case, perspiration and inspiration combined to produce the magnificent result above!

Like Herron's Theorem, both the Pythagorean Theorem and Pappus' Theorem are independent and co-equal results. **Figure 3.12** shows the methods associated with Pappus' Theorem as they are used to prove the Pythagorean Theorem, providing another example of a Pythagorean shearing proof in addition to the one shown in Section 2.11. The second set of equally spaced sheer lines on the left side, are used to cut the smaller square into two pieces so that a fitting within the rails can quite literally occur.

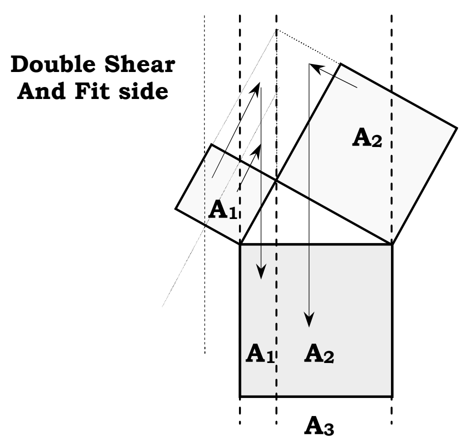


Figure 3.12: Pappus Meets Pythagoras

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We already have introduced Thabit ibn Kurrah in Section 2.4 with respect to his Bride's Chair and associated transformer proof. Kurrah also investigated non-right triangles and was able to generalize the Pythagorean Proposition as follows.

Kurrah's Theorem: Suppose a general triangle ΔAED has three sides whose lengths are $\overline{AD}$, $\overline{AE}$, and $\overline{ED}$. Let the vertex angle $\angle AED = \alpha$ as shown in **Figure 3.13**. Construct the two line segments $\overline{ED}$ and $\overline{EC}$ in such a fashion that both $\angle ABE = \alpha$ and $\angle DCE = \alpha$. Then we have that

$$\overline{AE}^2 + \overline{ED}^2 = \overline{AD}(\overline{AB} + \overline{CD}).$$

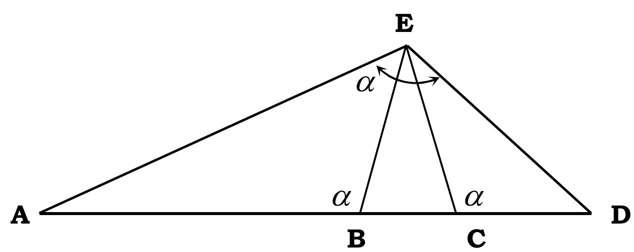

Figure 3.13: Diagram for Kurrah's Theorem

Proof of Kurrah's Theorem:

$\overset{1}{\mapsto}$: Establish similarity for three key triangles by noting that angle $\angle BAE$ is common to ΔABE and ΔAED. Likewise $\angle EDC$ is common to ΔECD and ΔAED. This and the fact that $\angle AEB = \angle ABE = \angle DCE$ implies $\Delta AED \approx \Delta ABE \approx \Delta ECD$.

$\overset{2}{\mapsto}$: Set up and solve two proportional relationships

$$\frac{\overline{AE}}{\overline{AB}} = \frac{\overline{AD}}{\overline{AE}} \Rightarrow \overline{AE}^2 = \overline{AB} \cdot \overline{AD} \, \&$$
$$\frac{\overline{ED}}{\overline{CD}} = \frac{\overline{AD}}{\overline{ED}} \Rightarrow \overline{ED}^2 = \overline{CD} \cdot \overline{AD}$$

$\overset{3}{\mapsto}$: Add the two results above and simplify to complete the proof.

$$\overline{AE}^2 + \overline{ED}^2 = \overline{AB} \cdot \overline{AD} + \overline{CD} \cdot \overline{AD} \Rightarrow$$
$$\overline{AE}^2 + \overline{ED}^2 = \overline{AD}\left(\overline{AB} + \overline{CD}\right) \therefore$$

Kurrah's Theorem can be used to prove the Pythagorean Theorem for the special case $\alpha = 90^0$ via the following very simple logic sequence.

For, if the angle α happens to be a right angle, then we have a merging of the line segment $\overline{EB}$ with the line segment $\overline{EC}$. This in turn implies that the line segment $\overline{BC}$ has zero length and $\overline{AB} + \overline{CD} = \overline{AD}$. Substituting into Kurrah's result gives

$$\overline{AE}^2 + \overline{ED}^2 = \overline{AD}^2 \therefore,$$

Note: Yet another proof of the Pythagorean Theorem!

***

Mathew Stewart (1717-1785) was a professor at the University of Edinburgh and a Fellow of the Royal Society. The famous theorem as stated on the next page—although attributed to Stewart—has geometric similarities to theorems originally discovered and proved by Apollonius.

Some historians believe Stewart's Theorem can be traced as far back as Archimedes. Other historians believe that Simpson (one of Stewart's academic mentors) actually proved 'Stewart's Theorem'. Controversy aside, Stewart was a brilliant geometer in his own right. Like Euclid, Stewart was an expert organizer of known and useful results in the burgeoning new area of mathematical physics—especially orbital mechanics.

Unlike the three previous theorems, Stewart's Theorem is not co-equal to the Pythagorean Theorem. This means that Stewart's Theorem requires the use of the Pythagorean Theorem in order to construct a proof. Independent proofs of Stewart's Theorem are not possible and, thus, any attempt to prove the Pythagorean Theorem using Stewart's Theorem becomes an exercise in circular logic. We now proceed with the theorem statement

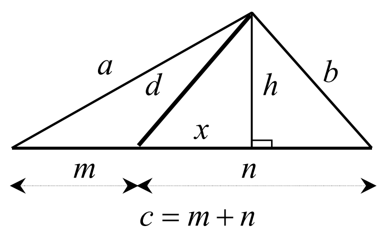

Figure 3.14: Diagram for Stewart's Theorem

Stewart's Theorem: Suppose a general triangle (a triangle having no special angles such as right angles) has sides with lengths a, b, and c. Let a line segment of *unknown length* d be drawn from the vertex of the triangle to the horizontal side as shown in **Figure 3.14**, cutting c into two portions m and n where $c = m + n$. *Then, the following relationship holds among the six line segments whose lengths are* a, b, c, d, m, and n:

$$na^2 + mb^2 = c[d^2 + mn].$$

Proof of Stewart's Theorem

$\overset{1}{\mapsto}$: Using the Pythagorean Theorem, create three equations $E_1, E_2, \& E_3$ in the three unknowns d, h, and x.

$$E_1 : b^2 = h^2 + (n-x)^2$$
$$E_2 : a^2 = h^2 + (m+x)^2$$
$$E_3 : d^2 = h^2 + x^2$$

$\overset{2}{\mapsto}$: Solve E_3 for h^2 Substitute result in E_1 and E_2, simplify.

$$E_1 : b^2 = d^2 - x^2 + (n-x)^2$$
$$E_2 : a^2 = d^2 - x^2 + (m+x)^2 \Rightarrow$$
$$E_1 : b^2 = d^2 + n^2 - 2nx$$
$$E_2 : a^2 = d^2 + m^2 + 2mx$$

$\overset{3}{\mapsto}$: Formulate $mE_1 + nE_2$ and simplify using $c = m + n$.

$$mE_1 : mb^2 = md^2 + mn^2 - 2mnx$$
$$nE_2 : na^2 = nd^2 + nm^2 + 2nmx \Rightarrow$$
$$mE_1 + nE_2 : mb^2 + na^2 = md^2 + nd^2 + mn^2 + nm^2 \Rightarrow$$
$$mb^2 + na^2 = (m+n)d^2 + mn(m+n) \Rightarrow$$
$$mb^2 + na^2 = cd^2 + mnc \Rightarrow$$
$$mb^2 + na^2 = c(d^2 + mn) \therefore$$

Stewart's result can be used to solve for the unknown line-segment length d in terms of the known lengths a, b, c, m and n.

In pre-computer days, both all four theorems in this section had major value to the practicing mathematician and scientist. Today, they may be more of an academic curiosity. However, this in no way detracts from those before us whose marvelous and clever reasoning led to these four historic landmarks marking our continuing Pythagorean journey.

Table 3.7 summarizes the four theorems presented in this section. A reversible proof in the context below is a proof that is can be independently derived and the used to prove the Pythagorean Theorem.

WHO?	**WHAT?**	**PROOF REVERSIBLE?**	**PROOF SHOWN?**
Heron: Greek	Triangular Area in Terms of Sides	Yes	No
Pappus: Greek	Generalization for Parallelograms Erected on sides	Yes	Yes
Kurrah: Turkish	Generalization for Squares of Side Lengths	Yes	Yes
Stewart: Scottish	Length Formula for Line Segment Emanating from Apex	No	Not Possible

Table 3.7: Pythagorean Generalizations

3.7) The Five Pillars of Trigonometry

Language is an innate human activity, and mathematics can be defined as the language of measurement! This definition makes perfect sense, for humans have been both measuring and speaking/writing for a very long time. Truly, the need to measure is in our 'blood' just as much as the need to communicate. Trigonometry *initially* can be thought of as the mathematics of 'how far', or 'how wide', or 'how deep'. All of the preceding questions concern measurement, in particular, the measurement of distance. Hence, trigonometry, as originally conceived by the ancients, is primarily the mathematical science of measuring distance. In modern times, trigonometry has been found to be useful in scores of other applications such as the mathematical modeling and subsequent analysis of any reoccurring or cyclic pattern.

Geometry, particularly right-triangle or Pythagorean geometry, was the forerunner to modern trigonometry. Again, the ancients noticed that certain proportions amongst the three sides of two similar triangles (triangles having equal interior angles) were preserved—no matter the size difference between the two triangles. These proportions were tabulated for various angles. They were then used to figure out the dimensions of a large triangle using the dimensions of a smaller, similar triangle. This one technique alone allowed many powerful things to be done before the Common Era: e.g. construction of the Great Pyramid, measurement of the earth's circumference (25,000 miles in today's terms), estimation of the distance from the earth to the moon, and the precise engineering of roadways, tunnels, and aqueducts. 'Nascent trigonometry', in the form of right-triangle geometry, was one of the backbones of ancient culture.

In modern times trigonometry has grown far beyond its right triangle origins. It can now be *additionally* described as the mathematics describing periodic or cyclic processes.

One example of a periodic process is the time/distance behavior of a piston in a gasoline engine as it repeats the same motion pattern some 120,000 to 200,000 times in a normal hour of operation. Our human heart also exhibits repetitive, steady, and cyclic behavior when in good health. Thus, the heart and its beating motion can be analyzed and/or described using ideas from modern trigonometry as can any electromagnetic wave form.

Note: On a recent trip to Lincoln, Nebraska, I calculated that the wheel on our Toyota would revolve approximately 1,000,000 times in a twelve-hour journey—very definitely a cyclic, repetitive process.

Returning to measurement of distance, look up to the night sky and think 'how far to the stars'—much like our technical ancestors did in ancient Greece, Rome, Babylon, etc. Trigonometry has answered that question in modern times using the powerful parallax technique. The *parallax technique* is a marriage of modern and old: careful, precise measurement of known distances/angles extrapolated across vast regions of space to calculate the distance to the stars. The Greeks, Romans, and Babylonians would have marveled! Now look down at your GPS hand-held system. Turn it on, and, within a few seconds, you will know your precise location on planet earth. This fabulous improvement on the compass operates using satellites, electronics, and *basic trigonometry* as developed from *right triangles* and the associated *Pythagorean Theorem*. And if you do not have a GPS device, you surely have a cell phone. Every fascinating snippet of cellular technology will have a mathematical foundation in trigonometry and the Pythagorean Theorem.

Trigonometry rests on five pillars that are constructed using direct Pythagorean principles or derivates thereof. These five pillars serve as the foundation for the whole study of trigonometry, and, from these pillars, one can develop the subject in its entirety.

Note: In 1970, while I was in graduate school, a mathematics professor stated that he could teach everything that there is to know about trigonometry in two hours. I have long since realized that he is right. The five Pythagorean pillars make this statement so.

To start a formal exploration of trigonometry, first construct a unit circle of radius $r = 1$ and center $(0,0)$. Let t be the counterclockwise arc length along the rim from the point $(1,0)$ to the point $P(t) = (x, y)$. Positive values for t correspond to rim distances measured in the counterclockwise direction from $(1,0)$ to $P(t) = (x, y)$ whereas negative values for t correspond to rim distances measured in the clockwise direction from $(1,0)$ to $P(t) = (x, y)$. Let α be the angle subtended by the arc length t, and draw a right triangle with hypotenuse spanning the length from $(0,0)$ to (x, y) as shown in **Figure 3.15**.

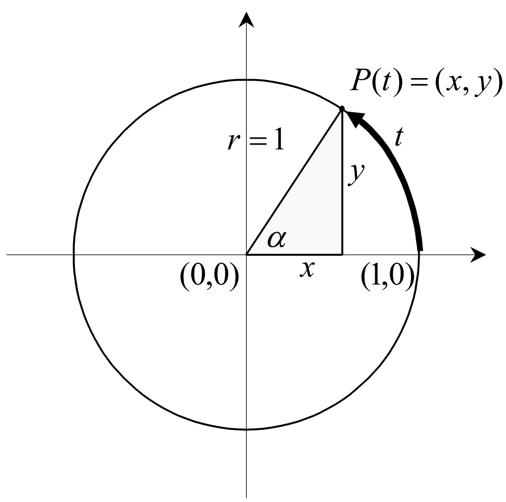

Figure 3.15: Trigonometry via Unit Circle

The first pillar is the collection definition for the *six trigonometric functions*. Each of the six functions has as their independent or input variable either α or t. Each function has as the associated dependent or output variable some combination of the two right-triangular sides whose lengths are x and y.

The six trigonometric functions are defined as follows:

1] $\sin(\alpha) = \sin(t) = y$ 2] $\csc(\alpha) = \csc(t) = \dfrac{1}{y}$

3] $\cos(\alpha) = \cos(t) = x$ 4] $\sec(\alpha) = \sec(t) = \dfrac{1}{x}$

5] $\tan(\alpha) = \tan(t) = \dfrac{y}{x}$ 6] $\cot(\alpha) = \cot(t) = \dfrac{x}{y}$

As the independent variable t increases, one eventually returns to the point $(1,0)$ and circles around the rim a second time, a third time, and so on. Additionally, any fixed point (x_0, y_0) on the rim is passed over many times as we continuously spin around the circle in a positive or negative direction. Let t_0 or α_0 be such that $P(t_0) = P(\alpha_0) = (x_0, y_0)$. Since $r = 1$, one complete revolution around the rim of the circle is equivalent to 2π units and, consequently, $P(t_0) = P(t_0 + 2n\pi) : n = \pm1, \pm2, \pm3\ldots$. We also have that $P(\alpha_0) = P(360^0 - \alpha_0)$. For $\sin(t_0)$, this translates to $\sin(t_0) = \sin(t_0 + 2n\pi)$ or $\sin(\alpha_0) = \sin(360^0 - \alpha_0)$. Identical behavior is exhibited by the five remaining trigonometric functions.

In general, trigonometric functions cycle through the same values over and over again as the independent variable t indefinitely increases on the interval $[0, \infty)$ or, by the same token, indefinitely decreases on the interval $(-\infty, 0]$, corresponding to repeatedly revolving around the rim of the unit circle in ***Figure 3.15***.

This one characteristic alone suggests that (at least in theory) trigonometric functions can be used to model the behavior of any natural or contrived phenomena having an oscillating or repeating action over time such as the induced ground-wave motion of earthquakes or the induced atmospheric-wave motion due to various types of music or voice transmission.

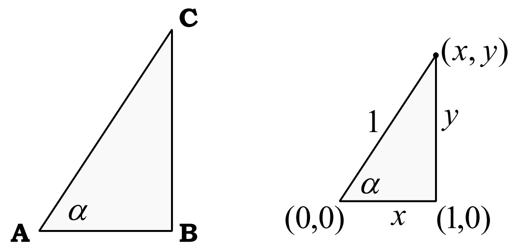

Figure 3.16: Trigonometry via General Right Triangle

The six trigonometric functions also can be defined using a general right triangle. Let ΔABC be an arbitrary right triangle with angle α as shown on the previous page in **Figure 3.16**. The triangle ΔABC is similar to triangle $\Delta\{(0,0),(1,0),(x,y)\}$. This right-triangle similarity guarantees:

1] $\sin(\alpha) = \frac{y}{1} = \frac{\overline{BC}}{\overline{AC}}$ 2] $\csc(\alpha) = \frac{1}{y} = \frac{\overline{AC}}{\overline{BC}}$

3] $\cos(\alpha) = \frac{x}{1} = \frac{\overline{AB}}{\overline{AC}}$ 4] $\sec(\alpha) = \frac{1}{x} = \frac{\overline{AC}}{\overline{AB}}$

5] $\tan(\alpha) = \frac{y}{x} = \frac{\overline{BC}}{\overline{AB}}$ 6] $\cot(\alpha) = \frac{x}{y} = \frac{\overline{AB}}{\overline{AC}}$

The following four relationships are immediately evident from the above six definitions. These four relationships will comprise our *primitive trigonometric identities*.

1] $\csc(\alpha) = \dfrac{1}{\sin(\alpha)}$ 2] $\sec(\alpha) = \dfrac{1}{\cos(\alpha)}$

3] $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ 4] $\cot(\alpha) = \dfrac{\cos(\alpha)}{\sin(\alpha)}$

Since the four trigonometric functions $\csc(\alpha)$, $\sec(\alpha)$, $\tan(\alpha)$, and $\cot(\alpha)$ reduce to nothing more than combinations of sines and cosines, one only needs to know the value of $\sin(\alpha)$ and $\cos(\alpha)$ for a given α in order to evaluate these remaining four functions. In practice, trigonometry is accomplished by table look up. Tables for the six trigonometric function values versus α are set up for angles from 0^0 to 90^0 in increments of $10'$ or less. Various trigonometric relationships due to embedded symmetry in the unit circle were used to produce trigonometric functional values corresponding to angles ranging from 90^0 to 360^0. Prior to 1970, these values were manually created using a book of mathematical tables. Nowadays, the painstaking operation of table lookup is transparent to the user via the instantaneous modern electronic calculator.

Our second pillar is a group of three identities, collectively called the *Pythagorean identities*. The Pythagorean identities are a direct consequence of the Pythagorean Theorem. From the Pythagorean Theorem and the definitions of $\sin(t)$ and $\cos(t)$, we immediately see that

$$\sin^2(t) + \cos^2(t) = 1 .$$

This is the first Pythagorean identity. Dividing the identity $\sin^2(t)+\cos^2(t)=1$, first by $\cos^2(t)$, and a second time by $\sin^2(t)$ gives

$$\tan^2(t)+1=\sec^2(t)\ \& \\ \cot^2(t)+1=\csc^2(t).$$

These are the second and third Pythagorean identities. All three Pythagorean identities are extensively used in trigonometric analysis in order to convert from one functional form to another on an as-needed basis when dealing with various real-world problems involving angles and measures of distances. In Section 4.3, we will explore a few of these fascinating real-world applications.

The *addition formulas* for $\cos(\alpha\pm\beta)$, $\sin(\alpha\pm\beta)$ and $\tan(\alpha\pm\beta)$ in terms of $\cos(\alpha)$, $\sin(\alpha)$, $\tan(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$, and $\tan(\beta)$ comprise as a group the third pillar of trigonometry.

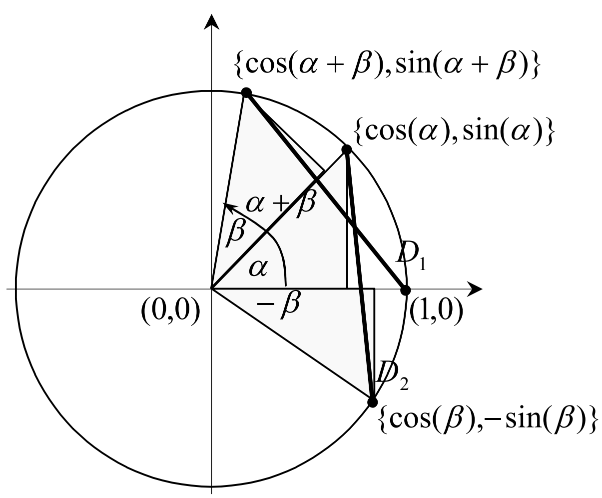

Figure 3.17: The Cosine of the Sum

As with any set of identities, the addition formulas provide important conversion tools for manipulation and transforming trigonometric expressions into needed forms. **Figure 3.17** on the previous page is our starting point for developing the addition formula for the quantity $\cos(\alpha+\beta)$, which states

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta).$$

Using **Figure 3.17**, we develop the addition formula via five steps.

$\overset{1}{\mapsto}$: Since the angle $\alpha+\beta$ in the first quadrant is equal to the angle $\alpha+(-\beta)$ bridging the first and fourth quadrants, we have the distance equality

$$D_1 = D_2.$$

$\overset{2}{\mapsto}$: Use the Pythagorean Theorem (expressed in analytic geometry form via the *2-D Distance Formula*) to capture this equality in algebraic language.

$$D_1 = D_2 \Rightarrow$$
$$D[\{\cos(\alpha+\beta),\sin(\alpha+\beta)\},(1,0)] =$$
$$D[\{\cos(\alpha),\sin(\alpha)\},\{\cos(\beta),-\sin(\beta)\}] \Rightarrow$$
$$\sqrt{\{\cos(\alpha+\beta)-1\}^2+\{\sin(\alpha+\beta)-0\}^2} =$$
$$\sqrt{\{\cos(\alpha)-\cos(\beta)\}^2+\{\sin(\alpha)-[-\sin(\beta)]\}^2}$$

$\overset{3}{\mapsto}$: Square both sides of the last expression:

$$\{\cos(\alpha+\beta)-1\}^2+\{\sin(\alpha+\beta)-0\}^2 =$$
$$\{\cos(\alpha)-\cos(\beta)\}^2+\{\sin(\alpha)-[-\sin(\beta)]\}^2$$

$\overset{4}{\mapsto}$: Square where indicated.

$$\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) =$$
$$\cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) +$$
$$\sin^2(\alpha) + 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)$$

$\overset{5}{\mapsto}$: Use the Pythagorean identity $\sin^2(\phi)+\cos^2(\phi)=1$, *true for any common angle* ϕ, to simply to completion.

$$[\cos^2(\alpha+\beta) + \sin^2(\alpha+\beta)] - 2\cos(\alpha+\beta) + 1 =$$
$$[\cos^2(\alpha) + \cos^2(\beta)] - 2\cos(\alpha)\cos(\beta) +$$
$$[\sin^2(\alpha) + \sin^2(\beta)] + 2\sin(\alpha)\sin(\beta) \Rightarrow$$
$$[1] - 2\cos(\alpha+\beta) + 1 =$$
$$[1] - 2\cos(\alpha)\cos(\beta) + [1] + 2\sin(\alpha)\sin(\beta) \Rightarrow$$
$$2 - 2\cos(\alpha+\beta) = 2 - 2\cos(\alpha)\cos(\beta) +$$
$$2\sin(\alpha)\sin(\beta) \Rightarrow$$
$$-2\cos(\alpha+\beta) = -2\cos(\alpha)\cos(\beta)$$
$$+2\sin(\alpha)\sin(\beta) \Rightarrow$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \therefore$$

The last development, though lengthy, illustrates the combined power of algebra and existing trigonometric identities in order to produce new relationships for the trigonometric functions. Replacing β with $-\beta$ and noting that

$$\cos(\beta) = \cos(-\beta) \;\&$$
$$\sin(\beta) = -\sin(\beta)$$

from **Figure 3.17** immediately gives the companion formula

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \therefore$$

Figure 3.17 can be used as a jumping-off point for an alternate method for simultaneously developing addition formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. In **Figure 3.18**, the point $\{\cos(\alpha+\beta), \sin(\alpha+\beta)\}$ is decomposed into components.

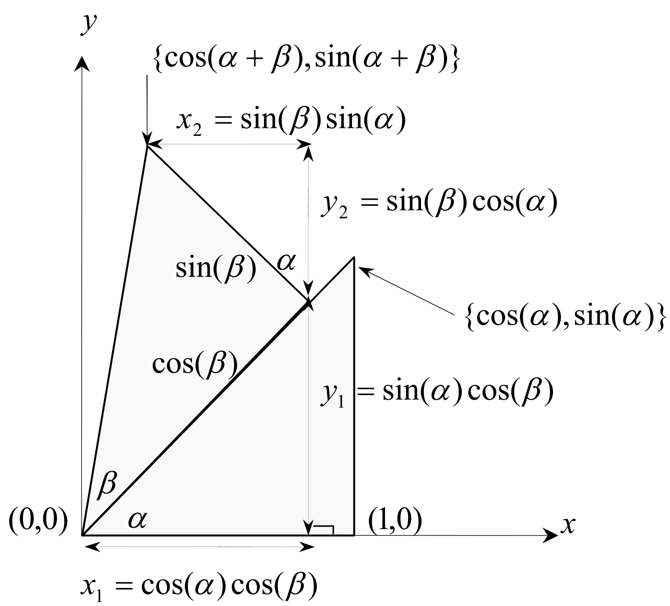

Figure 3.18: An Intricate Trigonometric Decomposition

This is done by using the fundamental definitions of $\sin(\gamma)$ and $\cos(\gamma)$ based on both general right triangles and right triangles having a hypotenuse of unit length as shown in **Figure 3.16**. *The reader is asked to fill in the details on how the various side-lengths in* ***Figure 3.18*** *are obtained—a great practice exercise for facilitating understanding of elementary trigonometric concepts.*

From **Figure 3.18**, we have that

$$\cos(\alpha+\beta)=x_1-x_2 \Rightarrow$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \therefore$$

and

$$\sin(\alpha+\beta)=y_1+y_2 \Rightarrow$$
$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha) \therefore$$

Using the trigonometric relationships $\cos(\beta)=\cos(-\beta)$ and $\sin(\beta)=-\sin(\beta)$ a second time quickly leads to the companion formula for $\sin(\alpha-\beta)$:

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha) \therefore .$$

The addition formula for $\tan(\alpha+\beta)$ is obtained from the addition formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in the following fashion

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \Rightarrow$$
$$\tan(\alpha+\beta)=\frac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \Rightarrow$$
$$\tan(\alpha+\beta)=\frac{\dfrac{\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta)}}{\dfrac{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \Rightarrow$$
$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} \therefore$$

We will leave it to the reader to develop the companion addition formula for $\tan(\alpha-\beta)$.

The remaining two pillars are the *Law of Sines* and the *Law of Cosines*. Both laws are extensively used in surveying work and remote measuring of inaccessible distances, applications to be discussed in Section 4.3.

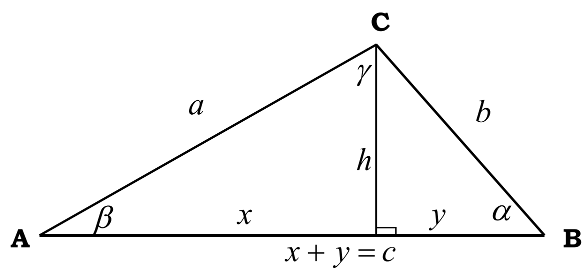

Figure 3.19: Setup for the Law of Sines and Cosines

Figure 3.19 is the setup diagram for both the Law of Sines and the Law of Cosines. Let ΔABC be a general triangle and drop a perpendicular from the apex as shown. Then, for the Law of Sines we have—by the fundamental definition of $\sin(\alpha)$ and $\sin(\beta)$ based on a general right triangle—that

$$\overset{1}{\mapsto}: \frac{h}{b} = \sin(\alpha) \Rightarrow$$

$$h = b\sin(\alpha)$$

$$\overset{2}{\mapsto}: \frac{h}{a} = \sin(\beta) \Rightarrow$$

$$h = a\sin(\beta)$$

$$\overset{3}{\mapsto}: b\sin(\alpha) = a\sin(\beta) \Rightarrow$$

$$\frac{b}{\sin(\beta)} = \frac{a}{\sin(\alpha)} \therefore$$

The last equality is easily extended to include the third angle γ within ΔABC leading to our final result.

Law of Sines

$$\frac{b}{\sin(\beta)} = \frac{a}{\sin(\alpha)} = \frac{c}{\sin(\gamma)}$$

The ratio of the sine of the angle to the side opposite the angle remains constant within a general triangle.

To develop the Law of Cosines, we proceed as follows using the same triangle ΔABC as a starting point and recalling that $h = b\sin(\alpha)$.

$\overset{1}{\mapsto}$: Solve for y and x in terms of the angle α

$$\frac{y}{b} = \cos(\alpha) \Rightarrow y = b\cos(\alpha) \Rightarrow$$
$$x = c - y = c - b\cos(\alpha)$$

$\overset{2}{\mapsto}$: Use the Pythagorean Theorem to complete the development.

$$x^2 + h^2 = a^2 \Rightarrow$$
$$[c - b\cos(\alpha)]^2 + [b\sin(\alpha)]^2 = a^2 \Rightarrow$$
$$c^2 - 2bc\cos(\alpha) + b^2\cos^2(\alpha) + b^2\sin^2(\alpha) = a^2 \Rightarrow$$
$$c^2 - 2bc\cos(\alpha) + b^2 = a^2 \Rightarrow$$
$$a^2 = c^2 + b^2 - 2bc\cos(\alpha) \therefore$$

The last equality is easily extended to include the third angle γ, leading to our final result on the next page.

Law of Cosines

$$a^2 = c^2 + b^2 - 2bc\cos(\alpha)$$

The square of the side opposite the angle is equal to the sum of the squares of the two sides bounding the angle minus twice their product multiplied by the cosine of the bounded angle.

Similar expressions can be written for the remaining two sides. We have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\gamma) \\ b^2 &= a^2 + c^2 - 2ac\cos(\beta) \end{aligned}.$$

The Law of Cosines serves as a generalized form of the Pythagorean Theorem. For if any one of the three angles (say α in particular) is equal to 90^0, then $\cos(\alpha) = 0$ implying that

$$\begin{aligned} a^2 &= c^2 + b^2 - 2bc\{0\} \Rightarrow \\ a^2 &= c^2 + b^2 \end{aligned}$$

In closing we will say that trigonometry in itself is a vast topic that justifies its own course. This is indeed how it is taught throughout the world. A student is first given a course in elementary trigonometry somewhere in high school or early college. From there, advanced topics—*such as Fourier analysis of waveforms and transmission phenomena*—are introduced on an as-needed basis throughout college and graduate school. *Yet, no matter how advanced either the subject of trigonometry or associated applications may become, all levels of modern trigonometry can be directly traced back to the Pythagorean Theorem.*

3.8) Fermat's Line in the Sand

Chapters 1, 2, and 3 have turned the Pythagorean Theorem 'every which way but loose'. For in these pages, we have examined it from a variety of different aspects, proved it in a number of different ways, and developed some far-reaching applications and extensions. In order to provide a fitting capstone, we close Chapter 4 with a mathematical line in the sand, a world-renown result that not only states *that which is impossible*, but also subtlety implies that the Pythagorean Theorem is the only example of *the possible*.

Pierre de Fermat, a Frenchman, was born in the town of Montauban in 1601. Fermat died in 1665. Like Henry Perigal whom we met in Section 2.8, Fermat was a dedicated amateur mathematician by passion, with his full-time job being that of a mid-level administrator for the French government. It was Fermat's passion—over and above his official societal role—that produced one of the greatest mathematical conjectures of all time, *Fermat's Last Theorem*. This theorem was to remain unproved until the two-year period 1993-1995 when Dr. Andrew Wiles, an English mathematician working at Princeton, finally verified the general result via collaboration with Dr. Richard N. Taylor of The University of California at Irvine.

Note: A PBS special on Fermat's Last Theorem revealed that Dr. Wiles had a life-long passion, starting at the age of 12, to be the mathematician to prove Fermat's Last Theorem. Dr. Wiles was in his early forties at the time of program taping. Wiles proof of Fermat's Last Theorem is 121 pages long, which exceed the combined length of Chapters 2 through 4.

So what exactly is *Fermat's Last Theorem*? Before we answer this question, we will provide a little more historical background. A Diophantine equation is an algebra equation whose potential solutions are integers and integers alone. They were first popularized and studied extensively by the Greek mathematician Diophantus (250 ACE) who wrote an ancient text entitled Arithmetica.

An example of a Diophantine equation is the Pythagorean relationship $x^2 + y^2 = z^2$ where the solutions x, y, z are restricted to integers. We examined the problem of finding integer solutions to $x^2 + y^2 = z^2$ in Section 4.2 when we studied Pythagorean Triples. Likewise, Diophantus himself examined Pythagorean Triples and wrote about them in his Arithmetica. The subject of Pythagorean Triples was the focus of Fermat's studies sometime in 1637 (as the story goes) when he pondered a natural extension.

Fermat's Fundamental Question

Since many sets of three integers x, y, z exist where $x^2 + y^2 = z^2$, could it be that there exist a set of three integers x, y, z such that $x^3 + y^3 = z^3$? What about $x^4 + y^4 = z^4$, etc.

For the cubic relationship $x^3 + y^3 = z^3$, one can come tantalizingly close as the following five examples show:

$$\begin{aligned} &6^3 + 8^3 = 9^3 - 1 \\ &71^3 + 138^3 = 144^3 - 1 \\ &135^3 + 138^3 = 172^3 - 1 \\ &791^3 + 812^3 = 1010^3 - 1 \\ &2676^3 + 3230^3 = 3753^3 - 1 \end{aligned}$$

But as we say in present times, almost does not count except when one is tossing horseshoes or hand grenades. Fermat must have quickly come to the realization that $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$ and companions constituted a series of Diophantine impossibilities, for he formulated the following in the margin of his copy of Arithmetica.

Fermat's Last Theorem

The Diophantine equation

$$x^n + y^n = z^n$$

where x, y, z, and n are all integers,

has no nonzero solutions for $n > 2$.

Fermat also claimed to have proof, but, alas, it was too large to fit in the margin of his copy of Arithmetica! Fermat's Theorem and the tantalizing reference to a proof were not to be discovered until after Fermat's death in 1665. Fermat's son, Clement-Samuel, discovered his father's work concerning Diophantine equations and published an edition of Arithmetica in 1670 annotated with his father's notes and observations.

The impossible integer relationship $x^n + y^n = z^n : n > 2$ became known as Fermat's Last Theorem—a theorem that could not be definitively proved or disproved by counterexample for over 300 years until Wiles closed this chapter of mathematical history in the two-year span 1993-1995. To be fair, according to current historians, Fermat probably had a proof for the case $n = 3$ and the case $n = 4$. But a general proof for all $n > 2$ was probably something way out of reach with even the best mathematical knowledge available in Fermat's day.

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The Pythagorean relationship $x^2 + y^2 = z^2$ forever stands as the only solvable Diophantine equation having exactly three variables and like powers. We will call this *'Fermat's Line in the Sand'*. Then again, suppose we add more terms on the left-hand side. Is Diophantine equality possible for power sums having a common exponent $n > 2$, for example, a nontrivial quartet of integers x, y, z, w where $x^3 + y^3 + z^3 = w^3$?

The German mathematician Leonhard Euler (1707,1783) studied this issue and made an educated guess. In his famous *Euler's Conjecture*, he simply stated that the number of terms on the left-hand side needed to guarantee a Diophantine solution was equal to the power involved.

Euler's Conjecture

Suppose $\{x_i : i = 1, k\}$ is a solution set to the Diophantine equation $\sum_{i=1}^{k} x_i^{\,n} = z^n$ where x_i are all integers.

Then we must have that $k \geq n$.

At first glance, Euler's Conjecture seems reasonable. More terms provide the additional 'degrees of freedom' or 'wiggle room' needed to accommodate the more restrictive higher powers, better raising (no pun intended) one's chances of producing an equality. And indeed, equalities are found. For we have

$$3^2 + 4^2 = 5^2$$
$$3^3 + 4^3 + 5^3 = 6^3$$
$$30^4 + 120^4 + 315^4 + 272^4 = 353^4 \quad ,$$
$$43^5 + 57^5 + 7^5 + 80^5 + 100^5 = 107^5$$

all examples of Euler's Conjecture.

However, Euler's Conjecture did not stand the test of time as Fermat's Last Theorem did. In 1966, Lander and Parker found a counterexample for $n = 5$:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Two counterexamples for $n = 4$ followed in 1988. The first was discovered by Noam Elkies of Harvard. The second, the smallest possible for a quartet of numbers raised to the fourth power, was discovered by Roger Frye of Thinking Machines Corporation.

$$2{,}682{,}440^4 + 15{,}465{,}639^4 + 187{,}960^4 = 20{,}615{,}673^4$$

$$95{,}800^4 + 217{,}519^4 + 414{,}560^4 = 422{,}481^4$$

Today, power sums—both equal and non-equal—provide a source of mathematical recreation of serious and not-so-serious amateurs alike. A typical 'challenge problem' might be as follows:

Find six positive integers u, v, w, x, y, z *satisfying the power sum* $u^7 + v^7 + w^7 + x^7 = y^7 + z^7$ *where the associated linear sum* $u + v + w + x + y + z$ *is minimal.*

Table 3.7 on the next page displays just a few of the remarkable examples of the various types of *power sums*. With this table, we close Chapter 3 and our examination of select key spin-offs of the Pythagorean Theorem.

→

#	EXPRESSION
1	$6^3 = 3^3 + 4^3 + 5^3$
2	$371 = 3^3 + 7^3 + 1^3$
3	$407 = 4^3 + 0^3 + 7^3$
4	$135 = 1^1 + 3^2 + 5^3$
5	$175 = 1^1 + 7^2 + 5^3$
6	$1634 = 1^4 + 6^4 + 3^4 + 4^4$
7	$3435 = 3^3 + 4^4 + 3^3 + 5^5$
8	$598 = 5^1 + 9^2 + 8^3$
9	$54{,}748 = 5^5 + 4^5 + 7^5 + 4^5 + 8^5$
10	$169 = 13^2$ and $961 = 31^2$
11	$2025 = 45^2$ and $20 + 25 = 45$
12	$4913 = 17^3$ and $4 + 9 + 1 + 3 = 17$
13	$1676 = 1^1 + 6^2 + 7^3 + 6^4 = 1^5 + 6^4 + 7^3 + 6^2$
14	$1233 = 12^2 + 33^2$
15	$990100 = 990^2 + 100^2$
16	$94122353 = 9412^2 + 2353^2$
17	$2646798 = 2^1 + 6^2 + 4^3 + 6^4 + 7^5 + 9^6 + 8^7$
18	$2427 = 2^1 + 4^2 + 2^3 + 7^4$
19	$221859 = 22^3 + 18^3 + 59^3$
20	$343 = (3 + 4)^3$

Table 3.7: Power Sums

4) Pearls of Fun and Wonder

Pearls of ancient mind and wonder,
Time will never pillage, plunder,
Or give your soul to the worm—
Or worse yet, for nerds to keep
With their mental treasures deep
Where secret squares are cut asunder.
So, come good fun, have a turn,
Bring your gold, build and learn!
April 2005

4.1) Sam Lloyd's Triangular Lake

Sam Lloyd was a famous American creator of puzzles, tricks and conundrums who produced most of his masterpieces in the late 1800s. Many of Lloyd's puzzles have survived and actually thrived, having found their way into modern puzzle collections. Martin Gardner, of Scientific American fame, has been a tremendous preserver of Lloyd's legacy to recreational mathematics. Recreational mathematics, one might classify that phrase as an oxymoron. Yet, back in the late 1800s, Lloyd's heyday, people actually worked puzzles for evening relaxation, much like we moderns watch TV or play video games. The need to relax has always been there; how people fulfill the need is more a function of the era in which we live and the available technology that enables us to recreate.

One of Sam Lloyd's famous creations is his Triangular Lake. Lloyd subtlety gives his readers two choices: solution by sweat and brute force, or solution by cleverness and minimal effort. The clever solution requires use of the Pythagorean Theorem. What follows is Lloyd's original statement:

*"The question I ask our puzzlists is to determine how many acres there would be in that triangular lake, surrounded as shown in **Figure 4.1** by square plots of 370, 116 and 74 acres.*

The problem is of peculiar interest to those of a mathematical turn, in that it gives a positive and definite answer to a proposition, which, according to usual methods produces one of those ever-decreasing, but never-ending decimal fractions.

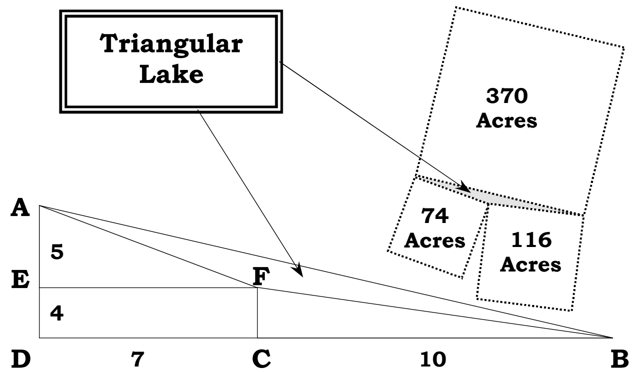


Figure 4.1: Triangle Lake and Solution

In the year 2006, the Triangular Lake does not require a great deal of original thought to solve. Nowadays, we would say it is a 'cookie cutter' problem and requires no sweat whatsoever! One simply needs to apply Heron's formula for triangular area, $A=\sqrt{s(s-a)(s-b)(s-a)}$ using a modern electronic calculator. The answer is available after a few keystrokes.

$$\overset{1}{\mapsto}: a=\sqrt{370}=19.23538, b=\sqrt{116}=10.77033,\&$$

$$c=\sqrt{74}=8.60233$$

$$\overset{2}{\mapsto}: s=\frac{a+b+c}{2}=\frac{\sqrt{370}+\sqrt{116}+\sqrt{74}}{2}\Rightarrow$$

$$s=19.30402$$

$$\overset{3}{\mapsto}: A=\sqrt{s(s-a)(s-b)(s-a)}\Rightarrow$$

$$A=\sqrt{19.30402(0.06864)(8.53369)(10.70169)}=11.00037$$

We have achieved five-digit accuracy in ten seconds. Electronic calculators are indeed wonderful, but how would one obtain this answer before an age of technology? Granted, Heron's formula was available in 1890, but the formula needed numbers to work, numbers phrased in terms of decimal equivalents for those wary of roots. To produce these decimal equivalents would require the manual, laborious extraction of the three square roots $\sqrt{370}$, $\sqrt{116}$, and $\sqrt{74}$, not to mention extraction the final root in Heron's Formula itself. Thus, a decimal approach was probably not a very good fireside option in 1890. One would have to inject a dose of cleverness, a lost art in today's brute-force electronic world.

Lloyd's original solution entails a masterful decomposition of three right triangles. In **Figure 4.1**, the area of the triangular lake is the area of ΔABF, which we denote as $\Delta_A ABF$.

Now:

$$\Delta_A ABF = \Delta_A ABD - \Delta_A AFE - \Delta_A FBC - R_A EFCD.$$

For ΔABD, we have that

$$\overline{AB}^2 = \overline{DA}^2 + \overline{DB}^2 \Rightarrow$$

$$\overline{AB}^2 = 9^2 + 17^2 = 370 \Rightarrow$$

$$\overline{AB} = \sqrt{370}$$

Notice Lloyd was able to cleverly construct a right triangle with two perpendicular sides (each of integral length) producing a hypotenuse of the needed length $\sqrt{370}$. This was not all he did!

For ΔAFE, we have that

$$\overline{AF}^2 = \overline{EA}^2 + \overline{EF}^2 \Rightarrow$$

$$\overline{AF}^2 = 5^2 + 7^2 = 74 \Rightarrow \overline{AF} = \sqrt{74}$$

Finally, for ΔFBC

$$\overline{FB}^2 = \overline{CF}^2 + \overline{CB}^2 \Rightarrow$$

$$\overline{FB}^2 = 4^2 + 10^2 = 116 \Rightarrow \overline{FB} = \sqrt{116}$$

Thus, our three triangles ΔABD, ΔAFE, and ΔFBC form the boundary of the Triangular Lake. The area of the Triangular Lake directly follows

$$\Delta_A ABF = \Delta_A ABD - \Delta_A AFE - \Delta_A FBC - R_A EFCD \Rightarrow$$
$$\Delta_A ABF = \tfrac{1}{2}(9)(17) - \tfrac{1}{2}(5)(7) - \tfrac{1}{2}(4)(10) - (4)(7) = 11 \therefore$$

The truth of Lloyd's problem statement is now evident: *to those of a mathematical turn, the number* 11 *is a very positive and definitive answer not sullied by an irrational decimal expansion.*

One might ask if it is possible to 'grind through' Heron's formula and arrive at 11 using the square roots as is. Obviously, due to the algebraic complexity, Lloyd was counting on the puzzler to give up on this more brute-force direct approach and resort to some sort of cleverness. However, it is possible to grind! Below is the computational sequence, an algebraic nightmare indeed.

Note: I happen to agree with Lloyd's 'forcing to cleverness' in that I have given students a similar computational exercise for years. In this exercise requiring logarithm use, electronic calculators are deliberately rendered useless due to overflow or underflow of derived numerical quantities. Students must resort to 'old fashion' clever use of logarithms in order to complete the computations involving extreme numbers.

$$\overset{1}{\mapsto}: a = \sqrt{370}, b = \sqrt{116}, \& c = \sqrt{74}$$

$$\overset{2}{\mapsto}: s = \frac{a+b+c}{2} = \frac{\sqrt{370}+\sqrt{116}+\sqrt{74}}{2}$$

$$\overset{3}{\mapsto}: A=\sqrt{s(s-a)(s-b)(s-a)} \Rightarrow$$

$$A=\frac{1}{4}\begin{Bmatrix}(\sqrt{116}+\sqrt{74}+\sqrt{370})(\sqrt{116}+\sqrt{74}-\sqrt{370})\cdot \\ (\sqrt{116}+\sqrt{370}-\sqrt{74})(\sqrt{370}+\sqrt{74}-\sqrt{116})\end{Bmatrix}^{\frac{1}{2}} \Rightarrow$$

$$A=\frac{1}{4}\begin{Bmatrix}([\sqrt{116}+\sqrt{74}]^2-370)\cdot \\ (\sqrt{370}+\sqrt{74}+\sqrt{116})(\sqrt{370}+\sqrt{74}-\sqrt{116})\end{Bmatrix}^{\frac{1}{2}} \Rightarrow$$

$$A=\frac{1}{4}\begin{Bmatrix}(116+2\sqrt{116}\sqrt{74}+74-370)\cdot \\ (370+\sqrt{370}\sqrt{74}-\sqrt{370}\sqrt{116} \\ -\sqrt{74}\sqrt{370}-74+\sqrt{74}\sqrt{116}+ \\ \sqrt{116}\sqrt{370}+\sqrt{116}\sqrt{74}-116)\end{Bmatrix}^{\frac{1}{2}} \Rightarrow$$

$$A=\frac{1}{4}\left\{(2\sqrt{116}\sqrt{74}-180)\cdot(2\sqrt{116}\sqrt{74}+180)\right\}^{\frac{1}{2}} \Rightarrow$$

$$A=\frac{1}{4}\left\{4(116)(74)-180^2\right\}^{\frac{1}{2}}=\frac{1}{4}\left\{34{,}336-32{,}400\right\}^{\frac{1}{2}} \Rightarrow$$

$$A=\frac{1}{4}\left\{1936\right\}^{\frac{1}{2}}=\frac{44}{4}=11\,\therefore$$

4.2) Pythagorean Magic Squares

A *magic square* is an array of counting numbers (positive integers) geometrically arranged in a square. **Figure 4.2** shows a 4X4 magic square. Where is the magic in this square?

1	15	6	12
8	10	3	13
11	5	16	2
14	4	9	7

Figure 4.2: Pure and Perfect 4X4 Magic Square

Answer: if one adds the four numbers in any one row, any one column, or along any one of the two diagonals—totaling 10 different ways—one will obtain the same number 34 for each sum so done, called the *magic sum*.

Normal or *Pure Magic Squares* are magic squares where the numbers in the little squares are consecutive counting numbers starting with one. *Perfect 4X4 Magic Squares* are magic squares having many additional four-number patterns that sum to 34, *such as the four corners of any smaller square embedded in the 4x4 square*. **Figure 4.3** depicts a sampling of four-number patterns that sum to 34 for the magic square shown in **Figure 4.2**.

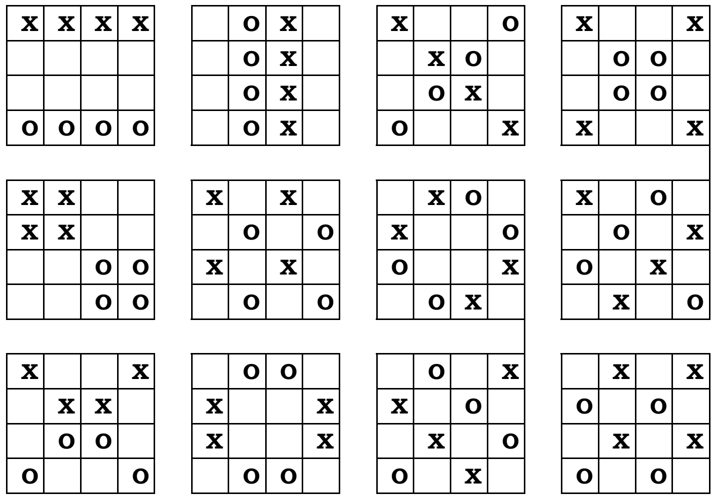


Figure 4.3: 4X4 Magic Patterns

The 4X4 magic square has within it several sum-of-squares and sum-of-cubes equalities that provide additional examples of power sums as discussed in Section 3.8.

1. The sum of the squares in the first row equals the sum of the squares in the fourth row: A similar equality holds for the second and third rows.

$$1^2 + 15^2 + 6^2 + 12^2 = 14^2 + 4^2 + 9^2 + 7^2$$

2. The sum of the squares in the first column equals the sum of the squares in the fourth column. A similar equality holds for the second and third columns.

$$1^2 + 8^2 + 11^2 + 14^2 = 12^2 + 13^2 + 2^2 + 7^2$$

3. The sum of the cubes for the numbers on the two diagonals equals the sum of the cubes for the numbers remaining within the square:

$$1^3 + 10^3 + 16^3 + 7^3 + 14^3 + 5^3 + 3^3 + 12^3 =$$

$$15^3 + 6^3 + 13^3 + 2^3 + 9^3 + 4^3 + 11^3 + 8^3$$

The reader is invited to verify all three power sums!

Magic squares of all types have intrigued math enthusiasts for decades. What you see below in **Figure 4.4** is a Pythagorean masterpiece that couples three magic squares of different sizes (or *orders*) with the truth of the Pythagorean Theorem. Royal Vale Heath, a well-known British puzzle maker, created this wonder in England prior to 1930. Of the fifty numbers used in total, none appears more than once.

<table>
<tr><th colspan="12">THE PYTHAGOREAN 3-4-5 WONDER SET OF THREE MAGIC SQUARES</th></tr>
<tr><td colspan="3" rowspan="3">3X3 magic sum is 174. Square the sum of nine numbers to obtain 272,484</td><td colspan="4" rowspan="2">4X4 magic sum is also 174. Square the sum of sixteen numbers to obtain 484,416</td><td colspan="5">5X5 magic sum is also 174. Square the sum of twenty-five numbers to obtain 756,900</td></tr>
<tr><td>16</td><td>22</td><td>28</td><td>34</td><td>74</td></tr>
<tr><td>36</td><td>43</td><td>48</td><td>47</td><td>33</td><td>73</td><td>20</td><td>21</td><td>27</td></tr>
<tr><td>61</td><td>54</td><td>59</td><td>49</td><td>46</td><td>37</td><td>42</td><td>25</td><td>26</td><td>32</td><td>72</td><td>19</td></tr>
<tr><td>56</td><td>58</td><td>60</td><td>39</td><td>40</td><td>51</td><td>44</td><td>71</td><td>18</td><td>24</td><td>30</td><td>31</td></tr>
<tr><td>57</td><td>62</td><td>55</td><td>50</td><td>45</td><td>38</td><td>41</td><td>29</td><td>35</td><td>70</td><td>17</td><td>23</td></tr>
<tr><td colspan="12">$3^2 + 4^2 = 5^2$ & $272{,}484 + 484{,}416 = 756{,}900$!</td></tr>
</table>

Figure 4.4: Pythagorean Magic Squares

4.3) Earth, Moon, Sun, and Stars

In this section, we move away from recreational use of the Pythagorean Theorem and back to the physical world and universe in which we live. In doing so, we will examine the power of trigonometry as a tool to measure distances. In particular, we are interested in *inaccessible or remote distances*, distances that we cannot 'reach out and touch' in order to measure directly.

The first example is determining the height **H** of the flagpole in front of the local high school. A plethora of American trigonometry students throughout the years have been sent outside to measure the height of the pole, obviously an inaccessible distance unless you entice the little guy to shimmy up the pole and drop a plumb bob. Be careful, for the principal may be looking!

Figure 4.5 illustrates how we measured that old schoolhouse flagpole using the elementary ideas of trigonometry. We walked out a known distance from the base of pole and sighted the angle from the horizontal to the top of the pole.

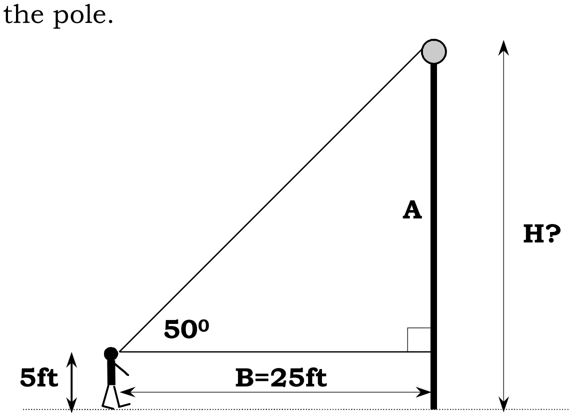


Figure 4.5: The Schoolhouse Flagpole

Note: Students usually performed this sighting with a hand-made device consisting of a protractor and a pivoting soda straw. In actual surveying, a sophisticated instrument called a theodolite is used to accomplish the same end.

Once these two measurements are taken, the height of the flagpole easily follows if one applies the elementary definition of tan, as given in Section 3.7, to the right triangle depicted in **Figure 4.5** to obtain the unknown length **A**.

$$\overset{1}{\mapsto}: \frac{A}{25\,ft} = \tan(50^0) \Rightarrow$$

$$A = (25\,ft)\cdot\tan(50^0) \Rightarrow$$

$$A = (25\,ft)\cdot(1.19176) = 29.79\,ft$$

$$\overset{2}{\mapsto}: H = 29.79\,ft + 5\,ft = 34.79\,ft \therefore$$

A common mistake is the failure to add the height from ground level to the elevation of the sighting instrument.

Figure 4.6 shows a marked increase in complexity over the previous example. Here the objective is to measure the height **H** of a historic windmill that has been fenced off from visitors. To accomplish this measurement via the techniques of trigonometry, two angular measurements are made 25 feet apart resulting in two triangles and associated base angles as shown.

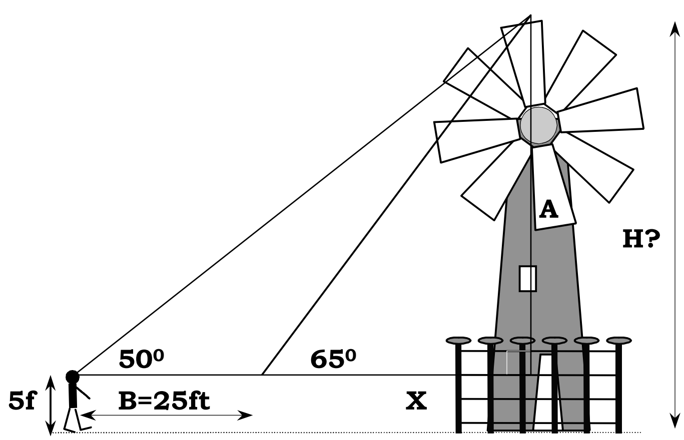


Figure 4.6: Off-Limits Windmill

The four-step solution follows. All linear measurements are in feet.

$$\overset{1}{\mapsto}: \frac{A}{25+X} = \tan(50^0) \Rightarrow$$

$$A = (25+X)\cdot\tan(50^0) \Rightarrow$$

$$A = (25+X)\cdot(1.19176) \Rightarrow$$

$$A = 29.79 + 1.19176X$$

$$\overset{2}{\mapsto}: \frac{A}{X} = \tan(65^0) \Rightarrow$$

$$A = 2.14451X \Rightarrow X = \frac{A}{2.14451}$$

$$\overset{3}{\mapsto}: A = 29.79 + 1.19176X \Rightarrow$$

$$A = 29.79 + 1.19176\left[\frac{A}{2.14451}\right] \Rightarrow$$

$$A = 29.79 + .55573A \Rightarrow$$

$$0.44427A = 29.79 \Rightarrow A = 67.05\,ft.$$

$$\overset{4}{\mapsto}: H = A + 5\,ft = 72.05\,ft \therefore$$

Our next *down-to-earth* example is to find the straight-line distance **L** through a patch of thorns and nettles—most definitely an inaccessible distance—as shown in **Figure 4.7**.

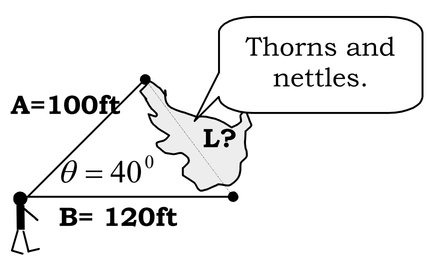


Figure 4.7: Across the Thorns and Nettles

Finding the inaccessible distance **L** requires the measuring of the two accessible distances **A** & **B** and included angle θ. Once these measurements are obtained, the Law of Cosines is directly utilized to find **L**.

$$\overset{1}{\mapsto}: L^2 = A^2 + B^2 - 2AB\cos(\theta) \Rightarrow$$
$$L^2 = 100^2 + 120^2 - 2(100)(120)\cos(40^0) \Rightarrow$$
$$L^2 = 6014.93 \Rightarrow L = 77.556\, ft \therefore$$

Our next measurement, much more ambitious, was first performed by Eratosthenes (275-194 BCE), the Director of the Alexandrian Library in Egypt. Eratosthenes invented an ingenious methodology for determining the earth's circumference. His methodology utilized basic trigonometric principles in conjunction with three underlying assumptions quite advanced for his time:

1) The earth was round
2) When sunrays finally reached the earth after traveling *across the unknown void*, they arrived as parallel beams.
3) Alexandria and the town of Syene (500 miles to the South) fell on the same meridian: an assumption not quite correct as shown in **Figure 4.8**.

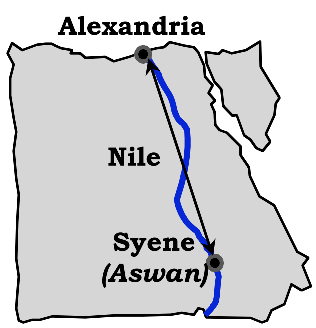


Figure 4.8: Eratosthenes' Egypt

Figure 4.9 depicts Eratosthenes' methodology. A mirror was placed at the bottom of a deep well at Syene. This mirror would reflect back the rays of the noonday sun to an observer during that time of year when the sun was shining directly overhead. This was time correlated with a second observation at Alexandria where the shadow length of an obelisk (of know height) was measured allowing determination of the sun's incident angle.

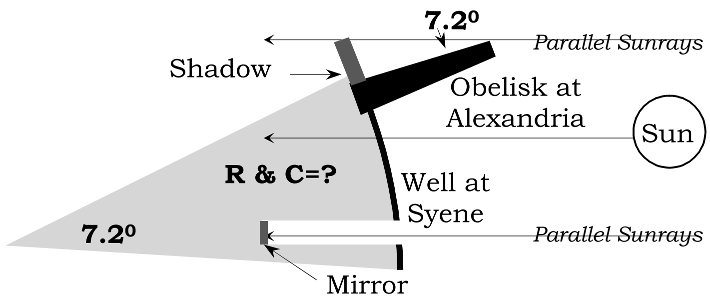


Figure 4.9: Eratosthenes Measures the Earth

Eratosthenes reasoned that the circumferential distance from the obelisk at Alexandria to the well at Syene (500 miles) represented 7.2^0 of the earth's circumference. A simple proportion was then solved allowing the determination of the earth's total circumference **C**. The earth's radius **R** easily followed.

$$\overset{1}{\mapsto}: \frac{7.2^O}{360^O} = \frac{500 miles}{C} \Rightarrow C = 25{,}000 miles$$

$$\overset{2}{\mapsto}: R = \frac{25{,}000 miles}{2\pi} = 3979 miles$$

Eratosthenes came very close to modern measurements (The earth's mean radius is $3959 miles$) by this ingenious method now well over 2000 years old. It is a solid example of mathematics and natural science teaming to produce a milestone of rational thought in the history of humanity.

Once we know the measurement of the earth's radius, we can immediately use the Pythagorean Theorem to calculate the view distance **V** to an unobstructed horizon as a function of the observer's height **H** above the earth's surface, **Figure 4.10**.

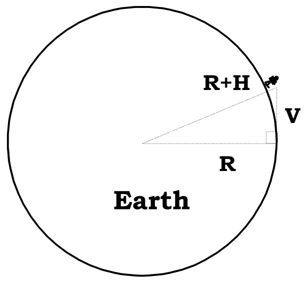


Figure 4.10: View Distance to Earth's Horizon

The view-distance calculation proceeds as follows.

$$V^2 + R^2 = (R+H)^2 \Rightarrow$$
$$V^2 + R^2 = R^2 + 2RH + H^2 \Rightarrow$$
$$V^2 = 2RH + H^2 \Rightarrow$$
$$V = \sqrt{2RH + H^2}$$

Table 4.1 gives select view distances to the horizon as a function of viewer altitude.

H	V		H	V
5 ft.	2.73 miles		100,000 ft	387.71 miles
100 ft	12.24 miles		150 miles	1100 miles
1000ft	38.73 miles		1000 miles	2986 miles
30,000 ft	212.18 miles		**R**=3959 miles	$\sqrt{3}R$

Table 4.1: View Distance versus Altitude

High above the earth, we see the moon hanging in the evening sky at an angle and inclination that varies according to latitude, date, and time. The ancient Greeks were the first to measure the distance to the moon and the moon's diameter. They actually did this using several different ways. We will follow suit by employing the modern Law of Sines, utilizing it one of the same geocentric setups that the Greeks used 2000 years ago in order to make these two astronomical determinations. Today, our instrument of choice would be a theodolite in order to make the needed precision angular measurements. The Greeks actually had an ancient version of the same instrument, called an astrolabe, which allowed them to make the needed measurements in their day. **Figure 4.11** shows our setup for both measurements.

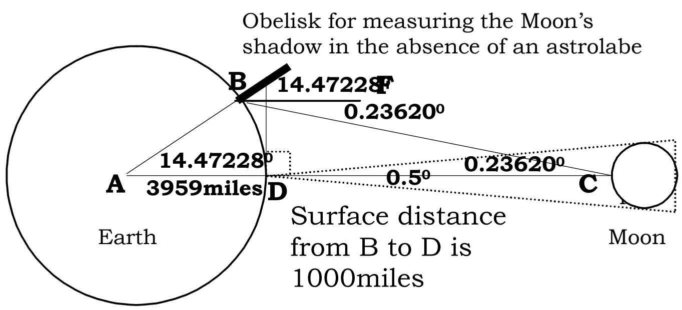


Figure 4.11: Measuring the Moon

In order to measure the distance to the moon, first pick two points **B** and **D** on the earth's surface a known distance apart. **Figure 4.11** suggests 1000 miles, roughly the distance that the Greeks used. Both points need to have either the same latitude or the same longitude. In the United States, two points of equal latitude (such as 40^0N) are probably a tad easily to locate than two points of similar longitude. Point **D** should correspond to a time of night where the *full moon* is straight overhead or squarely in the middle of the ecliptic as sighted by use of an astrolabe with the sighting point squarely in the center of the moon's disk.

Simultaneously, an observer at point **B** would sight the center of the moon's disk and ascertain the exterior angle $\angle FBC$ at shown. Modern email would make this an exciting exercise for amateur astronomers or high school students since these measurements can be communicated instantaneously. The ancient Greeks had no such luxury in that they had to pre-agree as to a date and time that two observers would make the needed measurements. Several weeks after that, information would be exchanged and the calculations performed.

By the Law of Sines, we can find the distance $\overline{BC}$ to the moon per the following computational sequence.

$$\frac{\overline{BC}}{\sin(\angle BAD)} = \frac{\overline{AB}}{\sin(\angle BCD)} \Rightarrow$$

$$\frac{\overline{BC}}{\sin(14.47228^0)} = \frac{3959 miles}{\sin(0.23620^0)} \Rightarrow$$

$$\overline{BC} = \left[\frac{3959 miles}{\sin(0.23620^0)}\right]\sin(14.47228^0) \Rightarrow$$

$$\overline{BC} = 240{,}002.5 miles$$

Notice that the ratio of the distance to the radius of the earth is given by the expression

$$\frac{\overline{BC}}{\overline{AB}} = 60.62\,,$$

a traditional value first obtained by the Greeks after several refinements and iterations.

Once we have the distance to the moon, we can easily calculate the radius of the moon. Stand at point **D** and measure the sweep angle between the moon's two limbs sighting through a diameter.

The commonly accepted value is about $\frac{1}{2}$ degree, obtained by holding out your thumb and, by doing so, barely covering the lunar disk. Taking exactly half of 0.5^0 to complete one very huge right triangle, we obtain

$$\frac{R_{moon}}{240{,}002.5 miles} = \tan(0.25^0) \Rightarrow$$
$$R_{moon} = 1047 miles \Rightarrow$$
$$D_{moon} = 2R_{moon} = 2094 miles$$

The calculated diameter $2094 miles$ is 66 miles shy of the true value. To illustrate the angular sensitivity associated with these results, suppose more sophisticated instrumentation indicates that the sweep angle between the two lunar limbs is actually 0.515^0, an increase of 3%. Revising the previous gives

$$\frac{R_{moon}}{240{,}002.5 miles} = \tan(0.2575^0) \Rightarrow$$
$$R_{moon} = 1078.63 miles \Rightarrow$$
$$D_{moon} = 2157.26 miles$$

now only a couple of miles shy of the true value. When dealing with tiny angles, precision instrumentation is the key to accurate results. However, the underlying Pythagorean infrastructure is still the same four millennia after its inception!

We will now use the distance to the moon to obtain the distance to the sun. Like a series of celestial stepping-stones, one inaccessible distance leads to a second inaccessible distance, each succeeding distance more extreme than the previous one.

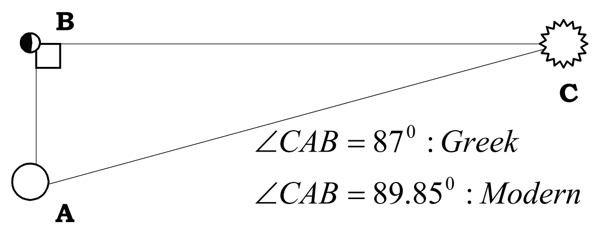


Figure 4.12: From Moon to Sun

Greek astronomers knew that the moon reflected sunlight in order to shine. Using the cosmological model shown in **Figure 4.12**, the Greeks surmised that $\angle ABC = 90^0$ during the first quarter moon (half dark-half light). By constructing the huge imaginary triangle ΔABC, the moon-to-sun distance $\overline{BC}$ could be obtained by the simple formula

$$\frac{\overline{BC}}{\overline{BA}} = \tan(\angle BAC)$$

assuming that the angle $\angle BAC$ could be accurately measured. The Greeks tried and obtained a best value of $\angle BAC = 87^0$, about six thumb widths away from the vertical. Modern instrumentation gives the value as $\angle BAC = 89.85^0$. The next computation is a comparison of both the ancient and modern values for the moon-to-sun distance using the two angular measurements in **Figure 4.12**.

Note: Staring directly into the sun is a very dangerous proposition, so the Greeks were quite naturally limited in their ability to refine 'six thumb widths'.

$$\overset{1}{\mapsto}: \frac{\overline{BC}}{\overline{BA}} = \tan(\angle BAC) \Rightarrow$$

$$\overline{BC} = \overline{BA}\tan(\angle BAC) \Rightarrow$$

$$\overline{BC} = (240{,}002.5)\cdot\tan(87^0) \Rightarrow$$

$$\overline{BC} = 4{,}579{,}520.5 miles$$

$$\overset{2}{\mapsto}: \overline{BC} = (240{,}002.5)\cdot\tan(89.85^0) \Rightarrow$$

$$\overline{BC} = 91{,}673{,}992.7 miles$$

Our 'back-of-the-envelope' computed value has the sun about 382 times further away from the earth than the moon. By contrast, the Greek estimate was about 19 times, a serious underestimate. However, even with an underestimate, it is important to understand that Greek geometry—again, Pythagorean geometry—was being used correctly. The failing was not having sophisticated instrumentation. Knowing the distance to the sun, one can easily obtain the sun's diameter, whose disk also spans approximately 0.5^0.

Note: The sun and moon have the same apparent size in the sky making solar eclipses possible.

$$\frac{R_{Sun}}{91{,}673{,}992.7 miles} = \tan(0.25^0) \Rightarrow$$

$$R_{sun} = 400{,}005.79 miles \Rightarrow$$

$$D_{sun} = 800{,}011.58 miles$$

*Using **Figure 4.11,** we invite the reader to calculate the earth-to-sun distance using the Pythagorean Theorem and the modern value of the moon-to-sun distance shown above.*

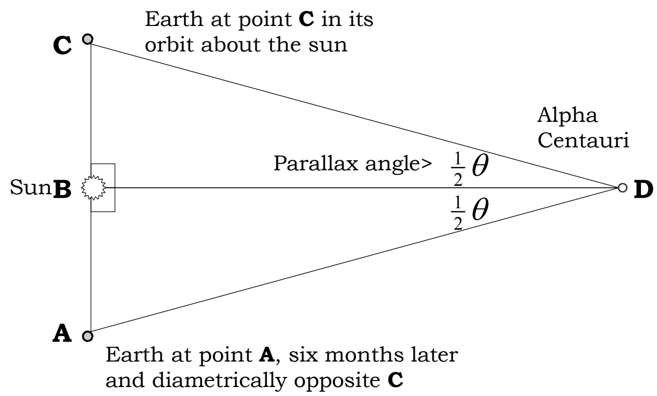


Figure 4.13: From Sun to Alpha Centauri

Earth, Moon, Sun, and Stars, the section title hints at a progressive journey using ever-increasing stepping stones. The Greeks were limited to the known solar system. Starting in the 1800s, increasingly sophisticated astronomical instrumentation made determination of stellar distances possible by allowing the measurement of very tiny angles just a few seconds in size, as depicted via the much magnified angle θ in **Figure 4.13**. From the measurement of tiny angles and the building of huge imaginary interstellar triangles, astronomers could ascertain tremendous distances using a Pythagorean-based method called the *parallax technique*. **Figure 4.13** illustrates the use of the parallax technique to find the distance to our nearest stellar neighbor, Alpha Centauri, about $4.2ly$ (light years) from the sun.

The position of the target star, in this case Alpha Centari, is measured from two diametrically opposite points on the earth's orbit. The difference in angular location against a backdrop of much farther 'fixed' stars is called the parallax angle θ. Half the parallax angle is then used to compute the distance to the star by the expression

$$\overline{BD} = \frac{\overline{AC}}{\tan(\frac{1}{2}\theta)}$$

Even for the closest star Alpha Centauri, parallax angles are extremely small. For $\overline{AC} = 184{,}000{,}000 miles$ and $\overline{BD} = 4.2ly$ or $2{,}463{,}592{,}320{,}000 miles$, we have

$$\tan(\tfrac{1}{2}\theta) = \frac{\overline{AC}}{\overline{BD}} \Rightarrow$$

$$\tan(\tfrac{1}{2}\theta) = \frac{184{,}000{,}000}{2{,}463{,}592{,}320{,}000} = 0.00000747 \Rightarrow$$

$$\tfrac{1}{2}\theta = \tan^{-1}(0.00000747) = 0.00042793^0 \Rightarrow$$

$$\theta = 0.00085586^0$$

The final parallax value is less than one-thousandth of a degree. It converts to just $2.4''$ seconds. Being able to determine angles this small and smaller is a testimony to the accuracy of modern astronomical instrumentation. As **Figure 4.13** would hint, Smaller values of parallax angles imply even greater distances. For example, a star where $\overline{BD} = 600ly$ (Betelgeuse in Orion is $522ly$ from our sun) requires the measurement of a incredibly small parallax angle whose value is $\theta = 0.00000599^0$.

The *instantaneous elliptical diameter* $\overline{AC}$ also needs precision measurement in conjunction with the parallax angle for a given star. In the preceding examples, we rounded $91{,}673{,}992.7$ to the nearest million and doubled, which resulted in $\overline{AC} = 184{,}000{,}000 miles$. For precise work in a research or academic environment, $\overline{AC}$ would need to be greatly refined.

4.4) Phi, PI, and Spirals

Our last section briefly introduces three topics that will allow the reader to decide upon options for further reading and exploration. The first is the Golden Ratio, or Phi. Coequal to the Pythagorean Theorem in terms of mathematical breadth and applicability to the natural world, the Golden Ratio deserves a book in its own right, and indeed several books have been written (see References). In the discussion that follows, we briefly introduce the Golden Ratio and explore two instances where the Golden Ratio finds its way into right and non-right triangles.

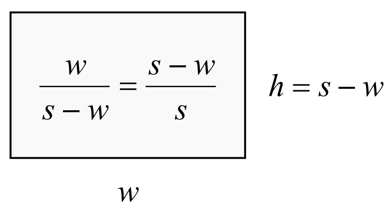


Figure 4.14: The Golden Ratio

Let s be the semi-perimeter of a rectangle whose width and height are in the proportion shown in **Figure 4.14**. This proportion defines the Golden Ratio, praised by artists and scientists alike. The equation in **Figure 4.14** reduces to a quadratic equation via the sequence

$$sw = (s-w)^2 \Rightarrow$$
$$sw = s^2 - 2sw + w^2 \Rightarrow$$
$$s^2 - 3sw + w^2 = 0$$

Solving (left to reader) gives the following values for the width and height:

$$w = \left[\frac{3-\sqrt{5}}{2}\right] s \Rightarrow w = 0.38196s \ \&$$

$$h = s - w = \left[\frac{\sqrt{5}-1}{2}\right] s \Rightarrow w = 0.61804s$$

The Golden Ratio, symbolized by the Greek letter Phi (ϕ), is the reciprocal of $\frac{\sqrt{5}-1}{2}$, given by $\frac{\sqrt{5}+1}{2} = 1.61804$.

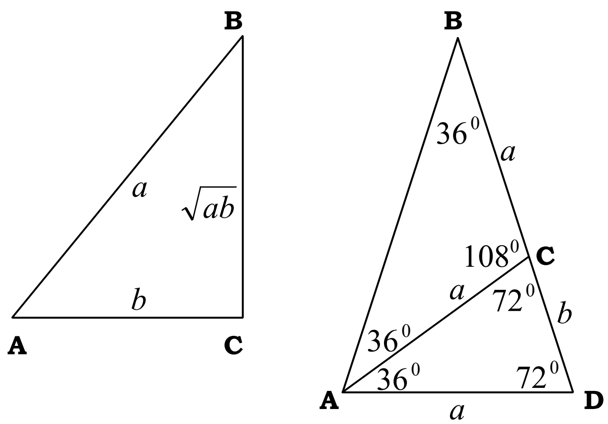


Figure 4.15: Two Golden Triangles

As stated, the Golden Ratio permeates mathematics and science to the same extent as the Pythagorean Theorem. **Figure 4.15** depicts two appearances in a triangular context, making them suitable inclusions for this book.

Triangle ΔABC on the left is a right triangle where the vertical side has length $\sqrt{ab}$ equal to the geometric mean of the hypotenuse a and the horizontal side b.

By the Pythagorean Theorem, we have:

$$a^2 = b^2 + (\sqrt{ab})^2 \Rightarrow$$
$$a^2 - ab - b^2 = 0 \Rightarrow$$
$$a = \left[\frac{1+\sqrt{5}}{2}\right] b \Rightarrow$$
$$a = \phi b \Rightarrow \frac{a}{b} = \phi \therefore$$

This last result can be summarized as follows.

If one short side of a right triangle is the geometric mean of the hypotenuse and the remaining short side, then the ratio of the hypotenuse to the remaining short side is the Golden Ratio ϕ.

In triangle ΔABD to the right, all three triangles ΔABD, ΔABC, and ΔACD are isosceles. In addition, the two triangles ΔABD and ΔACD are similar: $\Delta ABD \approx \Delta ACD$. We have by proportionality rules:

$$\frac{a+b}{a} = \frac{a}{b} \Rightarrow$$
$$a^2 - ab - b^2 = 0 \Rightarrow$$
$$a = \phi b$$

Thus, in **Figure 4.15**, ΔABD has been sectioned as to create the Golden Ratio ϕ between the slant height and base for the two similar triangles ΔABC and ΔACD. **Figure 4.16** aptly displays the inherent, unlabeled beauty of the Golden Ratio when applied to our two triangles.

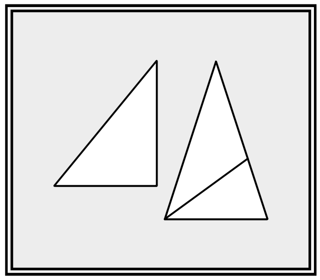


Figure 4.16: Triangular Phi

The history of PI, the *fixed ratio* of the circumference to the diameter for any circle, parallels the history of the Pythagorean Theorem itself. PI is denoted by the Greek letter π. Like the Golden Ratio Phi, PI is a topic of sufficient mathematical weight to warrant a complete book in its own right. As one might suspect, books on PI have already been written. In this volume, we will simply guide the reader to the two excellent 'PI works' listed in the References.

In this Section, we will illustrate just one of many different ways of computing Pi to any desired degree of accuracy. We will do so by way of a nested iterative technique that hinges upon *repeated use of the Pythagorean Theorem*.

Figure 4.17 on the next page shows a unit circle whose circumference is given by $C = 2\pi$. Let T_1 be an isosceles right triangle inscribed in the first quadrant whose hypotenuse is $h_1 = \sqrt{2}$. If we inscribe three right triangles congruent to T_1, one per each remaining quadrant, then a crude approximation to C is given by $4h_1 = 4\sqrt{2}$.

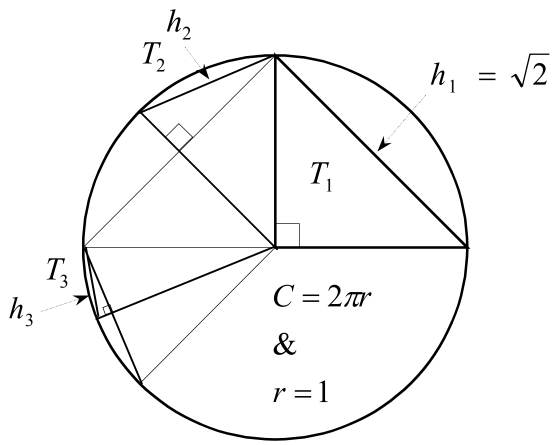


Figure 4.17: Pythagorean PI

By bisecting the hypotenuse h_1, we can create a smaller right triangle T_2 whose hypotenuse is h_2. Eight of these triangles can be symmetrical arranged within the unit circle leading to a second approximation to C given by $8h_2$. A third bisection leads to triangle T_3 and a third approximation $16h_3$. Since the 'gap' between the rim of the circle and the hypotenuse of the right triangle generated by our bisection process noticeably tightens with successive iterations, one might expect that the approximation for 2π can be generated to any degree of accuracy, given enough iterative cycles.

Note: Analysis is the branch of mathematics addressing 'endless behavior', such as our bisection process above, which can go on ad infinitum. Some situations studied in analysis run counter to intuitively predicted behavior. Happily, the bisection process was shown to converge (get as close as we like and stay there) to 2π in the early 1990s.

What remains to be done is to develop a formula for h_{i+1} that expresses h_{i+1} in terms of h_i. Such a formula is called an *iterative* or *recursive* formula. Given a recursive formula and the fact that $h_1 = \sqrt{2}$, we should be able to generate h_2, then h_3, and so on.

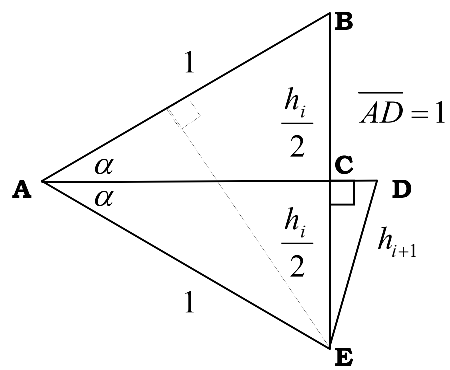


Figure 4.18: Recursive Hypotenuses

Figure 4.18 shows the relationship between two successive iterations and the associated hypotenuses. To develop our recursive formula, start with triangle ΔACE, which is a right triangle by construction since line segment $\overline{AD}$ bisects the angle $\angle BAE$. We have by the Pythagorean Theorem:

$$\left(\overline{AC}\right)^2 + \left(\overline{CE}\right)^2 = \left(\overline{AE}\right)^2 = (1)^2 \Rightarrow$$

$$\left(\overline{AC}\right)^2 = 1 - \left(\frac{h_i}{2}\right)^2 \Rightarrow$$

$$\left(\overline{AC}\right)^2 = \frac{4}{4} - \frac{h_i^{\,2}}{4}$$

$$\overline{AC} = \frac{\sqrt{4 - h_i^{\,2}}}{2}$$

Continuing, we form an expression for the side $\overline{CD}$ associated with triangle ΔCDE , also a right triangle.

$$\overline{CD} = 1 - \overline{AC} \Rightarrow$$

$$\overline{CD} = \frac{2 - \sqrt{4 - h_i^{\,2}}}{2}$$

Again, by the Pythagorean Theorem

$$\left(\overline{CD}\right)^2 + \left(\overline{CE}\right)^2 = \left(\overline{DE}\right)^2 \Rightarrow$$

$$\left(\frac{2 - \sqrt{4 - h_i^{\,2}}}{2}\right)^2 + \left(\frac{h_i}{2}\right)^2 = h_{i+1}^{\,2} \Rightarrow$$

$$h_{i+1}^{\,2} = \frac{\left(2 - \sqrt{4 - h_i^{\,2}}\right)^2 + h_i^{\,2}}{4}$$

$$h_{i+1} = \sqrt{2 - \sqrt{4 - h_i^{\,2}}}$$

After a good bit of algebraic simplification (left to reader).

The associated approximation for the actual circumference 2π is given by the formula $C_i = 2^{i+1} h_i$. To establish the iterative pattern, we perform four cycles of numeric calculation.

$$\overset{1}{\mapsto} : h_1 = \sqrt{2}, C_1 = 2^2$$

$$h_1 = 4\sqrt{2}$$

$$\overset{2}{\mapsto} : h_2 = \sqrt{2 - \sqrt{4 - h_1^{\,2}}} = \sqrt{2 - \sqrt{2}}$$

$$C_2 = 2^3 h_2 = 8\sqrt{2 - \sqrt{2}}$$

$$\overset{3}{\mapsto}: h_3 = \sqrt{2 - \sqrt{4 - h_2^{\;2}}} = \sqrt{2 - \sqrt{2 + \sqrt{2}}}$$

$$C_3 = 2^4 h_3 = 16\sqrt{2 - \sqrt{2 + \sqrt{2}}}$$

$$\overset{4}{\mapsto}: h_4 = \sqrt{2 - \sqrt{4 - h_3^{\;2}}} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$C_4 = 2^5 h_4 = 32\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

In words, to calculate each succeeding hypotenuse, simply add one more $\sqrt{2}$ to the innermost 2 within the nested radicals. To calculate each succeeding circumferential approximation, simply multiply the associated hypotenuse by 2 to one additional power.

i	h_i	c_i
1	1.414	5.6569
2	0.7654	6.1229
3	0.3902	6.2429
4	0.1960	6.2731
5	0.0981	6.2807
6	0.0491	6.2826
7	0.0245	6.2830
8	0.0123	6.2831
9	0.0061	6.2832
10	0.0031	6.2832

Table 4.2: Successive Approximations for 2π

Table 4.2 gives the results for the first ten iterations. As one can see, the approximation has stabilized to the fourth decimal place. Successive iterations would stabilize additional decimal places to the right of the decimal point. In this way, any degree of accuracy could be obtained if one had enough time and patience. The true value of 2π to nine decimal places is $2\pi = 6.283185307$.

If we stabilize one digit for every two iterations (which seems to be the indication by **Table 4.2**), then it would take about twenty iterations to stabilize our approximation to nine digits—a great closing challenge to the reader!

Our last topic in Section 4.4 is that of Pythagorean Spirals. More art than mathematics, Pythagorean Spirals are created by joining a succession of right triangles as shown in **Figure 4.19**. All nine triangles have outer sides equal in length. In addition, the longer non-hypotenuse side of a larger triangle is equal to the hypotenuse of the preceding triangle. The generator for the 'spiraling seashell' in **Figure 4.19** is an isosceles right triangle of side length one. As one might imagine, the stopping point is arbitrary. Pythagorean Spirals make great objects for computer graphics programs to generate where coloration and precise alignment can be brought into play. Give it a try using more sophisticated software than the Microsoft Word utility that I used to generate the spiraling seashell. Enjoy!

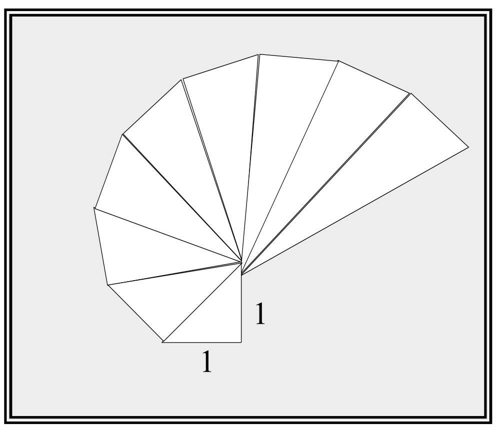


Figure 4.19: Pythagorean Spiral

Epilogue: The Crown and the Jewels

"**G**eometry has two great treasures;
One is the Theorem of Pythagoras:
The other, the division of a line into extreme and mean ratio.
The first we may compare to a measure of gold;
The second we may name a precious jewel." Johannes Kepler

Saint Paul says his First Epistle to the Corinthians, "For now we see through a glass, darkly…" Though the quote may be out of context, the thought freely standing on its own is carries the truth of the human condition. Humans are finite creatures limited by space, time, and the ability of a three-pond mass to perceive the wondrous workings of the Near Infinite…or Infinite. Even of that—Near Infinite or Infinite—we are not sure.

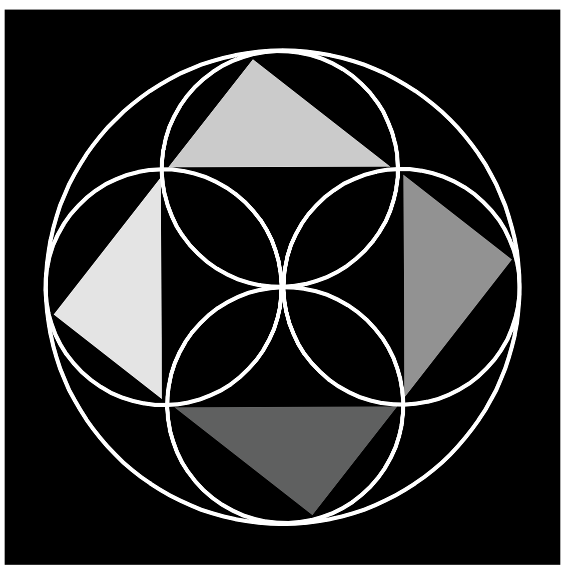


Figure E.1: Beauty in Order

Mathematics is one of the primary tools we humans use to see through the glass darkly. With it, we can both discover and describe ongoing order in the natural laws governing the physical universe. Moreover, within that order we find comfort, for order implies design and purpose—as opposed to chaos—through intelligence much grander than anything produced on a three-pound scale. The study of mathematics as a study of order is one way to perceive cosmic-level design and purpose. But no matter how good or how complete the human invention of mathematics can be, it still allows just a glimpse through the darkened glass.

Figure E.1, a mandala of sorts, hints at this *truth of order* as it applies to the Pythagorean Theorem—which is but a single pattern within the total synchronicity found throughout the cosmos. The four gradually shaded triangles speak of Pythagoras and one of the first Pythagorean proofs. Moreover, each triangle is golden via the ratio of hypotenuse to side, in deference to Kepler's statement regarding the two great geometric treasures. Four also numbers the primary waveforms in the visible spectrum—red, blue, green, and yellow— which can be mathematically modeled by advanced *trigonometric patterns* (not addressed in this general-readership volume). Since right triangles are intricately linked to circles, these two geometric figures are coupled in one-to-one fashion. A great circle, expressing unity, encompasses the pattern. White, as the synthesis of the visual rainbow, appropriately colors all five circles. But even within the beauty of pattern, linkage, and discovery as expressed by our mandala, there is black. Let this darkness represent that which remains to be discovered or that which is consigned to mystery. Either way, it becomes part of who we are as humans as our innate finiteness seeks to comprehend that which is much greater.

It is my hope that this book has allowed a brief glimpse through the glass, a glimpse at a superbly simple yet subtle geometric pattern called the Pythagorean Theorem that has endured for 4000 years or more of human history.

Not only has it endured, but the Pythagorean Theorem also has expanded in utility and application throughout the same 40 centuries, paralleling human intellectual progress during the same four millennia. In modern times, the Pythagorean Theorem has found its long-standing truth even more revered and revitalized as the theorem constantly "reinvents" itself in order to support new mathematical concepts.

Today, our ancient friend is the foundation for several branches of mathematics supporting a plethora of exciting applications—ranging from waveform analysis to experimental statistics to interplanetary ballistics—totally unheard of just two centuries ago. However, with a couple of exceptions, the book in your hands has explored more traditional trails. Like the color black in **Figure E.1**, there is much that you, the reader, can yet discover regarding the Pythagorean Proposition. The eighteen works cited in Appendix D provide excellent roadmaps for further explorations. We close with a classic geometric conundrum, Curry's Paradox, to send you on your way: Given two sets of four identical playing pieces as shown in **Figure E.2**, *how did the square in the top figure disappear?*

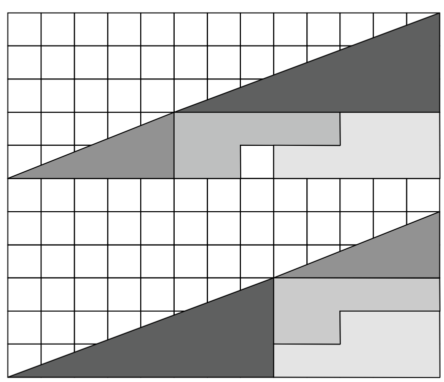


Figure E.2: Curry's Paradox

Appendices

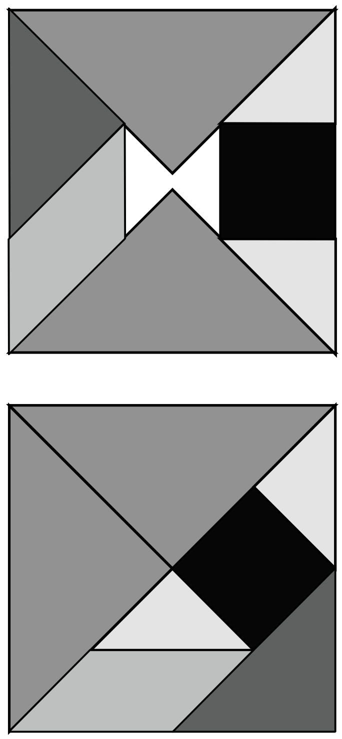


Figure A.0: The Tangram

Note: The Tangram is a popular puzzle currently marketed under various names. Shown above in two configurations, the Tangram has Pythagorean origins.

A] Greek Alphabet

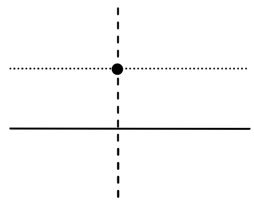
GREEK LETTER		ENGLISH NAME
Upper Case	Lower Case	
Α	α	Alpha
Β	β	Beta
Γ	γ	Gamma
Δ	δ	Delta
Ε	ε	Epsilon
Ζ	ζ	Zeta
Η	η	Eta
Θ	θ	Theta
Ι	ι	Iota
Κ	κ	Kappa
Λ	λ	Lambda
Μ	μ	Mu
Ν	ν	Nu
Ξ	ξ	Xi
Ο	ο	Omicron
Π	π	Pi
Ρ	ρ	Rho
Σ	σ	Sigma
Τ	τ	Tau
Υ	υ	Upsilon
Φ	φ	Phi
Χ	χ	Chi
Ψ	ψ	Psi
Ω	ω	Omega

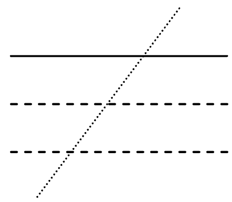
B] Mathematical Symbols

SYMBOL	MEANING
$+$	Plus or Add
-	Minus or Subtract or Take Away
$\pm$	Plus or Minus: do both for two results
$\div$ /	Divide
$\times$ $\cdot$	Multiply or Times
$\{\ \}$ or $[\ ]$ or $(\ )$	Parentheses
$=$	Is equal to
$\equiv$	Is defined as
$\neq$	Does not equal
$\cong$	Is approximately equal to
$\approx$	Is similar too
$>$	Is greater than
$\geq$	Is greater than or equal to
$<$	Is less than
$\leq$	Is less than or equal to
$\overset{1}{\mapsto}, \overset{2}{\mapsto}$	Step 1, Step 2, etc.
$\Rightarrow$	Implies the following
$A \Rightarrow B$	A implies B
$A \Leftarrow B$	B implies A
$A \Leftrightarrow B$	A implies B implies A
$\sqrt{\ }$	Sign for square root
$\sqrt[n]{\ }$	Symbol for n^{th} root
| |	Parallel
$\perp$	Perpendicular
$\angle$	Angle
$\neg$	Right angle
Δ	Triangle

C] Geometric Foundations

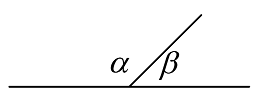
The Parallel Postulates



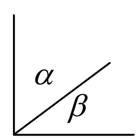


1. Let a point reside outside a given line. Then there is exactly one line passing through the point parallel to the given line.
2. Let a point reside outside a given line. Then there is exactly one line passing through the point perpendicular to the given line.
3. Two lines both parallel to a third line are parallel to each other.
4. If a transverse line intersects two parallel lines, then corresponding angles in the figures so formed are congruent.
5. If a transverse line intersects two lines and makes congruent, corresponding angles in the figures so formed, then the two original lines are parallel.

Angles and Lines



$\alpha + \beta = 180^0$

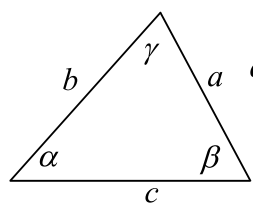


$\alpha + \beta = 90^0$

1. Complimentary Angles: Two angles α, β with $\alpha + \beta = 90^0$.

2. Supplementary Angles: Two angles α, β with $\alpha + \beta = 180^0$
3. Linear Sum of Angles: The sum of the two angles α, β formed when a straight line is intersected by a line segment is equal to 180^0
4. Acute Angle: An angle less than 90^0
5. Right Angle: An angle exactly equal to 90^0
6. Obtuse Angle: An angle greater than 90^0

Triangles



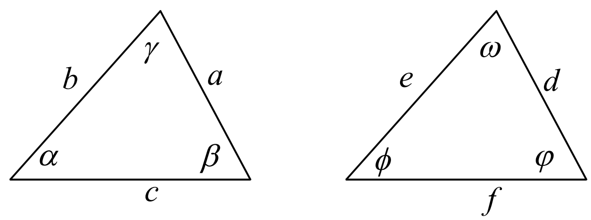
$$\alpha + \beta + \gamma = 180^0$$

1. Triangular Sum of Angles: The sum of the three interior angles α, β, γ in any triangle is equal to 180^0
2. Acute Triangle: A triangle where all three interior angles α, β, γ are acute
3. Right Triangle: A triangle where one interior angle from the triad α, β, γ is equal to 90^0
4. Obtuse Triangle: A triangle where one interior angle from the triad α, β, γ is greater than 90^0
5. Scalene Triangle: A triangle where no two of the three side-lengths a, b, c are equal to another
6. Isosceles Triangle: A triangle where exactly two of the side-lengths a, b, c are equal to each other
7. Equilateral Triangle: A triangle where all three side-lengths a, b, c are identical $a = b = c$ or all three angles α, β, γ are equal with $\alpha = \beta = \gamma = 60^0$

8. Congruent Triangles: Two triangles are congruent (equal) if they have identical interior angles and side-lengths
9. Similar Triangles: Two triangles are similar if they have identical interior angles
10. Included Angle: The angle that is between two given sides
11. Opposite Angle: The angle opposite a given side
12. Included Side: The side that is between two given angles
13. Opposite Side: The side opposite a given angle

Congruent Triangles

Given the congruent two triangles as shown below



1. Side-Angle-Side (**SAS**): If any two side-lengths and the included angle are identical, then the two triangles are congruent.

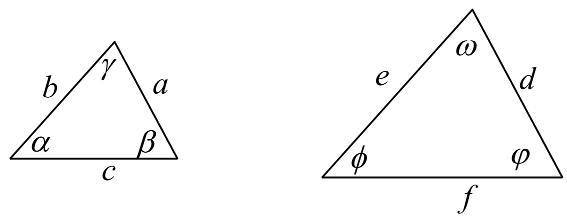
2. Angle-Side-Angle (**ASA**): If any two angles and the included side are identical, then the two triangles are congruent.

3. Side-Side-Side (**SSS**): If the three side-lengths are identical, then the triangles are congruent.

4. Three Attributes Identical: If any three attributes—side-lengths and angles—are equal with at least one attribute being a side-length, then the two triangles are congruent. These other cases are of the form Angle-Angle-Side (**AAS**) or Side-Side-Angle (**SSA**).

Similar Triangles

Given the two similar triangles as shown below



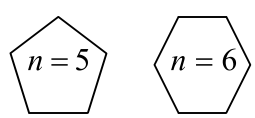
1. Minimal Condition for Similarity: If any two angles are identical (**AA**), then the triangles are similar.

2. Ratio laws for Similar Triangles: Given similar triangles as shown above, then $\frac{b}{e} = \frac{c}{f} = \frac{a}{d}$

Planar Figures

A is the planar area, P is the perimeter, n is the number of sides.

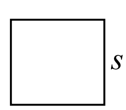
1. Degree Sum of Interior Angles in General Polygon:
   $D = 180^0[n-2]$



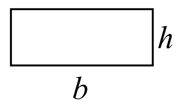
$n = 5 \Rightarrow D = 540^0$

$n = 6 \Rightarrow D = 720^0$

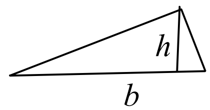
2. Square: $A = s^2 : P = 4s$, s is the length of a side



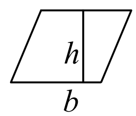
3. Rectangle: $A = bh : P = 2b + 2h$, b & h are the base and height



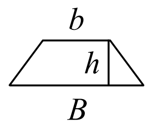
4. Triangle: $A = \frac{1}{2}bh$, b & h are the base and altitude



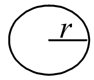
5. Parallelogram: $A = bh$, b & h are the base and altitude



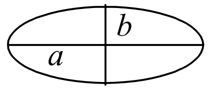
6. Trapezoid: $A = \frac{1}{2}(B + b)h$, B & b are the two parallel bases and h is the altitude



7. Circle: $A = \pi r^2 : P = 2\pi r$ where r is the radius, or $P = \pi d$ where $d = 2r$, the diameter.



8. Ellipse: $A = \pi ab$; a & b are the half lengths of the major & minor axes



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