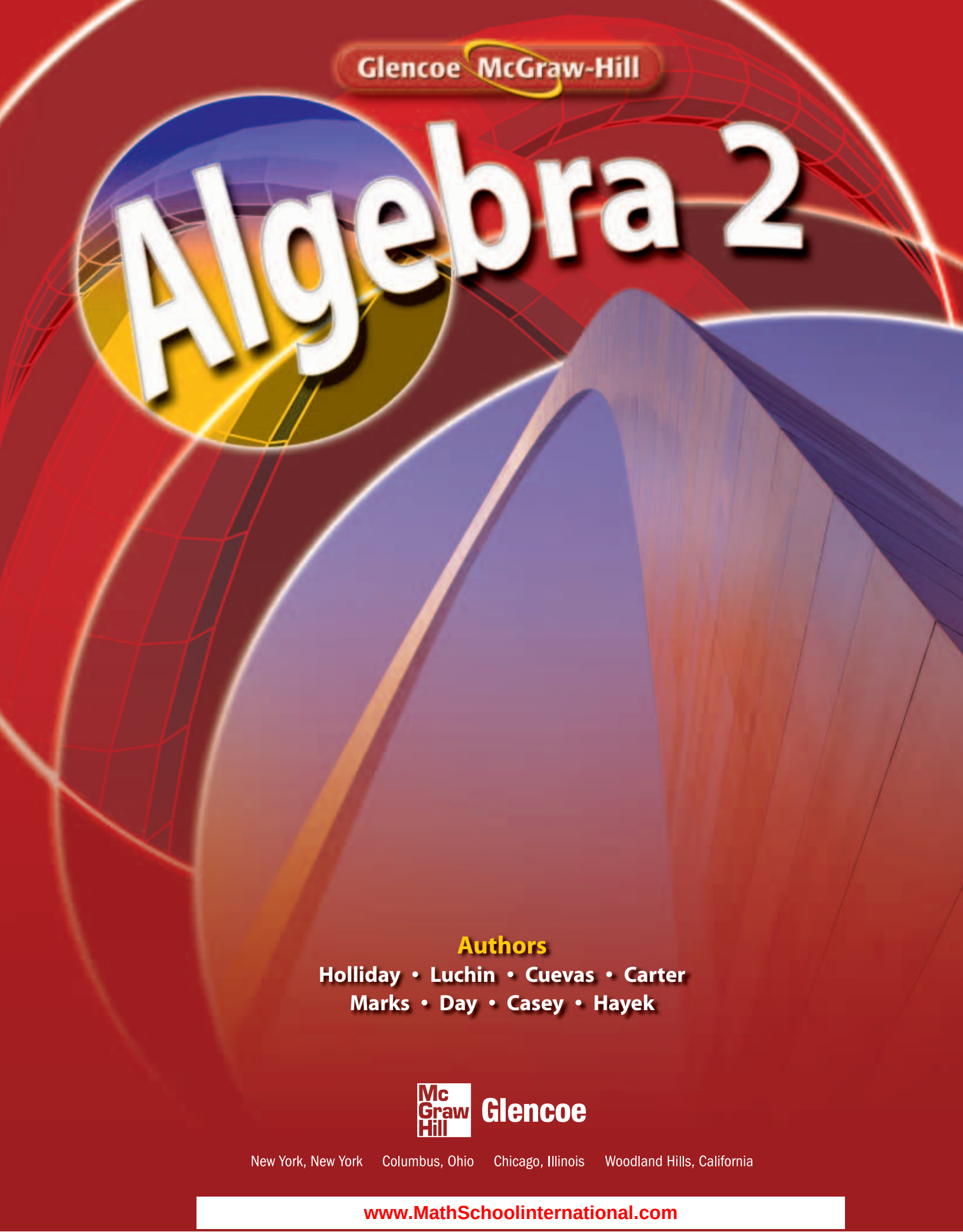

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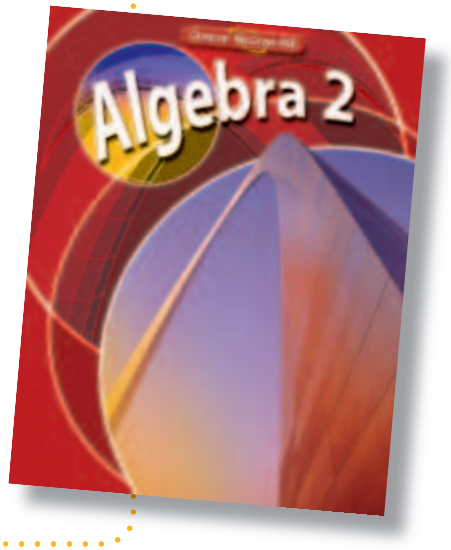
About the Cover

On a clear day, visitors to the top of the Gateway Arch in St. Louis, Missouri, can see up to thirty miles to the east or west. The Arch, towering 630 feet (192 meters) above the banks of the Mississippi River, commemorates the westward expansion of the United States in the 19th century. It takes the shape of a catenary curve, which can be approximated using a quadratic function. You will study quadratic functions in Chapter 5.



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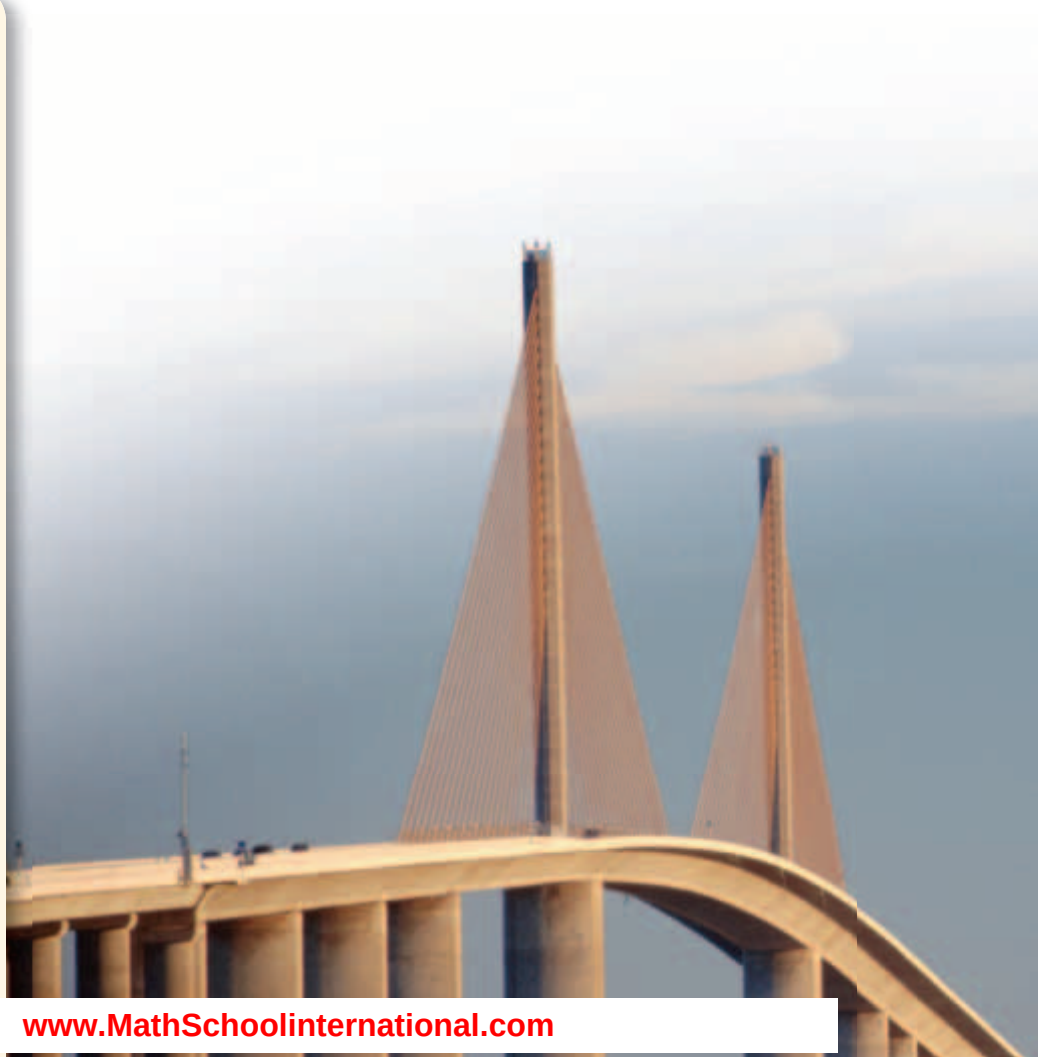

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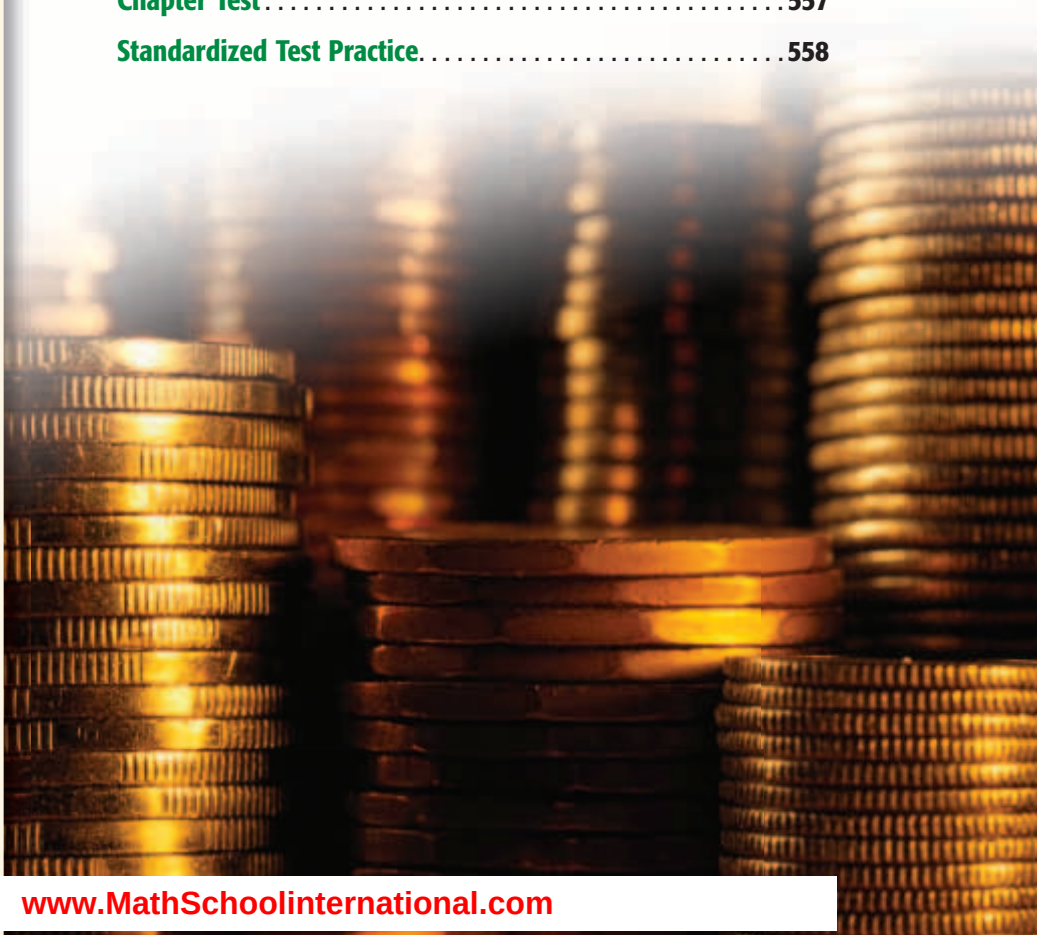

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Unit 5
Trigonometry

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Unit 1

First-Degree Equations and Inequalities

Focus

Use algebraic concepts and the relationships among them to better understand the structure of algebra.

CHAPTER 1

Equations and Inequalities

BIG Idea Manipulate symbols in order to solve problems and use algebraic skills to solve equations and inequalities in problem situations.

CHAPTER 2

Linear Relations and Functions

BIG Idea Use properties and attributes of functions and apply functions to problem situations.

BIG Idea Connect algebraic and geometric representations of functions.

CHAPTER 3

Systems of Equations and Inequalities

BIG Idea Formulate systems of equations and inequalities from problem situations, use a variety of methods to solve them, and analyze the solutions in terms of the situations.

CHAPTER 4

Matrices

BIG Idea Use matrices to organize data and solve systems of equations from problem situations.

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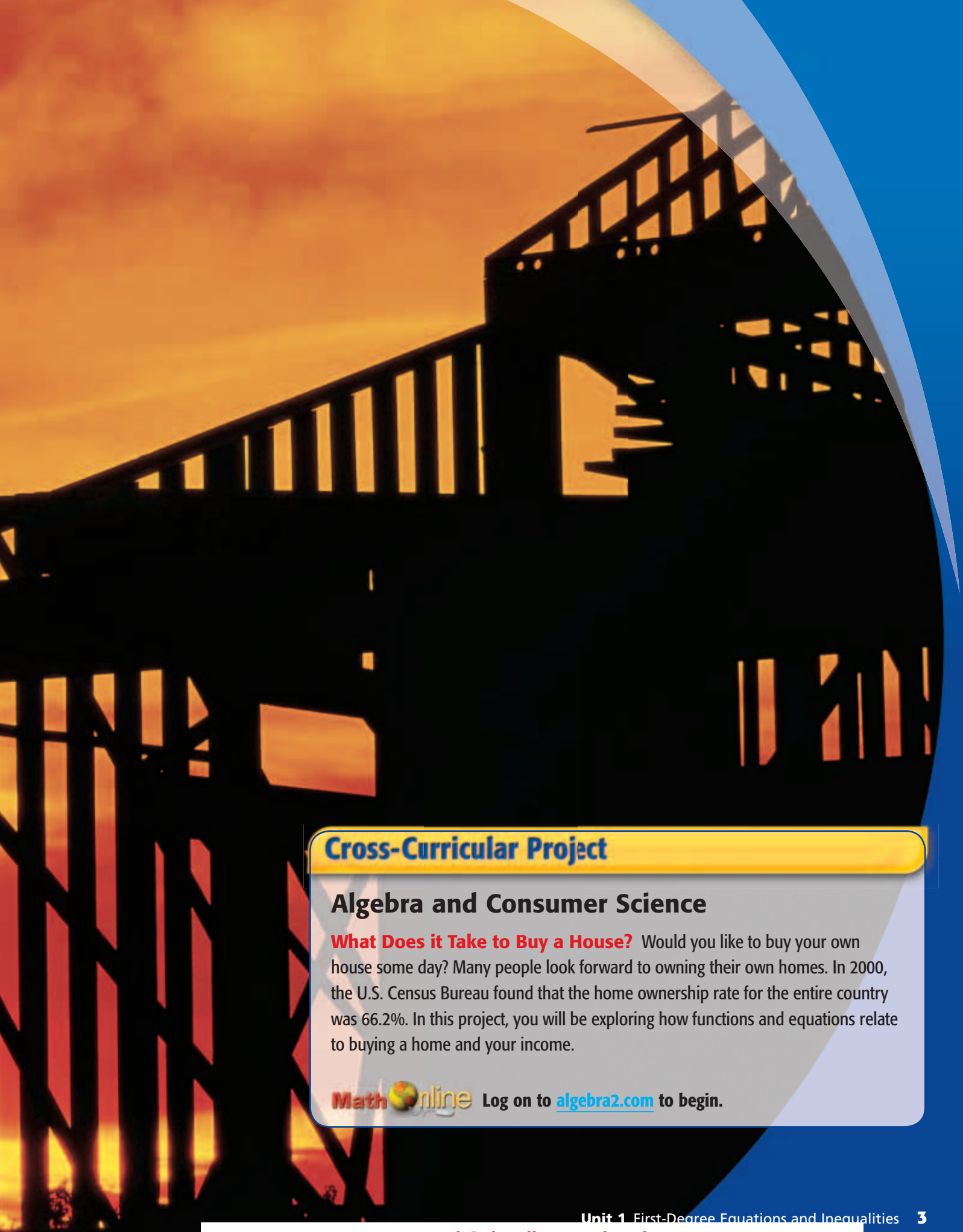

Cross-Curricular Project

Algebra and Consumer Science

What Does it Take to Buy a House? Would you like to buy your own house some day? Many people look forward to owning their own homes. In 2000, the U.S. Census Bureau found that the home ownership rate for the entire country was 66.2%. In this project, you will be exploring how functions and equations relate to buying a home and your income.

MathOnline **Log on to algebra2.com to begin.**

Bryan Peterson/Getty Images

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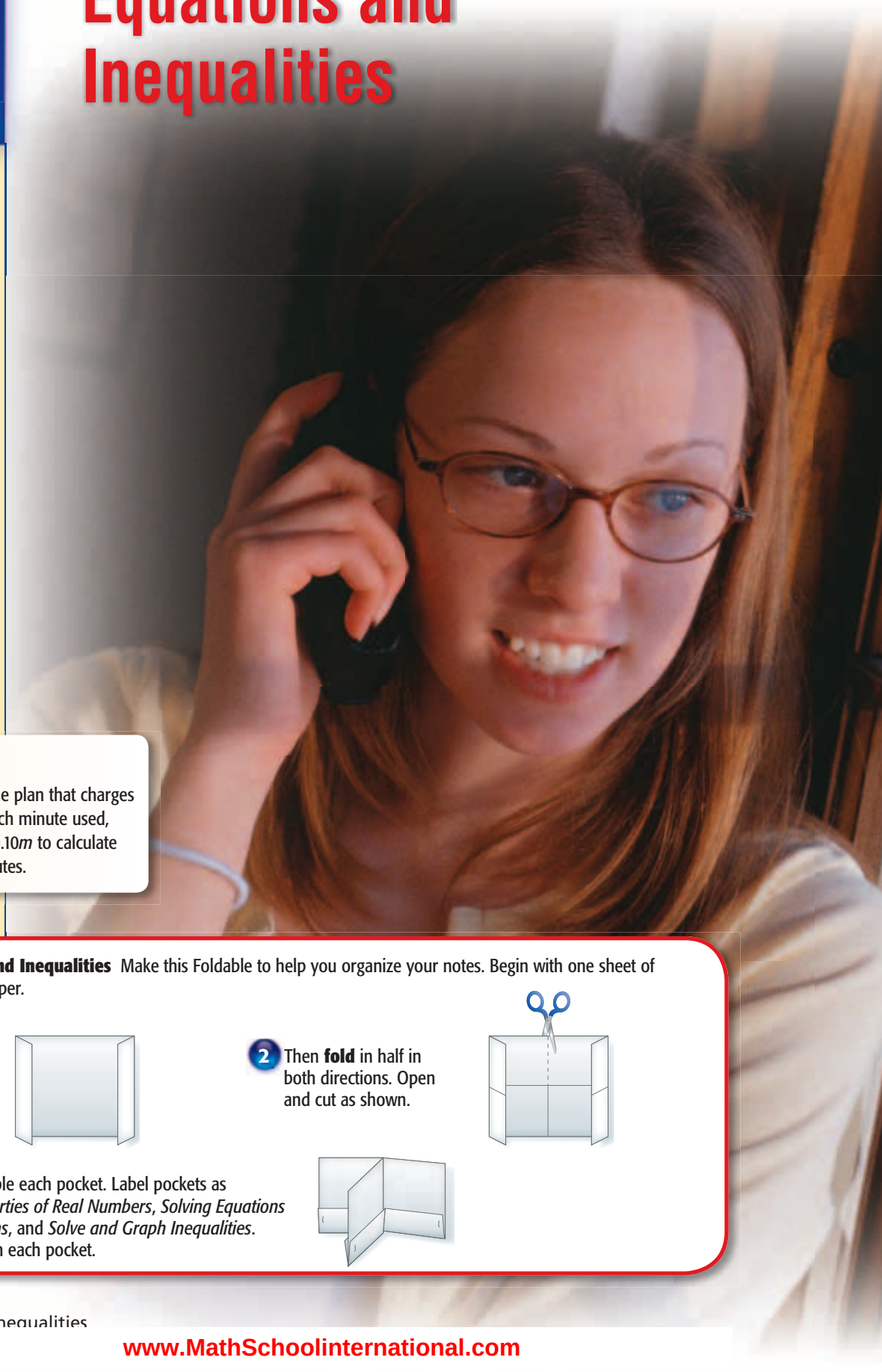
CHAPTER 1

Equations and Inequalities



BIG Ideas

- Simplify and evaluate algebraic expressions.
- Solve linear and absolute value equations.
- Solve and graph inequalities

Key Vocabulary

counterexample (p. 17)
equation (p. 18)
formula (p. 8)
solution (p. 19)

Real-World Link

Cell Phone Charges For a cell phone plan that charges a monthly fee of $10 plus $0.10 for each minute used, you can use the equation $C = 10 + 0.10m$ to calculate the monthly charges for using m minutes.

FOLDABLES Study Organizer

Equations and Inequalities Make this Foldable to help you organize your notes. Begin with one sheet of 11″ by 17″ paper.

1. **Fold** 2″ tabs on each of the short sides.

2. Then **fold** in half in both directions. Open and cut as shown.

3. **Refold** along the width. Staple each pocket. Label pockets as *Algebraic Expressions*, *Properties of Real Numbers*, *Solving Equations and Absolute Value Equations*, and *Solve and Graph Inequalities*. Place index cards for notes in each pocket.

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GET READY for Chapter 1

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Simplify. (Prerequisite Skill)

1. $20 - 0.16$
2. $12.2 + (-8.45)$
3. $\frac{1}{4} - \frac{2}{3}$
4. $\frac{3}{5} + (-6)$
5. $-7\frac{1}{2} + 5\frac{1}{3}$
6. $-11\frac{5}{8} - \left(-4\frac{3}{7}\right)$
7. $(0.15)(3.2)$
8. $2 \div (-0.4)$
9. $-4 \div \frac{3}{2}$
10. $\left(\frac{5}{4}\right)\left(-\frac{3}{10}\right)$
11. $\left(-2\frac{3}{4}\right)\left(-3\frac{1}{5}\right)$
12. $7\frac{1}{8} \div (-2)$
13. **LUNCH** Angela has \$11.56. She spends \$4.25 on lunch. How much money does Angela have left? (Prerequisite Skill)

Evaluate each power. (Prerequisite Skill)

14. 2^3
15. 5^3
16. $(-7)^2$
17. $(-1)^3$
18. $(-0.8)^2$
19. $-(1.2)^2$
20. $\left(\frac{2}{3}\right)^2$
21. $\left(\frac{5}{9}\right)^2$
22. $\left(-\frac{4}{11}\right)^2$
23. **GENEALOGY** In a family tree, you are generation "now." One generation ago, your 2 parents were born. Two generations ago your 4 grandparents were born. How many ancestors were born five generations ago? (Prerequisite Skill)

Identify each statement as *true* or *false*. (Prerequisite Skill)

24. $-5 < -7$
25. $6 > -8$
26. $-2 \geq -2$
27. $-3 \geq -3.01$
28. $-1 < -2$
29. $\frac{1}{5} < \frac{1}{8}$
30. $\frac{2}{5} \geq \frac{16}{40}$
31. $\frac{3}{4} > 0.8$

QUICKReview

EXAMPLE 1

Simplify $\left(-\frac{3}{5}\right)\left(\frac{13}{15}\right)$.

$\left(-\frac{3}{5}\right)\left(\frac{13}{15}\right) = -\frac{3(13)}{5(15)}$ Multiply the numerators and denominators.

$= -\frac{39}{75}$ Simplify.

$= -\frac{39 \div 3}{75 \div 3}$ Divide the numerator and denominator by their GCF, 3.

$= -\frac{13}{25}$ Simplify.

EXAMPLE 2

Evaluate $-(-10)^3$.

$-(-10)^3 = -[(-10)(-10)(-10)]$ $(-10)^3$ means -10 is a factor 3 times.

$= -[-1000]$ Evaluate inside the brackets.

$= 1000$ Simplify.

EXAMPLE 3

Identify $\frac{2}{7} < \frac{8}{28}$ as *true* or *false*.

$\frac{2}{7} \stackrel{?}{<} \frac{8 \div 4}{28 \div 4}$ Divide 8 and 28 by their GCF, 4.

$\frac{2}{7} \not< \frac{2}{7}$ Simplify.

False, $\frac{2}{7} \not< \frac{8}{28}$ because $\frac{2}{7} = \frac{8}{28}$.

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1-1 Expressions and Formulas

Main Ideas

- Use the order of operations to evaluate expressions.
- Use formulas.

New Vocabulary

variable
algebraic expression
order of operations
monomial
constant
coefficient
degree
power
polynomial
term
like terms
trinomial
binomial
formula

GET READY for the Lesson

Nurses setting up intravenous or IV fluids must control the flow rate F, in drops per minute. They use the formula $F = \frac{V \times d}{t}$, where V is the volume of the solution in milliliters, d is the drop factor in drops per milliliter, and t is the time in minutes.

Suppose 1500 milliliters of saline are to be given over 12 hours. Using a drop factor of 15 drops per milliliter, the expression $\frac{1500 \times 15}{12 \times 60}$ gives the correct IV flow rate.

Order of Operations

Variables are symbols, usually letters, used to represent unknown quantities. Expressions that contain at least one variable are called **algebraic expressions**. You can evaluate an algebraic expression by replacing each variable with a number and then applying the **order of operations**.

KEY CONCEPT *Order of Operations*

Step 1 Evaluate expressions inside grouping symbols.

Step 2 Evaluate all powers.

Step 3 Multiply and/or divide from left to right.

Step 4 Add and/or subtract from left to right.

An algebraic expression that is a number, a variable, or the product of a number and one or more variables is called a **monomial**. Monomials cannot contain variables in denominators, variables with exponents that are negative, or variables under radicals.

Monomials	Not Monomials
$5b$	$\frac{1}{n^4}$
$-w$	$\sqrt[3]{x}$
23	$x + 8$
x^2	a^{-1}
$\frac{1}{3}x^3y^4$	

Mark Harmel/Getty Images

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Constants are monomials that contain no variables, like 23 or -1. The numerical factor of a monomial is the **coefficient** of the variable(s). For example, the coefficient of m in $-6m$ is -6. The **degree** of a monomial is the sum of the exponents of its variables. For example, the degree of $12g^7h^4$ is $7 + 4$ or 11. The degree of a constant is 0. A **power** is an expression of the form x^n. The word *power* is also used to refer to the exponent itself.

A **polynomial** is a monomial or a sum of monomials. The monomials that make up a polynomial are called the **terms** of the polynomial. In a polynomial such as $x^2 + 2x + x + 1$, the two monomials $2x$ and x can be combined because they are **like terms**. The result is $x^2 + 3x + 1$. The polynomial $x^2 + 3x + 1$ is a **trinomial** because it has three unlike terms. A polynomial such as $xy + z^3$ is a **binomial** because it has two unlike terms.

EXAMPLE Evaluate Algebraic Expressions

1 **a. Evaluate $m + (n - 1)^2$ if $m = 3$ and $n = -4$.**

$$\begin{aligned} m + (n - 1)^2 &= 3 + (-4 - 1)^2 && \text{Replace } m \text{ with 3 and } n \text{ with } -4. \\ &= 3 + (-5)^2 && \text{Add } -4 \text{ and } -1. \\ &= 3 + 25 && \text{Find } (-5)^2. \\ &= 28 && \text{Add 3 and 25.} \end{aligned}$$

b. Evaluate $x^2 - y(x + y)$ if $x = 8$ and $y = 1.5$.

$$\begin{aligned} x^2 - y(x + y) &= 8^2 - 1.5(8 + 1.5) && \text{Replace } x \text{ with 8 and } y \text{ with 1.5.} \\ &= 8^2 - 1.5(9.5) && \text{Add 8 and 1.5.} \\ &= 64 - 1.5(9.5) && \text{Find } 8^2. \\ &= 64 - 14.25 && \text{Multiply 1.5 and 9.5.} \\ &= 49.75 && \text{Subtract 14.25 from 64.} \end{aligned}$$

c. Evaluate $\dfrac{a^3 + 2bc}{c^2 - 5}$ if $a = 2$, $b = -4$, and $c = -3$.

$$\begin{aligned} \frac{a^3 + 2bc}{c^2 - 5} &= \frac{2^3 + 2(-4)(-3)}{(-3)^2 - 5} && a = 2, b = -4, \text{ and } c = -3 \\ &= \frac{8 + (-8)(-3)}{9 - 5} && \text{Evaluate the numerator and the denominator separately.} \\ &= \frac{8 + 24}{9 - 5} && \text{Multiply } -8 \text{ by } -3. \\ &= \frac{32}{4} \text{ or } 8 && \text{Simplify the numerator and the denominator. Then divide.} \end{aligned}$$

Study Tip

Fraction Bar

The fraction bar acts as both an operation symbol, indicating division, and as a grouping symbol. Evaluate the expressions in the numerator and denominator separately before dividing.

CHECK Your Progress

1A. Evaluate $m + (3 - n)^2$ if $m = 12$ and $n = -1$.

1B. Evaluate $x^2y + x(x - y)$ if $x = 4$ and $y = 0.5$.

1C. Evaluate $\dfrac{b^2 - 3a^2c}{b^3 + 2}$ if $a = -1$, $b = 2$, and $c = 8$.

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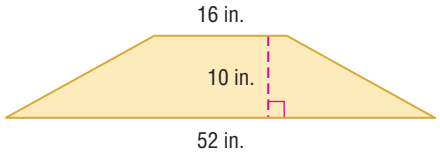
Formulas A **formula** is a mathematical sentence that expresses the relationship between certain quantities. If you know the value of every variable in the formula except one, you can find the value of the remaining variable.

EXAMPLE 2 Use a Formula

GEOMETRY **The formula for the area A of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, where h represents the height, and b_1 and b_2 represent the measures of the bases. Find the area of the trapezoid shown below.**



The height is 10 inches. The bases are 16 inches and 52 inches. Substitute each value given into the formula. Then evaluate the expression using the order of operations.

$A = \frac{1}{2}h(b_1 + b_2)$ Area of a trapezoid

$= \frac{1}{2}(10)(16 + 52)$ Replace h with 10, b_1 with 16, and b_2 with 52.

$= \frac{1}{2}(10)(68)$ Add 16 and 52.

$= 5(68)$ Multiply $\frac{1}{2}$ and 10.

$= 340$ Multiply 5 by 68.

The area of the trapezoid is 340 square inches.

CHECK Your Progress

2. The formula for the volume V of a rectangular prism is $V = \ell wh$, where ℓ represents the length, w represents the width, and h represents the height. Find the volume of a rectangular prism with a length of 4 feet, a width of 2 feet, and a height of 3.5 feet.

Online **Personal Tutor at** algebra2.com

CHECK Your Understanding

Example 1 (p. 7)

Evaluate each expression if $x = 4$, $y = -2$, and $z = 3.5$.

1. $z - x + y$

2. $x + (y - 1)^3$

3. $x + [3(y + z) - y]$

4. $\frac{x^2 - y}{z + 2.5}$

5. $\frac{x + 2y^2}{x - z}$

6. $\frac{y^3 + 2xz}{x^2 - z}$

Example 2 (p. 8)

BANKING **For Exercises 7 and 8, use the following information.**
Simple interest is calculated using the formula $I = prt$, where p represents the principal in dollars, r represents the annual interest rate, and t represents the time in years. Find the simple interest I given each set of values.

7. $p = \$1800$, $r = 6\%$, $t = 4$ years

8. $p = \$31{,}000$, $r = 2\frac{1}{2}\%$, $t = 18$ months

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–22	1
23, 24	2

Evaluate each expression if $w = 6$, $x = 0.4$, $y = \frac{1}{2}$, and $z = -3$.

9. $w + x + z$ **10.** $w + 12 \div z$ **11.** $w(8 - y)$

12. $z(x + 1)$ **13.** $w - 3x + y$ **14.** $5x + 2z$

Evaluate each expression if $a = 3$, $b = 0.3$, $c = \frac{1}{3}$, and $d = -1$.

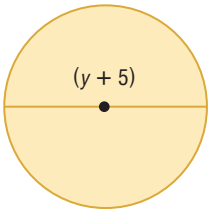
15. $\frac{a - d}{bc}$ **16.** $\frac{a + d}{c}$ **17.** $\frac{a^2c^2}{d}$

18. $\frac{a - 10b}{c^2d^2}$ **19.** $\frac{d + 4}{a^2 + 3}$ **20.** $\frac{1 - b}{3c - 3b}$

21. NURSING Determine the IV flow rate for the patient described at the beginning of the lesson by finding the value of $\frac{1500 \times 15}{12 \times 60}$.

22. BICYCLING Air pollution can be reduced by riding a bicycle rather than driving a car. To find the number of pounds of pollutants created by starting a typical car 10 times and driving it for 50 miles, find the value of the expression $\frac{(52.84 \times 10) + (5.955 \times 50)}{454}$.

23. GEOMETRY The formula for the area A of a circle with diameter d is $A = \pi\left(\frac{d}{2}\right)^2$. Write an expression to represent the area of the circle.



24. GEOMETRY The formula for the volume V of a right circular cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Write an expression for the volume of a cone with $r = 3x$ and $h = 2x$.

Real-World Link

To estimate the width w in feet of a firework burst, use the formula $w = 20At$. In this formula, A is the estimated viewing angle of the fireworks display, and t is the time in seconds from the instant you see the light until you hear the sound.

Source: efg2.com

Evaluate each expression if $a = \frac{2}{5}$, $b = -3$, $c = 0.5$, and $d = 6$.

25. $b^4 - d$ **26.** $(5 - d)^2 + a$ **27.** $\frac{5ad}{b}$

28. $\frac{2b - 15a}{3c}$ **29.** $(a - c)^2 - 2bd$ **30.** $\frac{1}{c} + \frac{1}{d}$

31. Find the value of ab^n if $n = 3$, $a = 2000$, and $b = -\frac{1}{5}$.

32. FIREWORKS Suppose you are about a mile from a fireworks display. You count 5 seconds between seeing the light and hearing the sound of the fireworks display. You estimate the viewing angle is about 4°. Using the information at the left, estimate the width of the firework display.

33. MONEY In 1960, the average price of a car was about $2500. This may sound inexpensive, but the average income in 1960 was much less than it is now. To compare dollar amounts over time, use the formula $V = \frac{A}{S}C$, where A is the old dollar amount, S is the starting year's Consumer Price Index (CPI), C is the converting year's CPI, and V is the current value of the old dollar amount. Buying a car for $2500 in 1960 was like buying a car for how much money in 2004?

Year	1960	1970	1980	1990	2000	2004
Average CPI	29.6	38.8	82.4	130.7	172.2	188.9

Source: U.S. Department of Labor

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EXTRA PRACTICE
See pages 891, 926.
MathOnline
Self-Check Quiz at algebra2.com

34. MEDICINE A patient must take blood pressure medication that is dispensed in 125-milligram tablets. The dosage is 15 milligrams per kilogram of body weight and is given every 8 hours. If the patient weighs 25 kilograms, how many tablets would be needed for a 30-day supply? Use the formula $n = [15b \div (125 \times 8)] \times 24d$, where n is the number of tablets, d is the number of days the supply should last, and b is body weight in kilograms.

35. QB RATING The formula for quarterback efficiency rating in the National Football League is $\left(\frac{\frac{C}{A} - 0.3}{0.2} + \frac{\frac{Y}{A} - 3}{4} + \frac{\frac{T}{A}}{0.05} + \frac{0.095 - \frac{I}{A}}{0.04}\right) \times \frac{100}{6}$, where C is the number of passes completed, A is the number of passes attempted, Y is passing yardage, T is the number of touchdown passes, and I is the number of interceptions. In 2005, Ben Roethlisberger of the Pittsburgh Steelers completed 168 of the 268 passes he attempted for 2385 yards. He threw 17 touchdowns and 9 interceptions. Find his efficiency rating for 2005.

H.O.T. Problems

36. OPEN ENDED Write an algebraic expression in which subtraction is performed before division, and the symbols ( ), [ ], or { } are not used.

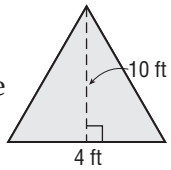
37. CHALLENGE Write expressions having values from one to ten using exactly four 4s. You may use any combination of the operation symbols +, −, ×, ÷, and/or grouping symbols, but no other digits are allowed. An example of such an expression with a value of zero is $(4 + 4) - (4 + 4)$.

38. REASONING Explain how to evaluate $a + b[(c + d) \div e]$, if you were given the values for a, b, c, d, and e.

39. *Writing in Math* Use the information about IV flow rates on page 6 to explain how formulas are used by nurses. Explain why a formula for the flow rate of an IV is more useful than a table of specific IV flow rates and describe the impact of using a formula, such as the one for IV flow rate, incorrectly.

STANDARDIZED TEST PRACTICE

40. ACT/SAT The following are the dimensions of four rectangles. Which rectangle has the same area as the triangle at the right?



A 1.6 ft by 25 ft
B 5 ft by 16 ft
C 3.5 ft by 4 ft
D 0.4 ft by 50 ft

41. REVIEW How many cubes that are 3 inches on each edge can be placed completely inside a box that is 9 inches long, 6 inches wide, and 27 inches tall?

F 12
G 36
H 54
J 72

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.**

42. $\sqrt{9}$
43. $\sqrt{16}$
44. $\sqrt{100}$
45. $\sqrt{169}$
46. $-\sqrt{4}$
47. $-\sqrt{25}$
48. $\sqrt{\frac{4}{9}}$
49. $\sqrt{\frac{36}{49}}$

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1-2 Properties of Real Numbers

Main Ideas

- Classify real numbers.
- Use the properties of real numbers to evaluate expressions.

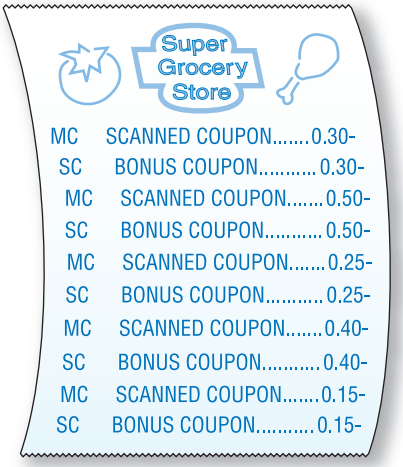
New Vocabulary

real numbers
rational numbers
irrational numbers

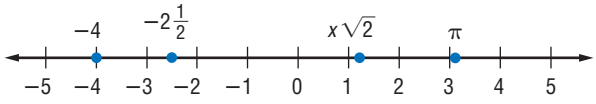
GET READY for the Lesson

Manufacturers often offer coupons to get consumers to try their products. Some grocery stores try to attract customers by doubling the value of manufacturers' coupons.

You can use the Distributive Property to calculate these savings.



Real Numbers The numbers that you use in everyday life are **real numbers**. Each real number corresponds to exactly one point on the number line, and every point on the number line represents exactly one real number.



Real numbers can be classified as either **rational** or **irrational**.

Review Vocabulary

Ratio the comparison of two numbers by division

KEY CONCEPT — *Real Numbers*

Words	A rational number can be expressed as a ratio $\frac{m}{n}$, where m and n are integers and n is not zero. The decimal form of a rational number is either a terminating or repeating decimal.
Examples	$\frac{1}{6}$, 1.9, 2.575757…, −3, $\sqrt{4}$, 0
Words	A real number that is not rational is irrational. The decimal form of an irrational number neither terminates nor repeats.
Examples	$\sqrt{5}$, π, 0.010010001…

The sets of natural numbers, {1, 2, 3, 4, 5, …}, whole numbers, {0, 1, 2, 3, 4, …}, and integers, {…, −3,−2,−1, 0, 1, 2, …} are all subsets of the rational numbers. The whole numbers are a subset of the rational numbers because every whole number n is equal to $\frac{n}{1}$.

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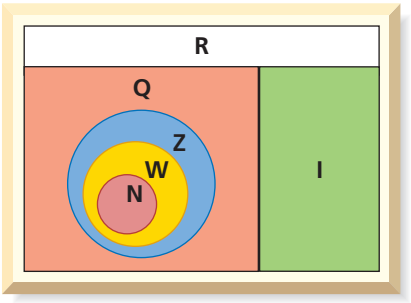

The Venn diagram shows the relationships among these sets of numbers.

R = reals Q = rationals

I = irrationals Z = integers

W = wholes N = naturals

The square root of any whole number is either a whole number or it is irrational. For example, $\sqrt{36}$ is a whole number, but $\sqrt{35}$ is irrational and lies between 5 and 6.

Study Tip

Common Misconception

Do not assume that a number is irrational because it is expressed using the square root symbol. Find its value first.

EXAMPLE Classify Numbers

1 **Name the sets of numbers to which each number belongs.**

a. $\sqrt{16}$

$\sqrt{16} = 4$ naturals (N), wholes (W), integers (Z), rationals (Q), reals (R)

b. -18 integers (Z), rationals (Q), and reals (R)

c. $\sqrt{20}$ irrationals (I) and reals (R)

$\sqrt{20}$ lies between 4 and 5 so it is not a whole number.

d. $-\frac{7}{8}$ rationals (Q) and reals (R)

e. $0.\overline{45}$ rationals (Q) and reals (R)

The bar over the 45 indicates that those digits repeat forever.

CHECK Your Progress

1A. -185 **1B.** $-\sqrt{49}$ **1C.** $\sqrt{95}$

Properties of Real Numbers Some of the properties of real numbers are summarized below.

KEY CONCEPT *Real Number Properties*

For any real numbers *a*, *b*, and *c*:

Property	Addition	Multiplication
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = 1 \cdot a$
Inverse	$a + (-a) = 0 = (-a) + a$	If $a \neq 0$, then $a \cdot \frac{1}{a} = 1 = \frac{1}{a} \cdot a$.
Distributive	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$	

Reading Math

Opposites

$-a$ is read *the opposite of a*.

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EXAMPLE 2 Identify Properties of Real Numbers

Name the property illustrated by $(5 + 7) + 8 = 8 + (5 + 7)$.

Commutative Property of Addition

The Commutative Property says that the order in which you add does not change the sum.

CHECK Your Progress

2. Name the property illustrated by $2(x + 3) = 2x + 6$.

EXAMPLE 3 Additive and Multiplicative Inverses

Identify the additive inverse and multiplicative inverse for $-1\frac{3}{4}$.

Since $-1\frac{3}{4} + \left(1\frac{3}{4}\right) = 0$, the additive inverse of $-1\frac{3}{4}$ is $1\frac{3}{4}$.

Since $-1\frac{3}{4} = -\frac{7}{4}$ and $\left(-\frac{7}{4}\right)\left(-\frac{4}{7}\right) = 1$, the multiplicative inverse of $-1\frac{3}{4}$ is $-\frac{4}{7}$.

CHECK Your Progress

Identify the additive inverse and multiplicative inverse for each number.

3A. 1.25 **3B.** $2\frac{1}{2}$

Concepts in Motion
Animation algebra2.com

You can model the Distributive Property using algebra tiles.

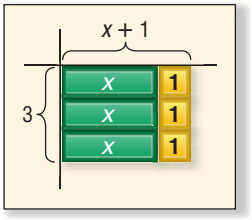
ALGEBRA LAB

Distributive Property

Step 1 A 1-tile is a square that is 1 unit wide and 1 unit long. Its area is 1 square unit. An x-tile is a rectangle that is 1 unit wide and x units long. Its area is x square units.

Step 2 To find the product $3(x + 1)$, model a rectangle with a width of 3 and a length of $x + 1$. Use your algebra tiles to mark off the dimensions on a product mat. Then make the rectangle with algebra tiles.

Step 3 The rectangle has 3 x-tiles and 3 1-tiles. The area of the rectangle is $x + x + x + 1 + 1 + 1$ or $3x + 3$. Thus, $3(x + 1) = 3x + 3$.



MODEL AND ANALYZE

Tell whether each statement is *true* or *false*. Justify your answer with algebra tiles and a drawing.

1. $4(x + 2) = 4x + 2$

2. $3(2x + 4) = 6x + 7$

3. $2(3x + 5) = 6x + 10$

4. $(4x + 1)5 = 4x + 5$

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Real-World EXAMPLE

4 **FOOD SERVICE** **A restaurant adds a 20% tip to the bills of parties of 6 or more people. Suppose a server waits on five such tables. The bill without the tip for each party is listed in the table. How much did the server make in tips during this shift?**

Party 1	Party 2	Party 3	Party 4	Party 5
$185.45	$205.20	$195.05	$245.80	$262.00

There are two ways to find the total amount of tips received.

Method 1 Multiply each dollar amount by 20% or 0.2 and then add.

$$\begin{aligned} T &= 0.2(185.45) + 0.2(205.20) + 0.2(195.05) + 0.2(245.80) + 0.2(262) \\ &= 37.09 + 41.04 + 39.01 + 49.16 + 52.40 \\ &= 218.70 \end{aligned}$$

Method 2 Add all of the bills and then multiply the total by 0.2.

$$\begin{aligned} T &= 0.2(185.45 + 205.20 + 195.05 + 245.80 + 262) \\ &= 0.2(1093.50) \\ &= 218.70 \end{aligned}$$

The server made $218.70 during this shift.

Notice that both methods result in the same answer.

Real-World Link

Leaving a "tip" began in 18th century English coffee houses and is believed to have originally stood for "To Insure Promptness." Today, the American Automobile Association suggests leaving a 15% tip.

Source: Market Facts, Inc.

CHECK Your Progress

4. Kayla makes $8 per hour working at a grocery store. The number of hours Kayla worked each day in one week are 3, 2.5, 2, 1, and 4. How much money did Kayla earn this week?

Online Personal Tutor at algebra2.com

The properties of real numbers can be used to simplify algebraic expressions.

EXAMPLE Simplify an Expression

5 **Simplify $2(5m + n) + 3(2m - 4n)$.**

$$\begin{aligned} &2(5m + n) + 3(2m - 4n) \\ &= 2(5m) + 2(n) + 3(2m) - 3(4n) && \text{Distributive Property} \\ &= 10m + 2n + 6m - 12n && \text{Multiply.} \\ &= 10m + 6m + 2n - 12n && \text{Commutative Property (+)} \\ &= (10 + 6)m + (2 - 12)n && \text{Distributive Property} \\ &= 16m - 10n && \text{Simplify.} \end{aligned}$$

CHECK Your Progress

5. Simplify $3(4x - 2y) - 2(3x + y)$.

Amy C. Etra/PhotoEdit

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CHECK Your Understanding

Example 1 (p. 12) **Name the sets of numbers to which each number belongs.**

1. -4 **2.** 45 **3.** $6.\overline{23}$

Example 2 (p. 13) **Name the property illustrated by each question.**

4. $\frac{2}{3} \cdot \frac{3}{2} = 1$ **5.** $(a + 4) + 2 = a + (4 + 2)$ **6.** $4x + 0 = 4x$

Example 3 (p. 13) **Identify the additive inverse and multiplicative inverse for each number.**

7. -8 **8.** $\frac{1}{3}$ **9.** 1.5

Example 4 (p. 14) **FUND-RAISING For Exercises 10 and 11, use the table.**

Catalina is selling candy for $1.50 each to raise money for the band.

10. Write an expression to represent the total amount of money Catalina raised during this week.

11. Evaluate the expression from Exercise 10 by using the Distributive Property.

Catalina's Sales for One Week

Day	Bars Sold
Monday	10
Tuesday	15
Wednesday	12
Thursday	8
Friday	19
Saturday	22
Sunday	31

Example 5 (p. 14) **Simplify each expression.**

12. $3(5c + 4d) + 6(d - 2c)$

13. $\frac{1}{2}(16 - 4a) - \frac{3}{4}(12 + 20a)$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–21	1
22–27	2
28–33	3
34, 35	4
36–43	5

Name the sets of numbers to which each number belongs.

14. $-\frac{2}{9}$ **15.** -4.55 **16.** $-\sqrt{10}$ **17.** $\sqrt{19}$

18. -31 **19.** $\frac{12}{2}$ **20.** $\sqrt{121}$ **21.** $-\sqrt{36}$

Name the property illustrated by each equation.

22. $5a + (-5a) = 0$ **23.** $-6xy + 0 = -6xy$

24. $[5 + (-2)] + (-4) = 5 + [-2 + (-4)]$ **25.** $(2 + 14) + 3 = 3 + (2 + 14)$

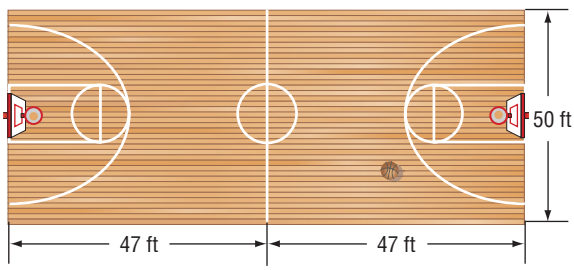
26. $\left(1\frac{2}{7}\right)\left(\frac{7}{9}\right) = 1$ **27.** $2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3}$

Identify the additive inverse and multiplicative inverse for each number.

28. -10 **29.** 2.5 **30.** -0.125

31. $-\frac{5}{8}$ **32.** $\frac{4}{3}$ **33.** $-4\frac{3}{5}$

34. BASKETBALL Illustrate the Distributive Property by writing two expressions for the area of the NCAA basketball court. Then find the area of the basketball court.



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35. BAKING Mitena is making two types of cookies. The first recipe calls for $2\frac{1}{4}$ cups of flour, and the second calls for $1\frac{1}{8}$ cups of flour. If she wants to make 3 batches of the first recipe and 2 batches of the second recipe, how many cups of flour will she need? Use the properties of real numbers to show how Mitena could compute this amount mentally. Justify each step.

Simplify each expression.

36. $7a + 3b - 4a - 5b$

37. $3x + 5y + 7x - 3y$

38. $3(15x - 9y) + 5(4y - x)$

39. $2(10m - 7a) + 3(8a - 3m)$

40. $8(r + 7t) - 4(13t + 5r)$

41. $4(14c - 10d) - 6(d + 4c)$

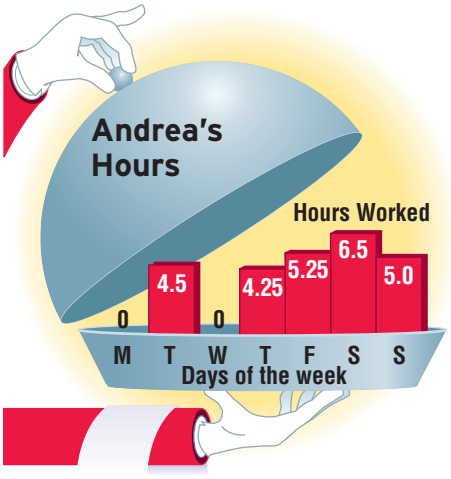
42. $4(0.2m - 0.3n) - 6(0.7m - 0.5n)$

43. $7(0.2p + 0.3q) + 5(0.6p - q)$

WORK **For Exercises 44 and 45, use the information below and in the graph.**
Andrea works in a restaurant and is paid every two weeks.

44. If Andrea earns \$6.50 an hour, illustrate the Distributive Property by writing two expressions representing Andrea's pay last week.

45. Find the mean or average number of hours Andrea worked each day, to the nearest tenth of an hour. Then use this average to predict her pay for a two-week pay period.



Real-World Link
Pythagoras (572–497 B.C.) was a Greek philosopher whose followers came to be known as the Pythagoreans. It was their knowledge of what is called the Pythagorean Theorem that led to the first discovery of irrational numbers.

Source: *A History of Mathematics*

NUMBER THEORY **For Exercises 46–49, use the properties of real numbers to answer each question.**

46. If $m + n = m$, what is the value of n?

47. If $m + n = 0$, what is the value of n? What is n's relationship to m?

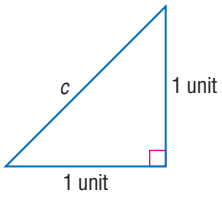
48. If $mn = 1$, what is the value of n? What is n's relationship to m?

49. If $mn = m$ and $m \neq 0$, what is the value of n?

MATH HISTORY **For Exercises 50–52, use the following information.**
The Greek mathematician Pythagoras believed that all things could be described by numbers. By *number* he meant a positive integer.

50. To what set of numbers was Pythagoras referring when he spoke of *numbers*?

51. Use the formula $c = \sqrt{2s^2}$ to calculate the length of the hypotenuse c, or longest side, of this right triangle using s, the length of one leg.



52. Explain why Pythagoras could not find a "number" for the value of c.

Name the sets of numbers to which each number belongs.

53. 0

54. $\frac{3\pi}{2}$

55. $-2\sqrt{7}$

56. Name the sets of numbers to which all of the following numbers belong. Then arrange the numbers in order from least to greatest.

$$2.\overline{49},\ 2.4\overline{9},\ 2.4,\ 2.49,\ 2.\overline{9}$$

EXTRA PRACTICE
See pages 891, 926.

Math Online
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H.O.T. Problems

OPEN ENDED **Give an example of a number that satisfies each condition.**

57. integer, but not a natural number

58. integer with a multiplicative inverse that is an integer

CHALLENGE Determine whether each statement is *true* or *false*. If *false*, give a counterexample. A **counterexample** is a specific case that shows that a statement is false.

59. Every whole number is an integer.

60. Every integer is a whole number.

61. Every real number is irrational.

62. Every integer is a rational number.

63. REASONING Is the Distributive Property also true for division? In other words, does $\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}$, $a \neq 0$? If so, give an example and explain why it is true. If not true, give a counterexample.

64. *Writing in Math* Use the information about coupons on page 11 to explain how the Distributive Property is useful in calculating store savings. Include an explanation of how the Distributive Property could be used to calculate the coupon savings listed on a grocery receipt.

STANDARDIZED TEST PRACTICE

65. ACT/SAT If a and b are natural numbers, then which of the following must also be a natural number?

I. $a - b$ II. ab III. $\frac{a}{b}$

A I only **C** III only

B II only **D** I and II only

66. REVIEW Which equation is equivalent to $4(9 - 3x) = 7 - 2(6 - 5x)$?

F $8x = 41$ **H** $22x = 41$

G $8x = 24$ **J** $22x = 24$

Spiral Review

Evaluate each expression. (Lesson 1-1)

67. $9(4 - 3)^5$

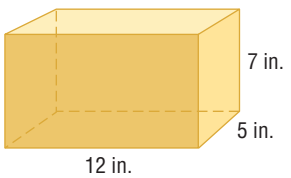
68. $5 + 9 \div 3(3) - 8$

Evaluate each expression if $a = -5$, $b = 0.25$, $c = \frac{1}{2}$, and $d = 4$. (Lesson 1-1)

69. $a + 2b - c$

70. $b + 3(a + d)^3$

71. GEOMETRY The formula for the surface area SA of a rectangular prism is $SA = 2\ell w + 2\ell h + 2wh$, where ℓ represents the length, w represents the width, and h represents the height. Find the surface area of the rectangular prism. (Lesson 1-1)



GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression if $a = 2$, $b = -\frac{3}{4}$, and $c = 1.8$.** (Lesson 1-1)

72. $8b - 5$

73. $\frac{2}{5}b + 1$

74. $1.5c - 7$

75. $-9(a - 6)$

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1-3 Solving Equations

Main Ideas

- Translate verbal expressions into algebraic expressions and equations, and vice versa.
- Solve equations using the properties of equality.

New Vocabulary

open sentence
equation
solution

GET READY for the Lesson

An important statistic for pitchers is the earned run average (ERA). To find the ERA, divide the number of earned runs allowed R by the number of innings pitched I. Then multiply the quotient by 9.

$$\text{ERA} = \frac{R \text{ runs}}{I \text{ innings}} \times \frac{9 \text{ innings}}{1 \text{ game}}$$
$$= \frac{9R}{I} \text{ runs per game}$$

Verbal Expressions to Algebraic Expressions Verbal expressions can be translated into algebraic or mathematical expressions. Any letter can be used as a variable to represent a number that is not known.

EXAMPLE 1 Verbal to Algebraic Expression

Write an algebraic expression to represent each verbal expression.

a. three times the square of a number $3x^2$

b. twice the sum of a number and 5 $2(y + 5)$

CHECK Your Progress

1A. the cube of a number increased by 4 times the same number

1B. three times the difference of a number and 8

A mathematical sentence containing one or more variables is called an **open sentence**. A mathematical sentence stating that two mathematical expressions are equal is called an **equation**.

EXAMPLE 2 Algebraic to Verbal Sentence

Write a verbal sentence to represent each equation.

a. $n + (-8) = -9$ The sum of a number and -8 is -9.

b. $\frac{n}{6} = n^2$ A number divided by 6 is equal to that number squared.

CHECK Your Progress

2A. $g - 5 = -2$

2B. $2c = c^2 - 4$

Andy Lyons/Getty Images

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Open sentences are neither true nor false until the variables have been replaced by numbers. Each replacement that results in a true sentence is called a **solution** of the open sentence.

Properties of Equality To solve equations, we can use properties of equality. Some of these properties are listed below.

KEY CONCEPT *Properties of Equality*

Property	Symbols	Examples
Reflexive	For any real number a, $a = a$.	$-7 + n = -7 + n$
Symmetric	For all real numbers a and b, if $a = b$, then $b = a$.	If $3 = 5x - 6$, then $5x - 6 = 3$.
Transitive	For all real numbers a, b, and c, if $a = b$ and $b = c$, then $a = c$.	If $2x + 1 = 7$ and $7 = 5x - 8$, then $2x + 1 = 5x - 8$.
Substitution	If $a = b$, then a may be replaced by b and b may be replaced by a.	If $(4 + 5)m = 18$, then $9m = 18$.

Vocabulary Link
Symmetric
Everyday Use having two identical sides
Math Use The two sides of an equation are equal, so the sides can be switched.

EXAMPLE **Identify Properties of Equality**

3 **Name the property illustrated by each statement.**

a. **If $3m = 5n$ and $5n = 10p$, then $3m = 10p$.**
Transitive Property of Equality

b. **If $12m = 24$, then $(2 \cdot 6)m = 24$.**
Substitution

CHECK Your Progress

3. If $-11a + 2 = -3a$, then $-3a = -11a + 2$.

Sometimes an equation can be solved by adding the same number to each side, or by subtracting the same number from each side, or by multiplying or dividing each side by the same number.

KEY CONCEPT *Properties of Equality*

Addition and Subtraction

Symbols For any real numbers a, b, and c, if $a = b$, then $a + c = b + c$ and $a - c = b - c$.

Examples If $x - 4 = 5$, then $x - 4 + 4 = 5 + 4$.
If $n + 3 = -11$, then $n + 3 - 3 = -11 - 3$.

Multiplication and Division

Symbols For any real numbers a, b, and c, if $a = b$, then $a \cdot c = b \cdot c$, and if $c \neq 0$, $\frac{a}{c} = \frac{b}{c}$.

Examples If $\frac{m}{4} = 6$, then $4 \cdot \frac{m}{4} = 4 \cdot 6$. If $-3y = 6$, then $\frac{-3y}{-3} = \frac{6}{-3}$.

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EXAMPLE 4 Solve One-Step Equations

Solve each equation. Check your solution.

a. $a + 4.39 = 76$

$$a + 4.39 = 76 \quad \text{Original equation}$$
$$a + 4.39 - 4.39 = 76 - 4.39 \quad \text{Subtract 4.39 from each side.}$$
$$a = 71.61 \quad \text{Simplify.}$$

The solution is 71.61.

CHECK
$$a + 4.39 = 76 \quad \text{Original equation}$$
$$71.61 + 4.39 \stackrel{?}{=} 76 \quad \text{Substitute 71.61 for } a.$$
$$76 = 76 \checkmark \quad \text{Simplify.}$$

b. $-\frac{3}{5}d = 18$

$$-\frac{3}{5}d = 18 \quad \text{Original equation}$$
$$-\frac{5}{3}\left(-\frac{3}{5}\right)d = -\frac{5}{3}(18) \quad \text{Multiply each side by } -\frac{5}{3}\text{, the multiplicative inverse of } -\frac{3}{5}.$$
$$d = -30 \quad \text{Simplify.}$$

The solution is -30.

CHECK
$$-\frac{3}{5}d = 18 \quad \text{Original equation}$$
$$-\frac{3}{5}(-30) \stackrel{?}{=} 18 \quad \text{Substitute } -30 \text{ for } d.$$
$$18 = 18 \checkmark \quad \text{Simplify.}$$

Study Tip

Multiplication and Division Properties of Equality

Example 4b could also have been solved using the Division Property of Equality. Note that dividing each side of the equation by $-\frac{3}{5}$ is the same as multiplying each side by $-\frac{5}{3}$.

CHECK Your Progress

4A. $x - 14.29 = 25$ **4B.** $\frac{2}{3}y = -18$

EXAMPLE 5 Solve a Multi-Step Equation

Solve $2(2x + 3) - 3(4x - 5) = 22$.

$$2(2x + 3) - 3(4x - 5) = 22 \quad \text{Original equation}$$
$$4x + 6 - 12x + 15 = 22 \quad \text{Apply the Distributive Property.}$$
$$-8x + 21 = 22 \quad \text{Simplify the left side.}$$
$$-8x = 1 \quad \text{Subtract 21 from each side to isolate the variable.}$$
$$x = -\frac{1}{8} \quad \text{Divide each side by } -8.$$

The solution is $-\frac{1}{8}$.

CHECK Your Progress

Solve each equation.

5A. $-10x + 3(4x - 2) = 6$ **5B.** $2(2x - 1) - 4(3x + 1) = 2$

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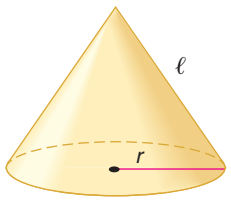
You can use properties to solve an equation or formula for a variable.

EXAMPLE Solve for a Variable

6 **GEOMETRY** **The formula for the surface area S of a cone is $S = \pi r\ell + \pi r^2$, where ℓ is the slant height of the cone and r is the radius of the base. Solve the formula for ℓ.**



$S = \pi r\ell + \pi r^2$ Surface area formula

$S - \pi r^2 = \pi r\ell + \pi r^2 - \pi r^2$ Subtract πr^2 from each side.

$S - \pi r^2 = \pi r\ell$ Simplify.

$\frac{S - \pi r^2}{\pi r} = \frac{\pi r\ell}{\pi r}$ Divide each side by πr.

$\frac{S - \pi r^2}{\pi r} = \ell$ Simplify.

CHECK Your Progress

6. The formula for the surface area S of a cylinder is $S = 2\pi r^2 + 2\pi rh$, where r is the radius of the base, and h is the height of the cylinder. Solve the formula for h.

STANDARDIZED TEST EXAMPLE Apply Properties of Equality

7 **If $3n - 8 = \frac{9}{5}$, what is the value of $3n - 3$?**

A $\frac{34}{5}$ **B** $\frac{49}{15}$ **C** $-\frac{16}{5}$ **D** $-\frac{27}{5}$

Test-Taking Tip

Using Properties
If a problem seems to require lengthy calculations, look for a shortcut. There may be a quicker way to solve it. Try using properties of equality.

Read the Test Item

You are asked to find the value of $3n - 3$. Your first thought might be to find the value of n and then evaluate the expression using this value. Notice that you are *not* required to find the value of n. Instead, you can use the Addition Property of Equality.

Solve the Test Item

$3n - 8 = \frac{9}{5}$ Original equation

$3n - 8 + 5 = \frac{9}{5} + 5$ Add 5 to each side.

$3n - 3 = \frac{34}{5}$ $\frac{9}{5} + 5 = \frac{9}{5} + \frac{25}{5}$ or $\frac{34}{5}$

The answer is A.

CHECK Your Progress

7. If $5y + 2 = \frac{8}{3}$, what is the value of $5y - 6$?

F $\frac{-20}{3}$ **G** $\frac{-16}{3}$ **H** $\frac{16}{3}$ **J** $\frac{32}{3}$

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To solve a word problem, it is often necessary to define a variable and write an equation. Then solve by applying the properties of equality.

Real-World Link

Previously occupied homes account for approximately 85% of all U.S. home sales. Most homeowners remodel within 18 months of purchase. The top two remodeling projects are kitchens and baths.

Source: National Association of Remodeling Industry

Real-World EXAMPLE Write an Equation

8 **HOME IMPROVEMENT** **Josh spent $425 of his $1685 budget for home improvements. He would like to replace six interior doors next. What can he afford to spend on each door?**

Explore Let c represent the cost to replace each door.

Plan Write and solve an equation to find the value of c.

The number of doors	times	the cost to replace each door	plus	previous expenses	equals	the total cost.
6	$\cdot$	c	$+$	425	$=$	1685

Solve

$6c + 425 = 1685$ Original equation

$6c + 425 - 425 = 1685 - 425$ Subtract 425 from each side.

$6c = 1260$ Simplify.

$\frac{6c}{6} = \frac{1260}{6}$ Divide each side by 6.

$c = 210$ Simplify.

Josh can afford to spend $210 on each door.

Check The total cost to replace six doors at $210 each is 6(210) or $1260. Add the other expenses of $425 to that, and the total home improvement bill is 1260 + 425 or $1685. Thus, the answer is correct.

CHECK Your Progress

8. A radio station had 300 concert tickets to give to its listeners as prizes. After 1 week, the station had given away 108 tickets. If the radio station wants to give away the same number of tickets each day for the next 8 days, how many tickets must be given away each day?

MathOnline **Problem Solving Handbook at** algebra2.com

CHECK Your Understanding

Example 1 (p. 18) **Write an algebraic expression to represent each verbal expression.**

1. five increased by four times a number

2. twice a number decreased by the cube of the same number

Example 2 (p. 18) **Write a verbal expression to represent each equation.**

3. $9n - 3 = 6$

4. $5 + 3x^2 = 2x$

Example 3 (p. 19) **Name the property illustrated by each statement.**

5. $(3x + 2) - 5 = (3x + 2) - 5$

6. If $4c = 15$, then $4c + 2 = 15 + 2$.

Michael Newman/PhotoEdit

www.MathSchoolinternational.com

Examples 4–5 (p. 20)

Solve each equation. Check your solution.

7. $y + 14 = -7$ **8.** $3x = 42$ **9.** $16 = -4b$

10. $4(q - 1) - 3(q + 2) = 25$ **11.** $1.8a - 5 = -2.3$ **12.** $-\frac{3}{4}n + 1 = -11$

Example 6 (p. 21)

Solve each equation or formula for the specified variable.

13. $4y - 2n = 9$, for y **14.** $I = prt$, for p

Example 7 (p. 21)

15. STANDARDIZED TEST PRACTICE If $4x + 7 = 18$, what is the value of $12x + 21$?

A 2.75 **B** 32 **C** 33 **D** 54

Example 8 (p. 22)

16. BASEBALL During the 2005 season, Jacque Jones and Matthew LeCroy of the Minnesota Twins hit a combined total of 40 home runs. Jones hit 6 more home runs than LeCroy. How many home runs did each player hit? Define a variable, write an equation, and solve the problem.

Exercises

HOMEWORK HELP

For Exercises	See Examples
17–22	1
23–26	2
27–30	3
31, 32	4
33–36	5
37–40	6
41	7
42, 43	8

Write an algebraic expression to represent each verbal expression.

17. the sum of 5 and three times a number

18. seven more than the product of a number and 10

19. four less than the square of a number

20. the product of the cube of a number and -6

21. five times the sum of 9 and a number

22. twice the sum of a number and 8

Write a verbal expression to represent each equation.

23. $x - 5 = 12$ **24.** $2n + 3 = -1$

25. $y^2 = 4y$ **26.** $3a^3 = a + 4$

Name the property illustrated by each statement.

27. If $[3(-2)]z = 24$, then $-6z = 24$. **28.** If $5 + b = 13$, then $b = 8$.

29. If $2x = 3d$ and $3d = -4$, then $2x = -4$. **30.** If $y - 2 = -8$, then $3(y - 2) = 3(8)$.

Solve each equation. Check your solution.

31. $2p = 14$ **32.** $-14 + n = -6$

33. $7a - 3a + 2a - a = 16$ **34.** $x + 9x - 6x + 4x = 20$

35. $27 = -9(y + 5) + 6(y + 8)$ **36.** $-7(p + 7) + 3(p - 4) = -17$

Solve each equation or formula for the specified variable.

37. $d = rt$, for r **38.** $x = \frac{-b}{2a}$, for a

39. $V = \frac{1}{3}\pi r^2 h$, for h **40.** $A = \frac{1}{2}h(a + b)$, for b

41. If $3a + 1 = \frac{13}{3}$, what is the value of $3a - 3$?

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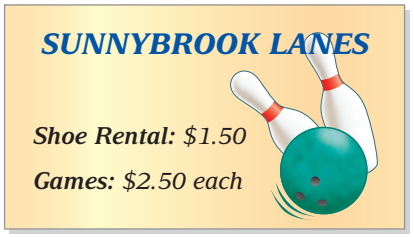
For Exercises 42 and 43, define a variable, write an equation, and solve the problem.

42. BOWLING Omar and Morgan arrive at Sunnybrook Lanes with $16.75. What is the total number of games they can afford if they each rent shoes?

43. GEOMETRY The perimeter of a regular octagon is 124 inches. Find the length of each side.



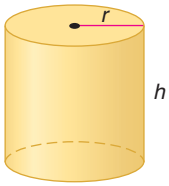
Write an algebraic expression to represent each verbal expression.

44. the square of the quotient of a number and 4

45. the cube of the difference of a number and 7

GEOMETRY For Exercises 46 and 47, use the following information.

The formula for the surface area of a cylinder with radius r and height h is π times twice the product of the radius and height plus twice the product of π and the square of the radius.



46. Write this as an algebraic expression.

47. Write an equivalent expression using the Distributive Property.

Write a verbal expression to represent each equation.

48. $\frac{b}{4} = 2(b + 1)$

49. $7 - \frac{1}{2}x = \frac{3}{x^2}$

Solve each equation or formula for the specified variable.

50. $\frac{a(b-2)}{c-3} = x$, for b

51. $x = \frac{y}{y+4}$, for y

Solve each equation. Check your solution.

52. $\frac{1}{9} - \frac{2}{3}b = \frac{1}{18}$

53. $3f - 2 = 4f + 5$

54. $4(k + 3) + 2 = 4.5(k + 1)$

55. $4.3n + 1 = 7 - 1.7n$

56. $\frac{3}{11}a - 1 = \frac{7}{11}a + 9$

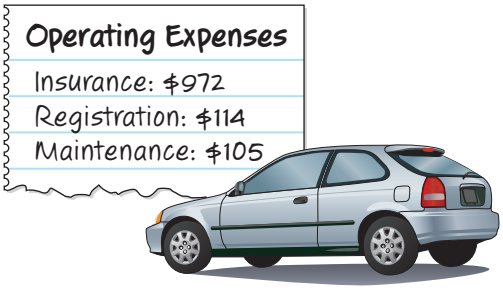
57. $\frac{2}{5}x + \frac{3}{7} = 1 - \frac{4}{7}x$

Cross-Curricular Project

You can write and solve equations to determine the monthly payment for a home. Visit algebra2.com to continue work on your project.

For Exercises 58–63, define a variable, write an equation, and solve the problem.

58. CAR EXPENSES Benito spent $1837 to operate his car last year. Some of these expenses are listed at the right. Benito's only other expense was for gasoline. If he drove 7600 miles, what was the average cost of the gasoline per mile?



59. SCHOOL A school conference room can seat a maximum of 83 people. The principal and two counselors need to meet with the school's student athletes to discuss eligibility requirements. If each student must bring a parent with them, how many students can attend each meeting?

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Real-World Career

Industrial Design

Industrial designers use research on product use, marketing, materials, and production methods to create functional and appealing packaging designs.

Math Online

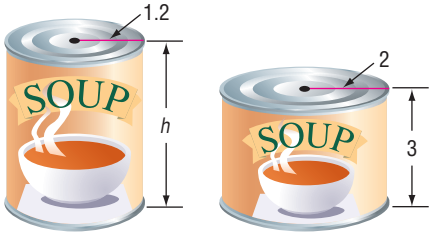
For more information, go to algebra2.com.

60. AGES Chun-Wei's mother is 8 more than twice his age. His father is three years older than his mother is. If the three family members have lived a total of 94 years, how old is each family member?

61. SCHOOL TRIP A Parent Teacher Organization has raised $1800 to help pay for a trip to an amusement park. They ask that there be one adult for every five students attending. Adult tickets are $45 and student tickets are $30. If the group wants to take 50 students, how much will each student need to pay so that adults agreeing to chaperone pay nothing?

62. BUSINESS A trucking company is hired to deliver 125 lamps for $12 each. The company agrees to pay $45 for each lamp that is broken during transport. If the trucking company needs to receive a minimum payment of $1364 for the shipment to cover their expenses, find the maximum number of lamps they can afford to break during the trip.

63. PACKAGING Two designs for a soup can are shown at the right. If each can holds the same amount of soup, what is the height of can A?



Can A Can B

RAILROADS **For Exercises 64–66, use the following information.**

The First Transcontinental Railroad was built by two companies. The Central Pacific began building eastward from Sacramento, California, while the Union Pacific built westward from Omaha, Nebraska. The two lines met at Promontory, Utah, in 1869, approximately 6 years after construction began.

64. The Central Pacific Company laid an average of 9.6 miles of track per month. Together the two companies laid a total of 1775 miles of track. Determine the average number of miles of track laid per month by the Union Pacific Company.

65. About how many miles of track did each company lay?

66. Why do you think the Union Pacific was able to lay track so much more quickly than the Central Pacific?

EXTRA PRACTICE
See pages 891, 926.
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67. MONEY Allison is saving money to buy a video game system. In the first week, her savings were $8 less than $\frac{2}{5}$ the price of the system. In the second week, she saved 50 cents more than $\frac{1}{2}$ the price of the system. She was still $37 short. Find the price of the system.

H.O.T. Problems

68. FIND THE ERROR Crystal and Jamal are solving $C = \frac{5}{9}(F - 32)$ for F. Who is correct? Explain your reasoning.

Crystal

$C = \frac{5}{9}(F - 32)$

$C + 32 = \frac{5}{9}F$

$\frac{9}{5}(C + 32) = F$

Jamal

$C = \frac{5}{9}(F - 32)$

$\frac{9}{5}C = F - 32$

$\frac{9}{5}C + 32 = F$

Robert Llewellyn/Imagestate
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69. OPEN ENDED Write a two-step equation with a solution of -7.

70. REASONING Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

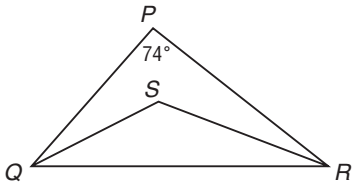
Dividing each side of an equation by the same expression produces an equivalent equation.

71. CHALLENGE Compare and contrast the Symmetric Property of Equality and the Commutative Property of Addition.

72. *Writing in Math* Use the information about ERA on page 18 to find the number of earned runs allowed for a pitcher who has an ERA of 2.00 and who has pitched 180 innings. Explain when it would be desirable to solve a formula like the one given for a specified variable.

STANDARDIZED TEST PRACTICE

73. ACT/SAT In triangle PQR, $\overline{QS}$ and $\overline{SR}$ are angle bisectors and angle $P = 74°$. How many degrees are there in angle QSR?



A 106 **C** 125

B 121 **D** 127

74. REVIEW Which of the following best describes the graph of the equations below?

$8y = 2x + 13$

$24y = 6x + 13$

F The lines have the same y-intercept.

G The lines have the same x-intercept.

H The lines are perpendicular.

J The lines are parallel.

Spiral Review

Simplify each expression. (Lesson 1-2)

75. $2x + 9y + 4z - y - 8x$

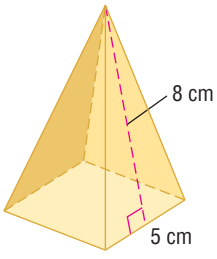
76. $4(2a + 5b) - 3(4b - a)$

Evaluate each expression if $a = 3$, $b = -2$, and $c = 1.2$. (Lesson 1-1)

77. $a - [b(a - c)]$

78. $c^2 - ab$

79. GEOMETRY The formula for the surface area S of a regular pyramid is $S = \frac{1}{2}P\ell + B$, where P is the perimeter of the base, ℓ is the slant height, and B is the area of the base. Find the surface area of the square pyramid at the right. (Lesson 1-1)



GET READY for the Next Lesson

PREREQUISITE SKILL **Identify the additive inverse for each number or expression.** (Lesson 1-2)

80. 2.5 **81.** $\frac{1}{4}$ **82.** $-3x$ **83.** $5 - 6y$

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1-4 Solving Absolute Value Equations

Main Ideas

- Evaluate expressions involving absolute values.
- Solve absolute value equations.

New Vocabulary

absolute value

empty set

GET READY for the Lesson

Seismologists use the Richter scale to express the magnitudes of earthquakes. This scale ranges from 1 to 10, with 10 being the highest. The uncertainty in the estimate of a magnitude E is about plus or minus 0.3 unit. This means that an earthquake with a magnitude estimated at 6.1 on the Richter scale might actually have a magnitude as low as 5.8 or as high as 6.4. These extremes can be described by the absolute value equation $|E - 6.1| = 0.3$.

Absolute Value Expressions The **absolute value** of a number is its distance from 0 on the number line. Since distance is nonnegative, the absolute value of a number is always nonnegative. The symbol $|x|$ is used to represent the absolute value of a number x.

KEY CONCEPT *Absolute Value*

Words For any real number a, if a is positive or zero, the absolute value of a is a. If a is negative, the absolute value of a is the opposite of a.

Symbols For any real number a, $|a| = a$ if $a \geq 0$, and $|a| = -a$ if $a < 0$.

When evaluating expressions, absolute value bars act as a grouping symbol. Perform any operations inside the absolute value bars first.

EXAMPLE **Evaluate an Expression with Absolute Value**

1 **Evaluate $1.4 + |5y - 7|$ if $y = -3$.**

$1.4 + |5y - 7| = 1.4 + |5(-3) - 7|$ Replace y with -3.

$= 1.4 + |-15 - 7|$ Simplify $5(-3)$ first.

$= 1.4 + |-22|$ Subtract 7 from -15.

$= 1.4 + 22$ $|-22| = 22$

$= 23.4$ Add.

CHECK Your Progress

1A. Evaluate $\left|4x + 3\right| - 3\frac{1}{2}$ if $x = -2$.

1B. Evaluate $1\frac{1}{3} - \left|2y + 1\right|$ if $y = -\frac{2}{3}$.

Robert Yager/Getty Images

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Absolute Value Equations Some equations contain absolute value expressions. The definition of absolute value is used in solving these equations. For any real numbers a and b, where $b \geq 0$, if $|a| = b$, then $a = b$ or $-a = b$. This second case is often written as $a = -b$.

EXAMPLE Solve an Absolute Value Equation

2 **Solve $|x - 18| = 5$. Check your solutions.**

Case 1 $a = b$ or **Case 2** $a = -b$

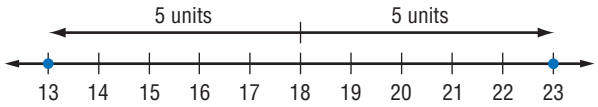
$$x - 18 = 5 \qquad x - 18 = -5$$
$$x - 18 + 18 = 5 + 18 \qquad x - 18 + 18 = -5 + 18$$
$$x = 23 \qquad x = 13$$

CHECK

$$|x - 18| = 5 \qquad |x - 18| = 5$$
$$|23 - 18| \stackrel{?}{=} 5 \qquad |13 - 18| \stackrel{?}{=} 5$$
$$|5| \stackrel{?}{=} 5 \qquad |-5| \stackrel{?}{=} 5$$
$$5 = 5 \checkmark \qquad 5 = 5 \checkmark$$

The solutions are 23 and 13. Thus, the solution set is {13, 23}.

On the number line, we can see that each answer is 5 units away from 18.



CHECK Your Progress

Solve each equation. Check your solutions.

2A. $9 = |x + 12|$ **2B.** $8 = |y + 5|$

Study Tip

Symbols

The empty set is symbolized by { } or ∅.

Because the absolute value of a number is always positive or zero, an equation like $|x| = -5$ is never true. Thus, it has no solution. The solution set for this type of equation is the **empty set**.

EXAMPLE No Solution

3 **Solve $|5x - 6| + 9 = 0$.**

$|5x - 6| + 9 = 0$ Original equation

$|5x - 6| = -9$ Subtract 9 from each side.

This sentence is *never* true. So the solution set is ∅.

CHECK Your Progress

3A. Solve $-2|3a - 2| = 6$. **3B.** Solve $|4b + 1| + 8 = 0$.

It is important to check your answers when solving absolute value equations. Even if the correct procedure for solving the equation is used, the answers may not be actual solutions of the original equation.

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EXAMPLE One Solution

4 **Solve $|x + 6| = 3x - 2$. Check your solutions.**

Case 1 $a = b$ or **Case 2** $a = -b$

$$x + 6 = 3x - 2 \qquad x + 6 = -(3x - 2)$$
$$6 = 2x - 2 \qquad x + 6 = -3x + 2$$
$$8 = 2x \qquad 4x + 6 = 2$$
$$4 = x \qquad 4x = -4$$
$$x = -1$$

There appear to be two solutions, 4 and -1.

CHECK Substitute each value in the original equation.

$$|x + 6| = 3x - 2 \qquad |x + 6| = 3x - 2$$
$$|4 + 6| \stackrel{?}{=} 3(4) - 2 \qquad |-1 + 6| \stackrel{?}{=} 3(-1) - 2$$
$$|10| \stackrel{?}{=} 12 - 2 \qquad |5| \stackrel{?}{=} -3 - 2$$
$$10 = 10 \ \checkmark \qquad 5 \neq -5$$

Since $5 \neq -5$, the only solution is 4. Thus, the solution set is $\{4\}$.

CHECK Your Progress

Solve each equation. Check your solutions.

4A. $2|x + 1| - x = 3x - 4$ **4B.** $3|2x + 2| - 2x = x + 3$

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CHECK Your Understanding

Example 1 (p. 27)

Evaluate each expression if $a = -4$ and $b = 1.5$.

1. $|a + 12|$ **2.** $|-6b|$ **3.** $-|a + 21| + 6.2$

Example 2 (p. 28)

FOOD For Exercises 4–6, use the following information.
Most meat thermometers are accurate to within plus or minus 2°F.

4. If a meat thermometer reads 160°F, write an equation to determine the least and greatest possible temperatures of the meat.

5. Solve the equation you wrote in Exercise 4.

6. Ham needs to reach an internal temperature of 160°F to be fully cooked. To what temperature reading should you cook a ham to ensure that the minimum temperature is reached? Explain.

Examples 2–4 (pp. 28–29)

Solve each equation. Check your solutions.

7. $|x + 4| = 17$ **8.** $|b + 15| = 3$

9. $20 = |a - 9|$ **10.** $34 = |y - 2|$

11. $|2w + 3| + 6 = 2$ **12.** $|3n + 2| + 4 = 0$

13. $|c - 2| = 2c - 10$ **14.** $|h - 5| = 3h - 7$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
15–22	1
23–32	2–4
33–34	2

Evaluate each expression if $a = -5$, $b = 6$, and $c = 2.8$.

15. $|-3a|$ **16.** $|-4b|$ **17.** $|a + 5|$ **18.** $|2 - b|$

19. $|2b - 15|$ **20.** $|4a + 7|$ **21.** $-|18 - 5c|$ **22.** $-|2c - a|$

Solve each equation. Check your solutions.

23. $|x - 25| = 17$ **24.** $|y + 9| = 21$

25. $33 = |a + 12|$ **26.** $11 = |3x + 5|$

27. $8|w - 7| = 72$ **28.** $2|b + 4| = 48$

29. $0 = |2z - 3|$ **30.** $|6c - 1| = 0$

31. $-12|9x + 1| = 144$ **32.** $1 = |5x + 9| + 6$

33. COFFEE Some say that to brew an excellent cup of coffee, you must have a brewing temperature of 200°F, plus or minus 5 degrees. Write and solve an equation describing the maximum and minimum brewing temperatures for an excellent cup of coffee.

34. SURVEYS Before an election, a company conducts a telephone survey of likely voters. Based on their survey data, the polling company states that an amendment to the state constitution is supported by 59% of the state's residents and that 41% of the state's residents do not approve of the amendment. According to the company, the results of their survey have a margin of error of 3%. Write and solve an equation describing the maximum and minimum percent of the state's residents that support the amendment.

Solve each equation. Check your solutions.

35. $35 = 7|4x - 13|$ **36.** $-9 = -3|2n + 5|$

37. $-6 = |a - 3| - 14$ **38.** $3|p - 5| = 2p$

39. $3|2a + 7| = 3a + 12$ **40.** $|3x - 7| - 5 = -3$

41. $16t = 4|3t + 8|$ **42.** $-2m + 3 = |15 + m|$

Evaluate each expression if $x = 6$, $y = 2.8$, and $z = -5$.

43. $9 - |-2x + 8|$ **44.** $3|z - 10| + |2z|$ **45.** $|z - x| - |10y - z|$

46. MANUFACTURING A machine fills bags with about 16 ounces of sugar each. After the bags are filled, another machine weighs them. If the bag weighs 0.3 ounce more or less than the desired weight, the bag is rejected. Write an equation to find the heaviest and lightest bags the machine will approve.

47. METEOROLOGY The *troposphere* is the layer of atmosphere closest to Earth. The average upper boundary of the layer is about 13 kilometers above Earth's surface. This height varies with latitude and with the seasons by as much as 5 kilometers. Write and solve an equation describing the maximum and minimum heights of the upper bound of the troposphere.

EXTRA PRACTICE
See pages 892, 926.
MathOnline
Self-Check Quiz at algebra2.com

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H.O.T. Problems

48. OPEN ENDED Write an absolute value equation and graph the solution set.

CHALLENGE For Exercises 49–51, determine whether each statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

49. If a and b are real numbers, then $|a + b| = |a| + |b|$.

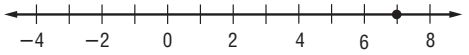
50. If a, b, and c are real numbers, then $c|a + b| = |ca + cb|$.

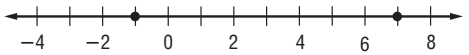
51. For all real numbers a and b, $a \neq 0$, the equation $|ax + b| = 0$ will have exactly one solution.

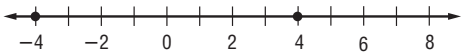
52. ***Writing in Math*** Use the information on page 27 to explain how an absolute value equation can describe the magnitude of an earthquake. Include a verbal and graphical explanation of how $|E - 6.1| = 0.3$ describes the possible magnitudes.

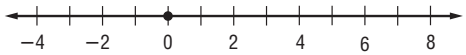
STANDARDIZED TEST PRACTICE

53. ACT/SAT Which graph represents the solution set for $|x - 3| - 4 = 0$?

A 

B 

C 

D 

54. REVIEW For a party, Lenora bought several pounds of cashews and several pounds of almonds. The cashews cost \$8 per pound, and the almonds cost \$6 per pound. Lenora bought a total of 7 pounds and paid a total of \$48. How many pounds of cashews did she buy?

F 2 pounds
H 4 pounds
G 3 pounds
J 5 pounds

Spiral Review

Solve each equation. Check your solution. (Lesson 1-3)

55. $3x + 6 = 22$

56. $7p - 4 = 3(4 + 5p)$

57. $\frac{5}{7}y - 3 = \frac{3}{7}y + 1$

Name the property illustrated by each equation. (Lesson 1-2)

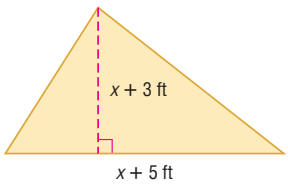
58. $(5 + 9) + 13 = 13 + (5 + 9)$

59. $m(4 - 3) = m \cdot 4 - m \cdot 3$

GEOMETRY For Exercises 60 and 61, use the following information.
The formula for the area A of a triangle is $A = \frac{1}{2}bh$, where b is the measure of the base and h is the measure of the height. (Lesson 1-1)

60. Write an expression to represent the area of the triangle.

61. Evaluate the expression you wrote in Exercise 60 for $x = 23$.



GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation. (Lesson 1-3)

62. $14y - 3 = 25$

63. $4.2x + 6.4 = 40$

64. $7w + 2 = 3w - 6$

65. $2(a - 1) = 8a - 6$

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CHAPTER 1

Mid-Chapter Quiz

Lessons 1-1 through 1-4

Evaluate each expression if $a = -2$, $b = \frac{1}{3}$, and $c = -12$. (Lesson 1-1)

1. $a^3 + b(9 - c)$
2. $b(a^2 - c)$
3. $\frac{3ab}{c}$
4. $\frac{a - c}{a + c}$
5. $\frac{a^3 - c}{b^2}$
6. $\frac{c + 3}{ab}$

7. **ELECTRICITY** Find the amount of current I (in amperes) produced if the electromotive force E is 2.5 volts, the circuit resistance R is 1.05 ohms, and the resistance r within a battery is 0.2 ohm. Use the formula $I = \frac{E}{R + r}$. (Lesson 1-1)

Name the sets of numbers to which each number belongs. (Lesson 1-2)

8. 3.5
9. $\sqrt{100}$

Name the property illustrated by each equation. (Lesson 1-2)

10. $bc + (-bc) = 0$
11. $\left(\frac{4}{7}\right)\left(1\frac{3}{4}\right) = 1$
12. $3 + (x - 1) = (3 + x) + (-1)$

Name the additive inverse and multiplicative inverse for each number. (Lesson 1-2)

13. $\frac{6}{7}$
14. $-\frac{4}{3}$

15. Simplify $4(14x - 10y) - 6(x + 4y)$. (Lesson 1-2)

Write an algebraic expression to represent each verbal expression. (Lesson 1-3)

16. twice the difference of a number and 11
17. the product of the square of a number and 5

Solve each equation. Check your solution. (Lesson 1-3)

18. $-2(a + 4) = 2$
19. $2d + 5 = 8d + 2$
20. $4y - \frac{1}{10} = 3y + \frac{4}{5}$

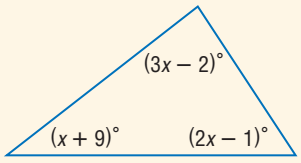
21. Solve $s = \frac{1}{2}gt^2$ for g. (Lesson 1-3)

22. **MULTIPLE CHOICE** Karissa has \$10 per month to spend text messaging on her cell phone. The phone company charges \$4.95 for the first 100 messages and \$0.10 for each additional message. How many text messages can Karissa afford to send each month? (Lesson 1-3)

A 50 **C** 150

B 100 **D** 151

23. **GEOMETRY** Use the information in the figure to find the value of x. Then state the degree measures of the three angles of the triangle. (Lesson 1-3)



Solve each equation. Check your solutions. (Lesson 1-4)

24. $|a + 4| = 3$
25. $|3x + 2| = 1$
26. $|3m - 2| = -4$
27. $|2x + 5| - 7 = 4$
28. $|h + 6| + 9 = 8$
29. $|5x - 2| - 6 = -3$

30. **CARNIVAL GAMES** Julian will win a prize if the carnival worker cannot guess his weight to within 3 pounds. Julian weighs 128 pounds. Write an equation to find the highest and lowest weights that the carnival guesser can guess to keep Julian from winning a prize. (Lesson 1-4)

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1-5 Solving Inequalities

Main Ideas

- Solve inequalities with one operation.
- Solve multi-step inequalities.

New Vocabulary

set-builder notation

GET READY for the Lesson

Kuni is trying to decide between two rate plans offered by a wireless phone company.

	Plan 1	Plan 2
Monthly Access Fee	$35.00	$55.00
Minutes Included	400	650
Additional Minutes	40¢	35¢

To compare these two rate plans, we can use inequalities. The monthly access fee for Plan 1 is less than the fee for Plan 2, $35 < $55. However, the additional minutes fee for Plan 1 is greater than that of Plan 2, $0.40 > $0.35.

Solve Inequalities with One Operation For any two real numbers, a and b, exactly one of the following statements is true.

$$a < b \qquad a = b \qquad a > b$$

This is known as the **Trichotomy Property.**

Adding the same number to, or subtracting the same number from, each side of an inequality does not change the truth of the inequality.

KEY CONCEPT *Properties of Inequality*

Addition Property of Inequality

Words For any real numbers, a, b, and c:
If $a > b$, then $a + c > b + c$.
If $a < b$, then $a + c < b + c$.

Example
$3 < 5$
$3 + (-4) < 5 + (-4)$
$-1 < 1$

Subtraction Property of Inequality

Words For any real numbers, a, b, and c:
If $a > b$, then $a - c > b - c$.
If $a < b$, then $a - c < b - c$.

Example
$2 > -7$
$2 - 8 > -7 - 8$
$-6 > -15$

These properties are also true for $\leq$, $\geq$, and $\neq$.

These properties can be used to solve inequalities. The solution sets of inequalities in one variable can then be graphed on number lines. Graph using a circle with an arrow to the left for $<$ and an arrow to the right for $>$. Graph using a dot with an arrow to the left for $\leq$ and an arrow to the right for $\geq$.

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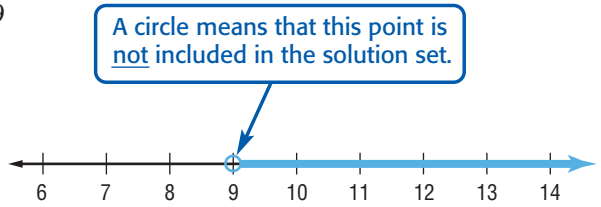
EXAMPLE 1 Solve an Inequality Using Addition or Subtraction

Solve $7x - 5 > 6x + 4$. Graph the solution set on a number line.

$7x - 5 > 6x + 4$	Original inequality
$7x - 5 + (-6x) > 6x + 4 + (-6x)$	Add $-6x$ to each side.
$x - 5 > 4$	Simplify.
$x - 5 + 5 > 4 + 5$	Add 5 to each side.
$x > 9$	Simplify.

Any real number greater than 9 is a solution of this inequality. The graph of the solution set is shown at the right.



CHECK Substitute a number greater than 9 for x in $7x - 5 > 6x + 4$. The inequality should be true.

CHECK Your Progress

1. Solve $4x + 7 \leq 3x + 9$. Graph the solution set on a number line.

Multiplying or dividing each side of an inequality by a positive number does not change the truth of the inequality. However, multiplying or dividing each side of an inequality by a *negative* number requires that the order of the inequality be *reversed*. For example, to reverse $\leq$, replace it with $\geq$.

KEY CONCEPT — Properties of Inequality

Multiplication Property of Inequality

Words		Examples
For any real numbers, a, b, and c, where		
c is positive:	if $a > b$, then $ac > bc$.	$-2 < 3$
	if $a < b$, then $ac < bc$.	$4(-2) < 4(3)$
		$-8 < 12$
c is negative:	if $a > b$, then $ac < bc$.	$5 > -1$
	if $a < b$, then $ac > bc$.	$(-3)(5) < (-3)(21)$
		$-15 < 3$

Division Property of Inequality

Words		Examples
For any real numbers, a, b, and c, where		
c is positive:	if $a > b$, then $\frac{a}{c} > \frac{b}{c}$.	$-18 < -9$
	if $a < b$, then $\frac{a}{c} < \frac{b}{c}$.	$\frac{-18}{3} < \frac{-9}{3}$
		$-6 < -3$
c is negative:	if $a > b$, then $\frac{a}{c} < \frac{b}{c}$.	$12 > 8$
	if $a < b$, then $\frac{a}{c} > \frac{b}{c}$.	$\frac{12}{-2} < \frac{8}{-2}$
		$-6 < -4$

These properties are also true for $\leq$, $\geq$, and $\neq$.

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Reading Math

Set-Builder Notation
$\{x \mid x > 9\}$ is read *the set of all numbers x such that x is greater than 9.*

The solution set of an inequality can be expressed by using **set-builder notation.** For example, the solution set in Example 1 can be expressed as $\{x \mid x > 9\}$.

EXAMPLE 2 Solve an Inequality Using Multiplication or Division

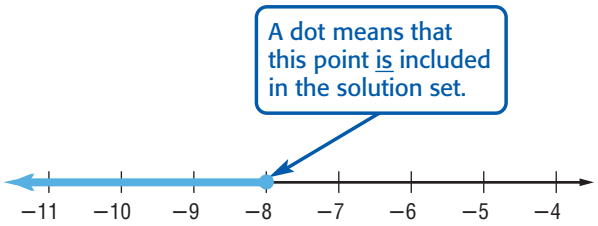
Solve $-0.25y \geq 2$. Graph the solution set on a number line.

$-0.25y \geq 2$ Original inequality

$\frac{-0.25y}{-0.25} \leq \frac{2}{-0.25}$ Divide each side by -0.25, reversing the inequality symbol.

$y \leq -8$ Simplify.

The solution set is $\{y \mid y \leq -8\}$. The graph of the solution set is shown below.



CHECK Your Progress

2. Solve $-\frac{1}{3}x < 1$. Graph the solution set on a number line.

Study Tip

Solutions to Inequalities

When solving an inequality,

- if you arrive at a false statement, such as $3 > 5$, then the solution set for that inequality is the empty set, $\emptyset$.
- if you arrive at a true statement such as $3 > -1$, then the solution set for that inequality is the set of all real numbers.

Solve Multi-Step Inequalities Solving multi-step inequalities is similar to solving multi-step equations.

EXAMPLE 3 Solve a Multi-Step Inequality

Solve $-m \leq \frac{m+4}{9}$. Graph the solution set on a number line.

$-m \leq \frac{m+4}{9}$ Original inequality

$-9m \leq m + 4$ Multiply each side by 9.

$-10m \leq 4$ Add $-m$ to each side.

$m \geq -\frac{4}{10}$ Divide each side by -10, reversing the inequality symbol.

$m \geq -\frac{2}{5}$ Simplify.

The solution set is $\left\{m \mid m \geq -\frac{2}{5}\right\}$ and is graphed below.

CHECK Your Progress

3. Solve $3(2q - 4) > 6$. Graph the solution set on a number line.

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Real-World EXAMPLE Write an Inequality

4 **DELIVERIES** **Craig is delivering boxes of paper. Each box weighs 64 pounds, and Craig weighs 160 pounds. If the maximum capacity of the elevator is 2000 pounds, how many boxes can Craig safely take on each trip?**

Explore Let b = the number of boxes Craig can safely take on each trip. A maximum capacity of 2000 pounds means that the total weight must be less than or equal to 2000.

Plan The total weight of the boxes is $64b$. Craig's weight plus the total weight of the boxes must be less than or equal to 2000. Write an inequality.

Craig's weight		plus		the weight of the boxes		is less than or equal to		2000.
160		+		$64b$		$\leq$		2000

Solve

$160 + 64b \leq 2000$ Original inequality

$64b \leq 1840$ Subtract 160 from each side.

$b \leq 28.75$ Divide each side by 64.

Check Since Craig cannot take a fraction of a box, he can take no more than 28 boxes per trip and still meet the safety requirements.

Study Tip

Math Symbols

- $<$ is less than; is fewer than
- $>$ is greater than; is more than
- $\leq$ is at most; is no more than; is less than or equal to
- $\geq$ is at least; is no less than; is greater than or equal to

CHECK Your Progress

4. Sophia's goal is to score at least 200 points this basketball season. If she has already scored 122 points, how many points does Sophia have to score on average for the last 6 games to reach her goal?

Online Personal Tutor at algebra2.com

You can use a graphing calculator to solve inequalities.

GRAPHING CALCULATOR LAB

Solving Inequalities

The inequality symbols in the TEST menu on the TI-83/84 Plus are called *relational operators.* They compare values and return 1 if the test is true or 0 if the test is false.

You can use these relational operators to solve an inequality in one variable.

THINK AND DISCUSS

1. Clear the Y= list. Enter $11x + 3 \geq 2x - 6$ as Y1. Put your calculator in DOT mode. Then, graph in the standard viewing window. Describe the graph.
2. Using the TRACE function, investigate the graph. What values of x are on the graph? What values of y are on the graph?
3. Based on your investigation, what inequality is graphed?
4. Solve $11x + 3 \geq 2x - 6$ algebraically. How does your solution compare to the inequality you wrote in Exercise 3?

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CHECK Your Understanding

Examples 1–3 (pp. 34–35)

Solve each inequality. Then graph the solution set on a number line.

1. $a + 2 < 3.5$

2. $11 - c \leq 8$

3. $5 \geq 3x$

4. $-0.6p < -9$

5. $2w + 19 < 5$

6. $4y + 7 > 31$

7. $n \leq \frac{n - 4}{5}$

8. $\frac{3z + 6}{11} < z$

Example 4 (p. 36)

9. SCHOOL The final grade for a class is calculated by taking 75% of the average test score and adding 25% of the score on the final exam. If all scores are out of 100 and a student has a 76 test average, what score does the student need on the final exam to have a final grade of at least 80?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10, 11	1
12–15	2
16–26	3
27–32	4

Solve each inequality. Then graph the solution set on a number line.

10. $n + 4 \geq -7$

11. $b - 3 \leq 15$

12. $5x < 35$

13. $\frac{d}{2} > -4$

14. $\frac{g}{-3} \geq -9$

15. $-8p \geq 24$

16. $13 - 4k \leq 27$

17. $14 > 7y - 21$

18. $-27 < 8m + 5$

19. $6b + 11 \geq 15$

20. $2(4t + 9) \leq 18$

21. $90 \geq 5(2r + 6)$

22. $\frac{3t + 6}{2} < 3t + 6$

23. $\frac{k + 7}{3} - 1 < 0$

24. $\frac{2n - 6}{5} + 1 > 0$

25. PART-TIME JOB David earns \$6.40 an hour working at Box Office Videos. Each week 25% of his total pay is deducted for taxes. If David wants his take-home pay to be at least \$120 a week, solve $6.4x - 0.25(6.4x) \geq 120$ to determine how many hours he must work.

26. STATE FAIR Admission to a state fair is \$12 per person. Bus parking costs \$20. Solve $12n + 20 \leq 600$ to determine how many people can go to the fair if a group has \$600 and uses only one bus.

Define a variable and write an inequality for each problem. Then solve.

27. The product of 12 and a number is greater than 36.

28. Three less than twice a number is at most 5.

29. The sum of a number and 8 is more than 2.

30. The product of -4 and a number is at least 35.

31. The difference of one half of a number and 7 is greater than or equal to 5.

32. One more than the product of -3 and a number is less than 16.

Solve each inequality. Then graph the solution set on a number line.

33. $14 - 8n \leq 0$

34. $-4(5w - 8) < 33$

35. $0.02x + 5.58 < 0$

36. $1.5 - 0.25c < 6$

37. $6d + 3 \geq 5d - 2$

38. $9z + 2 > 4z + 15$

39. $2(g + 4) < 3g - 2(g - 5)$

40. $3(a + 4) - 2(3a + 4) \leq 4a - 1$

41. $y < \frac{-y + 2}{9}$

42. $\frac{1 - 4p}{5} < 0.2$

43. $\frac{4x + 2}{6} < \frac{2x + 1}{3}$

44. $12\left(\frac{1}{4} - \frac{n}{3}\right) \leq -6n$

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CAR SALES **For Exercises 45 and 46, use the following information.**
Mrs. Lucas earns a salary of $34,000 per year plus 1.5% commission on her sales. If the average price of a car she sells is $30,500, about how many cars must she sell to make an annual income of at least $50,000?

45. Write an inequality to describe this situation.

46. Solve the inequality and interpret the solution.

Define a variable and write an inequality for each problem. Then solve.

47. Twice the sum of a number and 5 is no more than 3 times that same number increased by 11.

48. 9 less than a number is at most that same number divided by 2.

49. CHILD CARE By Ohio law, when children are napping, the number of children per childcare staff member may be as many as twice the maximum listed at the right. Write and solve an inequality to determine how many staff members are required to be present in a room where 17 children are napping and the youngest child is 18 months old.

Maximum Number of Children Per Child Care Staff Member
At least one child care staff member caring for:
Every 5 infants less than 12 months old (or 2 for every 12)
Every 6 infants who are at least 12 months old, but less than 18 months old
Every 7 toddlers who are at least 18 months old, but less than 30 months old
Every 8 toddlers who are at least 30 months old, but less than 3 years old

Source: Ohio Department of Job and Family Services

EXTRA PRACTICE
See pages 892, 926.
MathOnline
Self-Check Quiz at algebra2.com

TEST GRADES **For Exercises 50 and 51, use the following information.**
Flavio's scores on the first four of five 100-point history tests were 85, 91, 89, and 94.

50. If a grade of at least 90 is an A, write an inequality to find the score Flavio must receive on the fifth test to have an A test average.

51. Solve the inequality and interpret the solution.

Graphing Calculator

Use a graphing calculator to solve each inequality.

52. $-5x - 8 < 7$ **53.** $-4(6x - 3) \leq 60$ **54.** $3(x + 3) \geq 2(x + 4)$

H.O.T. Problems

55. OPEN ENDED Write an inequality for which the solution set is the empty set.

56. REASONING Explain why it is not necessary to state a division property for inequalities.

57. CHALLENGE Which of the following properties hold for inequalities? Explain your reasoning or give a counterexample.

a. Reflexive **b.** Symmetric **c.** Transitive

58. CHALLENGE Write a multi-step inequality requiring multiplication or division, the solution set is graphed below.

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59. ***Writing in Math*** Use the information about phone rate plans on page 33 to explain how inequalities can be used to compare phone plans. Include an explanation of how Kuni might determine when Plan 2 might be cheaper than Plan 1 if she typically uses more than 400 but less than 650 minutes.

STANDARDIZED TEST PRACTICE

60. ACT/SAT If $a < b$ and $c < 0$, which of the following are true?

I. $ac > bc$

II. $a + c < b + c$

III. $a - c > b - c$

A I only

B II only

C III only

D I and II only

61. REVIEW What is the complete solution to the equation $|8 - 4x| = 40$?

F $x = 8; x = 12$

G $x = 8; x = -12$

H $x = -8; x = -12$

J $x = -8; x = 12$

Spiral Review

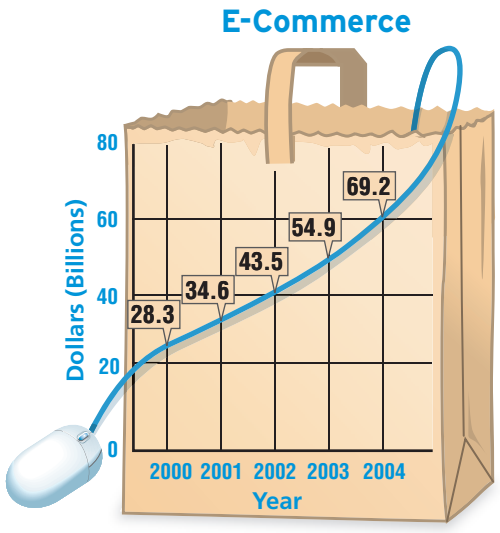
Solve each equation. Check your solutions. (Lesson 1-4)

62. $|x - 3| = 17$

63. $8|4x - 3| = 64$

64. $|x + 1| = x$

65. E-COMMERCE On average, by how much did the amount spent on online purchases increase each year from 2000 to 2004? Define a variable, write an equation, and solve the problem. (Lesson 1-3)



Name the sets of numbers to which each number belongs. (Lesson 1-2)

66. 31

67. $-4.\overline{2}$

68. $\sqrt{7}$

69. BABY-SITTING Jenny baby-sat for $5\frac{1}{2}$ hours on Friday night and 8 hours on Saturday. She charges $4.25 per hour. Use the Distributive Property to write two equivalent expressions that represent how much money Jenny earned. (Lesson 1-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Check your solutions.** (Lesson 1-4)

70. $|x| = 7$

71. $|x + 5| = 18$

72. $|5y - 8| = 12$

73. $14 = |2x - 36|$

74. $10 = 2|w + 6|$

75. $|x + 4| + 3 = 17$

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READING MATH

Interval Notation

The solution set of an inequality can be described by using **interval notation.** The infinity symbols below are used to indicate that a set is unbounded in the positive or negative direction, respectively.

Read as *positive infinity.* → $+\infty$ $-\infty$ ← Read as *negative infinity.*

To indicate that an endpoint is *not* included in the set, a parenthesis, (or), is used.

$x < 2$

interval notation
$(-\infty, 2)$

A bracket is used to indicate that the endpoint, −2, *is* included in the solution set below. Parentheses are always used with the symbols $+\infty$ and $-\infty$, because they do not include endpoints.

$x \geq -2$

interval notation
$[-2, +\infty)$

In interval notation, the symbol for the union of the two sets is ∪. The solution set of the compound inequality $y \leq -7$ or $y > -1$ is written as $(-\infty, -7] \cup (-1, +\infty)$.

Reading to Learn

Describe each set using interval notation.

1. $\{a|a \leq -3\}$

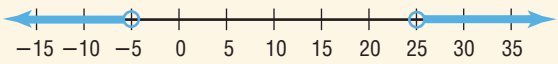
2. $\{n|n > -8\}$

3. $\{y|y < 2 \text{ or } y \geq 14\}$

4. $\{b|b \leq -9 \text{ or } b > 1\}$

5.

6.



Graph each solution set on a number line.

7. $(-1, +\infty)$

8. $(-\infty, 4]$

9. $(-\infty, 5] \cup (7, +\infty)$

10. Write in words the meaning of $(-\infty, 3) \cup [10, +\infty)$. Then write the compound inequality that has this solution set.

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1-6 Solving Compound and Absolute Value Inequalities

Main Ideas

- Solve compound inequalities.
- Solve absolute value inequalities.

New Vocabulary

compound inequality
intersection
union

GET READY for the Lesson

One test used to determine whether a patient is diabetic is a glucose tolerance test. Patients start the test in a *fasting state*, meaning they have had no food or drink except water for at least 10, but no more than 16, hours. The acceptable number of hours h for fasting can be described by the following compound inequality.

$$h \geq 10 \text{ and } h \leq 16$$

Compound Inequalities A **compound inequality** consists of two inequalities joined by the word *and* or the word *or*. To solve a compound inequality, you must solve each part of the inequality. The graph of a compound inequality containing *and* is the **intersection** of the solution sets of the two inequalities. Compound inequalities involving the word *and* are called *conjunctions*. Compound inequalities involving the word *or* are called *disjunctions*.

Vocabulary Link
Intersection
Everyday Use the place where two streets meet
Math Use the set of elements common to two sets

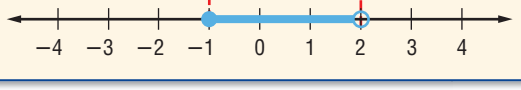
KEY CONCEPT "And" Compound Inequalities

Words A compound inequality containing the word *and* is true if and only if *both* inequalities are true.

Example $x \geq -1$

$x < 2$

$x \geq -1$ and $x < 2$



Another way of writing $x \geq -1$ and $x < 2$ is $-1 \leq x < 2$.
Both forms are read *x is greater than or equal to −1 and less than 2.*

EXAMPLE 1 Solve an "and" Compound Inequality

Solve $13 < 2x + 7 \leq 17$. Graph the solution set on a number line.

Method 1

Write the compound inequality using the word *and*. Then solve each inequality.

$13 < 2x + 7$ and $2x + 7 \leq 17$
$6 < 2x$ $2x \leq 10$
$3 < x$ $x \leq 5$
$3 < x \leq 5$

Method 2

Solve both parts at the same time by subtracting 7 from each part. Then divide each part by 2.

$13 < 2x + 7 \leq 17$
$6 < 2x \leq 10$
$3 < x \leq 5$

(continued on the next page)

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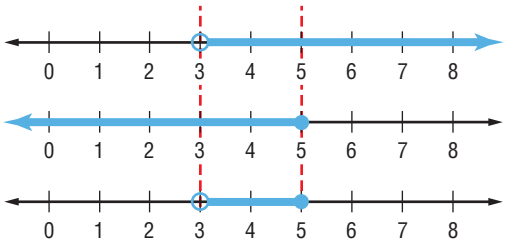
Graph the solution set for each inequality and find their intersection.



The solution set is $\{x|3 < x \leq 5\}$.

Concepts in Motion
Animation algebra2.com

CHECK Your Progress

1. Solve $8 \leq 3x - 4 < 11$. Graph the solution set on a number line.

The graph of a compound inequality containing *or* is the **union** of the solution sets of the two inequalities.

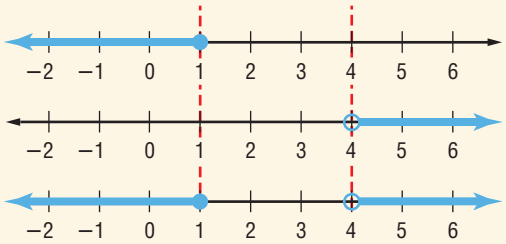
KEY CONCEPT "Or" Compound Inequalities

Words A compound inequality containing the word *or* is true if one or more of the inequalities is true.

Examples $x \leq 1$

$x > 4$

$x \leq 1$ or $x > 4$



Vocabulary Link
Union
Everyday Use something formed by combining parts or members
Math Use the set of elements belonging to one or more of a group of sets

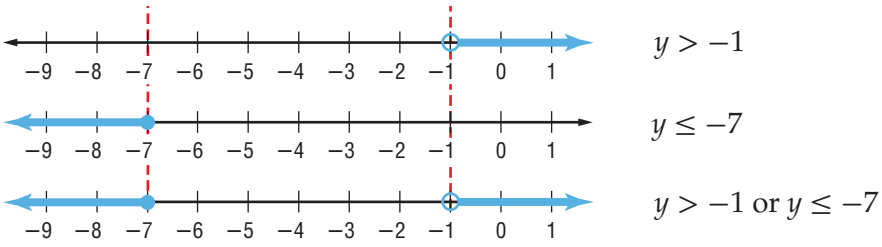
EXAMPLE 2 Solve an "or" Compound Inequality

Solve $y - 2 > -3$ or $y + 4 \leq -3$. Graph the solution set on a number line.

Solve each inequality separately.

$y - 2 > -3$ or $y + 4 \leq -3$

$y > -1$ $\qquad$ $y \leq -7$



The solution set is $\{y|y > -1 \text{ or } y \leq -7\}$.

CHECK Your Progress

2. Solve $y + 5 \leq 7$ or $y - 6 > 2$. Graph the solution set on a number line.

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Reading Math

When solving problems involving inequalities,

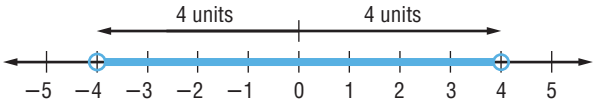
- *within* is meant to be inclusive. Use $\leq$ or $\geq$.
- *between* is meant to be exclusive. Use $<$ or $>$.

Absolute Value Inequalities In Lesson 1-4, you learned that the absolute value of a number is its distance from 0 on the number line. You can use this definition to solve inequalities involving absolute value.

EXAMPLE Solve an Absolute Value Inequality (<)

3 **Solve $|a| < 4$. Graph the solution set on a number line.**

$|a| < 4$ means that the distance between a and 0 on a number line is less than 4 units. To make $|a| < 4$ true, substitute numbers for a that are fewer than 4 units from 0.



Notice that the graph of $|a| < 4$ is the same as the graph of $a > -4$ and $a < 4$.

All of the numbers between -4 and 4 are less than 4 units from 0. The solution set is $\{a \mid -4 < a < 4\}$.

CHECK Your Progress

3. Solve $|x| \leq 3$. Graph the solution set on a number line.

Study Tip

Absolute Value Inequalities

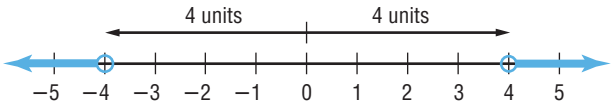
Because the absolute value of a number is never negative,

- the solution of an inequality like $|a| < -4$ is the empty set.
- the solution of an inequality like $|a| > -4$ is the set of all real numbers.

EXAMPLE Solve an Absolute Value Inequality (>)

4 **Solve $|a| > 4$. Graph the solution set on a number line.**

$|a| > 4$ means that the distance between a and 0 on a number line is greater than 4 units.



Notice that the graph of $|a| > 4$ is the same as the graph of $\{a > 4$ *or* $a < -4\}$.

The solution set is $\{a \mid a > 4 \text{ or } a < -4\}$.

CHECK Your Progress

4. Solve $|x| \geq 3$. Graph the solution set on a number line.

An absolute value inequality can be solved by rewriting it as a compound inequality.

KEY CONCEPT *Absolute Value Inequalities*

Symbols For all real numbers a and b, $b > 0$, the following statements are true.
1. If $|a| < b$, then $-b < a < b$.
2. If $|a| > b$, then $a > b$ or $a < -b$

Examples If $|2x + 1| < 5$, then $-5 < 2x + 1 < 5$
If $|2x + 1| > 5$, then $2x + 1 > 5$ or $2x + 1 < -5$.

These statements are also true for $\leq$ and $\geq$, respectively.

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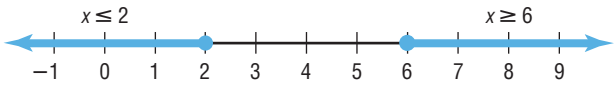
EXAMPLE Solve a Multi-Step Absolute Value Inequality

5 **Solve $|3x - 12| \geq 6$. Graph the solution set on a number line.**

$|3x - 12| \geq 6$ is equivalent to $3x - 12 \geq 6$ or $3x - 12 \leq -6$.
Solve the inequality.

$3x - 12 \geq 6$	or	$3x - 12 \leq -6$	Rewrite the inequality.
$3x \geq 18$		$3x \leq 6$	Add 12.
$x \geq 6$		$x \leq 2$	Divide by 3.

The solution set is $\{x \mid x \geq 6 \text{ or } x \leq 2\}$.



CHECK Your Progress

5. Solve $|3x + 4| < 10$. Graph the solution set on a number line.

Real-World Link

When executives in a recent survey were asked to name one quality that impressed them the most about a candidate during a job interview, 32 percent said honesty and integrity.

Source: careerexplorer.net

Real-World EXAMPLE Write an Absolute Value Inequality

6 **JOB HUNTING** **To prepare for a job interview, Megan researches the position's requirements and pay. She discovers that the average starting salary for the position is $38,500, but her actual starting salary could differ from the average by as much as $2450.**

a. Write an absolute value inequality to describe this situation.
Let x equal Megan's starting salary.

Her starting salary could differ from the average	by as much as	$2450.
$\lvert 38{,}500 - x \rvert$	$\leq$	2450

b. Solve the inequality to find the range of Megan's starting salary.
Rewrite the absolute value inequality as a compound inequality. Then solve for x.

$$\begin{aligned} -2450 \leq\ & 38{,}500 - x \leq 2450 \\ -2450 - 38{,}500 \leq\ & 38{,}500 - x - 38{,}500 \leq 2450 - 38{,}500 \\ -40{,}950 \leq\ & -x \leq -36{,}050 \\ 40{,}950 \geq\ & x \geq 36{,}050 \end{aligned}$$

The solution set is $\{x \mid 36{,}050 \leq x \leq 40{,}950\}$. Thus, Megan's starting salary will fall within $36,050 and $40,950.

CHECK Your Progress

6. The ideal pH value for water in a swimming pool is 7.5. However, the pH may differ from the ideal by as much as 0.3 before the water will cause discomfort to swimmers or damage to the pool. Write an absolute value inequality to describe this situation. Then solve the inequality to find the range of acceptable pH values for the water.

Online Personal Tutor at algebra2.com

Andrew Ward/Life File/Getty Images

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CHECK Your Understanding

Examples 1–5 (pp. 41–44)

Solve each inequality. Graph the solution set on a number line.

1. $3 < d + 5 < 8$

2. $-4 \leq 3x - 1 < 14$

3. $y - 3 > 1$ or $y + 2 < 1$

4. $p + 6 < 8$ or $p - 3 > 1$

5. $|a| \geq 5$

6. $|w| \geq -2$

7. $|h| < 3$

8. $|b| < -2$

9. $|4k - 8| < 20$

10. $|g + 4| \leq 9$

Example 6 (p. 44)

11. FLOORING Deion is considering several types of flooring for his kitchen. He estimates that he will need between 55 and 60 12-inch by 12-inch tiles to retile the floor. The table below shows the price per tile for each type of tile Deion is considering.

Tile Type	Price per Tile
Vinyl	\$0.99
Slate	\$2.34
Porcelain	\$3.88
Marble	\$5.98

Write a compound inequality to determine how much he could be spending.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12, 13	1
14, 15	2
16, 17	3
18, 19	4
20, 21	5
22, 23	6

Solve each inequality. Graph the solution set on a number line.

12. $9 < 3t + 6 < 15$

13. $-11 < -4x + 5 < 13$

14. $3p + 1 \leq 7$ or $2p - 9 \geq 7$

15. $2c - 1 < -5$ or $3c + 2 \geq 5$

16. $|g| \leq 9$

17. $|3k| < 0$

18. $|2m| \geq 8$

19. $|b - 4| > 6$

20. $|3w + 2| \leq 5$

21. $|6r - 3| < 21$

SPEED LIMITS **For Exercises 22 and 23, use the following information.**
On some interstate highways, the maximum speed a car may drive is 65 miles per hour. A tractor-trailer may not drive more than 55 miles per hour. The minimum speed for all vehicles is 45 miles per hour.

22. Write an inequality to represent the allowable speed for a car on an interstate highway.

23. Write an inequality to represent the speed at which a tractor-trailer may travel on an interstate highway.

Solve each inequality. Graph the solution set on a number line.

24. $-4 < 4f + 24 < 4$

25. $a + 2 > -2$ or $a - 8 < 1$

26. $|-5y| < 35$

27. $|7x| + 4 < 0$

28. $|n| \geq n$

29. $|n| \leq n$

30. $\frac{|2n - 7|}{3} \leq 0$

31. $\frac{|n - 3|}{2} < n$

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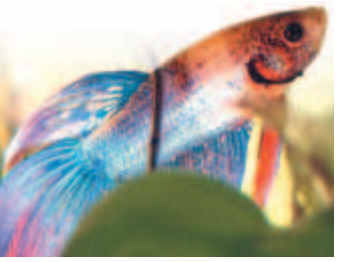


Real-World Link

Adult Male Size: 3 inches

Water pH: 6.8–7.4

Temperature: 75–86°F

Diet: omnivore, prefers live foods

Tank Level: top dweller

Difficulty of Care: easy to intermediate

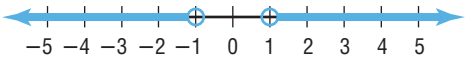
Life Span: 2–3 years

Source: www.about.com

32. FISH A Siamese Fighting Fish, better known as a Betta fish, is one of the most recognized and colorful fish kept as a pet. Using the information at the left, write a compound inequality to describe the acceptable range of water pH levels for a male Betta.

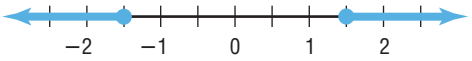
Write an absolute value inequality for each graph.

33.



34.

35.



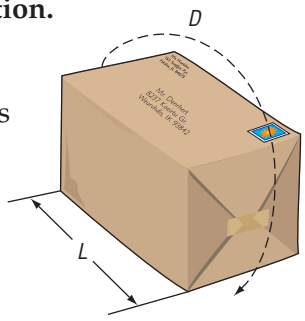
36.

37.

38.

39. HEALTH *Hypothermia* and *hyperthermia* are similar words but have opposite meanings. Hypothermia is defined as a lowered body temperature. Hyperthermia means an extremely high body temperature. Both conditions are potentially dangerous and occur when a person's body temperature fluctuates by more than 8° from the normal body temperature of 98.6°F. Write and solve an absolute value inequality to describe body temperatures that are considered potentially dangerous.

MAIL For Exercises 40 and 41, use the following information.
The U.S. Postal Service defines an oversized package as one for which the length L of its longest side plus the distance D around its thickest part is more than 108 inches and less than or equal to 130 inches.



40. Write a compound inequality to describe this situation.

41. If the distance around the thickest part of a package you want to mail is 24 inches, describe the range of lengths that would classify your package as oversized.

AUTO RACING For Exercises 42 and 43, use the following information.
The shape of a car used in NASCAR races is determined by NASCAR rules. The rules stipulate that a car must conform to a set of 32 templates, each shaped to fit a different contour of the car. The biggest template fits over the center of the car from front to back. When a template is placed on a car, the gap between it and the car cannot exceed the specified tolerance. Each template is marked on its edge with a colored line that indicates the tolerance for the template.

42. Suppose a certain template is 24.42 inches long. Use the information in the table at the right to write an absolute value inequality for templates with each line color.

43. Find the acceptable lengths for that part of a car if the template has each line color.

Line Color	Tolerance (in.)
Red	0.07
Blue	0.25
Green	0.5

Rudi Von Briel/PhotoEdit

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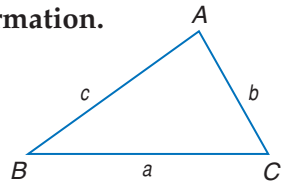
GEOMETRY **For Exercises 44 and 45, use the following information.**
The *Triangle Inequality Theorem* states that the sum of the measures of any two sides of a triangle is greater than the measure of the third side.



44. Write three inequalities to express the relationships among the sides of $\triangle ABC$.

45. Write a compound inequality to describe the range of possible measures for side c in terms of a and b. Assume that $a > b > c$. (*Hint:* Solve each inequality you wrote in Exercise 44 for c.)

Graphing Calculator

LOGIC MENU **For Exercises 46–49, use the following information.**
You can use the operators in the LOGIC menu on the TI-83/84 Plus to graph compound and absolute value inequalities. To display the LOGIC menu, press 2nd [TEST] ▶.

46. Clear the Y= list. Enter $(5x + 2 > 12)$ and $(3x - 8 < 1)$ as Y1. With your calculator in DOT mode and using the standard viewing window, press GRAPH. Make a sketch of the graph displayed.

47. Using the TRACE function, investigate the graph. Based on your investigation, what inequality is graphed?

48. Write the expression you would enter for Y1 to find the solution set of the compound inequality $5x + 2 \geq 3$ or $5x + 2 \leq -3$. Then use the graphing calculator to find the solution set.

49. A graphing calculator can also be used to solve absolute value inequalities. Write the expression you would enter for Y1 to find the solution set of the inequality $|2x - 6| > 10$. Then use the graphing calculator to find the solution set. (*Hint:* The absolute value operator is item 1 on the MATH NUM menu.)

EXTRA PRACTICE
See pages 892, 926.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

50. **OPEN ENDED** Write a compound inequality for which the graph is the empty set.

51. **FIND THE ERROR** Sabrina and Isaac are solving $|3x + 7| > 2$. Who is correct? Explain your reasoning.

Sabrina

$|3y + 7| > 2$

$3y + 7 > 2$ or $3y + 7 < -2$

$3y > -5$ $\quad$ $3y < -9$

$y > -\frac{5}{3}$ $\quad$ $y < -3$

Isaac

$|3x + 7| > 2$

$-2 < 3x + 7 < 2$

$-9 < 3x < -5$

$-3 < x < -\frac{5}{3}$

52. **CHALLENGE** **Graph each set on a number line.**

a. $-2 < x < 4$

b. $x < -1$ or $x > 3$

c. $(-2 < x < 4)$ and $(x < -1$ or $x > 3)$ (*Hint:* This is the intersection of the graphs in part **a** and part **b**.)

d. Solve $3 < |x + 2| \leq 8$. Explain your reasoning and graph the solution set.

53. ***Writing in Math*** Use the information about fasting on page 41 to explain how compound inequalities are used in medicine. Include an explanation of an acceptable number of hours for this fasting state and a graph to support your answer.

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STANDARDIZED TEST PRACTICE

54. ACT/SAT If $5 < a < 7 < b < 14$, then which of the following best describes $\frac{a}{b}$?

A $\frac{5}{7} < \frac{a}{b} < \frac{1}{2}$

B $\frac{5}{14} < \frac{a}{b} < \frac{1}{2}$

C $\frac{5}{7} < \frac{a}{b} < 1$

D $\frac{5}{14} < \frac{a}{b} < 1$

55. REVIEW What is the solution set of the inequality $-20 < 4x - 8 < 12$?

F $-7 < x < 1$

G $-3 < x < 5$

H $-7 < x < 5$

J $-3 < x < 1$

Spiral Review

Solve each inequality. Then graph the solution set on a number line. (Lesson 1-5)

56. $2d + 15 \geq 3$ **57.** $7x + 11 > 9x + 3$ **58.** $3n + 4\,(n + 3) < 5(n + 2)$

59. CONTESTS To get a chance to win a car, you must guess the number of keys in a jar to within 5 of the actual number. Those who are within this range are given a key to try in the ignition of the car. Suppose there are 587 keys in the jar. Write and solve an equation to determine the highest and lowest guesses that will give contestants a chance to win the car. (Lesson 1-4)

Solve each equation. Check your solutions. (Lesson 1-4)

60. $5\,|x - 3| = 65$ **61.** $|2x + 7| = 15$ **62.** $|8c + 7| = -4$

Name the property illustrated by each statement. (Lesson 1-3)

63. If $3x = 10$, then $3x + 7 = 10 + 7$.

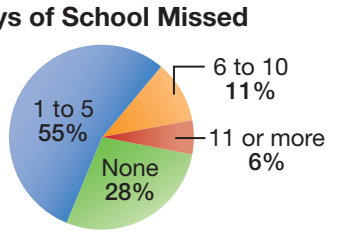
64. If $-5 = 4y - 8$, then $4y - 8 = -5$.

65. If $-2x - 5 = 9$ and $9 = 6x + 1$, then $-2x - 5 = 6x + 1$.

SCHOOL For Exercises 66 and 67, use the graph at the right.

66. Illustrate the Distributive Property by writing two expressions to represent the number of students at a high school who missed 5 or fewer days of school if the school enrollment is 743.

67. Evaluate the expressions from Exercise 66.



Source: Centers for Disease Control and Prevention

Simplify each expression. (Lesson 1-2)

68. $6a - 2b - 3a + 9b$ **69.** $-2(m - 4n) - 3(5n + 6)$

Find the value of each expression. (Lesson 1-1)

70. $6(5 - 8) \div 9 + 4$ **71.** $(3 + 7)^2 - 16 \div 2$ **72.** $\frac{7(1 - 4)}{8 - 5}$

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CHAPTER 1

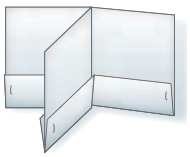
Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Expressions and Formulas (Lesson 1-1)

- Use the order of operations and the properties of equality to solve equations.

Properties of Real Numbers (Lesson 1-2)

- Real numbers can be classified as rational (Q) or irrational (I). Rational numbers can be classified as natural numbers (N), whole numbers (W), integers (Z), and/or quotients of these.

Solving Equations (Lesson 1-3 and 1-4)

- Verbal expressions can be translated into algebraic expressions.
- The absolute value of a number is the number of units it is from 0 on a number line.
- For any real numbers a and b, where $b \geq 0$, if $|a| = b$, then $a = b$ or $-a = b$.

Solving Inequalities (Lessons 1-5 and 1-6)

- Adding or subtracting the same number from each side of an inequality does not change the truth of the inequality.
- When you multiply or divide each side of an inequality by a negative number, the direction of the inequality symbol must be *reversed*.
- The graph of an *and* compound inequality is the intersection of the solution sets of the two inequalities. The graph of an *or* compound inequality is the union of the solution sets of the two inequalities.
- An *and* compound inequality can be expressed in two different ways. For example, $-2 \leq x \leq 3$ is equivalent to $x \geq -2$ and $x \leq 3$.
- For all real numbers a and b, where $b > 0$, the following statements are true.
  1. If $|a| < b$ then $-b < a < b$.
  2. If $|a| > b$ then $a > b$ or $a < -b$.

Key Vocabulary

absolute value (p. 27)
algebraic expression (p. 6)
coefficient (p. 7)
counterexample (p. 17)
empty set (p. 28)
equation (p. 18)
formula (p. 8)
intersection (p. 41)
irrational numbers (p. 11)
like terms (p. 7)
monomial (p. 6)
polynomial (p. 7)
rational numbers (p. 11)
real numbers (p. 11)
solution (p. 19)
trinomial (p. 7)
union (p. 42)

Vocabulary Check

Choose the term from the list above that best completes each statement.

1. The _____ contains no elements.
2. A polynomial with exactly three terms is called a _____.
3. The set of _____ includes terminating and repeating decimals but does not include π.
4. _____ can be combined by adding or subtracting their coefficients.
5. The _____ of a number is never negative.
6. The set of _____ contains the rational and the irrational numbers.
7. The _____ of the term $-6xy$ is -6.
8. A(n) _____ to an equation is a value that makes the equation true.
9. A(n) _____ is a statement that two expressions have the same value.
10. $\sqrt{2}$ belongs to the set of _____ but $\frac{1}{2}$ does not.

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Lesson-by-Lesson Review

1–1 Expressions and Formulas (pp. 6–10)

Evaluate each expression.

11. $10 + 16 \div 4 + 8$

12. $[21 - (9 - 2)] \div 2$

13. $\frac{1}{2}(5^2 + 3)$

14. $\frac{14(8 - 15)}{2}$

Evaluate each expression if $a = 12$, $b = 0.5$, $c = -3$, and $d = \frac{1}{3}$.

15. $6b - 5c$

16. $c^3 + ad$

17. $\frac{9c + ab}{c}$

18. $a[b^2(b + a)]$

19. DISTANCE The formula to evaluate distance is $d = r \times t$, where d is distance, r is rate, and t is time. How far can Tosha drive in 4 hours if she is driving at 65 miles per hour?

Example 1 Evaluate $(10 - 2) \div 2^2$.

$(10 - 2) \div 2^2 = 8 \div 2^2$ First subtract 2 from 10.

$= 8 \div 4$ Then square 2.

$= 2$ Finally, divide 8 by 4.

Example 2 Evaluate $\frac{y^3}{3ab + 2}$ if $y = 4$, $a = -2$, and $b = -5$.

$\frac{y^3}{3ab + 2} = \frac{4^3}{3(-2)(-5) + 2}$ $y = 4$, $a = -2$, and $b = -5$

$= \frac{64}{3(10) + 2}$ Evaluate the numerator and denominator separately.

$= \frac{64}{32}$ or 2 Simplify.

1–2 Properties of Real Numbers (pp. 11–17)

Name the sets of numbers to which each value belongs.

20. $-\sqrt{9}$

21. $1.\overline{6}$

22. $\sqrt{18}$

Simplify each expression.

23. $2m + 7n - 6m - 5n$

24. $-5(a - 4b) + 4b$

25. $2(5x + 4y) - 3(x + 8y)$

CLOTHING **For Exercises 26 and 27, use the following information.**
A department store sells shirts for $12.50 each. Dalila buys 2, Latisha buys 3, and Pilar buys 1.

26. Illustrate the Distributive Property by writing two expressions to represent the cost of these shirts.

27. Use the Distributive Property to find how much money the store received from selling these shirts.

Example 3 Name the sets of numbers to which $\sqrt{25}$ belongs.

$\sqrt{25} = 5$ naturals (N), wholes (W), integers (Z), rationals (Q), and reals (R)

Example 4 Simplify $3(x + 2) + 4x - 3y$.

$3(x + 2) + 4x - 3y$

$= 3(x) + 3(2) + 4x - 3y$ Distributive Property

$= 3x + 6 + 4x - 3y$ Multiply.

$= 7x - 3y + 6$ Simplify.

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Mixed Problem Solving
For mixed problem-solving practice, see page 926.

1–3 Solving Equations (pp. 18–26)

Solve each equation. Check your solution.

28. $x - 6 = -20$

29. $-\frac{2}{3}a = 14$

30. $7 + 5n = -58$

31. $3w + 14 = 7w + 2$

32. $\frac{n}{4} + \frac{n}{3} = \frac{1}{2}$

33. $5y + 4 = 2(y - 4)$

34. MONEY If Tabitha has 98 cents and you know she has 2 quarters, 1 dime, and 3 pennies, how many nickels does she have?

Solve each equation or formula for the specified variable.

35. $Ax + By = C$ for x

36. $\frac{a - 4b^2}{2c} = d$ for a

37. $A = p + prt$ for p

38. $d = b^2 - 4ac$ for c

39. GEOMETRY Alex wants to find the radius of the circular base of a cone. He knows the height of the cone is 8 inches and the volume of the cone is 18.84 cubic inches. Use the formula for volume of a cone, $V = \frac{1}{3}\pi r^2 h$, to find the radius.

Example 5 Solve $4(a + 5) - 2(a + 6) = 3$.

$4(a + 5) - 2(a + 6) = 3$	Original equation
$4a + 20 - 2a - 12 = 3$	Distributive Property
$4a - 2a + 20 - 12 = 3$	Commutative Property
$2a + 8 = 3$	Distributive and Substitution Properties
$2a = -5$	Subtraction Property
$a = -2.5$	Division Property

Example 6 Solve $A = \frac{h(a + b)}{2}$ for b.

$2A = h(a + b)$	Multiply each side by 2.
$\frac{2A}{h} = a + b$	Divide each side by h.
$\frac{2A}{h} - a = b$	Subtract a from each side.

1–4 Solving Absolute Value Equations (pp. 27–31)

Solve each equation. Check your solution.

40. $|x + 11| = 42$

41. $3|x + 6| = 36$

42. $|4x - 5| = -25$

43. $|x + 7| = 3x - 5$

44. $|y - 5| - 2 = 10$

45. $4|3x + 4| = 4x + 8$

46. BIKING Paloma's training goal is to ride four miles on her bicycle in 15 minutes. If her actual time is always within plus or minus 3 minutes of her preferred time, how long are her shortest and longest rides?

Example 7 Solve $|2x + 9| = 11$.

Case 1: $a = b$	**Case 2:** $a = -b$
$2x + 9 = 11$	$2x + 9 = -11$
$2x = 2$	$2x = -20$
$x = 1$	$x = -10$

The solutions are 1 and -10.

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1–5 Solving Inequalities (pp. 33–39)

Solve each inequality. Describe the solution set using set builder notation. Then graph the solution set on a number line.

47. $-7w > 28$

48. $3x + 4 \geq 19$

49. $\frac{n}{12} + 5 \leq 7$

50. $3(6 - 5a) < 12a - 36$

51. $2 - 3z \geq 7(8 - 2z) + 12$

52. $8(2x - 1) > 11x - 17$

53. PIZZA A group has \$75 to order 6 large pizzas each with the same amount of toppings. Each pizza costs \$9 plus \$1.25 per topping. Write and solve an inequality to determine how many toppings the group can order on each pizza.

Example 8 **Solve $5 - 4a > 8$. Graph the solution set on a number line.**

$5 - 4a > 8$ Original inequality

$-4a > 3$ Subtract 5 from each side.

$a < -\frac{3}{4}$ Divide each side by -4, reversing the inequality symbol.

The solution set is $\left\{a \mid a < -\frac{3}{4}\right\}$.

The graph of the solution set is shown below.

1–6 Solving Compound and Absolute Value Inequalities (pp. 41–48)

Solve each inequality. Graph the solution set on a number line.

54. $4x + 3 < 11$ or $2x - 1 > 9$

55. $-1 < 3a + 2 < 14$

56. $-1 < 3(d - 2) \leq 9$

57. $5y - 4 > 16$ or $3y + 2 < 1$

58. $|x| + 1 > 12$

59. $|2y - 9| \leq 27$

60. $|5n - 8| > -4$

61. $|3b + 11| > 1$

62. FENCING Don is building a fence around a rectangular plot and wants the perimeter to be between 17 and 20 yards. The width of the plot is 5 yards. Write and solve a compound inequality to describe the range of possible measures for the length of the fence.

Example 9 **Solve each inequality. Graph the solution set on a number line.**

a. $-19 < 4d - 7 \leq 13$

$-19 < 4d - 7 \leq 13$ Original inequality

$-12 < 4d \leq 20$ Add 7 to each part.

$-3 < d \leq 5$ Divide each part by 4.

The solution set is $\{d \mid -3 < d \leq 5\}$.

b. $|2x + 4| \geq 12$

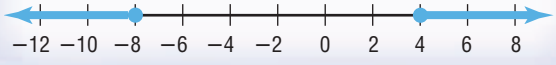
$|2x + 4| \geq 12$ is equivalent to $2x + 4 \geq 12$ or $2x + 4 \leq -12$.

$2x + 4 \geq 12$ or $2x + 4 \leq -12$

$2x \geq 8$ $\quad$ $2x \leq -16$ Subtract.

$x \geq 4$ $\quad$ $x \leq -8$ Divide.

The solution set is $\{x \mid x \geq 4 \text{ or } x \leq -8\}$.



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CHAPTER 1

Practice Test

Find the value of each expression.

1. $[(3 + 6)^2 \div 3] \times 4$
2. $\frac{20 + 4 \times 3}{11 - 3}$
3. $0.5(2.3 + 25) \div 1.5$

Evaluate each expression if $a = -9$, $b = \frac{2}{3}$, $c = 8$, and $d = -6$.

4. $\frac{db + 4c}{a}$
5. $\frac{a}{b^2} + c$

Name the sets of numbers to which each number belongs.

6. $\sqrt{17}$
7. 0.86
8. $\sqrt{64}$

Name the property illustrated by each equation or statement.

9. $(7 \cdot s) \cdot t = 7 \cdot (s \cdot t)$
10. If $(r + s)t = rt + st$, then $rt + st = (r + s)t$.
11. $\left(3 \cdot \frac{1}{3}\right) \cdot 7 = \left(3 \cdot \frac{1}{3}\right) \cdot 7$
12. $(6 - 2)a - 3b = 4a - 3b$
13. $(4 + x) + y = y + (4 + x)$
14. If $5(3) + 7 = 15 + 7$ and $15 + 7 = 22$, then $5(3) + 7 = 22$.

Solve each equation. Check your solution(s).

15. $5t - 3 = -2t + 10$
16. $2x - 7 - (x - 5) = 0$
17. $5m - (5 + 4m) = (3 + m) - 8$
18. $|8w + 2| + 2 = 0$
19. $12\left|\frac{1}{2}y + 3\right| = 6$
20. $2|2y - 6| + 4 = 8$

Solve each inequality. Then graph the solution set on a number line.

21. $4 > b + 1$
22. $3q + 7 \geq 13$
23. $|5 + k| \leq 8$
24. $-12 < 7d - 5 \leq 9$

Solve each inequality. Then graph the solution set on a number line.

25. $|3y - 1| > 5$
26. $5(3x - 5) + x < 2(4x - 1) + 1$

For Exercises 27 and 28, define a variable, write an equation or inequality, and solve the problem.

27. **CAR RENTAL** Ms. Denney is renting a car that gets 35 miles per gallon. The rental charge is \$19.50 a day plus 18¢ per mile. Her company will reimburse her for \$33 of this portion of her travel expenses. Suppose Ms. Denney rents the car for 1 day. Find the maximum number of miles that will be paid for by her company.

28. **SCHOOL** To receive a B in his English class, Nick must have an average score of at least 80 on five tests. What must he score on the last test to receive a B in the class?

Test	Score
1	87
2	89
3	76
4	77

29. **MULTIPLE CHOICE** If $\frac{a}{b} = 8$ and $ac - 5 = 11$, then $bc =$

A 93

B 2

C $\frac{5}{8}$

D cannot be determined

30. **MULTIPLE CHOICE** At a veterinarian's office, 2 cats and 4 dogs are seen in a random order. What is the probability that the 2 cats are seen in a row?

F $\frac{1}{3}$

G $\frac{2}{3}$

H $\frac{1}{2}$

J $\frac{3}{5}$

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CHAPTER 1

Standardized Test Practice

Chapter 1

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Lucas determined that the total cost C to rent a car for the weekend could be represented by the equation $C = 0.35m + 125$, where m is the number of miles that he drives. If the total cost to rent the car was \$363, how many miles did he drive?

A 125

B 238

C 520

D 680

TEST-TAKING TIP

Question 1 On multiple choice questions, try to compute the answer first. Then compare your answer to the given answer choices. If you don't find your answer among the choices, check your calculations.

2. Leo sells T-shirts at a local swim meet. It costs him \$250 to set up the stand and rent the machine. It costs him an additional \$5 to make each T-shirt. If he sells each T-shirt for \$15, how many T-shirts does he have to sell before he can make a profit?

F 10

G 15

H 25

J 50

3. GRIDDABLE Malea sells engraved necklaces over the Internet. She purchases 50 necklaces for \$400, and it costs her an additional \$3 for each personalized engraving. If she charges \$20 each, how many necklaces will she need to sell in order to make a profit of at least \$225?

4. If the surface area of a cube is increased by a factor of 9, what is the change in the length of the sides of the cube?

A The length is 2 times the original length.

B The length is 3 times the original length.

C The length is 6 times the original length.

D The length is 9 times the original length.

5. The profit p that Selena's Shirt store makes in a day can be represented by the inequality $10t + 200 < p < 15t + 250$, where t represents the number of shirts sold. If the store sold 45 shirts on Friday, which of the following is a reasonable amount that the store made?

F \$200.00

G \$625.00

H \$850.00

J \$950.00

6. Solve the equation $4x - 5 = 2x + 5 - 3x$ for x.

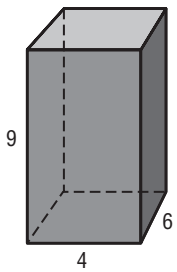
A -2

B -1

C 1

D 2

7. Which set of dimensions corresponds to a rectangular prism that is similar to the one shown below?



F 12 units by 18 units by 27 units

G 12 units by 18 units by 18 units

H 8 units by 12 units by 9 units

J 8 units by 10 units by 18 units

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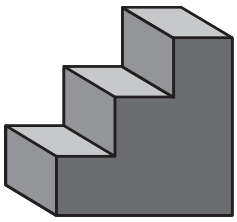
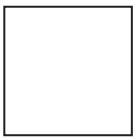
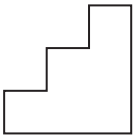
8. Which of the following best represents the side view of the solid shown below?



A

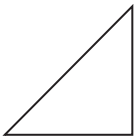


C



B

D



9. Given: Two angles are complementary. The measure of one angle is 10 less than the measure of the other angle.
Conclusion: The measures of the angles are 85 degrees and 95 degrees.
This conclusion:

F is contradicted by the first statement given.

G is verified by the first statement given.

H invalidates itself because there is no angle complementary to an 85 degree angle

J verifies itself because one angle is 10 degrees less than the other

10. A rectangle has a width of 8 inches and a perimeter of 30 inches. What is the perimeter, in inches, of a similar rectangle with a width of 12 inches?

A 40 **C** 48

B 45 **D** 360

11. Marvin and his younger brother like to bike together. Marvin rides his bike at a speed of 21 miles per hour and can ride his training loop 10 times in the time that it takes his younger brother to complete the training loop 8 times. Which is a reasonable estimate for Marvin's younger brother's speed?

F between 14 mph and 15 mph

G between 15 mph and 16 mph

H between 16 mph and 17 mph

J between 17 mph and 18 mph

Pre-AP

Record your answers on a sheet of paper Show your work.

12. Amanda's hours at her summer job for one week are listed in the table below. She earns $6 per hour.

Amanda's Work Hours	
Sunday	0
Monday	6
Tuesday	4
Wednesday	0
Thursday	2
Friday	6
Saturday	8

a. Write an expression for Amanda's total weekly earnings.

b. Evaluate the expression from Part **a** by using the Distributive Property.

c. Michael works with Amanda and also earns $6 per hour. If Michael's earnings were $192 this week, write and solve an equation to find how many more hours Michael worked than Amanda.

NEED EXTRA HELP?												
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12
Go to Lesson...	1-3	1-3	1-5	1-2	1-5	1-3	1-1	1-1	1-3	1-1	1-3	1-3

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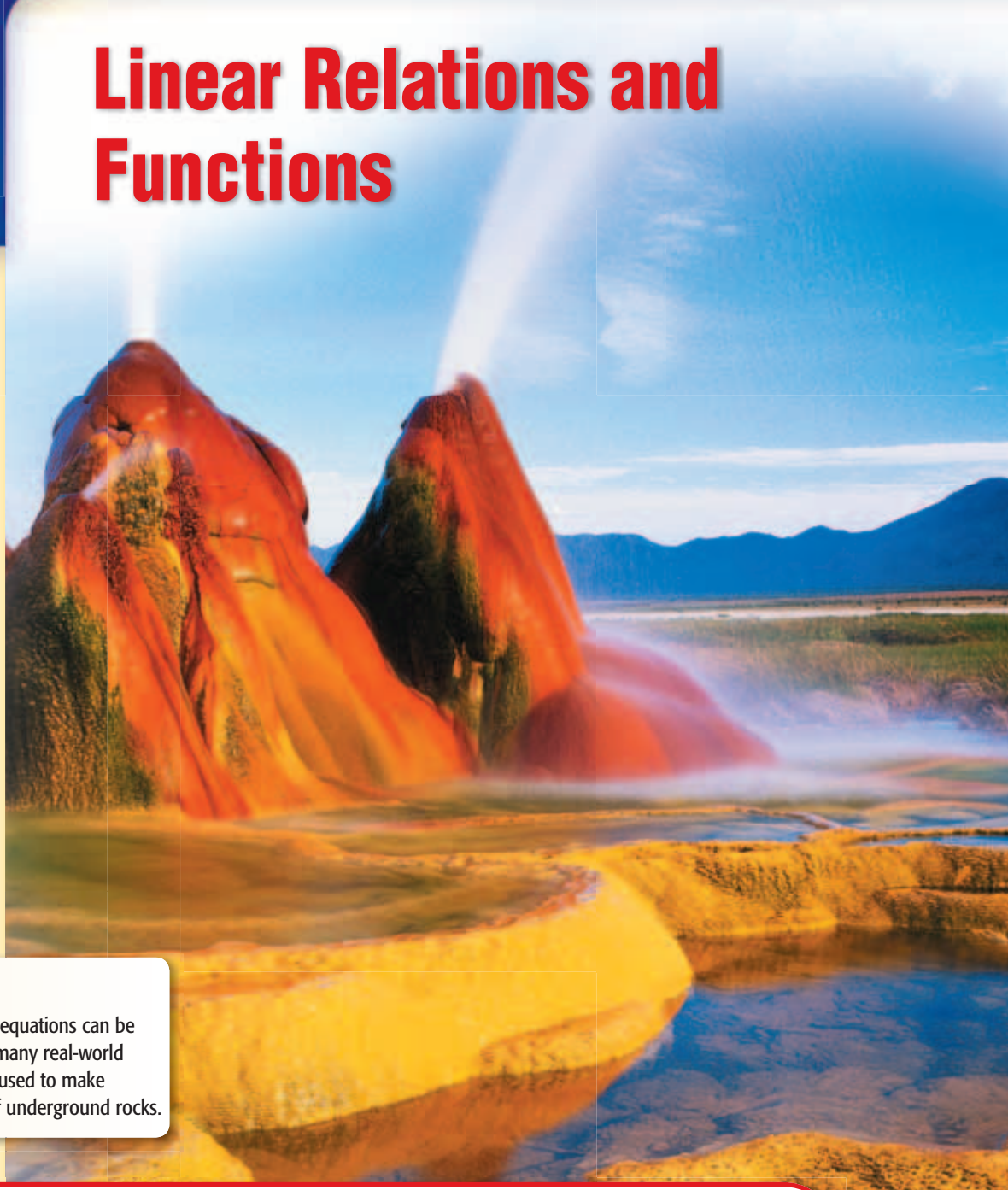
CHAPTER 2

Linear Relations and Functions



BIG Ideas

- Analyze relations and functions.
- Identify, graph, and write linear equations.
- Find the slope of a line.
- Draw scatter plots and find prediction equations.
- Graph special functions, linear inequalities, and absolute value inequalities.

Key Vocabulary

dependent variable (p. 61)
domain (p. 58)
function (p. 58)
independent variable (p. 61)
relation (p. 58)

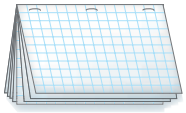
Real-World Link

Underground Temperature Linear equations can be used to model relationships between many real-world quantities. The equations can then be used to make predictions such as the temperature of underground rocks.

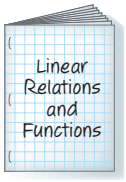
FOLDABLES Study Organizer

Linear Relations and Functions Make this Foldable to help you organize your notes. Begin with four sheets of grid paper.

1 **Fold** in half along the width and staple along the fold.



2 **Turn** the fold to the left and write the title of the chapter on the front. On each left-hand page of the booklet, write the title of a lesson from the chapter.



Jack Dykinga/Getty Images

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GET READY for Chapter 2

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

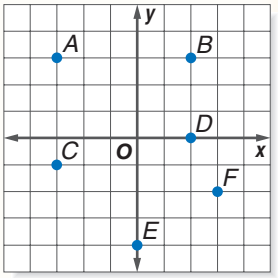
Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

QUICKCheck

Write the ordered pair for each point. (Prerequisite Skill)

1. A
2. B
3. C
4. D
5. E
6. F



ANIMALS A blue whale's heart beats 9 times a minute.

7. Make a table of ordered pairs in which the x-coordinate represents the number of minutes and the y-coordinate represents the number of heartbeats. (Prerequisite Skill)
8. Graph the ordered pairs. (Prerequisite Skill)

Evaluate each expression if $a = -1$, $b = 3$, $c = -2$, and $d = 0$. (Prerequisite Skill)

9. $c + d$
10. $4c - b$
11. $a^2 - 5a + 3$
12. $2b^2 + b + 7$
13. $\frac{a - b}{c - d}$
14. $\frac{a + c}{b + c}$

Simplify each expression. (Prerequisite Skill)

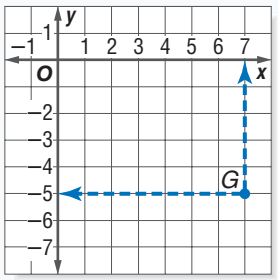
15. $x - (-1)$
16. $x - (-5)$
17. $2[x - (-3)]$
18. $4[x - (-2)]$

19. **TRAVEL** Joan travels 65 miles per hour for x hours on Monday. On Tuesday she drives 55 miles per hour for $(x + 3)$ hours. Write a simplified expression for the sum of the distances traveled. (Prerequisite Skill)

QUICKReview

Example 1 **Write the ordered pair for point G.**

Step 1 Follow a vertical line through the point to find the x-coordinate on the x-axis. The x-coordinate is 7.



Step 2 Follow a horizontal line through the point to find the y-coordinate on the y-axis. The y-coordinate is -5.

Step 3 The ordered pair for point G is $(7, -5)$. It can also be written as $G(7, -5)$.

Example 2 **Evaluate $d(a^2 + 2ab + b^2) - c$ if $a = -1$, $b = 3$, $c = -2$, and $d = 0$.**

$0[(-1)^2 + 2(-1)(3) + 3^2] - (-2)$ Substitute -1 for a, 3 for b, -2 for c, and 0 for d.

$= 0 - (-2)$ Multiplication Property of Zero.

$= 2$ Subtract.

Example 3 **Simplify $\frac{2}{5}[x - (-10)]$.**

$\frac{2}{5}[x - (-10)]$

$= \frac{2}{5}(x + 10)$ Simplify.

$= \frac{2}{5}(x) + \frac{2}{5}(10)$ Distributive Property

$= \frac{2}{5}x + 4$ Simplify.

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2-1 Relations and Functions

Main Ideas

- Analyze and graph relations.
- Find functional values.

New Vocabulary

ordered pair
Cartesian coordinate plane
quadrant
relation
domain
range
function
mapping
one-to-one function
discrete function
continuous function
vertical line test
independent variable
dependent variable
function notation

GET READY for the Lesson

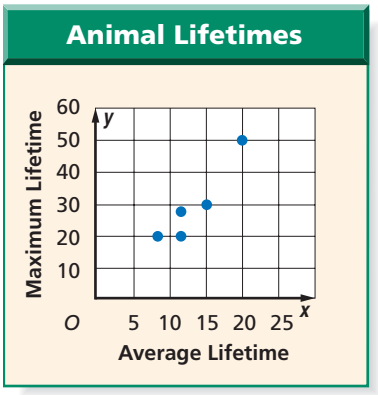
The table shows average and maximum lifetimes for some animals. The data can also be represented as the **ordered pairs** (12, 28), (15, 30), (8, 20), (12, 20), and (20, 50). The first number in each ordered pair is the average lifetime, and the second number is the maximum lifetime.

Animal	Average Lifetime (years)	Maximum Lifetime (years)
Cat	12	28
Cow	15	30
Deer	8	20
Dog	12	20
Horse	20	50

Source: *The World Almanac*

(12, 28)
average lifetime
maximum lifetime

Graph Relations You can graph the ordered pairs above on a *coordinate system*. Remember that each point in the coordinate plane can be named by exactly one ordered pair and every ordered pair names exactly one point in the coordinate plane.

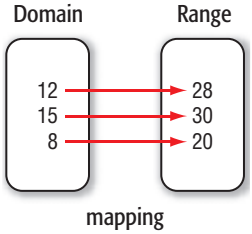


The graph of the animal lifetime data lies in the part of the Cartesian coordinate plane with all positive coordinates. The **Cartesian coordinate plane** is composed of the *x-axis* (horizontal) and the *y-axis* (vertical), which meet at the *origin* (0, 0) and divide the plane into four **quadrants**. In general, any ordered pair in the coordinate plane can be written in the form (x, y).

A **relation** is a set of ordered pairs, such as the one for the longevity of animals. The **domain** of a relation is the set of all first coordinates (x-coordinates) from the ordered pairs, and the **range** is the set of all second coordinates (y-coordinates) from the ordered pairs. The domain of the function above is {8, 12, 15, 20}, and the range is {20, 30, 28, 50}.

A **function** is a special type of relation in which each element of the domain is paired with *exactly one* element of the range. A **mapping** shows how the members are paired. A function like the one represented by the mapping in which each element of the range is paired with exactly one element of the domain is called a **one-to-one function.**

{(12, 28), (15, 30), (8, 20)}



William J. Weber
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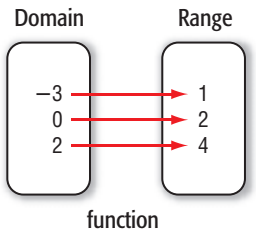
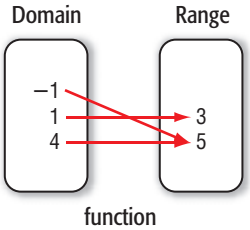
The first two relations shown below are functions. The third relation is not a function because the −3 in the domain is paired with both 0 and 6 in the range.

{(−3, 1), (0, 2), (2, 4)}



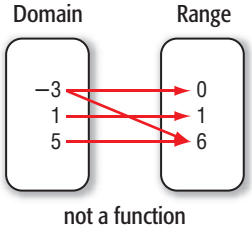
function

{(−1, 5), (1, 3), (4, 5)}



function

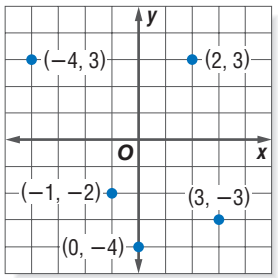
{(5, 6), (−3, 0), (1, 1), (−3, 6)}



not a function

EXAMPLE 1 Domain and Range

State the domain and range of the relation shown in the graph. Is the relation a function?



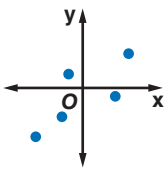
The relation is {(−4, 3), (−1, −2), (0, −4), (2, 3), (3, −3)}.
The domain is {−4, −1, 0, 2, 3}.
The range is {−4, −3, −2, 3}.

Each member of the domain is paired with exactly one member of the range, so this relation is a function.

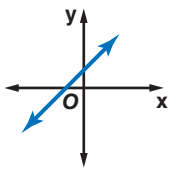
CHECK Your Progress

1. State the domain and range of the relation {(−2, 2), (1, 4), (3, 0), (−2, −4), (0, 3)}. Is the relation a function?

A relation in which the domain is a set of individual points, like the relation in Example 1, is said to be **discrete.** Notice that its graph consists of points that are not connected. When the domain of a relation has an infinite number of elements and the relation can be graphed with a line or smooth curve, the relation is **continuous.** With both discrete and continuous graphs, you can use the **vertical line test** to determine whether the relation is a function.

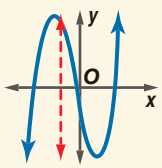
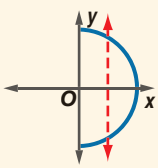


Discrete Relation



Continuous Relation

KEY CONCEPT *Vertical Line Test*

Words	If no vertical line intersects a graph in more than one point, the graph represents a function.	If some vertical line intersects a graph in two or more points, the graph does not represent a function.
Models		

Study Tip

Continuous Relations

You can draw the graph of a continuous relation without lifting your pencil from the paper.

In Example 1, there is no vertical line that contains more than one of the points. Therefore, the relation is a function.

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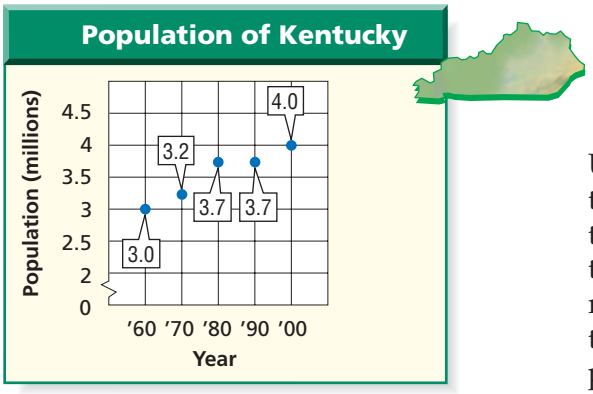
EXAMPLE Vertical Line Test

2 **GEOGRAPHY** **The table shows the population of the state of Kentucky over the last several decades. Graph this information and determine whether it represents a function. Is the relation *discrete* or *continuous*?**

Year	Population (millions)
1960	3.0
1970	3.2
1980	3.7
1990	3.7
2000	4.0

Source: U.S. Census Bureau



Use the vertical line test. Notice that no vertical line can be drawn that contains more than one of the data points. Therefore, this relation is a function. Because the graph consists of distinct points, the relation is discrete.

Study Tip

Vertical Line Test

You can use a pencil to represent a vertical line. Slowly move the pencil to the right across the graph to see if it intersects the graph at more than one point.

CHECK Your Progress

2. The number of employees a company had in each year from 1999 to 2004 were 25, 28, 34, 31, 27, and 29. Graph this information and determine whether it represents a function. Is the relation *discrete* or *continuous*?

Equations of Functions and Relations Relations and functions can also be represented by equations. The solutions of an equation in x and y are the set of ordered pairs (x, y) that make the equation true.

Consider the equation $y = 2x - 6$. Since x can be any real number, the domain has an infinite number of elements. To determine whether an equation represents a function, it is often simplest to look at the graph of the relation.

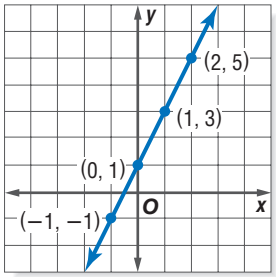
EXAMPLE Graph a Relation

3 **Graph each equation and find the domain and range. Then determine whether the equation is a function and state whether it is *discrete* or *continuous*.**

a. $y = 2x + 1$

Make a table of values to find ordered pairs that satisfy the equation. Choose values for x and find the corresponding values for y. Then graph the ordered pairs.

x	y
-1	-1
0	1
1	3
2	5



Since x can be any real number, there is an infinite number of ordered pairs that can be graphed. All of them lie on the line shown. Notice that every real number is the x-coordinate of some point on the line. Also, every real number is the y-coordinate of some point on the line. So the domain and range are both all real numbers, and the relation is continuous.

This graph passes the vertical line test. For each x-value, there is exactly one y-value, so the equation $y = 2x + 1$ represents a function.

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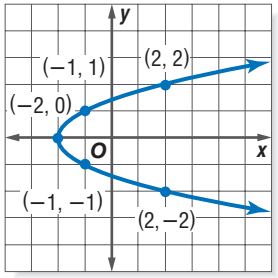
b. $x = y^2 - 2$

Make a table. In this case, it is easier to choose y values and then find the corresponding values for x. Then sketch the graph, connecting the points with a smooth curve.

x	y
2	−2
−1	−1
−2	0
−1	1
2	2



Every real number is the y-coordinate of some point on the graph, so the range is all real numbers. But, only real numbers greater than or equal to −2 are x-coordinates of points on the graph. So the domain is $\{x|x \geq -2\}$. The relation is continuous.

You can see from the table and the vertical line test that there are two y values for each x value except $x = -2$. Therefore, the equation $x = y^2 - 2$ does not represent a function.

CHECK Your Progress

3A. Graph the relation represented by $y = x^2 + 1$.

3B. Find the domain and range. Determine if the relation is *discrete* or *continuous*.

3C. Determine whether the relation is a function.

Online Personal Tutor at algebra2.com

Reading Math

Functions Suppose you have a job that pays by the hour. Since your pay *depends* on the number of hours you work, you might say that your pay is a *function* of the number of hours you work.

When an equation represents a function, the variable, usually x, whose values make up the domain is called the **independent variable**. The other variable, usually y, is called the **dependent variable** because its values depend on x.

Equations that represent functions are often written in **function notation**. The equation $y = 2x + 1$ can be written as $f(x) = 2x + 1$. The symbol $f(x)$ replaces the y and is read "f of x." The f is just the name of the function. It is not a variable that is multiplied by x. Suppose you want to find the value in the range that corresponds to the element 4 in the domain of the function. This is written as $f(4)$ and is read "f of 4." The value $f(4)$ is found by substituting 4 for each x in the equation. Therefore, $f(4) = 2(4) + 1$ or 9. *Letters other than f can be used to represent a function. For example, $g(x) = 2x + 1$.*

EXAMPLE **Evaluate a Function**

4 **Given $f(x) = x^2 + 2$, find each value.**

a. $f(-3)$

$f(x) = x^2 + 2$	Original function
$f(-3) = (-3)^2 + 2$	Substitute.
$= 9 + 2$ or 11	Simplify.

b. $f(3z)$

$f(x) = x^2 + 2$	Original function
$f(3z) = (3z)^2 + 2$	Substitute.
$= 9z^2 + 2$	$(ab)^2 = a^2b^2$

CHECK Your Progress

Given $g(x) = 0.5x^2 - 5x + 3.5$, find each value.

4A. $g(2.8)$ **4B.** $g(4a)$

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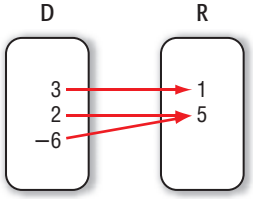
CHECK Your Understanding

Examples 1, 2 (pp. 59–60)

State the domain and range of each relation. Then determine whether each relation is a function. Write *yes* or *no*.

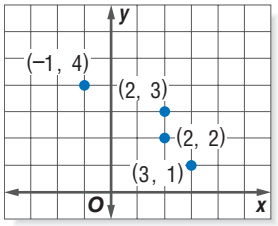
1.



2.

x	y
5	2
10	−2
15	−2
20	−2

3.



WEATHER **For Exercises 4–6, use the table that shows the record high temperatures (°F) for January and July for four states.**

4. Identify the domain and range. Assume that the January temperatures are the domain.

5. Write a relation of ordered pairs for the data.

6. Graph the relation. Is this relation a function?

State	Jan.	July
California	97	134
Illinois	78	117
North Carolina	86	109
Texas	98	119

Source: U.S. National Oceanic and Atmospheric Administration

Examples 2, 3 (pp. 60–61)

Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function and state whether it is *discrete* or *continuous*.

7. $\{(7, 8), (7, 5), (7, 2), (7, -1)\}$

8. $\{(6, 2.5), (3, 2.5), (4, 2.5)\}$

9. $y = -2x + 1$

10. $x = y^2$

Example 4 (p. 61)

11. Find $f(5)$ if $f(x) = x^2 - 3x$.

12. Find $h(-2)$ if $h(x) = x^3 + 1$.

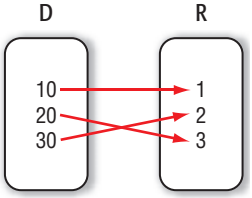
Exercises

HOMEWORK HELP

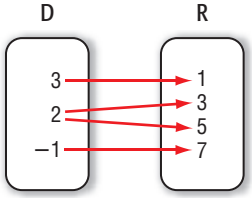
For Exercises	See Examples
13–28	1, 2
29–34	3
35–42	4

State the domain and range of each relation. Then determine whether each relation is a function. Write *yes* or *no*.

13.



14.



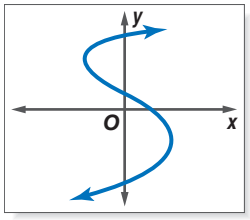
15.

x	y
0.5	−3
2	0.8
0.5	8

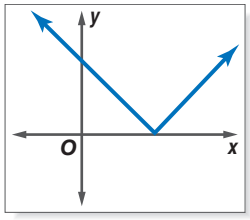
16.

x	y
2000	$4000
2001	$4300
2002	$4600
2003	$4500

17.

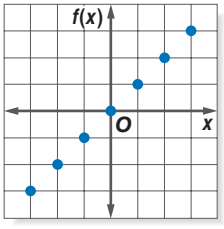


18.

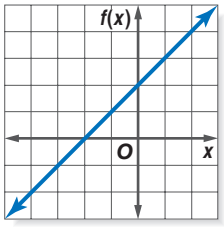


Determine whether each function is *discrete* or *continuous*.

19.



20.



21. $\{(-3, 0), (-1, 1), (1, 3)\}$

22. $y = -x + 4$

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Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function and state whether it is *discrete* or *continuous*.

23. $\{(2, 1), (-3, 0), (1, 5)\}$

24. $\{(4, 5), (6, 5), (3, 5)\}$

25. $\{(-2, 5), (3, 7), (-2, 8)\}$

26. $\{(3, 4), (4, 3), (6, 5), (5, 6)\}$

27. $\{(0, -1.1), (2, -3), (1.4, 2), (-3.6, 8)\}$

28. $\{(-2.5, 1), (-1, -1), (0, 1), (-1, 1)\}$

29. $y = -5x$

30. $y = 3x$

31. $y = 3x - 4$

32. $y = 7x - 6$

33. $y = x^2$

34. $x = 2y^2 - 3$

Find each value if $f(x) = 3x - 5$ and $g(x) = x^2 - x$.

35. $f(-3)$

36. $g(3)$

37. $g\left(\frac{1}{3}\right)$

38. $f\left(\frac{2}{3}\right)$

39. $f(a)$

40. $g(5n)$

41. Find the value of $f(x) = -3x + 2$ when $x = 2$.

42. What is $g(4)$ if $g(x) = x^2 - 5$?

Real-World Link

The major league record for runs batted in (RBIs) is 191 by Hack Wilson.

Source: www.baseball-almanac.com

SPORTS **For Exercises 43–45, use the table that shows the leading home run and runs batted in totals in the National League for 2000–2004.**

Year	2000	2001	2002	2003	2004
HR	50	73	49	47	48
RBI	147	160	128	141	131

Source: *The World Almanac*

43. Make a graph of the data with home runs on the horizontal axis and runs batted in on the vertical axis.

44. Identify the domain and range.

45. Does the graph represent a function? Explain your reasoning.

STOCKS **For Exercises 46–49, use the table that shows a company's stock price in recent years.**

Year	Price
2002	$39
2003	$43
2004	$48
2005	$55
2006	$61
2007	$52

46. Write a relation to represent the data.

47. Graph the relation.

48. Identify the domain and range.

49. Is the relation a function? Explain your reasoning.

GOVERNMENT **For Exercises 50–53, use the table below that shows the number of members of the U.S. House of Representatives with 30 or more consecutive years of service in Congress from 1991 to 2003.**

Year	1991	1993	1995	1997	1999	2001	2003
Representatives	11	12	9	6	3	7	9

Source: *Congressional Directory*

50. Write a relation to represent the data.

51. Graph the relation.

52. Identify the domain and range. Determine whether the relation is *discrete* or *continuous*.

53. Is the relation a function? Explain your reasoning.

54. AUDIO BOOK DOWNLOADS Chaz has a collection of 15 audio books. After he gets a part-time job, he decides to download 3 more audio books each month. The function $A(t) = 15 + 3t$ counts the number of audio books $A(t)$ he has after t months. How many audio books will he have after 8 months?

EXTRA PRACTICE
See pages 893, 927.
MathOnline
Self-Check Quiz at algebra2.com

Bettmann/CORBIS
www.MathSchoolinternational.com

H.O.T. Problems

55. OPEN ENDED Write a relation of four ordered pairs that is *not* a function. Explain why it is not a function.

56. FIND THE ERROR Teisha and Molly are finding $g(2a)$ for the function $g(x) = x^2 + x - 1$. Who is correct? Explain your reasoning.

Teisha
$g(2a) = 2(a^2 + a - 1)$
$= 2a^2 + 2a - 2$

Molly
$g(2a) = (2a)^2 + 2a - 1$
$= 4a^2 + 2a - 1$

57. CHALLENGE If $f(3a - 1) = 12a - 7$, find one possible expression for $f(x)$.

58. *Writing in Math* Use the information about animal lifetimes on page 58 to explain how relations and functions apply to biology. Include an explanation of how a relation can be used to represent data and a sentence that includes the words *average lifetime*, *maximum lifetime*, and *function*.

STANDARDIZED TEST PRACTICE

59. ACT/SAT If $g(x) = x^2$, which expression is equal to $g(x + 1)$?

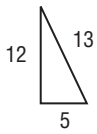
A 1

B $x^2 + 1$

C $x^2 + 2x + 1$

D $x^2 - x$

60. REVIEW Which set of dimensions represent a triangle similar to the triangle shown below?



F 7 units, 11 units, 12 units

G 10 units, 23 units, 24 units

H 20 units, 48 units, 52 units

J 1 unit, 2 units, 3 units

Spiral Review

Solve each inequality. (Lessons 1-5 and 1-6)

61. $|y + 1| < 7$

62. $|5 - m| < 1$

63. $x - 5 < 0.1$

64. SHOPPING Javier had $25.04 when he went to the mall. His friend Sally had $32.67. Javier wanted to buy a shirt for $27.89. How much money did Javier borrow from Sally? How much money did that leave Sally? (Lesson 1-3)

Simplify each expression. (Lessons 1-1 and 1-2)

65. $32(22 - 12) + 42$

66. $3(5a + 6b) + 8(2a - b)$

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation. Check your solution. (Lesson 1-3)

67. $x + 3 = 2$

68. $-4 + 2y = 0$

69. $0 = \frac{1}{2}x - 3$

70. $\frac{1}{3}x - 4 = 1$

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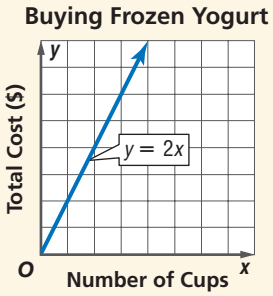
READING MATH

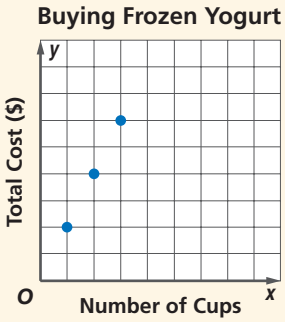
Discrete and Continuous Functions in the Real World

A cup of frozen yogurt costs \$2 at the Yogurt Shack. We might describe the cost of x cups of yogurt using the *continuous* function $y = 2x$, where y is the total cost in dollars. The graph of that function is shown at the right.



From the graph, you can see that 2 cups of yogurt cost \$4, 3 cups cost \$6, and so on. The graph also shows that 1.5 cups of yogurt cost 2(1.5) or \$3. However, the Yogurt Shack probably will not sell partial cups of yogurt. This function is more accurately modeled with a *discrete* function.

The graph of the discrete function at the right also models the cost of buying cups of frozen yogurt. The domain in this graph makes sense in this situation.

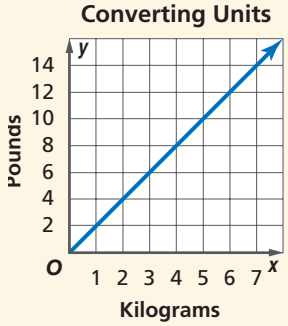


When choosing a discrete function or a continuous function to model a real-world situation, be sure to consider whether all real numbers are reasonable as part of the domain.

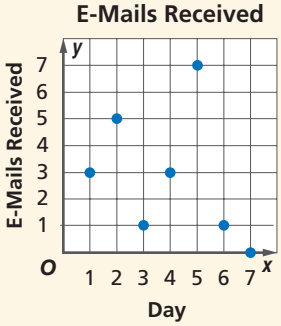
Reading to Learn

Determine whether each function is better modeled using a *discrete* or *continuous* function. Explain your reasoning.

1.



2.



3. y represents the distance a car travels in x hours.

4. y represents the total number of riders who have ridden a roller coaster after x rides.

5. Give an example of a real-world function that is discrete and a real-world function that is continuous. Explain your reasoning.

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2-2 Linear Equations

Main Ideas

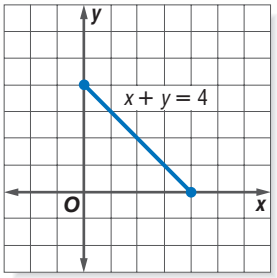
- Identify linear equations and functions.
- Write linear equations in standard form and graph them.

New Vocabulary

linear equation
linear function
standard form
y-intercept
x-intercept

GET READY for the Lesson

Lolita has 4 hours after dinner to study and do homework. She has brought home math and chemistry. If she spends x hours on math and y hours on chemistry, a portion of the graph of the equation $x + y = 4$ can be used to relate how much time she spends on each.



Identify Linear Equations and Functions An equation such as $x + y = 4$ is called a linear equation. A **linear equation** has no operations other than addition, subtraction, and multiplication of a variable by a constant. The variables may not be multiplied together or appear in a denominator. A linear equation does not contain variables with exponents other than 1. The graph of a linear equation is always a line.

Linear equations	Not linear equations
$5x - 3y = 7$	$7a + 4b^2 = -8$
$x = 9$	$y = \sqrt{x + 5}$
$6s = -3t - 15$	$x + xy = 1$
$y = \frac{1}{2}x$	$y = \frac{1}{x}$

A **linear function** is a function whose ordered pairs satisfy a linear equation. Any linear function can be written in the form $f(x) = mx + b$, where m and b are real numbers.

EXAMPLE 1 Identify Linear Functions

State whether each function is a linear function. Explain.

a. $f(x) = 10 - 5x$ This is a linear function because it can be written as $f(x) = -5x + 10$. $m = -5$, $b = 10$

b. $g(x) = x^4 - 5$ This is not a linear function because x has an exponent other than 1.

c. $h(x, y) = 2xy$ This is not a linear function because the two variables are multiplied together.

CHECK Your Progress

1A. $f(x) = \frac{5}{x + 6}$

1B. $g(x) = -\frac{3}{2}x + \frac{1}{3}$

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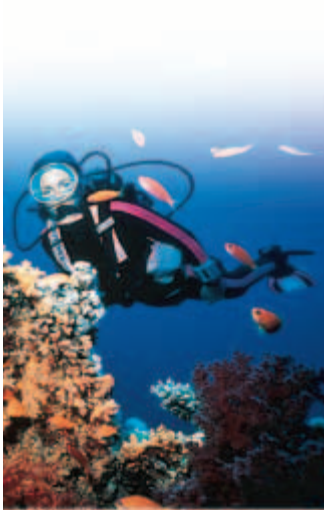

Real-World Link

To avoid decompression sickness, it is recommended that divers ascend no faster than 30 feet per minute.

Source: www.emedicine.com

Real-World EXAMPLE Evaluate a Linear Function

2 **WATER PRESSURE** The linear function $P(d) = 62.5d + 2117$ can be used to find the pressure (lb/ft^2) d feet below the surface of the water.

a. Find the pressure at a depth of 350 feet.

$P(d) = 62.5d + 2117$ Original function

$P(\mathbf{350}) = 62.5(\mathbf{350}) + 2117$ Substitute.

$= 23{,}992$ Simplify.

The pressure at a depth of 350 feet is about 24,000 lb/ft^2.

b. The term 2117 in the function represents the atmospheric pressure at the surface of the water. How many times as great is the pressure at a depth of 350 feet as the pressure at the surface?

Divide the pressure 350 feet down by the pressure at the surface.

$\frac{23{,}992}{2117} \approx 11.33$ Use a calculator.

The pressure at that depth is more than 11 times that at the surface.

CHECK Your Progress

2. At what depth is the pressure 33,367 lb/ft^2?

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Standard Form Many linear equations can be written in **standard form**, $Ax + By = C$, where A, B, and C are integers whose greatest common factor is 1.

KEY CONCEPT *Standard Form of a Linear Equation*

The standard form of a linear equation is $Ax + By = C$, where A, B, and C are integers whose greatest common factor is 1, $A \geq 0$, and A and B are not both zero.

EXAMPLE Standard Form

3 **Write each equation in standard form. Identify A, B, and C.**

a. $y = -2x + 3$

$y = -2x + 3$ Original equation

$2x + y = 3$ Add $2x$ to each side.

So, $A = 2$, $B = 1$, and $C = 3$.

b. $-\frac{3}{5}x = 3y - 2$

$-\frac{3}{5}x = 3y - 2$ Original equation

$-\frac{3}{5}x - 3y = -2$ Subtract $3y$ from each side.

$3x + 15y = 10$ Multiply each side by -5 so that the coefficients are integers and $A \geq 0$.

So, $A = 3$, $B = 15$, and $C = 10$.

CHECK Your Progress

3A. $2y = 4x + 5$ **3B.** $3x - 6y - 9 = 0$

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Study Tip

Vertical and Horizontal Lines

An equation of the form $x = C$ represents a vertical line, which has only an x-intercept. $y = C$ represents a horizontal line, which has only a y-intercept.

Since two points determine a line, one way to graph a linear equation or function is to find the points at which the graph intersects each axis and connect them with a line. The y-coordinate of the point at which a graph crosses the y-axis is called the ***y*-intercept**. Likewise, the x-coordinate of the point at which it crosses the x-axis is the ***x*-intercept**.

EXAMPLE 4 Use Intercepts to Graph a Line

Find the x-intercept and the y-intercept of the graph of $3x - 4y + 12 = 0$. Then graph the equation.

The x-intercept is the value of x when $y = 0$.

$3x - 4y + 12 = 0$	Original equation
$3x - 4(0) + 12 = 0$	Substitute 0 for y.
$3x = -12$	Subtract 12 from each side.
$x = -4$	Divide each side by 3.

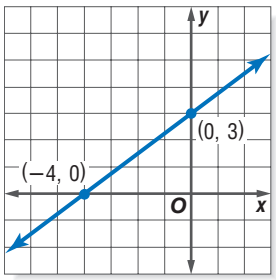
The x-intercept is -4. The graph crosses the x-axis at $(-4, 0)$.

Likewise, the y-intercept is the value of y when $x = 0$.

$3x - 4y + 12 = 0$	Original equation
$3(0) - 4y + 12 = 0$	Substitute 0 for x.
$-4y = -12$	Subtract 12 from each side.
$y = 3$	Divide each side by -4.

The y-intercept is 3. The graph crosses the y-axis at $(0, 3)$.

Use these ordered pairs to graph the equation.



CHECK Your Progress

4. Find the x-intercept and the y-intercept of the graph of $2x + 5y - 10 = 0$. Then graph the equation.

CHECK Your Understanding

Example 1 (p. 66)

State whether each equation or function is linear. Write *yes* or *no*. If no, explain your reasoning.

1. $x^2 + y^2 = 4$ **2.** $h(x) = 1.1 - 2x$

Example 2 (p. 67)

ECONOMICS For Exercises 3 and 4, use the following information.

On January 1, 1999, the euro became legal tender in 11 participating countries in Europe. Based on the exchange rate on one particular day, the linear function $d(x) = 0.8881x$ could be used to convert x euros to U.S. dollars.

3. On that day, what was the value in U.S. dollars of 200 euros?

4. On that day, what was the value in euros of 500 U.S. dollars?

Example 3 (p. 67)

Write each equation in standard form. Identify A, B, and C.

5. $y = 3x - 5$ **6.** $4x = 10y + 6$ **7.** $y = \frac{2}{3}x + 1$

Example 4 (p. 68)

Find the x-intercept and the y-intercept of the graph of each equation. Then graph the equation.

8. $y = -3x - 5$ **9.** $x - y - 2 = 0$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–17	1
18–21	2
22–27	3
28–33	4

State whether each equation or function is linear. Write *yes* or *no*. If no, explain your reasoning.

10. $x + y = 5$

11. $f(x) = 6x - 19$

12. $f(x) = 7x^5 + x - 1$

13. $h(x) = 2x^3 - 4x^2 + 5$

14. $g(x) = 10 + \frac{2}{x^2}$

15. $\frac{1}{x} + 3y = -5$

16. $x + \sqrt{y} = 4$

17. $y = \sqrt{2x - 5}$

PHYSICS **For Exercises 18 and 19, use the following information.**
When a sound travels through water, the distance y in meters that the sound travels in x seconds is given by the equation $y = 1440x$.

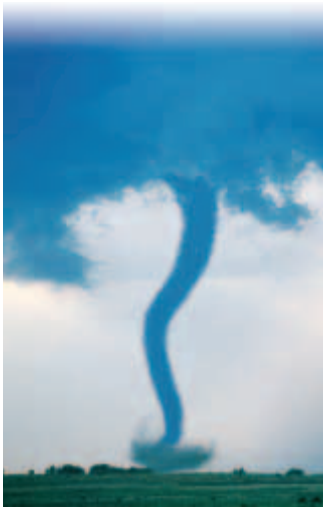
18. How far does a sound travel underwater in 5 seconds?

19. In air, the equation is $y = 343x$. Does sound travel faster in air or water? Explain.

ATMOSPHERE **For Exercises 20 and 21, use the following information.**
Suppose the temperature T in °F above the Earth's surface is given by $T(h) = -3.6h + 68$, where h is the height (in thousands of feet).

20. Find the temperature at a height of 10,000 feet.

21. Find the height if the temperature is −58°F.



Real-World Link
The troposphere is the lowest layer of the atmosphere. All weather events take place in the troposphere.

Write each equation in standard form. Identify *A*, *B*, and *C*.

22. $y = -3x + 4$

23. $y = 12x$

24. $x = 4y - 5$

25. $x = 7y + 2$

26. $5y = 10x - 25$

27. $4x = 8y - 12$

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation.

28. $5x + 3y = 15$

29. $2x - 6y = 12$

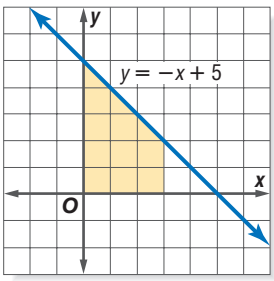
30. $3x - 4y - 10 = 0$

31. $2x + 5y - 10 = 0$

32. $y = x$

33. $y = 4x - 2$

34. **GEOMETRY** Find the area of the shaded region in the graph. (*Hint:* The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$.)



Write each equation in standard form. Identify *A*, *B*, and *C*.

35. $\frac{1}{2}x + \frac{1}{2}y = 6$

36. $\frac{1}{3}x - \frac{1}{3}y = -2$

37. $0.5x = 3$

38. $0.25y = 10$

39. $\frac{5}{6}x + \frac{1}{15}y = \frac{3}{10}$

40. $0.25x = 0.1 + 0.2y$

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation.

41. $y = -2$

42. $y = 4$

43. $x = 8$

44. $3x + 2y = 6$

45. $x = 1$

46. $f(x) = 4x - 1$

47. $g(x) = 0.5x - 3$

48. $4x + 8y = 12$

49. **ATMOSPHERE** Graph the linear function in Exercises 20 and 21.

CORBIS
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COMMISSION **For Exercises 50–52, use the following information.**
Latonya earns a commission of $1.75 for each magazine subscription she sells and $1.50 for each newspaper subscription she sells. Her goal is to earn a total of $525 in commissions in the next two weeks.

50. Write an equation that is a model for the different numbers of magazine and newspaper subscriptions that can be sold to meet the goal.

51. Graph the equation. Does this equation represent a function? Explain.

52. If Latonya sells 100 magazine subscriptions and 200 newspaper subscriptions, will she meet her goal? Explain.

EXTRA PRACTICE
See pages 893, 927.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

53. **OPEN ENDED** Write an equation of a line with an x-intercept of 2.

54. **REASONING** Explain why $f(x) = \frac{x+2}{2}$ is a linear function.

CHALLENGE **For Exercises 55 and 56, use $x + y = 0$, $x + y = 5$, and $x + y = -5$.**

55. Graph the equations. Then compare and contrast the graphs.

56. Write a linear equation whose graph is between the graphs of $x + y = 0$ and $x + y = 5$.

57. **REASONING** Explain why the graph of $x + 3y = 0$ has only one intercept.

58. ***Writing in Math*** Use the information about study time on page 66 to explain how linear equations relate to time spent studying. Explain why only the part of the graph in the first quadrant is shown and an interpretation of the graph's intercepts in terms of the situation.

STANDARDIZED TEST PRACTICE

59. **ACT/SAT** Which function is linear?

A $f(x) = x^2$

B $g(x) = 2.7$

C $f(x) = \sqrt{9 - x^2}$

D $g(x) = \sqrt{x - 1}$

60. **REVIEW** What is the complete solution to the equation?

$$|9 - 3x| = 18$$

F $x = -9; x = 3$ H $x = -3; x = 9$

G $x = -9; x = -3$ J $x = 3; x = 9$

Spiral Review

State the domain and range of each relation. Then graph the relation and determine whether it is a function. (Lesson 2-1)

61. $\{(-1, 5), (1, 3), (2, -4), (4, 3)\}$

62. $\{(0, 2), (1, 3), (2, -1), (1, 0)\}$

Solve each inequality. (Lesson 1-6)

63. $-2 < 3x + 1 < 7$

64. $|x + 4| > 2$

65. **TAX** Including a 6% sales tax, a paperback book costs $8.43. What is the price before tax? (Lesson 1-3)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the reciprocal of each number.**

66. -4 **67.** $\frac{1}{2}$ **68.** $3\frac{3}{4}$ **69.** -1.25

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2-3 Slope

Main Ideas

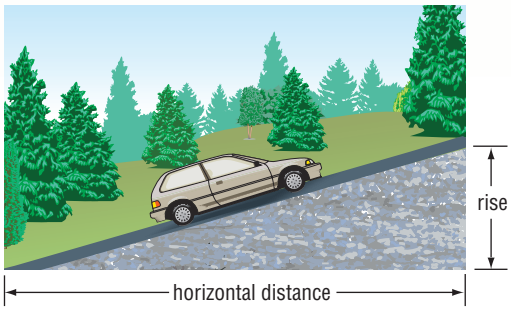
- Find and use the slope of a line.
- Graph parallel and perpendicular lines.

New Vocabulary

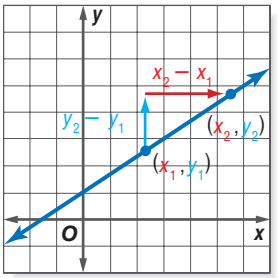
rate of change
slope
family of graphs
parent graph
oblique

GET READY for the Lesson

The grade of a road is a percent that measures the steepness of the road. It is found by dividing the amount the road rises by the corresponding horizontal distance.



Slope A **rate of change** measures how much a quantity changes, on average, relative to the change in another quantity, often time. The idea of rate of change can be applied to points in the coordinate plane to determine the steepness of the line between the points. The **slope** of a line is the ratio of the change in y-coordinates to the corresponding change in x-coordinates. Suppose a line passes through points at (x_1, y_1) and (x_2, y_2).



$$\text{slope} = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

KEY CONCEPT — *Slope of a Line*

Words The slope of a line is the ratio of the change in y-coordinates to the change in x-coordinates.

Symbols The slope m of the line passing through (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, where $x_1 \neq x_2$.

Study Tip

Slope
The formula for slope is often remembered as *rise over run*, where the rise is the difference in y-coordinates and the run is the difference in x-coordinates.

EXAMPLE **Find Slope and Use Slope to Graph**

1 **Find the slope of the line that passes through (−1, 4) and (1, −2). Then graph the line.**

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$= \frac{-2 - 4}{1 - (-1)}$ $(x_1, y_1) = (-1, 4), (x_2, y_2) = (1, -2)$

$= \frac{-6}{2}$ or -3 The slope is -3.

(continued on the next page)

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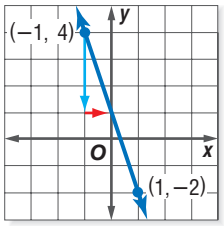
Graph the two ordered pairs and draw the line. Use the slope to check your graph by selecting any point on the line. Then go down 3 units and right 1 unit or go up 3 units and left 1 unit. This point should also be on the line.



CHECK Your Progress

1. Find the slope of the line that passes through (1, −3) and (3, 5). Then graph the line.

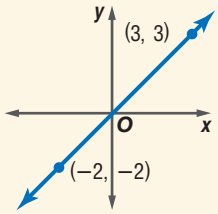
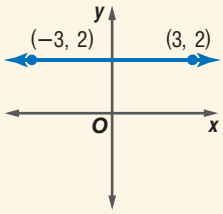
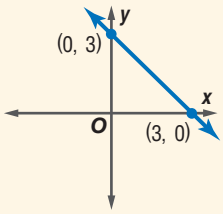
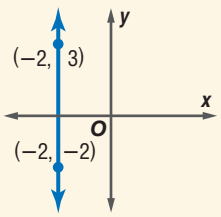
The slope of a line tells the direction in which it rises or falls.

Study Tip

Slope is Constant

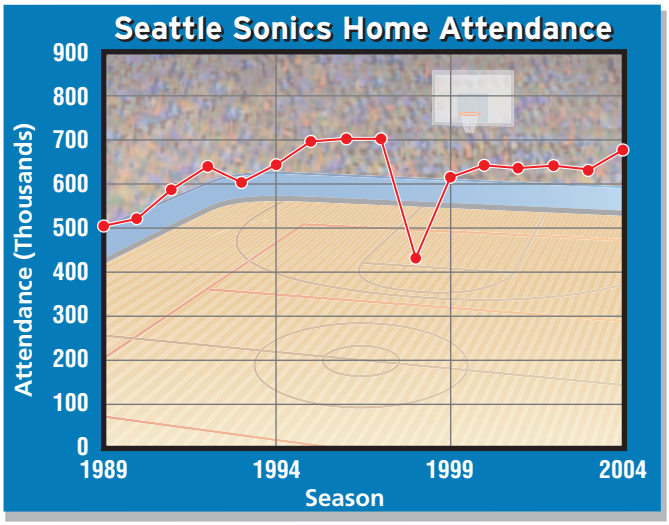
The slope of a line is the same, no matter what two points on the line are used.

CONCEPT SUMMARY *Slope*

If the line rises to the right, then the slope is *positive*.	If the line is horizontal, then the slope is *zero*.	If the line falls to the right, then the slope is *negative*.	If the line is vertical, then the slope is *undefined*.
			
$m = \frac{3-(-2)}{3-(-2)}$ $= 1$	$m = \frac{2-2}{3-(-3)}$ $= 0$	$m = \frac{0-3}{3-0}$ $= -1$	$x_1 = x_2$, so m is undefined.

Real-World EXAMPLE

2 **BASKETBALL** **Refer to the graph at the right. Find the rate of change of the number of people attending Seattle Sonics home games from 1993 to 1996.**



Source: Kenn.com

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{700 - 601}{1996 - 1993}$$ Substitute.

$$\approx 33$$ Simplify.

Between 1993 and 1996, the number of people attending Seattle Sonics home games increased at an average rate of about 33(1000) or 33,000 people per year.

CHECK Your Progress

2. In 1999, 45,616 students applied for admission to UCLA. In 2004, 56,878 students applied. Find the rate of change in the number of students applying for admission from 1999 to 2004.

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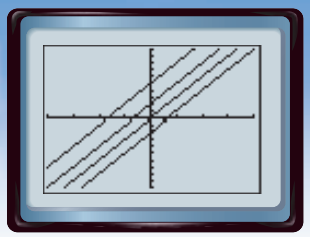
Parallel and Perpendicular Lines A **family of graphs** is a group of graphs that displays one or more similar characteristics. The **parent graph** is the simplest of the graphs in a family.

GRAPHING CALCULATOR LAB

Lines with the Same Slope

The calculator screen shows the graphs of $y = 3x$, $y = 3x + 2$, $y = 3x - 2$, and $y = 3x + 5$.



[−4, 4] scl: 1 by [−10, 10] scl: 1

THINK AND DISCUSS

1. What is similar about the graphs? What is different about the graphs?
2. Write another function that has the same characteristics as these graphs. Check by graphing.

In the Lab, you saw that lines that have the same slope are parallel.

Study Tip

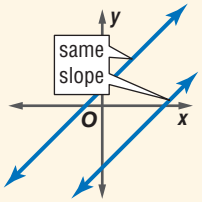
Horizontal Lines

All horizontal lines are parallel because they all have a slope of 0.

KEY CONCEPT — *Parallel Lines*

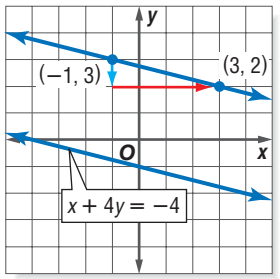
Words In a plane, nonvertical lines with the same slope are parallel. All vertical lines are parallel.

Model



EXAMPLE Parallel Lines

3 **Graph the line through (−1, 3) that is parallel to the line with equation $x + 4y = -4$.**



The x-intercept is −4, and the y-intercept is −1. Use the intercepts to graph $x + 4y = -4$.

The line falls 1 unit for every 4 units it moves to the right, so the slope is $-\frac{1}{4}$.

Now use the slope and the point at (−1, 3) to graph the line parallel to the graph of $x + 4y = -4$.

CHECK Your Progress

3. Graph the line through (−2, 4) that is parallel to the line with equation $x - 3y = 3$.

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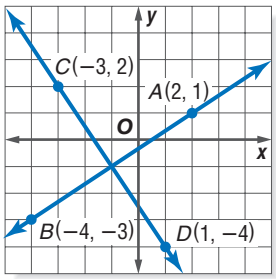
The graphs of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular.



slope of line AB slope of line CD

$$\frac{-3-1}{-4-2} = \frac{-4}{-6} \text{ or } \frac{2}{3} \qquad \frac{-4-2}{1-(-3)} = \frac{-6}{4} \text{ or } -\frac{3}{2}$$

The slopes are opposite reciprocals of each other. The product of the slopes of two perpendicular lines is always −1.

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Reading Math

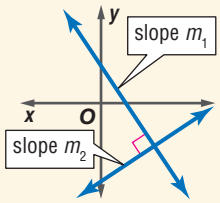
Oblique
An oblique line is a line that is neither horizontal nor vertical.

KEY CONCEPT *Perpendicular Lines*

Words In a plane, two oblique lines are perpendicular if and only if the product of their slopes is -1.

Symbols Suppose m_1 and m_2 are the slopes of two oblique lines. Then the lines are perpendicular if and only if $m_1 m_2 = -1$, or $m_1 = -\frac{1}{m_2}$.

Model



Any vertical line is perpendicular to any horizontal line.

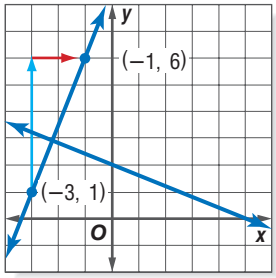
EXAMPLE 4 Perpendicular Lines

Graph the line through $(-3, 1)$ that is perpendicular to the line with equation $2x + 5y = 10$.

The x-intercept is 5, and the y-intercept is 2. Use the intercepts to graph $2x + 5y = 10$.

The line falls 2 units for every 5 units it moves to the right, so the slope is $-\frac{2}{5}$. The slope of the perpendicular line is the opposite reciprocal of $-\frac{2}{5}$, or $\frac{5}{2}$.

Start at $(-3, 1)$ and go up 5 units and right 2 units. Use this point and $(-3, 1)$ to graph the line.



CHECK Your Progress

4. Graph the line through $(-6, 2)$ that is perpendicular to the line with equation $3x - 2y = 6$.

CHECK Your Understanding

Example 1 (pp. 71–72)

Find the slope of the line that passes through each pair of points.

1. $(-2, -1), (2, -3)$ **2.** $(2, 2), (4, 2)$ **3.** $(4, 5), (-1, 0)$

Graph the line passing through the given point with the given slope.

4. $(2, -1), -3$ **5.** $(-3, -4), \frac{3}{2}$

Example 2 (p. 72)

WEATHER For Exercises 6–8, use the table that shows the temperatures at different times on the same day.

Time	8:00 A.M.	10:00 A.M.	12:00 P.M.	2:00 P.M.	4:00 P.M.
Temp (°F)	36	47	55	58	60

6. What was the average rate of change of the temperature from 8:00 A.M. to 10:00 A.M.?

7. What was the average rate of change of the temperature from 12:00 P.M. to 4:00 P.M.?

8. During what 2-hour period was the average rate of change of the temperature the least?

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Example 3 (p. 73)

Graph the line that satisfies each set of conditions.

9. passes through (0, 3), parallel to graph of $6y - 10x = 30$

10. passes through (1, 1) parallel to graph of $x + y = 5$

Example 4 (p. 74)

11. passes through (4, −2), perpendicular to graph of $3x - 2y = 6$

12. passes through (−1, 5), perpendicular to graph of $5x - 3y - 3 = 0$

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–24	1
25–29	2
30–37	3, 4

Find the slope of the line that passes through each pair of points.

13. (4, −1), (6, −6)

14. (−8, −3), (2, 3)

15. (8, 7), (7, −6)

16. (−2, −3), (0, −5)

17. (4, 9), (11, 9)

18. (4, −1.5), (4, 4.5)

Graph the line passing through the given point with the given slope.

19. $(-1, 4)$, $m = \frac{2}{3}$

20. $(-3, -1)$, $m = -\frac{1}{5}$

21. (3, −4), $m = 2$

22. (1, 2), $m = -3$

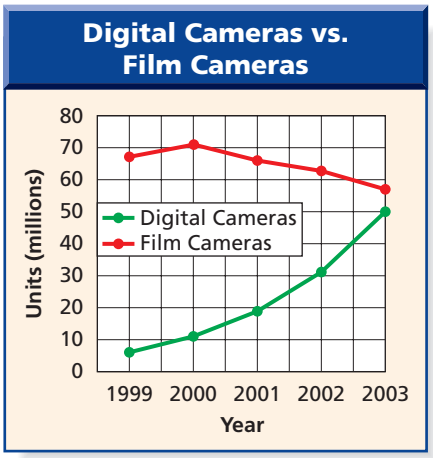
23. (6, 2), $m = 0$

24. (−2, −3), undefined

CAMERAS **For Exercises 25 and 26, refer to the graph that shows the number of digital still cameras and film cameras sold in recent years.**

25. Find the average rate of change of the number of digital cameras sold from 1999 to 2003.

26. Find the average rate of change of the number of film cameras sold from 1999 to 2003. What does the sign of the rate mean?



Source: Digital Photography Review

TRAVEL **For Exercises 27–29, use the following information.** Mr. and Mrs. Wellman are taking their daughter to college. The table shows their distance from home after various amounts of time.

Time (h)	Distance (mi)
0	0
1	55
2	110
3	165
4	165

27. Find the average rate of change of their distance from home between 1 and 3 hours after leaving home.

28. Find the average rate of change of their distance from home between 0 and 5 hours after leaving home.

29. What is another word for *rate of change* in this situation?

Graph the line that satisfies each set of conditions.

30. passes through (−2, 2), parallel to a line whose slope is −1

31. passes through (2, −5), parallel to graph of $x = 4$

32. passes through origin, parallel to graph of $x + y = 10$

33. passes through (2, −1), parallel to graph of $2x + 3y = 6$

34. passes through (2, −1), perpendicular to graph of $2x + 3y = 6$

35. passes through (−4, 1), perpendicular to a line whose slope is $-\frac{3}{2}$

36. passes through (3, 3), perpendicular to graph of $y = 3$

37. passes through (0, 0), perpendicular to graph of $y = -x$

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Find the slope of the line that passes through each pair of points.

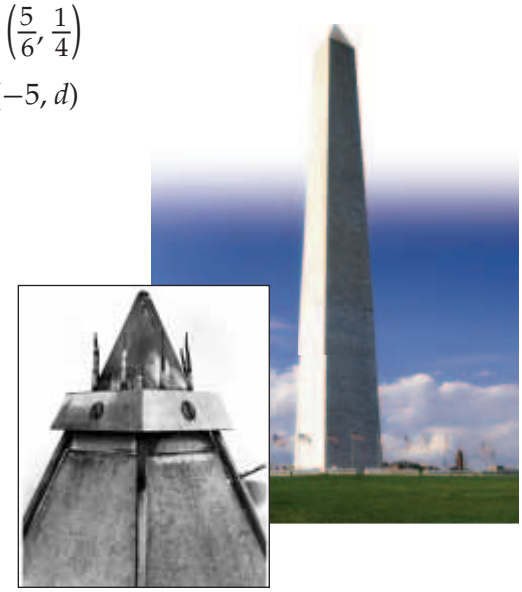
38. $\left(\frac{1}{2}, -\frac{1}{3}\right), \left(\frac{1}{4}, \frac{2}{3}\right)$

39. $\left(\frac{1}{2}, \frac{2}{3}\right), \left(\frac{5}{6}, \frac{1}{4}\right)$

40. $(c, 5), (c, -2)$

41. $(3, d), (-5, d)$

42. WASHINGTON MONUMENT The Washington Monument, in Washington, D.C., is 555 feet $5\frac{1}{8}$ inches tall and weighs 90,854 tons. The monument is topped by a square aluminum pyramid. The sides of the pyramid's base measure 5.6 inches, and the pyramid is 8.9 inches tall. Estimate the slope that a face of the pyramid makes with its base.



43. Determine the value of r so that the line through $(5, r)$ and $(2, 3)$ has slope 2.

44. Determine the value of r so that the line through $(6, r)$ and $(9, 2)$ has slope $\frac{1}{3}$.

EXTRA PRACTICE
See pages 893, 927.
Math Online
Self-Check Quiz at algebra2.com

Graph the line that satisfies each set of conditions.

45. perpendicular to graph of $3x - 2y = 24$, intersects that graph at its x-intercept

46. perpendicular to graph of $2x + 5y = 10$, intersects that graph at its y-intercept

47. GEOMETRY Determine whether quadrilateral $ABCD$ with vertices $A(-2, -1)$, $B(1, 1)$, $C(3, -2)$, and $D(0, -4)$ is a rectangle. Explain.

Graphing Calculator

For Exercises 48 and 49, use a graphing calculator to investigate the graphs of each set of equations. Explain how changing the slope affects the graph of the line.

48. $y = 2x + 3, y = 4x + 3, y = 8x + 3, y = x + 3$

49. $y = -3x + 1, y = -x + 1, y = -5x + 1, y = -7x + 1$

H.O.T. Problems

50. OPEN ENDED Write an equation of a line with slope 0. Describe the graph of the equation.

51. CHALLENGE If the graph of the equation $ax + 3y = 9$ is perpendicular to the graph of the equation $3x + y = -4$, find the value of a.

52. FIND THE ERROR Gabriel and Luisa are finding the slope of the line through $(2, 4)$ and $(-1, 5)$. Who is correct? Explain your reasoning.

Gabriel
$m = \frac{5-4}{2-(-1)}$ or $\frac{1}{3}$

Luisa
$m = \frac{4-5}{2-(-1)}$ or $-\frac{1}{3}$

53. REASONING Determine whether the statement *A line has a slope that is a real number* is *sometimes*, *always*, or *never* true. Explain your reasoning.

54. *Writing in Math* Use the information about the grade of a road on page 71 to explain how slope applies to the steepness of roads. Include a graph of $y = 0.08x$, which corresponds to a grade of 8%.

(inset)Alcoa, Brand X Pictures/Alamy Im

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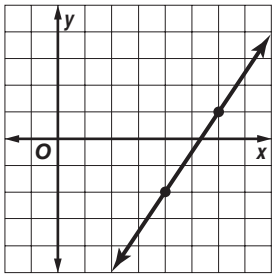
STANDARDIZED TEST PRACTICE

55. ACT/SAT What is the slope of the line shown in the graph?



A $-\frac{3}{2}$

B $-\frac{2}{3}$

C $\frac{2}{3}$

D $\frac{3}{2}$

56. REVIEW The table below shows the cost of bananas depending on the amount purchased. Which conclusion can be made based on information in the table?

Cost of Bananas	
Number of Pounds	Cost ($)
5	1.45
20	4.60
50	10.50
100	19.00

F The cost of 10 pounds of bananas would be more than $4.00.

G The cost of 200 pounds of bananas would be at most $38.00.

H The cost of bananas is always more than $0.20 per pound.

J The cost of bananas is always less than $0.28 per pound.

Spiral Review

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation. (Lesson 2-2)

57. $-2x + 5y = 20$ **58.** $4x - 3y + 8 = 0$ **59.** $y = 7x$

Find each value if $f(x) = 3x - 4$. (Lesson 2-1)

60. $f(-1)$ **61.** $f(3)$ **62.** $f\left(\frac{1}{2}\right)$ **63.** $f(a)$

Solve each inequality. (Lessons 1-5 and 1-6)

64. $5 < 2x + 7 < 13$ **65.** $2z + 5 \geq 1475$

66. SCHOOL A test has multiple-choice questions worth 4 points each and true-false questions worth 3 points each. Marco answers 14 multiple-choice questions correctly. How many true-false questions must he answer correctly to get at least 80 points total? (Lesson 1-5)

Simplify. (Lessons 1-1 and 1-2)

67. $\frac{1}{3}(15a + 9b) - \frac{1}{7}(28b - 84a)$ **68.** $3 + (21 \div 7) \times 8 \div 4$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation for *y*.** (Lesson 1-3)

69. $x + y = 9$ **70.** $4x + y = 2$ **71.** $-3x - y + 7 = 0$

72. $5x - 2y - 1 = 0$ **73.** $3x - 5y + 4 = 0$ **74.** $2x + 3y - 11 = 0$

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EXTEND 2-3

Graphing Calculator Lab

The Family of Linear Functions

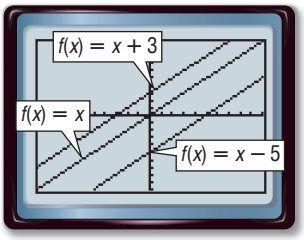
The parent function of the family of linear functions is $f(x) = x$. You can use a graphing calculator to investigate how changing the parameters m and b in $f(x) = mx + b$ affects the graphs as compared to the parent function.

ACTIVITY 1 b in $f(x) = mx + b$

Graph $f(x) = x$, $f(x) = x + 3$, and $f(x) = x - 5$ in the standard viewing window.

Enter the equations in the Y= list as Y1, Y2, and Y3. Then graph the equations.

KEYSTROKES: Y= X,T,θ,n ENTER X,T,θ,n + 3 ENTER X,T,θ,n − 5 ENTER



[−10, 10] scl:1 by [−10, 10] scl:1

1A. Compare and contrast the graphs.

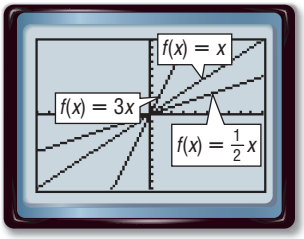
1B. How would you obtain the graphs of $f(x) = x + 3$ and $f(x) = x - 5$ from the graph of $f(x) = x$?

The parameter m in $f(x) = mx + b$ affects the graphs in a different way than b.

ACTIVITY 2 m in $f(x) = mx + b$

Graph $f(x) = x$, $f(x) = 3x$, and $f(x) = \frac{1}{2}x$ in the standard viewing window.

Enter the equations in the Y= list and graph.



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2A. How do the graphs compare?

2B. Which graph is steepest? Which graph is the least steep?

2C. Graph $f(x) = -x$, $f(x) = -3x$, and $f(x) = -\frac{1}{2}x$ in the standard viewing window. How do these graphs compare?

Analyze the Results

Graph each set of equations on the same screen. Describe the similarities or differences among the graphs.

1. $f(x) = 3x$
$f(x) = 3x + 1$
$f(x) = 3x - 2$

2. $f(x) = x + 2$
$f(x) = 5x + 2$
$f(x) = \frac{1}{2}x + 2$

3. $f(x) = x - 3$
$f(x) = 2x - 3$
$f(x) = 0.75x - 3$

4. What do the graphs of equations of the form $f(x) = mx + b$ have in common?

5. How do the values of b and m affect the graph of $f(x) = mx + b$ as compared to the parent function $f(x) = x$?

6. Summarize your results. How can knowing about the effects of m and b help you sketch the graph of a function?

Other Calculator Keystrokes at algebra2.com

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2-4 Writing Linear Equations

Main Ideas

- Write an equation of a line given the slope and a point on the line.
- Write an equation of a line parallel or perpendicular to a given line.

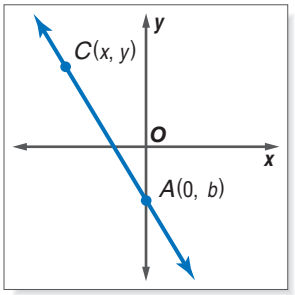
New Vocabulary

slope-intercept form
point-slope form

GET READY for the Lesson

When a company manufactures a product, they must consider two types of cost. There is the *fixed cost*, which they must pay no matter how many of the product they produce, and there is *variable cost*, which depends on how many of the product they produce. In some cases, the total cost can be found using a linear equation such as $y = 5400 + 1.37x$.

Forms of Equations Consider the graph at the right. The line passes through $A(0, b)$ and $C(x, y)$. Notice that b is the y-intercept of $\overleftrightarrow{AC}$. You can use these two points to find the slope of $\overleftrightarrow{AC}$. Substitute the coordinates of points A and C into the slope formula.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$m = \frac{y - b}{x - 0}$$ $(x_1, y_1) = (0, b), (x_2, y_2) = (x, y)$

$$m = \frac{y - b}{x}$$ Simplify.

Now solve the equation for y.

$$mx = y - b$$ Multiply each side by x.

$$mx + b = y$$ Add b to each side.

$$y = mx + b$$ Symmetric Property of Equality

When an equation is written in this form, it is in **slope-intercept form**.

Study Tip

Slope-Intercept Form

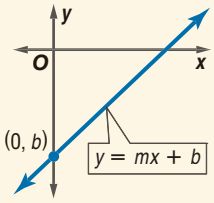
The equation of a vertical line cannot be written in slope-intercept form because its slope is undefined.

KEY CONCEPT *Slope-Intercept Form of a Linear Equation*

Words The slope-intercept form of the equation of a line is $y = mx + b$, where m is the slope and b is the y-intercept.

Symbols $y = mx + b$ (slope: m; y-intercept: b)

Model



If you are given the slope and y-intercept of a line, you can find an equation of the line by substituting the values of m and b into the slope-intercept form. You can also use the slope-intercept form to find an equation of a line if you know the slope and the coordinates of any point on the line.

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EXAMPLE Write an Equation Given Slope and a Point

Concepts in Motion
Interactive Lab
algebra2.com

1 **Write an equation in slope-intercept form for the line that has a slope of $-\frac{3}{2}$ and passes through $(-4, 1)$.**

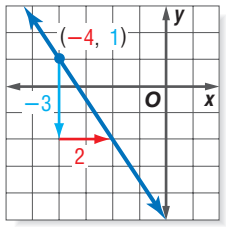
$y = mx + b$ Slope-intercept form

$1 = -\frac{3}{2}(-4) + b$ $(x, y) = (-4, 1), m = -\frac{3}{2}$

$1 = 6 + b$ Simplify.

$-5 = b$ Subtract 6 from each side.

The equation in slope-intercept form is $y = -\frac{3}{2}x - 5$.



CHECK Your Progress

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

1A. slope $\frac{4}{3}$, passes through $(3, 2)$ **1B.** slope -4, passes through $(-2, -2)$

If you are given the coordinates of two points on a line, you can use the **point-slope form** to find an equation of the line that passes through them.

KEY CONCEPT Point-Slope Form of a Linear Equation

Words The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$, where (x_1, y_1) are the coordinates of a point on the line and m is the slope of the line.

Symbols slope

$$y - y_1 = m(x - x_1)$$

coordinates of point on line

STANDARDIZED TEST EXAMPLE Write an Equation Given Two Points

2 What is an equation of the line through $(-1, 4)$ and $(-4, 5)$?

A $y = -\frac{1}{3}x + \frac{11}{3}$ **B** $y = \frac{1}{3}x + \frac{13}{3}$ **C** $y = -\frac{1}{3}x + \frac{13}{3}$ **D** $y = -3x + 1$

Read the Test Item

You are given the coordinates of two points on the line.

Solve the Test Item

First, find the slope of the line.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$= \frac{5 - 4}{-4 - (-1)}$ $(x_1, y_1) = (-1, 4)$, $(x_2, y_2) = (-4, 5)$

$= \frac{1}{-3}$ or $-\frac{1}{3}$ Simplify.

Then write an equation.

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 4 = -\frac{1}{3}[x - (-1)]$ $m = -\frac{1}{3}$; use either point for (x_1, y_1).

$y = -\frac{1}{3}x + \frac{11}{3}$ The answer is A.

Test-Taking Tip

To check your answer, substitute each ordered pair into your answer. Each should satisfy the equation.

CHECK Your Progress

2. What is an equation of the line through $(2, 3)$ and $(-4, -5)$?

F $y = \frac{4}{3}x + \frac{1}{3}$ **G** $y = \frac{4}{3x} + 8$ **H** $y = \frac{1}{3}x + \frac{17}{3}$ **J** $y = \frac{1}{3}x - 8$

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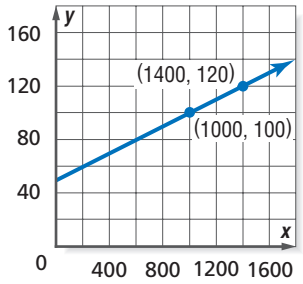
When changes in real-world situations occur at a linear rate, a linear equation can be used as a model for describing the situation.

Real-World EXAMPLE

3 **SALES** **As a salesperson, Eric Fu is paid a daily salary plus commission. When his sales are \$1000, he makes \$100. When his sales are \$1400, he makes \$120.**

a. Write a linear equation to model this situation.

Let x be his sales and let y be the amount of money he makes. Use the points (1000, 100) and (1400, 120) to make a graph to represent the situation.



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Slope formula

$$= \frac{120 - 100}{1400 - 1000}$$ $(x_1, y_1) = (1000, 100)$, $(x_2, y_2) = (1400, 120)$

$$= 0.05$$ Simplify.

Now use the slope and either of the given points with the point-slope form to write the equation.

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 100 = 0.05(x - 1000)$ $m = 0.05$, $(x_1, y_1) = (1000, 100)$

$y - 100 = 0.05x - 50$ Distributive Property

$y = 0.05x + 50$ Add 100 to each side.

The slope-intercept form of the equation is $y = 0.05x + 50$.

b. What are Mr. Fu's daily salary and commission rate?

The y-intercept of the line is 50. The y-intercept represents the money Eric would make if he had no sales. In other words, \$50 is his daily salary.

The slope of the line is 0.05. Since the slope is the coefficient of x, which is his sales, he makes 5% commission.

c. How much would Mr. Fu make in a day if his sales were \$2000?

Find the value of y when $x = 2000$.

$y = 0.05x + 50$ Use the equation you found in part **a**.

$= 0.05(2000) + 50$ Replace x with 2000.

$= 100 + 50$ or 150 Simplify.

Mr. Fu would make \$150 if his sales were \$2000.

Study Tip

Alternative Method

You could also find Mr. Fu's salary in part c by extending the graph. Then find the y-value when x is 2000.

CHECK Your Progress

SCHOOL CLUBS **For each meeting of the Putnam High School book club, \$25 is taken from the activities account to buy snacks and materials. After their sixth meeting, there will be \$350 left in the activities account.**

3A. If no money is put back into the account, what equation can be used to show how much money is left in the activities account after having x number of meetings?

3B. How much money was originally in the account?

3C. After how many meetings will there be no money left in the activities account?

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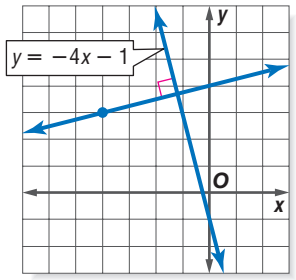
Parallel and Perpendicular Lines The slope-intercept and point-slope forms can be used to find equations of lines that are parallel or perpendicular to given lines.

EXAMPLE 4 Write an Equation of a Perpendicular Line

Write an equation for the line that passes through (−4, 3) and is perpendicular to the line whose equation is $y = -4x - 1$.



The slope of the given line is −4. Since the slopes of perpendicular lines are opposite reciprocals, the slope of the perpendicular line is $\frac{1}{4}$.

Use the point-slope form and the ordered pair (−4, 3).

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 3 = \frac{1}{4}[x - (-4)]$ $(x_1, y_1) = (-4, 3), m = \frac{1}{4}$

$y - 3 = \frac{1}{4}x + 1$ Distributive Property

$y = \frac{1}{4}x + 4$ Add 3 to each side.

CHECK Your Progress

4. Write an equation for the line that passes through (3, 7) and is perpendicular to the line whose equation is $y = \frac{3}{4}x - 5$.

CHECK Your Understanding

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

Example 1 (p. 80)

1. slope 0.5, passes through (6, 4)

2. slope $-\frac{3}{4}$, passes through $\left(2, \frac{1}{2}\right)$

3. slope 3, passes through (0, −6)

4. slope 0.25, passes through (0, 4)

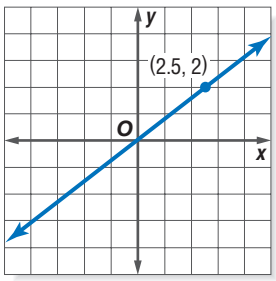
Example 2 (p. 80)

5. passes through (6, 1) and (8, −4)

6. passes through (−3, 5) and (2, 2)

Write an equation in slope-intercept form for each graph.

7.



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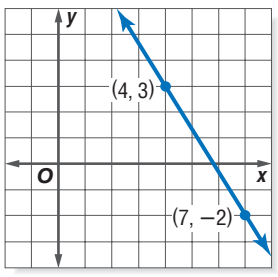

9. STANDARDIZED TEST PRACTICE What is an equation of the line through (2, −4) and (−3, −1)?

A $y = -\frac{3}{5}x + \frac{26}{5}$

B $y = -\frac{3}{5}x - \frac{14}{5}$

C $y = \frac{3}{5}x - \frac{26}{5}$

D $y = \frac{3}{5}x + \frac{14}{5}$

Example 3 (p. 81)

10. PART-TIME JOB Each week Carmen earns $15 plus $0.17 for every pamphlet that she delivers. Write an equation that can be used to find how much Carmen earns each week. How much will she earn the week she delivers 300 pamphlets?

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Example 4 (p. 82)

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

11. perpendicular to $y = \frac{3}{4}x - 2$, passes through (2, 0)

12. perpendicular to $y = \frac{1}{2}x + 6$, passes through (−5, 7)

Exercises

HOMEWORK HELP	
For Exercises	**See Examples**
13–16	1
17, 18, 21, 22	2
19, 20	4
23, 24	3

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

13. slope 3, passes through (0, −6)

14. slope 0.25, passes through (0, 4)

15. slope $-\frac{1}{2}$, passes through (1, 3)

16. slope $\frac{3}{2}$ passes through (−5, 1)

17. passes through (−2, 5) and (3, 1)

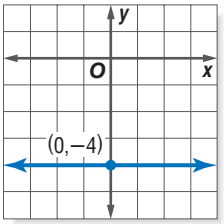
18. passes through (7, 1) and (7, 8)

19. passes through (4, 6), parallel to the graph of $y = \frac{2}{3}x + 5$

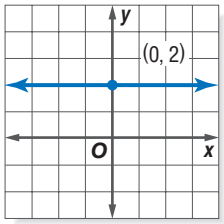
20. passes through (2, −5), perpendicular to the graph of $y = \frac{1}{4}x + 7$

Write an equation in slope-intercept form for each graph.

21.



22.



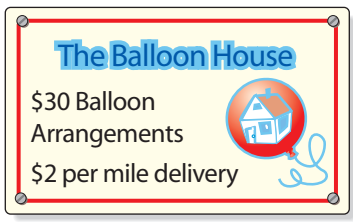
Real-World Link

The number of whitetail deer in the United States increased from about half a million in the early 1900s to 25 to 30 million in 2005.

Source: espn.com

23. ECOLOGY A park ranger at Creekside Woods estimates there are 6000 deer in the park. She also estimates that the population will increase by 75 deer each year to come. Write an equation that represents how many deer will be in the park in x years.

24. BUSINESS For what distance do the two stores charge the same amount for a balloon arrangement?



GEOMETRY For Exercises 25–27, use the equation $d = 180(c - 2)$ that gives the total number of degrees d in any convex polygon with c sides.

25. Write this equation in slope-intercept form.

26. Identify the slope and d-intercept.

27. Find the number of degrees in a pentagon.

SCIENCE For Exercises 28–30, use the following information.

Ice forms at a temperature of 0°C, which corresponds to a temperature of 32°F. A temperature of 100°C corresponds to a temperature of 212°F.

28. Write and graph the linear equation that gives the number y of degrees Fahrenheit in terms of the number x of degrees Celsius.

29. What temperature corresponds to 20°C?

30. What temperature is the same on both scales?

Cliff Keeler/Alamy Images

www.MathSchoolinternational.com

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

31. slope -0.5, passes through $(2, -3)$

32. slope 4, passes through the origin

33. x-intercept -4, y-intercept 4

34. x-intercept $\frac{1}{3}$, y-intercept $-\frac{1}{4}$

35. passes through $(6, -5)$, perpendicular to the line whose equation is $3x - \frac{1}{5}y = 3$

36. passes through $(-3, -1)$, parallel to the line that passes through $(3, 3)$ and $(0, 6)$

EXTRA PRACTICE
See pages 894, 927.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

37. OPEN ENDED Write an equation of a line in slope-intercept form.

38. REASONING What are the slope and y-intercept of the equation $cx + y = d$?

39. CHALLENGE Given $\triangle ABC$ with vertices $A(-6, -8)$, $B(6, 4)$, and $C(-6, 10)$, write an equation of the line containing the altitude from A. (*Hint:* The altitude from A is a segment that is perpendicular to $\overline{BC}$.)

40. *Writing in Math* Use the information on page 79 to explain how linear equations apply to business. Relate the terms *fixed cost* and *variable cost* to the equation $y = 5400 + 1.37x$, where y is the cost to produce x units of a product. Give the cost to produce 1000 units of the product.

STANDARDIZED TEST PRACTICE

41. ACT/SAT What is an equation of the line through $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$?

A $y = -2x - \frac{1}{2}$

B $y = -3x$

C $y = 2x - \frac{5}{2}$

D $y = \frac{1}{2}x + 1$

42. REVIEW The total cost c in dollars to go to a fair and ride n roller coasters is given by the equation

$$c = 15 + 3n.$$

If the total cost was \$33, how many roller coasters were ridden?

F 6

G 7

H 8

J 9

Spiral Review

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

43. $(7, 2)$, $(5, 6)$

44. $(1, -3)$, $(3, 3)$

45. $(-5, 0)$, $(4, 0)$

46. INTERNET A Webmaster estimates that the time (seconds) to connect to the server when n people are connecting is given by $t(n) = 0.005n + 0.3$. Estimate the time to connect when 50 people are connecting. (Lesson 2-2)

Solve each inequality. (Lessons 1-5 and 1-6)

47. $|x - 2| \leq -99$

48. $-4x + 7 \leq 31$

49. $2(r - 4) + 5 \geq 9$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the median of each set of numbers.** (Page 760)

50. $\{3, 2, 1, 3, 4, 8, 4\}$

51. $\{9, 3, 7, 5, 6, 3, 7, 9\}$

52. $\{138, 235, 976, 230, 412, 466\}$

53. $\{2.5, 7.8, 5.5, 2.3, 6.2, 7.8\}$

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CHAPTER 2

Mid-Chapter Quiz

Lessons 2-1 through 2-4

1. State the domain and range of the relation $\{(2, 5), (-3, 2), (2, 1), (-7, 4), (0, -2)\}$. Is the relation a function? Write *yes* or *no*. (Lesson 2-1)
2. Find $f(15)$ if $f(x) = 100x - 5x^2$. (Lesson 2-1)

For Exercises 3–5, use the table that shows a teacher's class size in recent years. (Lesson 2-1)

Year	Class Size
2002	27
2003	30
2004	29
2005	33

3. Graph the relation.
4. Identify the domain and range.
5. Is the relation a function? Explain your reasoning.

6. Write $y = -6x + 4$ in standard form. Identify A, B, and C. (Lesson 2-2)
7. Find the x-intercept and the y-intercept of the graph of $3x + 5y = 30$. Then graph the equation. (Lesson 2-2)
8. **MULTIPLE CHOICE** What is the y-intercept of the graph of $10 - x = 2y$? (Lesson 2-2)

   A 2 **B** 5 **C** 6 **D** 10

9. What is the slope of the line containing the points shown in the table? (Lesson 2-3)

x	y
1	−1
8	7
15	15

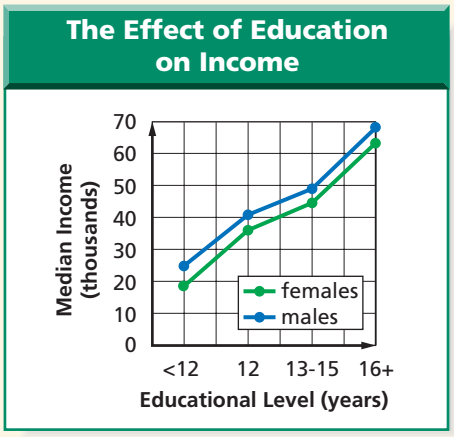
10. Graph the line that passes through $(4, -3)$ and is parallel to the line with equation $2x + 5y = 10$. (Lesson 2-3)

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

11. $(7, 3), (8, 5)$
12. $(12, 9), (9, 1)$
13. $(4, -4), (3, -7)$
14. $(0, 9), (4, 6)$

SCHOOL For Exercises 15 and 16, use the following information.

The graph shows the effect that education levels have on income. (Lesson 2-3)



Source: healthypeople.gov

15. Find the average rate of change of income for females that have 12 years of education to females that have 16+ years of education.
16. Find the average rate of change of income for males that have 12 years of education to males that have 16+ years of education.
17. Write an equation in slope-intercept form of the line with slope $-\frac{2}{3}$ that passes through the point $(-3, 5)$. (Lesson 2-4)
18. **MULTIPLE CHOICE** Find the equation of the line that passes through $(0, -3)$ and $(4, 1)$. (Lesson 2-4)

   F $y = -x + 3$

   G $y = -x - 3$

   H $y = x - 3$

   J $y = x + 3$

PART-TIME JOB Jesse is a pizza delivery driver. Each day his employer gives him \$20 plus \$0.50 for every pizza that he delivers. (Lesson 2-4)

19. Write an equation that can be used to determine how much Jesse earns each day if he delivers x pizzas.
20. How much will he earn the day he delivers 20 pizzas?

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2-5 Statistics: Using Scatter Plots

Main Ideas

- Draw scatter plots.
- Find and use prediction equations.

New Vocabulary

bivariate data
scatter plot
positive correlation
negative correlation
no correlation
line of fit
prediction equation

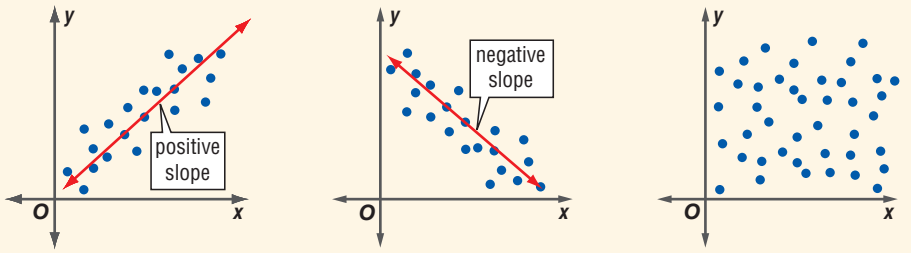
GET READY for the Lesson

The table shows the number of Calories burned per hour by a 140-pound person running at various speeds. A linear function can be used to model these data.

Speed (mph)	Calories
5	508
6	636
7	731
8	858

Scatter Plots Data with two variables, such as speed and Calories, is called **bivariate data**. A set of bivariate data graphed as ordered pairs in a coordinate plane is called a **scatter plot**. A scatter plot can show whether there is a **positive**, **negative**, or **no correlation** between the data.

KEY CONCEPT *Scatter Plots*



The more closely data can be approximated by a line, the stronger the correlation. Correlations are usually described as *strong* or *weak*.

Prediction Equations When you find a line that closely approximates a set of data, you are finding a **line of fit** for the data. An equation of such a line is often called a **prediction equation** because it can be used to predict one of the variables given the other variable.

To find a line of fit and a prediction equation for a set of data, select two points that appear to represent the data well. This is a matter of personal judgment, so your line and prediction equation may be different from someone else's.

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Study Tip

Choosing the Independent Variable

Letting x be the number of years since the first year in the data set sometimes simplifies the calculations involved in finding a function to model the data.

Real-World EXAMPLE **Find and Use a Prediction Equation**

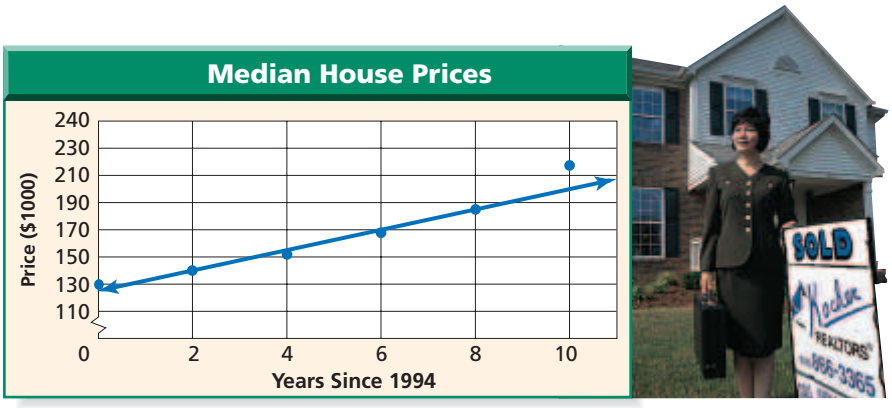
HOUSING **The table below shows the median selling price of new, privately-owned, one-family houses for some recent years.**

Year	1994	1996	1998	2000	2002	2004
Price ($1000)	130.0	140.0	152.5	169.0	187.6	219.6

Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development

a. Draw a scatter plot and a line of fit for the data. How well does the line fit the data?

Graph the data as ordered pairs, with the number of years since 1994 on the horizontal axis and the price on the vertical axis. The points (2, 140.0) and (8, 187.6) appear to represent the data well. Draw a line through these two points. Except for (10, 219.6), this line fits the data very well.



Reading Math

Predictions

When you are predicting for an x-value greater than or less than any in the data set, the process is known as **extrapolation**. When you are predicting for an x-value between the least and greatest in the data set, the process is known as **interpolation**.

b. Find a prediction equation. What do the slope and y-intercept indicate?

Find an equation of the line through (2, 140.0) and (8, 187.6).

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$= \frac{187.6 - 140.0}{8 - 2}$ Substitute.

≈ 7.93 Simplify.

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 140.0 = 7.93(x - 2)$ Substitute.

$y - 140.0 = 7.93x - 15.86$ Distribute.

$y = 7.93x + 124.14$ Simplify.

One prediction equation is $y = 7.93x + 124.14$. The slope indicates that the median price is increasing at a rate of about \$7930 per year. The y-intercept indicates that, according to the trend of the rest of the data, the median price in 1994 should have been about \$124,140.

c. Predict the median price in 2014.

The year 2014 is 20 years after 1994, so use the prediction equation to find the value of y when $x = 20$.

$y = 7.93x + 124.14$ Prediction equation

$= 7.93(20) + 124.14$ $x = 20$

$= 282.74$ Simplify.

The model predicts that the median price in 2014 will be about \$282,740.

(continued on the next page)

John Evans

www.MathSchoolinternational.com

d. How accurate does the prediction appear to be?

Except for the outlier, the line fits the data very well, so the predicted value should be fairly accurate.

CHECK Your Progress

1. The table shows the mean selling price of new, privately owned one-family homes for some recent years. Draw a scatter plot and a line of fit for the data. Then find a prediction equation and predict the mean price in 2014.

Year	1994	1996	1998	2000	2002	2004
Price ($1000)	154.5	166.4	181.9	207.0	228.7	273.5

Source: U.S. Census Bureau and U.S. Department of Housing and Urban Development

Online **Personal Tutor at algebra2.com**

ALGEBRA LAB

Head versus Height

COLLECT AND ORGANIZE THE DATA

Collect data from several of your classmates. Measure the circumference of each person's head and his or her height. Record the data as ordered pairs of the form (height, circumference).

ANALYZE THE DATA

1. Graph the data in a scatter plot and write a prediction equation.

2. Explain the meaning of the slope in the prediction equation.

3. Predict the head circumference of a person who is 66 inches tall.

4. Predict the height of an individual whose head circumference is 18 inches.

Study Tip

Outliers

If your scatter plot includes points that are far from the others on the graph, check your data before deciding that the point is an outlier. You may have made a graphing or recording mistake.

CHECK Your Understanding

Example (p. 87)

Complete parts a–c for each set of data in Exercises 1 and 2.

a. Draw a scatter plot and a line of fit, and describe the correlation.

b. Use two ordered pairs to write a prediction equation.

c. Use your prediction equation to predict the missing value.

1. SCIENCE The table shows the temperature in the atmosphere at various altitudes.

Altitude (ft)	0	1000	2000	3000	4000	5000
Temp (°C)	15.0	13.0	11.0	9.1	7.1	?

Source: NASA

2. TELEVISION The table shows the percentage of U.S. households with televisions that also had cable service in some recent years.

Year	1995	1997	1999	2001	2003	2015
Percent	65.7	67.3	68.0	69.2	68.0	?

Source: Nielsen Media Research

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Exercises

Complete parts a-c for each set of data in Exercises 3–6.

a. Draw a scatter plot and a line of fit, and describe the correlation.

b. Use two ordered pairs to write a prediction equation.

c. Use your prediction equation to predict the missing value.

3. SAFETY All states and the District of Columbia have enacted laws setting 21 as the minimum drinking age. The table shows the estimated cumulative number of lives these laws have saved by reducing traffic fatalities.

Year	1999	2000	2001	2002	2003	2015
Lives (1000s)	19.1	20.0	21.0	21.9	22.8	?

Source: National Highway Traffic Safety Administration

Cross-Curricular Project

A scatter plot of loan payments can help you analyze home loans. Visit algebra2.com to continue work on your project.

4. HOCKEY The table shows the number of goals and assists for some of the members of the Detroit Red Wings in a recent NHL season.

Goals	30	25	18	14	15	14	10	6	4	30	?
Assists	49	43	33	32	28	29	12	9	15	38	20

Source: www.detroitredwings.com

5. HEALTH The table shows the number of gallons of bottled water consumed per person in some recent years.

Year	1998	1999	2000	2001	2002	2003	2015
Gallons	15.0	16.4	17.4	18.8	20.7	22.0	?

Source: U.S. Department of Agriculture

6. THEATER The table shows the total revenue of all Broadway plays for recent seasons.

Season	1999-2000	2000-2001	2001-2002	2002-2003	2003-2004	2013-2014
Revenue ($ millions)	603	666	643	721	771	?

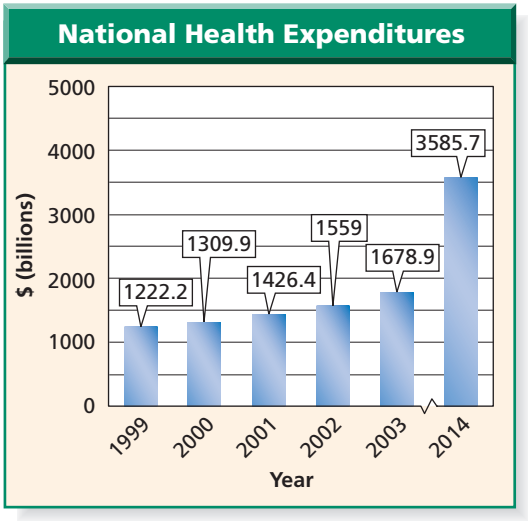
Source: The League of American Theatres and Producers, Inc.

MEDICINE For Exercises 7–9, use the graph that shows how much Americans spent on health care in some recent years and a prediction for how much they will spend in 2014.

7. Write a prediction equation from the data for 1999 to 2003.

8. Use your equation to predict the amount for 2014.

9. Compare your prediction to the one given in the graph.



Source: cms.hhs.gov

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FINANCE **For Exercises 10 and 11, use the following information.**
Della has $1000 that she wants to invest in the stock market. She is considering buying stock in either Company 1 or Company 2. The values of the stocks at the end of each of the last 4 months are shown in the tables below.

10. Based only on these data, which stock should Della buy? Explain.

11. Do you think investment decisions should be based on this type of reasoning? If not, what other factors should be considered?

Company 1	
Month	Share Price ($)
Aug.	25.13
Sept.	22.94
Oct.	24.19
Nov.	22.56

Company 2	
Month	Share Price ($)
Aug.	31.25
Sept.	32.38
Oct.	32.06
Nov.	32.44

Real-World Career
Financial Analyst

A financial analyst can advise people about how to invest their money and plan for retirement.

MathOnline
For more information, go to algebra2.com.

PLANETS **For Exercises 12–15, use the table below that shows the average distance from the Sun and average temperature for eight of the planets.**

Planet	Average Distance from the Sun (million miles)	Average Temperature (°F)
Mercury	36	333
Venus	67.2	867
Earth	93	59
Mars	141.6	−85
Jupiter	483.8	−166
Saturn	890.8	−200
Uranus	1784.8	−320
Pluto	3647.2	−375

Source: World Meteorological Association

12. Draw a scatter plot with average distance as the independent variable.

13. Write a prediction equation.

14. Predict the average temperature for Neptune, which has an average distance from the Sun of 2793.1 million miles.

15. Compare your prediction to the actual value of −330°F.

16. **RESEARCH** Use the Internet or other resource to look up the population of your community in several past years. Organize the data as ordered pairs. Then use an equation to predict the population in some future year.

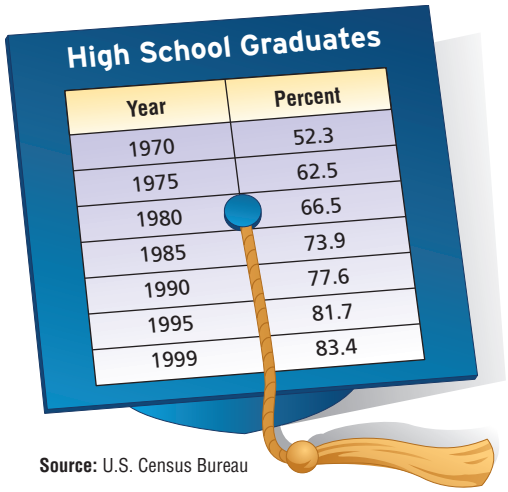
EXTRA PRACTICE
See pages 894, 927.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

CHALLENGE **For Exercises 17 and 18, use the table that shows the percent of people ages 25 and over with a high school diploma over the last few decades.**

17. Use a prediction equation to predict the percent in 2015.

18. Do you think your prediction is accurate? Explain.



Source: U.S. Census Bureau

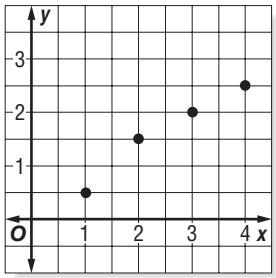
Paul Barton/CORBIS
www.MathSchoolinternational.com

19. OPEN ENDED Write a different prediction equation for the data in the example on page 87.

20. ***Writing in Math*** Use the information on page 86 to explain how a linear equation can model the number of Calories you burn while exercising. Include a scatter plot, a description of the correlation, and a prediction equation for the data. Then predict the number of Calories burned in an hour by a 140-pound person running at 9 miles per hour and compare your predicted value with the actual value of 953.

STANDARDIZED TEST PRACTICE

21. ACT/SAT Which line best fits the data in the graph?



A $y = x$

B $y = -0.5x + 4$

C $y = -0.5x - 4$

D $y = 0.5 + 0.5x$

22. REVIEW Anna took brownies to a club meeting. She gave half of her brownies to Sarah. Sarah gave a third of her brownies to Rob. Rob gave a fourth of his brownies to Trina. If Trina has 3 brownies, how many brownies did Anna have in the beginning?

F 12

G 36

H 72

J 144

Spiral Review

Write an equation in slope-intercept form that satisfies each set of conditions. (Lesson 2-4)

23. slope 4, passes through (0, 6)

24. passes through (5, −3) and (−2, 0)

TELEPHONES For Exercises 25 and 26, use the following information. (Lesson 2-4)

Namid is examining the calling card portion of his phone bill. A 4-minute call at the night rate cost $2.65. A 10-minute call at the night rate cost $4.75.

25. Write a linear equation to model this situation.

26. How much would it cost to talk for half an hour at the night rate?

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

27. (5, 4), (−3, 8)

28. (−1, −2), (4, −2)

29. (3, −4), (3, 16)

30. PROFIT Kara is planning to set up a booth at a local festival to sell her paintings. She determines that the amount of profit she will make is determined by the function $P(x) = 11x - 100$, where x is the number of paintings she sells. How much profit will Kara make if she sells 35 of her paintings? (Lesson 2-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Find each absolute value. (Lesson 1-4)

31. $|-3|$

32. $|11|$

33. $|0|$

34. $\left|-\frac{2}{3}\right|$

35. $|-1.5|$

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EXTEND 2-5

Graphing Calculator Lab

Lines of Regression

You can use a TI-83/84 Plus graphing calculator to find a function that best fits a set of data. The graph of a linear function that models a set of data is called a **regression line** or **line of best fit**. You can also use the calculator to draw scatter plots and make predictions.

Concepts in MOtion
Interactive Lab algebra2.com

ACTIVITY

INCOME **The table shows the median income of U.S. families for the period 1970–2002.**

Year	1970	1980	1985	1990	1995	1998	2000	2002
Income ($)	9867	21,023	27,735	35,353	40,611	46,737	50,732	51,680

Source: U.S. Census Bureau

Make a scatter plot of the data. Find a function and graph a regression line. Then use the function to predict the median income in 2015.

STEP 1 **Make a scatter plot.**

- Enter the years in L1 and the income in L2.

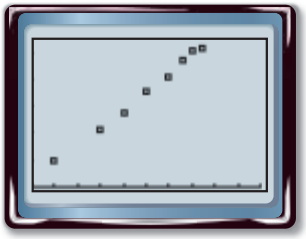
  KEYSTROKES: STAT ENTER 1970 ENTER 1980 ENTER …

- Set the viewing window to fit the data.

  KEYSTROKES: WINDOW 1965 ENTER 2015 ENTER 5 ENTER 0 ENTER 55000 ENTER 10000 ENTER

- Use STAT PLOT to graph a scatter plot.

  KEYSTROKES: 2nd [STAT PLOT] ENTER ENTER

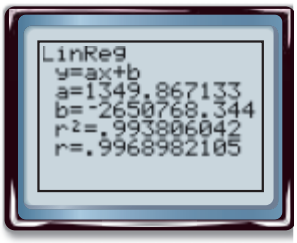


[1965, 2015] scl: 5 by [0, 55,000] scl: 10,000

STEP 2 **Find the equation of a regression line.**

- Find the regression equation by selecting LinReg(ax+ b) on the STAT CALC menu.

  KEYSTROKES: STAT ▶ 4 ENTER



The regression equation is about $y = 1349.87x - 2{,}650{,}768.34$. The slope indicates that family incomes were increasing at a rate of about $1350 per year.

The number r is called the **linear correlation coefficient.** The closer the value of r is to 1 or -1, the closer the data points are to the line. In this case, r is very close to 1 so the line fits the data well. *If the values of r^2 and r are not displayed, use DiagnosticOn from the* CATALOG *menu.*

Online **Other Calculator Keystrokes at algebra2.com**

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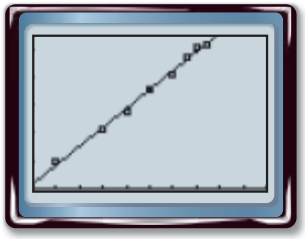
Step 3 **Graph the regression equation.**

- Copy the equation to the Y= list and graph.

KEYSTROKES: Y= VARS 5 ▶ ▶ 1 GRAPH



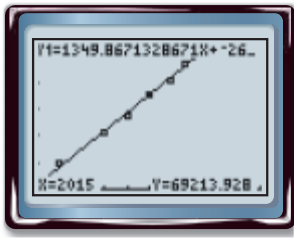
[1965, 2015] scl: 5 by [0, 55,000] scl: 10,000

The graph of the line will be displayed with the scatter plot. Notice that the regression line seems to pass through only one of the data points, but comes close to all of them. As the correlation coefficient indicated, the line fits the data very well.

Step 4 **Predict using the function.**

Find y when $x = 2015$. Use VALUE on the CALC menu.

KEYSTROKES: 2nd [CALC] 1 2015 ENTER



According to the function, the median family income in 2015 will be about \$69,214. Because the function is a very good fit to the data, the prediction should be quite accurate.

Exercises

BASEBALL **For Exercises 1–3, use the table at the right that shows the total attendance for minor league baseball in some recent years.**

1. Make a scatter plot of the data.
2. Find a regression equation for the data.
3. Predict the attendance in 2010.

Year	Attendance (millions)
1985	18.4
1990	25.2
1995	33.1
2000	37.6

Source: National Association of Professional Baseball Leagues

GOVERNMENT **For Exercises 4–6, use the table below that shows the population and the number of representatives in Congress for the most populous states.**

State	CA	TX	NY	FL	IL	PA	OH
Population (millions)	35.5	22.1	19.2	17.0	12.7	12.4	11.4
Representatives	53	32	29	25	19	19	18

Source: *World Almanac*

4. Make a scatter plot of the data.
5. Find a regression equation for the data.
6. Predict the number of representatives for South Carolina, which has a population of about 4.1 million.

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MUSIC **For Exercises 7–11, use the table at the right that shows the percent of music sales that were made in record stores in the United States for the period 1995–2004.**

7. Make a scatter plot of the data. Is the correlation of the data positive or negative? Explain.

8. Find a regression equation for the data.

9. According to the regression equation, what was the average rate of change of record store sales during the period?

10. Use the function to predict the percent of sales made in record stores in 2015.

11. How accurate do you think your prediction is? Explain.

Record Store Sales	
Year	Sales (percent)
1995	52
1996	49.9
1997	51.8
1998	50.8
1999	44.5
2000	42.4
2001	42.5
2002	36.8
2003	33.2
2004	32.5

Source: Recording Industry Association of America

RECREATION **For Exercises 12–16, use the table at the right that shows the amount of money spent on sporting footwear in some recent years.**

12. Find a regression equation for the data.

13. Use the regression equation to predict the sales in 2010.

14. Delete the outlier (1999, 12,546) from the data set and find a new regression equation for the data.

15. Use the new regression equation to predict the sales in 2010.

16. Compare the correlation coefficients for the two regression equations. Which function fits the data better? Which prediction would you expect to be more accurate?

Sporting Footwear Sales	
Year	Sales ($ millions)
1998	13,068
1999	12,546
2000	13,026
2001	13,814
2002	14,144
2003	14,446
2004	14,752

Source: National Sporting Goods Association

Extension

For Exercises 17–20, design and complete your own data analysis.

17. Write a question that could be answered by examining data. For example, you might estimate the number of students who will attend your school 5 years from now or predict the future cost of a piece of electronic equipment.

18. Collect and organize the data you need to answer the question you wrote. You may need to research your topic on the Internet or conduct a survey to collect the data you need.

19. Make a scatter plot and find a regression equation for your data. Then use the regression equation to answer the question.

20. Analyze your results. How accurate do you think your model is? Explain your reasoning.

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2-6 Special Functions

Main Ideas

- Identify and graph step, constant, and identity functions.
- Identify and graph absolute value and piecewise functions.

New Vocabulary

step function
greatest integer function
constant function
identity function
absolute value function
piecewise function

GET READY for the Lesson

The cost of the postage to mail a letter is a function of the weight of the letter. But the function is not linear. It is a special function called a **step function**.

For letters with weights between whole numbers, the cost "steps up" to the next higher cost. So the cost to mail a 1.5-ounce letter is the same as the cost to mail a 2-ounce letter, \$0.63.

Weight not over (ounces)	Price (\$)
1	0.39
2	0.63
3	0.87
4	1.11
...	...

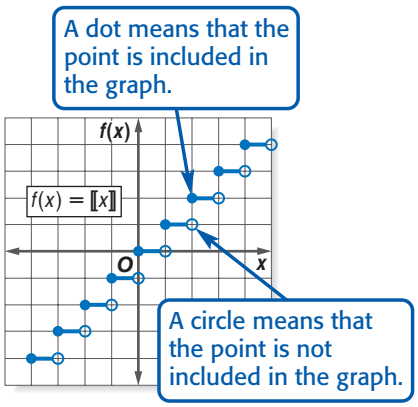
Step Functions, Constant Functions, and the Identity Function The graph of a step function is not linear. It consists of line segments or rays. The **greatest integer function**, written $f(x) = [\![x]\!]$, is an example of a step function. The symbol $[\![x]\!]$ means *the greatest integer less than or equal to x*. For example, $[\![7.3]\!] = 7$ and $[\![-1.5]\!] = -2$ because $-1 > -1.5$.

Study Tip

Greatest Integer Function

Notice that the domain of this step function is all real numbers and the range is all integers.

$f(x) = [\![x]\!]$	
x	$f(x)$
$-3 \le x < -2$	-3
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0
$1 \le x < 2$	1
$2 \le x < 3$	2
$3 \le x < 4$	3



Real-World EXAMPLE 1 Step Function

BUSINESS **The No Leak Plumbing Repair Company charges \$60 per hour or any fraction thereof for labor. Draw a graph that represents this situation.**

Explore The total labor charge must be a multiple of \$60, so the graph will be the graph of a step function.

Plan If the time spent on labor is greater than 0 hours, but less than or equal to 1 hour, then the labor cost is \$60. If the time is greater than 1 hour but less than or equal to 2 hours, then the labor cost is \$120, and so on.

(continued on the next page)

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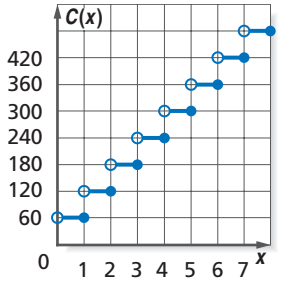
Solve Use the pattern of times and costs to make a table, where x is the number of hours of labor and $C(x)$ is the total labor cost. Then graph.

Concepts in Motion
Animation
algebra2.com

x	$C(x)$
$0 < x \leq 1$	\$60
$1 < x \leq 2$	\$120
$2 < x \leq 3$	\$180
$3 < x \leq 4$	\$240
$4 < x \leq 5$	\$300



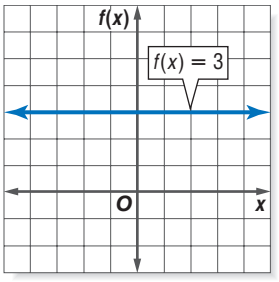
Check Since the company rounds any fraction of an hour up to the next whole number, each segment on the graph has a circle at the left endpoint and a dot at the right endpoint.

CHECK Your Progress

1. RECYCLING A recycling company pays \$5 for every full box of newspaper. They do not give any money for partial boxes. Draw a graph that shows the amount of money for the number of boxes brought to the center.

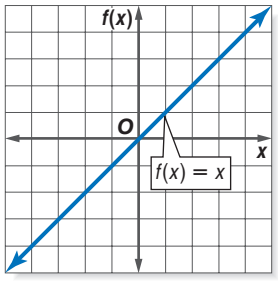
You learned in Lesson 2-4 that the slope-intercept form of a linear function is $y = mx + b$, or in function notation, $f(x) = mx + b$.

When $m = 0$, the value of the function is $f(x) = b$ for every x-value. So, $f(x) = b$ is called a **constant function**. *The function $f(x) = 0$ is called the zero function.*



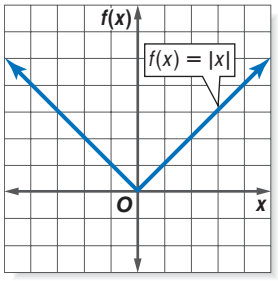
Another special case of slope-intercept form is $m = 1$, $b = 0$. This is the function $f(x) = x$. The graph is the line through the origin with slope 1.

Since the function does not change the input value, $f(x) = x$ is called the **identity function**.



Absolute Value and Piecewise Functions Another special function is the **absolute value function**, $f(x) = |x|$.

$f(x) = \lvert x \rvert$	
x	$f(x)$
−2	2
−1	1
0	0
1	1
2	2



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Study Tip

Absolute Value Function

Notice that the domain is all real numbers and the range is all nonnegative real numbers.

The absolute value function can be written as $f(x) = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$. A function that is written using two or more expressions is called a **piecewise function.**

Recall that a family of graphs displays one or more similar characteristics. The parent graph of most absolute value functions is $y = |x|$.

EXAMPLE 2 Absolute Value Functions

Graph $f(x) = |x| + 1$ and $g(x) = |x| - 2$ on the same coordinate plane. Determine the similarities and differences in the two graphs.

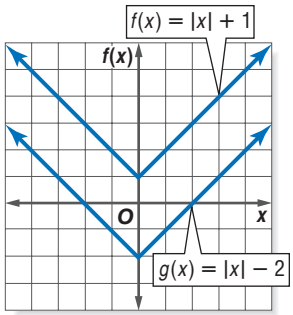
Find several ordered pairs for each function.

x	$\lvert x\rvert + 1$
-2	3
-1	2
0	1
1	2

x	$\lvert x\rvert - 2$
-2	0
-1	-1
0	-2
1	-1

Graph the points and connect them.

- The domain of each function is all real numbers.
- The range of $f(x) = |x| + 1$ is $\{y \mid y \geq 1\}$.
  The range of $g(x) = |x| - 2$ is $\{y \mid y \geq -2\}$.
- The graphs have the same shape, but different y–intercepts.
- The graph of $g(x) = |x| - 2$ is the graph of $f(x) = |x| + 1$ translated down 3 units.



CHECK Your Progress

2. Graph $f(x) = |x + 1|$ and $g(x) = |x - 2|$.

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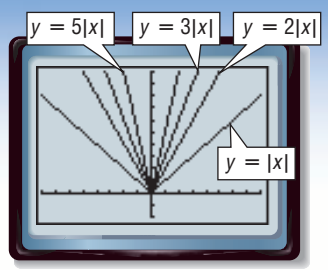
You can also use a graphing calculator to investigate families of absolute value graphs.

GRAPHING CALCULATOR LAB

Family of Absolute Value Graphs

The calculator screen shows the graphs of $y = |x|$, $y = 2|x|$, $y = 3|x|$, and $y = 5|x|$.



[−8, 8] scl: 1 by [−2, 10] scl: 1

THINK AND DISCUSS

1. What do these graphs have in common?
2. Describe how the graph of $y = a|x|$ changes as a increases. Assume $a > 0$.
3. Write an absolute value function whose graph is between the graphs of $y = 2|x|$ and $y = 3|x|$.
4. Graph $y = |x|$ and $y = -|x|$ on the same screen. Then graph $y = 2|x|$ and $y = -2|x|$ on the same screen. What is true in each case?
5. In general, what is true about the graph of $y = a|x|$ when $a < 0$?

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To graph other piecewise functions, examine the inequalities in the definition of the function to determine how much of each piece to include.

Study Tip

Graphs of Piecewise Functions

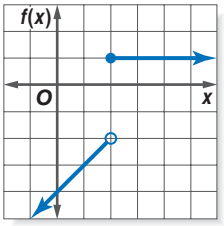
The graphs of each part of a piecewise function may or may not connect. A graph may stop at a given x value and then begin again at a different y value for the same x value.

EXAMPLE Piecewise Function

3 Graph $f(x) = \begin{cases} x - 4 \text{ if } x < 2 \\ 1 \text{ if } x \geq 2 \end{cases}$. **Identify the domain and range.**

Step 1 Graph the linear function $f(x) = x - 4$ for $x < 2$. Since 2 does not satisfy this inequality, stop with an open circle at $(2, -2)$.

Step 2 Graph the constant function $f(x) = 1$ for $x \geq 2$. Since 2 does satisfy this inequality, begin with a closed circle at $(2, 1)$ and draw a horizontal ray to the right.

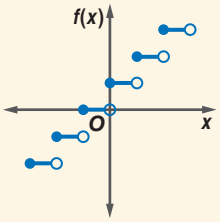
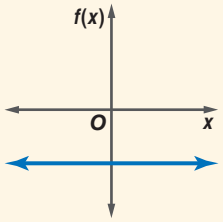
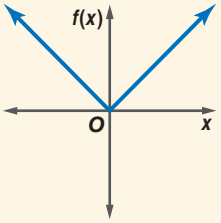
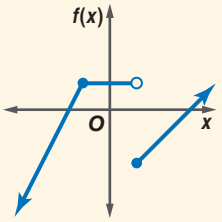


The function is defined for all values of x, so the domain is all real numbers. The values that are y-coordinates of points on the graph are 1 and all real numbers less than -2, so the range is $\{y | y < -2 \text{ or } y = 1\}$.

CHECK Your Progress

3. Graph $f(x) = \begin{cases} x + 2 \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$. Identify the domain and range.

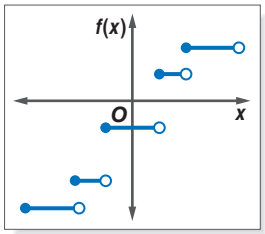
CONCEPT SUMMARY *Special Functions*

Step Function	Constant Function	Absolute Value Function	Piecewise Function
			
horizontal segments and/or rays	horizontal line	V-shape	different rays, segments, and curves

EXAMPLE Identify Functions

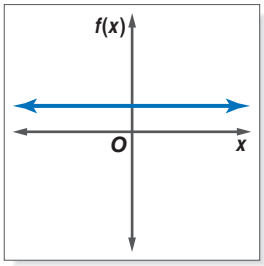
4 **Determine whether each graph represents a step function, a constant function, an absolute value function, or a piecewise function.**

a.



The graph has multiple horizontal segments. It represents a step function.

b.



The graph is a horizontal line. It represents a constant function.

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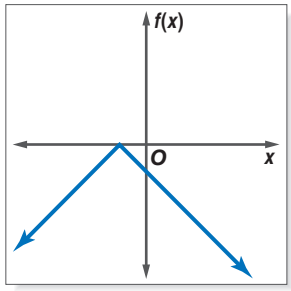
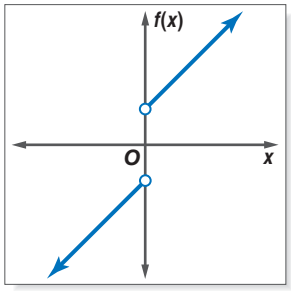
CHECK Your Progress

4A.



4B.



CHECK Your Understanding

Examples 1–3 (pp. 95–98)

Graph each function. Identify the domain and range.

1. $f(x) = -[\![x]\!]$

2. $g(x) = [\![2x]\!]$

3. $f(x) = 4$

4. $z(x) = -3$

5. $h(x) = |x| - 3$

6. $f(x) = |3x - 2|$

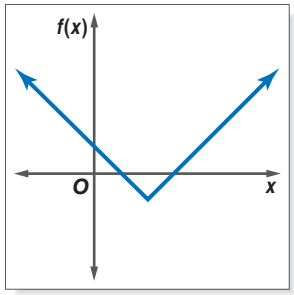
7. $g(x) = \begin{cases} -1 \text{ if } x < 0 \\ -x + 2 \text{ if } x \geq 0 \end{cases}$

8. $h(x) = \begin{cases} x + 3 \text{ if } x \leq -1 \\ 2x \text{ if } x > -1 \end{cases}$

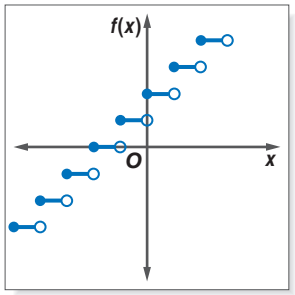
Example 4 (pp. 98–99)

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise.

9.



10.



PARKING **For Exercises 11–13, use the following information.**
A downtown parking lot charges \$2 for the first hour and \$1 for each additional hour or part of an hour.

11. What type of special function models this situation?

12. Draw a graph of a function that represents this situation.

13. Use the graph to find the cost of parking there for $4\frac{1}{2}$ hours.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–19	1
20–25	2
26–27	3
28–33	4

Graph each function. Identify the domain and range.

14. $f(x) = [\![x + 3]\!]$

15. $g(x) = [\![x - 2]\!]$

16. $f(x) = 2[\![x]\!]$

17. $h(x) = -3[\![x]\!]$

18. $g(x) = [\![x]\!] + 3$

19. $f(x) = [\![x]\!] - 1$

20. $f(x) = |2x|$

21. $h(x) = |-x|$

22. $g(x) = |x| + 3$

23. $g(x) = |x| - 4$

24. $h(x) = |x + 3|$

25. $f(x) = |x + 2|$

26. $f(x) = \begin{cases} -x \text{ if } x \leq 3 \\ 2 \text{ if } x > 3 \end{cases}$

27. $h(x) = \begin{cases} -1 \text{ if } x < -2 \\ 1 \text{ if } x > 2 \end{cases}$

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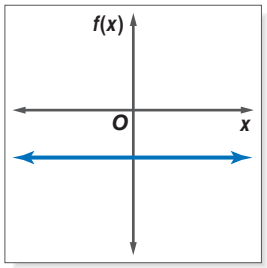
Identify each function as S for step, C for constant, A for absolute value, or P for piecewise.

28.



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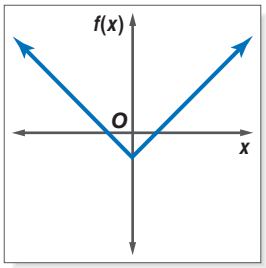
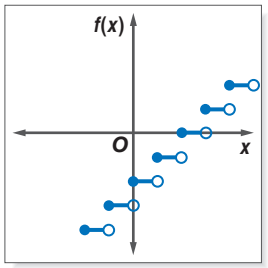
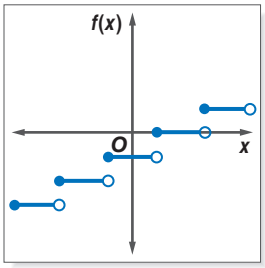


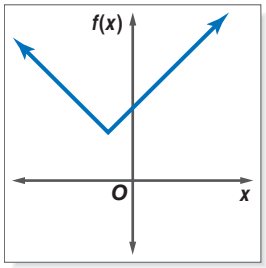
30.



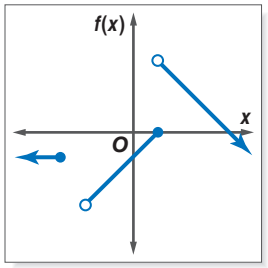
31.



32.



33.



Real-World Link

Good sources of vitamin C include citrus fruits and juices, cantaloupe, broccoli, brussels sprouts, potatoes, sweet potatoes, tomatoes, and cabbage.

Source: *The World Almanac*

34. THEATER Springfield High School's theater can hold 250 students. The drama club is performing a play in the theater. Draw a graph of a step function that shows the relationship between the number of tickets sold x and the minimum number of performances y that the drama club must do.

Graph each function. Identify the domain and range.

35. $f(x) = \left| x - \frac{1}{4} \right|$

36. $f(x) = \left| x + \frac{1}{2} \right|$

37. $f(x) = \begin{cases} x \text{ if } x < -3 \\ 2 \text{ if } -3 \le x < 1 \\ -2x + 2 \text{ if } x \ge 1 \end{cases}$

38. $g(x) = \begin{cases} -1 \text{ if } x \le -2 \\ x \text{ if } -2 < x < 2 \\ -x + 1 \text{ if } x \ge 2 \end{cases}$

39. $f(x) = [\![\, |x| \,]\!]$

40. $g(x) = |[\![x]\!]|$

TELEPHONE RATES **For Exercises 41 and 42, use the following information.**
Masao has a long-distance telephone plan where she pays 10¢ for each minute or part of a minute that she talks, regardless of the time of day.

41. Graph a step function that represents this situation.

42. How much would a call that lasts 9 minutes and 40 seconds cost?

NUTRITION **For Exercises 43–45, use the following information.**
The recommended dietary allowance for vitamin C is 2 micrograms per day.

43. Write an absolute value function for the difference between the number of micrograms of vitamin C you ate today x and the recommended amount.

44. What is an appropriate domain for the function?

45. Use the domain to graph the function.

EXTRA PRACTICE
See pages 894, 927.
MathOnline
Self-Check Quiz at algebra2.com

46. INSURANCE According to the terms of Lavon's insurance plan, he must pay the first $300 of his annual medical expenses. The insurance company pays 80% of the rest of his medical expenses. Write a function for how much the insurance company pays if x represents Lavon's annual medical expenses.

H.O.T. Problems

47. OPEN ENDED Write a function involving absolute value for which $f(-2) = 3$.

48. REASONING Find a counterexample to the statement *To find the greatest integer function of x when x is not an integer, round x to the nearest integer.*

49. CHALLENGE Graph $|x| + |y| = 3$.

David Ball/CORBIS

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50. ***Writing in Math*** Use the information on page 95 to explain how step functions apply to postage rates. Explain why a step function is the best model for this situation while your gas mileage as a function of time as you drive to the post office cannot be modeled with a step function. Then graph the function that represents the cost of a first-class letter.

STANDARDIZED TEST PRACTICE

51. ACT/SAT For which function does $f\left(-\frac{1}{2}\right) \neq -1$?

A $f(x) = 2x$

B $f(x) = |-2x|$

C $f(x) = [\![x]\!]$

D $f(x) = [\![2x]\!]$

52. ACT/SAT For which function is the range $\{y \mid y \leq 0\}$?

F $f(x) = -x$

G $f(x) = [\![x]\!]$

H $f(x) = |x|$

J $f(x) = -|x|$

53. REVIEW Solve: $5(x + 4) = x + 4$

Step 1: $5x + 20 = x + 4$

Step 2: $4x + 20 = 4$

Step 3: $4x = 24$

Step 4: $x = 6$

Which is the first *incorrrect* step in the solution shown above?

A Step 4

B Step 3

C Step 2

D Step 1

Spiral Review

HEALTH For Exercises 54–56, use the table that shows the life expectancy for people born in various years. (Lesson 2-5)

Year	1950	1960	1970	1980	1990	2000
Expectancy	68.2	69.7	70.8	73.7	75.4	77.0

Source: National Center for Health Statistics

54. Draw a scatter plot in which x is the number of years since 1940 and describe the correlation.

55. Find a prediction equation.

56. Predict the life expectancy of a person born in 2010.

Write an equation in slope-intercept form that satisfies each set of conditions. (Lesson 2-4)

57. slope 3, passes through $(-2, 4)$

58. passes through $(0, -2)$ and $(4, 2)$

Solve each inequality. Graph the solution set. (Lesson 1-3)

59. $3x - 5 \geq 4$

60. $28 - 6y < 23$

GET READY for the Next Lesson

PREREQUISITE SKILL Determine whether (0, 0) satisfies each inequality. Write *yes* or *no*. (Lesson 1-5)

61. $y < 2x + 3$

62. $y \geq -x + 1$

63. $y \leq \frac{3}{4}x - 5$

64. $2x + 6y + 3 > 0$

65. $y > |x|$

66. $|x| + y \leq 3$

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2-7 Graphing Inequalities

Main Ideas

- Graph linear inequalities.
- Graph absolute value inequalities.

New Vocabulary

boundary

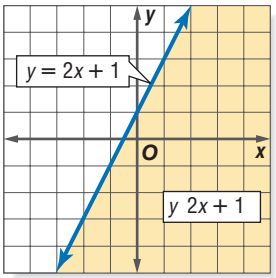
GET READY for the Lesson

Dana has Arizona Cardinals quarterback Kurt Warner as a player on his online fantasy football team. Dana gets 5 points for every yard on a completed pass and 100 points per touchdown pass that Warner makes. He considers 1000 points or more to be a good game. Dana can use a linear inequality to check whether certain combinations of yardage and touchdowns, such as those in the table, result in 1000 points or more.

	Yards	TDs
Game 1	168	3
Game 2	144	2
Game 3	136	1

Graph Linear Inequalities A linear inequality resembles a linear equation, but with an inequality symbol instead of an equals symbol. For example, $y \leq 2x + 1$ is a linear inequality and $y = 2x + 1$ is the related linear equation.

The graph of the inequality $y \leq 2x + 1$ is the shaded region. Every point in the shaded region satisfies the inequality. The graph of $y = 2x + 1$ is the **boundary** of the region. It is drawn as a solid line to show that points on the line satisfy the inequality. If the inequality symbol were $<$ or $>$, then points on the boundary would not satisfy the inequality, so the boundary would be drawn as a dashed line.



EXAMPLE 1 Dashed Boundary

Graph $2x + 3y > 6$.

The boundary is the graph of $2x + 3y = 6$. Since the inequality symbol is $>$, the boundary will be dashed.

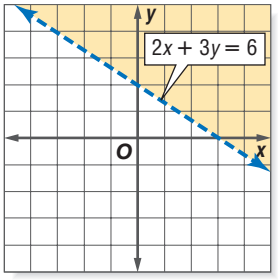
Now test the point (0, 0).

$2x + 3y > 6$ Original inequality

$2(0) + 3(0) > 6$ $(x, y) = (0, 0)$

$0 > 6$ false

Shade the region that does *not* contain (0, 0).



Study Tip

Mental Math

The point (0, 0) is usually a good point to test because it results in easy calculations that you can often perform mentally.

CHECK Your Progress

1A. Graph $3x + \frac{1}{2}y < 2$.

1B. Graph $-x + 2y > 4$.

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Real-World EXAMPLE Solid Boundary

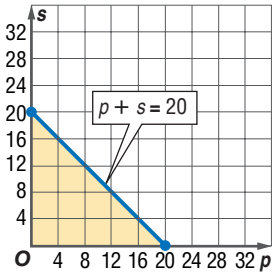
2 **BUSINESS** **A mail-order company is hiring temporary employees to help in its packing and shipping departments during their peak season.**

a. Write and graph an inequality to describe the number of employees that can be assigned to each department if the company has 20 temporary employees available.

Let p be the number of employees assigned to packing and let s be the number assigned to shipping. Since the company can assign *at most* 20 employees total to the two departments, use a $\leq$ symbol.

The employees for packing		the employees for shipping	are at most	twenty.
p	$+$	s	$\leq$	20

Since the inequality symbol is $\leq$, the graph of the related linear equation $p + s = 20$ is solid.



Test (0, 0).

$p + s \leq 20$ Original inequality

$0 + 0 \leq 20$ $(p, s) = (0, 0)$

$0 \leq 20$ true

Shade the region that contains (0, 0). *Since the variables cannot be negative, shade only the part in the first quadrant.*

b. Can the company assign 8 employees to packing and 10 to shipping?

The point (8, 10) is in the shaded region, so it satisfies the inequality. The company can assign 8 employees to packing and 10 to shipping.

Study Tip

Look Back

To review **translating verbal expressions to inequalities**, see Lesson 1-5.

CHECK Your Progress

2. Manuel has \$15 to spend at the fair. It costs \$5 for admission, \$0.75 for each ride ticket, and \$0.25 for each game ticket. Write and graph an inequality for the number of ride and game tickets that he can buy.

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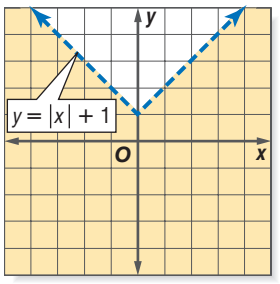
Graph Absolute Value Inequalities Graphing absolute value inequalities is similar to graphing linear inequalities.

EXAMPLE Absolute Value Inequality

3 **Graph $y < |x| + 1$.**

Since the inequality symbol is $<$, the boundary is dashed. Graph the equation. Then test (0, 0).



$y < |x| + 1$ Original inequality

$0 < |0| + 1$ $(x, y) = (0, 0)$

$0 < 0 + 1$ $|0| = 0$

$0 < 1$ true

Shade the region that includes (0, 0).

CHECK Your Progress

3. Graph $y > 2|x| - 3$.

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CHECK Your Understanding

Examples 1–3 (pp. 102–103)

Graph each inequality.

1. $y < 2$
2. $y > 2x - 3$
3. $x - y \geq 0$
4. $x - 2y \leq 5$
5. $y > |2x|$
6. $y \leq 3|x| - 1$

Example 2 (p. 103)

SHOPPING For Exercises 7–9, use the following information.
Gwen wants to buy some used CDs that cost \$10 each and some used DVDs that cost \$13 each. She has \$40 to spend.

7. Write an inequality to represent the situation, where c is the number of CDs she buys and d is the number of DVDs.
8. Graph the inequality.
9. Can she buy 2 CDs and 3 DVDs? Explain.

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–15	1
16–19, 22–26	2
20–21	3

Graph each inequality.

10. $x + y > -5$
11. $y > 6x - 2$
12. $y + 1 < 4$
13. $y - 2 < 3x$
14. $x - 6y + 3 > 0$
15. $y > \frac{1}{3}x + 5$
16. $y \geq 1$
17. $3 \geq x - 3y$
18. $x - 5 \leq y$
19. $y \geq -4x + 3$
20. $y \leq |x|$
21. $y > |4x|$

COLLEGE For Exercises 22 and 23, use the following information.
Rosa's professor says that the midterm exam will count for 40% of each student's grade and the final exam will count for 60%. A score of at least 90 is required for an A.

22. The inequality $0.4x + 0.6y \geq 90$ represents this situation, where x is the midterm score and y is the final exam score. Graph this inequality.
23. Refer to the graph. If she scores 85 on the midterm and 95 on the final, will Rosa get an A?

FINANCE For Exercises 24–26, use the following information.
Carl Talbert estimates that he will need to earn at least \$9000 per year combined in dividend income from the two stocks he owns to supplement his retirement plan.

Company	Dividend per Share
Able Records	\$1.20
Best Bakes	\$1.30

24. Write an inequality to represent this situation.
25. Graph the inequality.
26. Will he make enough from 3000 shares of each company?

27. Graph all the points on the coordinate plane to the left of the graph of $x = -2$. Write an inequality to describe these points.
28. Graph all the points on the coordinate plane below the graph of $y = 3x - 5$. Write an inequality to describe these points.

Real-World Link
A dividend is a payment from a company to an investor. It is a way to make money on a stock without selling it.

Graph each inequality.

29. $4x - 5y - 10 \leq 0$
30. $y \geq \frac{1}{2}x - 5$
31. $y + |x| < 3$
32. $y \geq |x - 1| - 2$
33. $|x + y| > 1$
34. $|x| \leq |y|$

Ken Reid/Cobalt Pictures
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Graphing Calculator

SHADE(COMMAND You can graph inequalities by using the SHADE(command located in the DRAW menu. Enter two functions.

- The first function defines the lower boundary of the shaded region. If the inequality is "$y \leq$," use the Ymin window value as the lower boundary.
- The second function defines the upper boundary of the region. If the inequality is "$y \geq$," use the Ymax window value as the upper boundary.

Graph each inequality.

35. $y \geq 3$ **36.** $y \leq x + 2$ **37.** $y \leq -2x - 4$ **38.** $x - 7 \leq y$

H.O.T. Problems

39. REASONING Explain how to determine which region to shade when graphing an inequality.

40. CHALLENGE Graph $|y| < x$.

41. *Writing in Math* Use the information on page 102 to write an inequality that defines a good game for Kurt Warner in Dana's fantasy football league, and explain how you obtained it.

EXTRA PRACTICE
See pages 895, 927.

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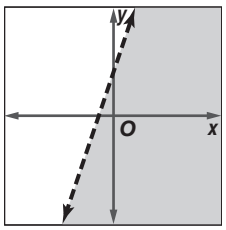
STANDARDIZED TEST PRACTICE

42. ACT/SAT Which could be the inequality for the graph?



A $y < 3x + 2$

B $y \leq 3x + 2$

C $y > 3x + 2$

D $y \geq 3x + 2$

43. REVIEW What is the solution set of the inequality?

$$6 - |x + 7| \leq -2$$

F $-15 \leq x + \leq 1$

G $-1 \leq x \leq 3$

H $x \leq -1$ or $x \geq 3$

J $x \leq -15$ or $x \geq 1$

Spiral Review

Graph each function. Identify the domain and range. (Lesson 2-6)

44. $f(x) = [\![x]\!] - 4$ **45.** $g(x) = |x| - 1$ **46.** $h(x) = |x - 3|$

SALARY For Exercises 47–49, use the table which shows the years of experience for eight computer programmers and their yearly salary. (Lesson 2-5)

Years	6	5	3	1	4	3	6	2
Salary ($)	55,000	53,000	45,000	42,000	48,500	46,500	53,000	43,000

47. Draw a scatter plot and describe the correlation.

48. Find a prediction equation.

49. Predict the salary for a representative with 9 years of experience.

Solve each equation. Check your solution. (Lesson 1-3)

50. $4x - 9 = 23$ **51.** $11 - 2y = 5$ **52.** $2z - 3 = -6z + 1$

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CHAPTER 2

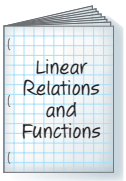
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Relations and Functions (Lesson 2-1)

- A relation is a set of ordered pairs. The domain is the set of all x-coordinates, and the range is the set of all y-coordinates.
- A function is a relation where each member of the domain is paired with exactly one member of the range.

Linear Equations and Slope (Lessons 2-2 to 2-4)

- A linear equation is an equation whose graph is a line.
- Slope is the ratio of the change in y-coordinates to the corresponding change in x-coordinates.
- Lines with the same slope are parallel. Lines with slopes that are opposite reciprocals are perpendicular.
- Standard Form: $Ax + By = C$, where A, B, and C are integers whose greatest common factor is 1, $A \geq 0$, and A and B are not both zero
- Slope-Intercept Form: $y = mx + b$
- Point-Slope Form: $y - y_1 = m(x - x_1)$

Using Scatter Plots (Lesson 2-5)

- A prediction equation can be used to predict the value of one of the variables given the value of the other variable.

Graphing Inequalities (Lesson 2-7)

- You can graph an inequality by following these steps.

  Step 1 Determine whether the boundary is solid or dashed. Graph the boundary.

  Step 2 Choose a point not on the boundary and test it in the inequality.

  Step 3 If a true inequality results, shade the region containing your test point. If a false inequality results, shade the other region.

Key Vocabulary

absolute value function (p. 96)
boundary (p. 102)
constant function (p. 96)
continuous function (p. 65)
coordinate plane (p. 58)
dependent variable (p. 61)
discrete function (p. 65)
domain (p. 58)
family of graphs (p. 73)
function (p. 58)
function notation (p. 61)
greatest integer function (p. 95)
identity function (p. 96)
independent variable (p. 61)
linear equation (p. 66)
linear function (p. 66)
line of fit (p. 86)
mapping (p. 58)
negative correlation (p. 86)
no correlation (p. 86)
one-to-one function (p. 58)
ordered pair (p. 58)
parent graph (p. 73)
piecewise function (p. 97)
point-slope form (p. 80)
positive correlation (p. 86)
prediction equation (p. 86)
quadrant (p. 58)
range (p. 58)
rate of change (p. 71)
relation (p. 58)
scatter plot (p. 86)
slope (p. 71)
slope-intercept form (p. 79)
standard form (p. 67)
step function (p. 95)
vertical line test (p. 59)
x-intercept (p. 68)

Vocabulary Check

Choose the correct term to complete each sentence.

1. The (constant, identity) function is a linear function described by $f(x) = x$.
2. The graph of the (absolute value, greatest integer) function forms a V-shape.
3. The (slope-intercept, standard) form of the equation of a line is $y = mx + b$.
4. Two lines in the same plane having the same slope are (parallel, perpendicular).
5. The (line of fit, vertical line test) can be used to determine if a relation is a function.
6. The (domain, range) of a relation is the set of all first coordinates from the ordered pairs which determine the relation.

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Lesson-by-Lesson Review

2–1 Relations and Functions (pp. 58–64)

Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. Is the relation *discrete* or *continuous*?

7. {(6, 3), (2, 1), (−2, 3)}

8. {(−5, 2), (2, 4), (1, 1), (−5, −2)}

9. $y = 0.5x$ **10.** $y = 2x + 1$

Find each value if $f(x) = 5x - 9$.

11. $f(6)$ **12.** $f(-2)$

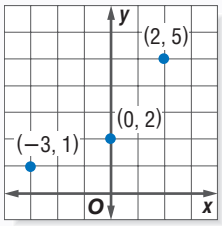
13. $f(y)$ **14.** $f(-2v)$

15. TAXI RIDE A taxi company charges $2.80 for the first mile and $1.60 for each additional mile. The amount a passenger will be charged can be expressed as $f(x) = 1.20 + 1.60x$, when $x \geq 1$. Graph this equation and find the domain and range. Then determine whether the equation is a function. Is the equation *discrete* or *continuous*?

Example 1 **Graph the relation {(−3, 1), (0, 2), (2, 5)} and find the domain and range. Then determine whether the relation is a function. Is the relation *discrete* or *continuous*?**

The domain is {−3, 0, 2}, and the range is {1, 2, 5}.

Since each x-value is paired with exactly one y-value, the relation is a function. The relation is discrete because the points are not connected.



2–2 Linear Equations (pp. 66–70)

State whether each equation or function is linear. Write *yes* or *no*. If no, explain your reasoning.

16. $2x + y = 11$ **17.** $h(x) = \sqrt{2x + 1}$

Write each equation in standard form. Identify *A*, *B*, and *C*.

18. $\frac{2}{3}x - \frac{3}{4}y = 6$ **19.** $0.5x = -0.2y - 0.4$

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation.

20. $-\frac{1}{5}y = x + 4$ **21.** $6x = -12y + 48$

22. CUBES Julián thinks that the equation for the volume of a cube, $V = s^3$, is a linear equation. Is he correct? Explain.

Example 2 **Write $2x - 6 = y + 8$ in standard form. Identify *A*, *B*, and *C*.**

$2x - 6 = y + 8$	Original equation
$2x - y - 6 = 8$	Subtract y from each side.
$2x - y = 14$	Add 6 to each side.

The standard form is $2x - y = 14$. So, $A = 2$, $B = -1$, and $C = 14$.

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2–3 Slope (pp. 71–77)

Find the slope of the line that passes through each pair of points.

23. $(-6, -3), (6, 7)$ **24.** $(5.5, -5.5), (11, -7)$

Graph the line passing through the given point with the given slope.

25. $(0, 1), m = 2$ **26.** $(-5, 2), m = -\frac{1}{4}$

Graph the line that satisfies each set of conditions.

27. passes through $(-1, -2)$, perpendicular to a line whose slope is $\frac{1}{2}$

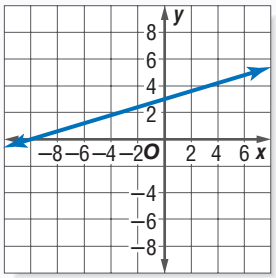
28. passes through $(-1, 2)$, parallel to the graph of $x - 3y = 14$

29. RAMPS Jack measures his bicycle ramp and finds that it is 5 feet long and 3 feet high. What is the slope of his ramp?

Example 3 **Graph the line passing through (3, 4) with slope $m = \frac{1}{3}$.**

Graph the ordered pair (3, 4). Then, according to the slope, go up 1 unit and right 3 units. Plot the new point at (6, 5). You can also go right 3 units and then up 1 unit to plot the new point.

Draw the line containing the points.



2–4 Writing Linear Equations (pp. 79–84)

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

30. slope $\frac{3}{4}$, passes through $(-6, 9)$

31. passes through $(-1, 2)$, parallel to the graph of $x - 3y = 14$

32. passes through $(3, -8)$ and $(-3, 2)$

33. passes through $(3, 2)$, perpendicular to the graph of $4x - 3y = 12$

34. LANDSCAPING Mr. Ryan is planning to plant rows of roses in a garden he is designing for a client. Before planting, he sketches out his plans on a coordinate grid. A row of white roses will be planted along the line with equation $y = 2x + 1$. A row of red roses will be parallel to the white roses and pass through the point (3, 5). What equation would represent the line for the row of red roses?

Example 4 **Write an equation in slope-intercept form for the line through (4, 5) that is parallel to the line through (−1, −3) and (2, −1).**

First, find the slope of the given line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - (-3)}{2 - (-1)}$$
$$= \frac{2}{3}$$

The parallel line will also have a slope of $\frac{2}{3}$.

$$y - y_1 = m(x - x_1)$$
$$y - 5 = \frac{2}{3}(x - 4)$$
$$y = \frac{2}{3}x + \frac{7}{3}$$

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2–5 Statistics: Using Scatter Plots (pp. 86–91)

HEALTH INSURANCE For Exercises 35 and 36 use the table that shows the number of people covered by private or government health insurance in the United States.

Year	People (millions)
1988	211
1992	218
1996	225
2000	240
2004	245

Source: U.S. Census

35. Draw a scatter plot and describe the correlation.

36. Use two ordered pairs to write a prediction equation. Then use your prediction equation to predict the number of people with health insurance in 2010.

GOLD PRODUCTION For Exercises 37 and 38, use the table that shows the number of ounces of gold produced in the United States for several years.

Year	Troy ounces (millions)
1998	11.8
1999	11.0
2000	11.3
2001	10.8
2002	9.6
2003	8.9

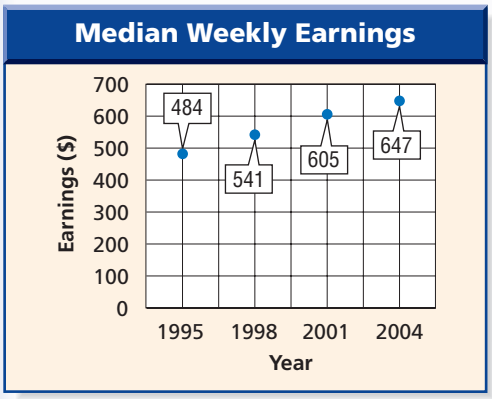
Source: *World Almanac*

37. Draw a scatter plot and describe the correlation.

38. Use two ordered pairs to write a prediction equation. Then use your prediction equation to predict the number of ounces of gold that will be produced in 2010.

Example 5 WEEKLY PAY The table below shows the median weekly earnings for American workers for the period 1985–1999. Predict the median weekly earnings for 2010.

Year	1995	1998	2001	2004	2010
Earnings ($)	484	541	605	647	?



Source: U.S. Bureau of Labor Statistics

Use (1995, 484) and (2004, 647) to find a prediction equation.

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula

$= \frac{647 - 484}{2004 - 1995}$ $(x_1, y_1) = (1995, 484)$, $(x_2, y_2) = (2004, 647)$,

$= \frac{163}{9}$ or about 18.1 Simplify.

$y - y_1 = m(x - x_1)$ Point-slope form

$y - 484 = 18.1(x - 1995)$ Substitute.

$y - 484 = 18.1x - 36{,}109.5$ Multiply.

$y = 18.1x - 35{,}625.5$ Add 484 to each side.

To predict earnings for 2010, substitute 2010 for x.

$y = 18.1(2010) - 35{,}625.5$ $x = 2010$

$= 755.5$ Simplify.

The model predicts median weekly earnings of $755.50 in 2010.

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2–6 Special Functions (pp. 95–101)

Graph each function. Identify the domain and range.

39. $f(x) = [\![x]\!] - 2$

40. $h(x) = [\![2x - 1]\!]$

41. $g(x) = |x| + 4$

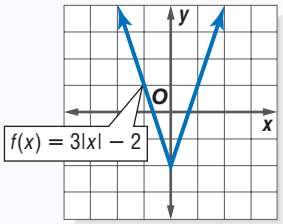
42. $h(x) = |x - 1| - 7$

43. $f(x) = \begin{cases} 2 \text{ if } x < -1 \\ -x - 1 \text{ if } x \geq -1 \end{cases}$

44. $g(x) = \begin{cases} -2x - 3 \text{ if } x < 1 \\ x - 4 \text{ if } x > 1 \end{cases}$

45. WIRELESS INTERNET A wireless Internet provider charges $40 a month plus an additional 30 cents a minute or any fraction thereof. Draw a graph that represents this situation.

Example 6 **Graph the function $f(x) = 3|x| - 2$. Identify the domain and range.**



The domain is all real numbers. The range is all real numbers greater than or equal to −2.

2–7 Graphing Inequalities (pp. 102–105)

Graph each inequality.

46. $y \leq 3x - 5$

47. $x > y - 1$

48. $y + 0.5x < 4$

49. $2x + y \geq 3$

50. $y \geq |x| + 2$

51. $y > |x - 3|$

52. BASEBALL The Cincinnati Reds must score more runs than their opponent to win a game. Write an inequality to represent this situation. Graph the inequality.

Example 7 **Graph $x + 4y \leq 4$.**

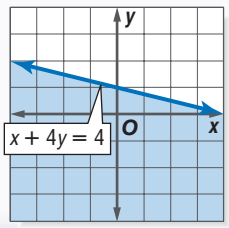
Since the inequality symbol is $\leq$, the graph of the boundary should be solid. Graph the equation.

Test (0, 0).

$x + 4y \leq 4$ Original inequality

$0 + 4(0) \leq 4$ $(x, y) = (0, 0)$

$0 \leq 4$ Shade the region that contains (0, 0).



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CHAPTER 2

Practice Test

Graph each relation and find the domain and range. Then determine whether the relation is a function.

1. $\{(-4, -8), (-2, 2), (0, 5), (2, 3), (4, -9)\}$
2. $y = 3x - 3$

Find each value.

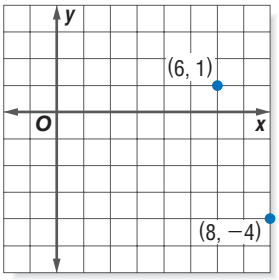
3. $f(3)$ if $f(x) = 7 - x^2$
4. $f(0)$ if $f(x) = x - 3x^2$

Graph each equation or inequality.

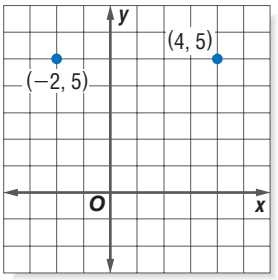
5. $y = \frac{3}{5}x - 4$
6. $4x - y = 2$
7. $x = -4$
8. $y = 2x - 5$
9. $f(x) = 3x - 1$
10. $f(x) = [\![3x]\!] + 3$
11. $g(x) = |x + 2|$
12. $y \leq 10$
13. $-2x + 5 \leq 3y$
14. $y < 4|x - 1|$
15. $h(x) = \begin{cases} x + 2 & \text{if } x < -2 \\ 2x - 1 & \text{if } x \geq -2 \end{cases}$

Find the slope of the line that passes through each pair of points.

16.



17.



18. (5, 7), (4, −6)
19. (1, 0), (3, 8)

Graph the line passing through the given point with the given slope.

20. (1, −3), 2
21. (−2, 2), $-\frac{1}{3}$
22. (3, −2), undefined

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

23. slope −5, y-intercept 11
24. x-intercept 9, y-intercept −4
25. passes through (−6, 15), parallel to the graph of $2x + 3y = 1$
26. passes through (5, 2), perpendicular to the graph of $x + 3y = 7$

RECREATION **For Exercises 27–29, use the table that shows the amount Americans spent on admission to spectator amusements in some recent years.**

Year	Amount (billion $)
2000	30.4
2001	32.2
2002	34.6
2003	35.6

Source: Bureau of Economic Analysis, U.S. Dept. of Commerce

27. Draw a scatter plot. Let x represents the number of years since 2000.
28. Write a prediction equation.
29. Predict the amount that will be spent on recreation in 2015.

30. **MULTIPLE CHOICE** What is the slope of a line parallel to $y - 2 = 4(x + 1)$?

A -4

B $-\frac{1}{4}$

C $\frac{1}{4}$

D 4

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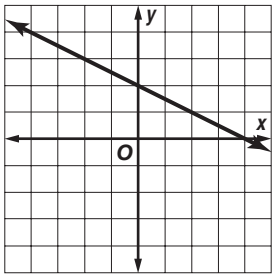
CHAPTER 2

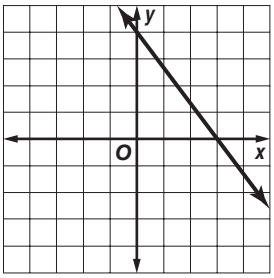
Standardized Test Practice

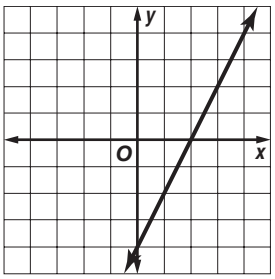
Cumulative, Chapters 1–2

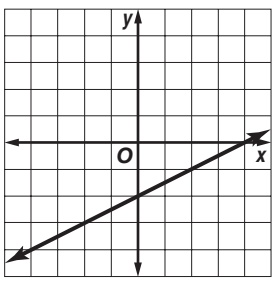
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Which graph best represents a line parallel to the line with equation $y = -\frac{4}{3}x + 1$?

A 

B 

C 

D 

2. GRIDDABLE Miranda traveled half of her trip by train. She then traveled one fourth of the rest of the distance by bus. She rented a car and drove the remaining 120 miles. How many miles away was her destination?

3. Rich's Pet Store sells cat food. The cost of two 5-pound bags is \$7.99. The total cost c of purchasing n bags can be found by—

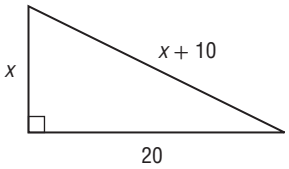
F multiplying n by c.

G multiplying n by 5.

H multiplying n by the cost of 1 bag.

J dividing n by c.

4. GRIDDABLE What is the value of x in the drawing below?



5. Peyton works as a nanny. She charges at least \$10 to drive to a home and \$10.50 an hour. Which best represents the relationship between the number of hours working n and the total charge c?

A $c \geq 10 + 10.50n$

B $c \geq 10.50 + 10n$

C $c \leq 10.50 + 10$

D $c \leq 10n + 10.50n$

TEST-TAKING TIP

Question 5 Watch for the phrases "at least" or "at most." Think logically about the conditions that make a value less than or greater than another variable. Notice what types of numbers are used—positive, even, prime, or integers.

6. Given the function $y = 2.24x + 16.45$, which statement best describes the effect of decreasing the y-intercept by 20.25?

F The x-intercept increases.

G The y-intercept increases.

H The new line has a greater rate of change.

J The new line is perpendicular to the original.

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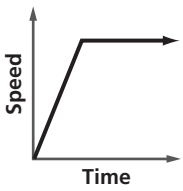
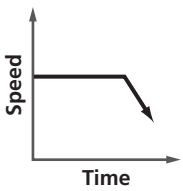
7. Stephen walks at a steady pace from his house. He then walks up a hill at a slower pace. Which graph best represents this situation?

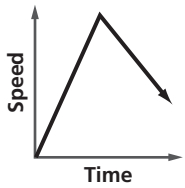
A



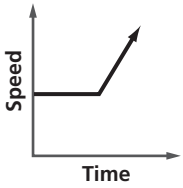
C



B



D



8. Use the table to determine the expression that best represents the sum of the degree measures of the interior angles of a polygon with n sides.

Polygon	Number of Sides	Sum of Measures
Triangle	3	180
Quadrilateral	4	360
Pentagon	5	540
Hexagon	6	720
Heptagon	7	900
Octagon	8	1080

F $180 + n$
G $180n$
H $180(n - 2)$
J $60n$

9. What are the coordinates of the x-intercept of the equation $2y = 4x + 3$?

A $\left(-\frac{1}{4}, 0\right)$
B $\left(-\frac{3}{4}, 0\right)$
C $\left(0, \frac{3}{2}\right)$
D $\left(0, \frac{7}{2}\right)$

10. Which two 3-dimensional figures have the same number of vertices?

F pentagonal prism and a rectangular pyramid.
G triangular prism and a pentagonal pyramid
H rectangular prism and a square pyramid
J triangular prism and a rectangular prism

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. The amount that a certain online retailer charges for shipping an electronics purchase is determined by the weight of the package. The charges for several different weights are given in the table.

Electronics Shipping Charges	
Weight (lb)	Shipping ($)
1	5.58
3	6.76
4	7.35
7	9.12
10	10.89
13	12.66
15	13.84

a. Write a relation to represent the data. Use weight as the independent variable and the shipping charges as the dependent variable.
b. Graph the relation on a coordinate plane.
c. Find the rate of change of the shipping charge per pound.
d. Write an equation that could be used to find the shipping charge y for a package that weights x pounds.
e. Find the shipping charge for a package that weighs 19 pounds.

NEED EXTRA HELP?											
If You Missed Question...	1	3	4	5	6	7	8	9	2	10	11
Go to Lesson or Page...	2-4	2-4	1-4	2-4	2-2	2-1	2-4	2-4	2-4	879	2-4

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CHAPTER 3

Systems of Equations and Inequalities

BIG Ideas

- Solve systems of linear equations in two or three variables.
- Solve systems of inequalities.
- Use linear programming to find minimum and maximum values of functions.

Key Vocabulary

elimination method (p. 125)
linear programming (p. 140)
ordered triple (p. 146)
system of equations (p. 116)

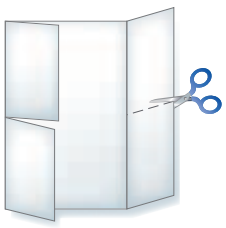
Real-World Link

Attendance Figures Nearly three hundred thousand people attend the annual Missouri State Fair in Sedalia. A system of equations can be used to determine how many children and how many adults attend if the total number of tickets sold and the income from the ticket sales are known.

FOLDABLES Study Organizer

Systems of Equations and Inequalities Make this Foldable to record information about systems of linear equations and inequalities. Begin with one sheet of 11" × 17" paper and four sheets of grid paper.

1 **Fold** the short sides of the 11" × 17" paper to meet in the middle. Cut each tab in half as shown.



2 **Cut** 4 sheets of grid paper in half and fold the half-sheets in half. Insert two folded half-sheets under each of the four tabs and staple along the fold. Label each tab as shown.

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GET READY for Chapter 3

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Graph each equation. (Lesson 2-1)

1. $2y = x$
2. $y = x - 4$
3. $y = 2x - 3$
4. $x + 3y = 6$
5. $2x + 3y = -12$
6. $4y - 5x = 10$

FUND-RAISING For Exercises 7–10, use the following information.

The Jackson Band Boosters sell beverages for $1.75 and candy for $1.50 at home games. Their goal is to have total sales of $525 for each game. (Lesson 2-3)

7. Write an equation that is a model for the different numbers of beverages and candy that can be sold to meet the goal.
8. Graph the equation.
9. Does this equation represent a function? Explain.
10. If they sell 100 beverages and 200 pieces of candy, will the Band Boosters meet their goal?

Graph each inequality. (Lesson 2-7)

11. $y \geq -2$
12. $x + y \leq 0$
13. $y < 2x - 2$
14. $x + 4y < 3$
15. $2x - y \geq 6$
16. $3x - 4y < 10$
17. **DRAMA** Tickets for the spring play cost $4 for adults and $3 for students. The club must make $2000 to cover expenses. Write and graph an inequality that describes this situation. (Lesson 2-7)

QUICKReview

EXAMPLE 1 Graph $3y - 15x = -15$.

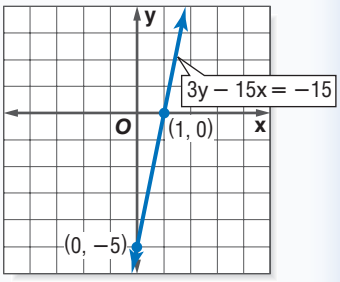
Find the x- and y-intercepts.

$3(0) - 15x = -15$ $\quad$ $3y - 15(0) = -15$

$-15x = -15$ $\quad$ $3y = -15$

$x = 1$ $\quad$ $y = -5$

The graph crosses the x-axis at (1, 0) and the y-axis at (0, −5). Use these ordered pairs to graph the equation.

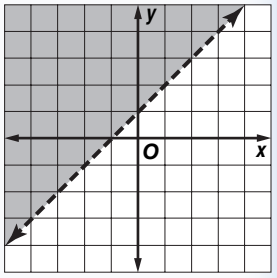


EXAMPLE 2 Graph $y > x + 1$.

The boundary is the graph of $y = x + 1$. Since the inequality symbol is $>$, the boundary will be dashed.
Test the point (0, 0).

$0 > 0 + 1$ $\quad$ $(x, y) = (0, 0)$

$0 > 1$ $\quad$ false



Shade the region that does *not* contain (0, 0).

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3-1 Solving Systems of Equations by Graphing

Main Ideas

- Solve systems of linear equations by graphing.
- Determine whether a system of linear equations is consistent and independent, consistent and dependent, or inconsistent.

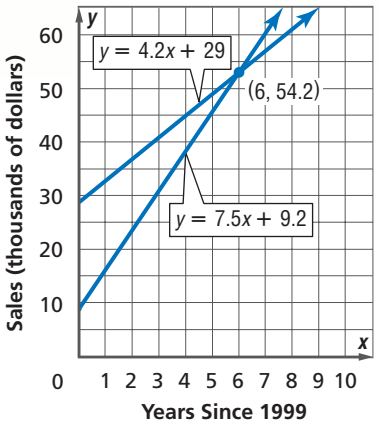
New Vocabulary

system of equations
consistent
inconsistent
independent
dependent

GET READY for the Lesson

Since 1999, the growth of in-store sales for Custom Creations can be modeled by $y = 4.2x + 29$. The growth of their online sales can be modeled by $y = 7.5x + 9.2$. In these equations, x represents the number of years since 1999, and y represents the amount of sales in thousands of dollars.

The equations $y = 4.2x + 29$ and $y = 7.5x + 9.2$ are called a system of equations.



Solve Systems Using Tables and Graphs A **system of equations** is two or more equations with the same variables. To solve a system of equations, find the ordered pair that satisfies all of the equations.

EXAMPLE 1 Solve the System of Equations by Completing a Table

Solve the system of equations by completing a table.

$-2x + 2y = 4$

$-4x + y = -1$

Write each equation in slope-intercept form.

$-2x + 2y = 4 \rightarrow y = x + 2$

$-4x + y = -1 \rightarrow y = 4x - 1$

Use a table to find the solution that satisfies both equations.

x	$y_1 = x + 2$	y_1	$y_2 = 4x - 1$	y_2	(x, y_1)	(x, y_2)
-1	$y_1 = (-1) + 2$	1	$y_2 = 4(-1) - 1$	-5	$(-1, 1)$	$(-1, -5)$
0	$y_1 = 0 + 2$	2	$y_2 = 4(0) - 1$	-1	$(0, 2)$	$(0, -1)$
1	$y_1 = (1) + 2$	3	$y_2 = 4(1) - 1$	3	$(1, 3)$	$(1, 3)$

The solution of the system is $(1, 3)$.

The solution of the system of equations is the ordered pair that satisfies both equations.

CHECK Your Progress

1A. $-3x + y = 4$
$2x + y = -6$

1B. $2x + 3y = 4$
$5x + 6y = 5$

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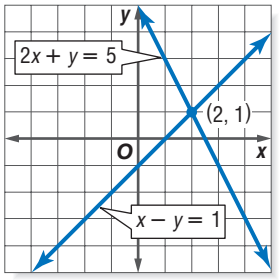
Another way to solve a system of equations is to graph the equations on the same coordinate plane. The point of intersection represents the solution.

EXAMPLE Solve by Graphing

2 **Solve the system of equations by graphing.**

$2x + y = 5$

$x - y = 1$



Write each equation in slope-intercept form.

$2x + y = 5 \rightarrow y = -2x + 5$

$x - y = 1 \rightarrow y = x - 1$

The graphs appear to intersect at (2, 1).

CHECK Substitute the coordinates into each equation.

$2x + y = 5$	$x - y = 1$	Original equations
$2(2) + 1 \stackrel{?}{=} 5$	$2 - 1 \stackrel{?}{=} 1$	Replace x with 2 and y with 1.
$5 = 5$ ✓	$1 = 1$ ✓	Simplify.

The solution of the system is (2, 1).

Study Tip

Checking Solutions

When using a graph to find a solution, always check the ordered pair in *both* original equations.

CHECK Your Progress

2A. $4x + \frac{1}{3}y = 8$
$3x + y = 6$

2B. $5x + 4y = 7$
$-x - 4y = -3$

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Systems of equations are used in businesses to determine the *break-even point*. The break-even point is the point at which the income equals the cost.

Real-World Link

Compact discs (CDs) store music digitally. The recorded sound is converted to a series of 1s and 0s. This coded pattern can then be read by an infrared laser in a CD player.

Real-World EXAMPLE Break-Even Point Analysis

3 **MUSIC** **The initial cost for Travis and his band to record their first CD was $1500. Each CD will cost $4 to produce. If they sell their CDs for $10 each, how many must they sell before they make a profit?**

Let x = the number of CDs and let y = the number of dollars.

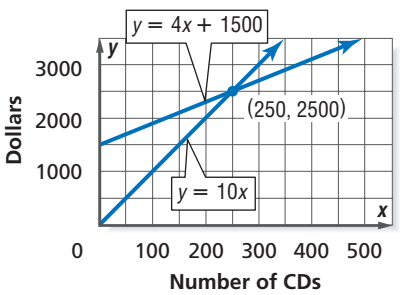
Costs of CDs is cost per CD plus start-up cost.

$y = 4x + 1500$

Income for CDs is price per CD times number sold.

$y = 10 \cdot x$

The graphs intersect at (250, 2500). This is the break-even point. If the band sells fewer than 250 CDs, they will lose money. If the band sells more than 250 CDs, they will make a profit.



Dave Starrett/Masterfile
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CHECK Your Progress

3A. RUNNING Curtis will run 4 miles the first week of training and increase the mileage by one mile each week. With another schedule, Curtis will run 1 mile the first week and increase his total mileage by 2 miles each week. During what week do the two schedules break even? How many miles will Curtis run during this week?

Study Tip

Graphs of Linear Systems

Graphs of systems of linear equations may be intersecting lines, parallel lines, or the same line.

Classify Systems of Equations A system of equations is **consistent** if it has at least one solution and **inconsistent** if it has no solutions. A consistent system is **independent** if it has exactly one solution or **dependent** if it has an infinite number of solutions.

EXAMPLE Intersecting Lines

4 **Graph the system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*.**

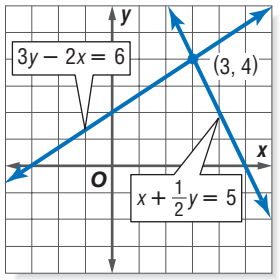
$x + \frac{1}{2}y = 5$

$3y - 2x = 6$

Write each equation in slope-intercept form.

$x + \frac{1}{2}y = 5 \rightarrow y = -2x + 10$

$3y - 2x = 6 \rightarrow y = \frac{2}{3}x + 2$



The graphs intersect at (3, 4). Since there is one solution, this system is *consistent and independent*.

CHECK Your Progress

4A. $2x - y = 5$
$x + 3y = 6$

4B. $2x - y = 5$
$y + \frac{1}{2x} = 5$

The graph of a system of linear equations that is consistent and dependent is one line.

EXAMPLE Same Line

5 **Graph the system of equations and describe it as *consistent* and *independent*, *consistent and dependent*, or *inconsistent*.**

$9x - 6y = 24$

$6x - 4y = 16$

Write each equation in slope-intercept form.

$9x - 6y = 24 \rightarrow y = \frac{3}{2}x - 4$

$6x - 4y = 16 \rightarrow y = \frac{3}{2}x - 4$

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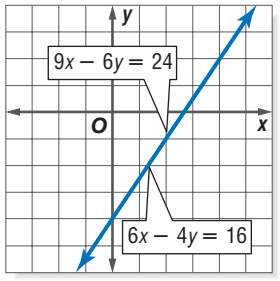
Since the equations are equivalent, their graphs are the same line. Any ordered pair representing a point on that line will satisfy both equations.

So, there are infinitely many solutions to this system. It is *consistent and dependent*.



CHECK Your Progress

5A. $5x - 3y = -2$
$4x + 2y = 5$

5B. $4x + 2y = 5$
$2x + y = \frac{5}{2}$

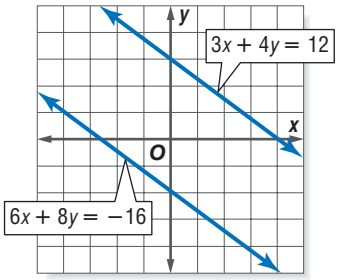
EXAMPLE 6 Parallel Lines

Graph the system of equations and describe it as *consistent and independent, consistent and dependent*, or *inconsistent*.

$3x + 4y = 12$
$6x + 8y = -16$

$3x + 4y = 12 \rightarrow y = -\frac{3}{4}x + 3$

$6x + 8y = -16 \rightarrow y = -\frac{3}{4}x - 2$



The lines do not intersect. Their graphs are parallel lines. So, there are no solutions that satisfy both equations. This system is *inconsistent*.

Study Tip

Parallel Lines

Notice from their equations that the lines have the same slope and different *y*-intercepts.

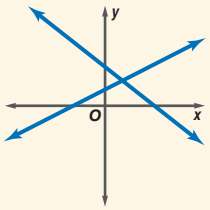
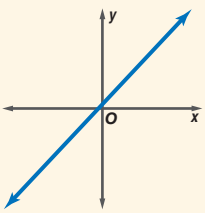
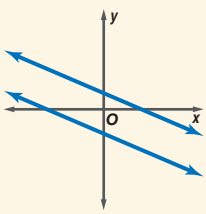
CHECK Your Progress

6A. $y - \frac{4}{3}x = -2$
$y + \frac{3}{4}x = -2$

6B. $y - \frac{4}{3}x = -2$
$y - \frac{4}{3}x = 3$

The relationship between the graph of a system of equations and the number of its solutions is summarized below.

CONCEPT SUMMARY

consistent and independent	consistent and dependent	inconsistent
		
intersecting lines; one solution	same line; infinitely many solutions	parallel lines; no solution

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CHECK Your Understanding

Example 1 (p. 116)

Solve each system of equations by completing a table.

1. $y = 2x + 9$
$y = -x + 3$

2. $3x + 2y = 10$
$2x + 3y = 10$

Example 2 (p. 117)

Solve each system of equations by graphing.

3. $4x - 2y = 22$
$6x + 9y = -3$

4. $y = 2x - 4$
$y = -3x + 1$

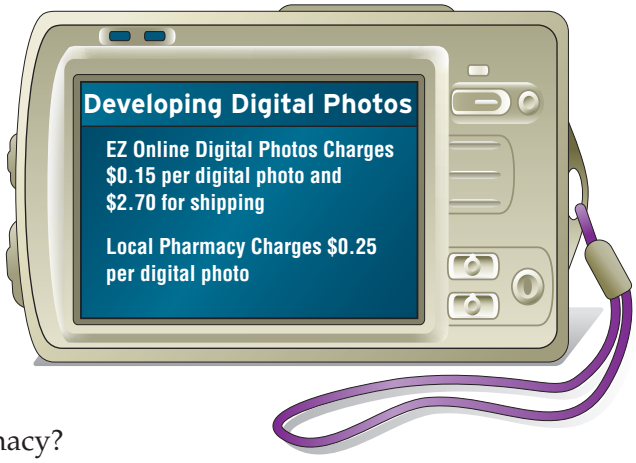
Example 3 (pp. 117–118)

DIGITAL PHOTOS **For Exercises 5–7, use the information in the graphic.**

5. Write equations that represent the cost of printing digital photos at each lab.

6. Under what conditions is the cost to print digital photos the same for either store?

7. When is it best to use EZ Online Digital Photos and when is it best to use the local pharmacy?



Examples 4–6 (pp. 118–119)

Graph each system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*.

8. $y = 6 - x$
$y = x + 4$

9. $x + 2y = 2$
$2x + 4y = 8$

10. $x - 2y = 8$
$\frac{1}{2}x - y = 4$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11, 12	1
13–18	2
19–26	4–6
27–32	3

Solve each system of linear equations by completing a table.

11. $y = 3x - 8$
$y = x - 8$

12. $x + 2y = 6$
$2x + y = 9$

Solve each system of linear equations by graphing.

13. $2x + 3y = 12$
$2x - y = 4$

14. $3x - 7y = -6$
$x + 2y = 11$

15. $5x - 11 = 4y$
$7x - 1 = 8y$

16. $2x + 3y = 7$
$2x - 3y = 7$

17. $8x - 3y = -3$
$4x - 2y = -4$

18. $\frac{1}{4}x + 2y = 5$
$2x - y = 6$

Graph each system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*.

19. $y = x + 4$
$y = x - 4$

20. $y = x + 3$
$y = 2x + 6$

21. $x + y = 4$
$-4x + y = 9$

22. $3x + y = 3$
$6x + 2y = 6$

23. $y - x = 5$
$2y - 2x = 8$

24. $4x - 2y = 6$
$6x - 3y = 9$

25. GEOMETRY The sides of an angle are parts of two lines whose equations are $2y + 3x = -7$ and $3y - 2x = 9$. The angle's vertex is the point where the two sides meet. Find the coordinates of the vertex of the angle.

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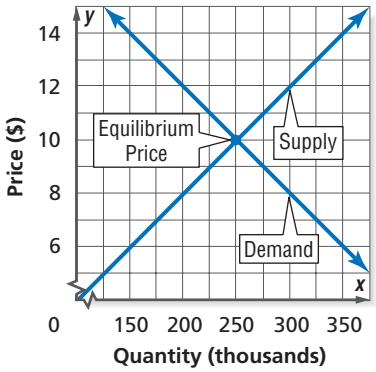
26. GEOMETRY The graphs of $y - 2x = 1$, $4x + y = 7$, and $2y - x = -4$ contain the sides of a triangle. Find the coordinates of the vertices of the triangle.

EXTRA PRACTICE
See pages 895, 928.
MathOnline
Self-Check Quiz at algebra2.com

ECONOMICS For Exercises 27–29, use the graph that shows the supply and demand curves for a new multivitamin.

In economics, the point at which the supply equals the demand is the *equilibrium price*. If the supply of a product is greater than the demand, there is a surplus and prices fall. If the supply is less than the demand, there is a shortage and prices rise.



27. If the price for vitamins is \$8.00 a bottle, what is the supply of the product and what is the demand? Will prices tend to rise or fall?

28. If the price for vitamins is \$12.00 a bottle, what is the supply of the product and what is the demand? Will prices tend to rise or fall?

29. At what quantity will the prices stabilize? What is the equilibrium price for this product?

Real-World Link

In the United States there is approximately one birth every 8 seconds and one death every 14 seconds.

Source: U.S. Census Bureau

ANALYZE TABLES For Exercises 30–32, use the table showing state populations.

Rank	State	Population 2003	Average Annual Gain (2000–2003)
1	California	25,484,000	567,000
2	Texas	22,118,000	447,000
3	New York	19,190,000	70,000
4	Florida	17,019,000	304,000
5	Illinois	12,653,000	80,000

Source: U.S. Census Bureau

30. Write equations that represent populations of Florida and New York x years after 2003. Assume that both states continue to gain the same number of residents every year. Let y equal the population.

31. Graph both equations for the years 2003 to 2020. Estimate when the populations of both states will be equal.

32. Do you think New York will overtake Texas as the second most populous state by 2010? by 2020? Explain your reasoning.

Solve each system of equations by graphing.

33. $\frac{2}{3}x + y = -3$
$y - \frac{1}{3}x = 6$

34. $\frac{1}{2}x - y = 0$
$\frac{1}{4}x + \frac{1}{2}y = -2$

35. $\frac{4}{3}x + \frac{1}{5}y = 3$
$\frac{2}{3}x - \frac{3}{5}y = 5$

Graph each system of equations and describe it as *consistent and independent, consistent and dependent, or inconsistent.*

36. $1.6y = 0.4x + 1$
$0.4y = 0.1x + 0.25$

37. $3y - x = -2$
$y - \frac{1}{3}x = 2$

38. $2y - 4x = 3$
$\frac{4}{3}x - y = -2$

Graphing Calculator

To use a TI-83/84 Plus to solve a system of equations, graph the equations. Then, select INTERSECT, which is option 5 under the CALC menu, to find the coordinates of the point of intersection to the nearest hundredth.

39. $y = 0.125x - 3.005$
$y = -2.58$

40. $3.6x - 2y = 4$
$-2.7x + y = 3$

41. $y = 0.18x + 2.7$
$y = -0.42x + 5.1$

H.O.T. Problems

42. OPEN ENDED Give an example of a system of equations that is consistent and independent.

43. REASONING Explain why a system of linear equations cannot have exactly two solutions.

Telegraph Colour Library/Getty Images
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44. CHALLENGE State the conditions for which the system below is: (a) consistent and dependent, (b) consistent and independent, and (c) inconsistent if none of the variables are equal to 0.

$ax + by = c$
$dx + ey = f$

45. *Writing in Math* Use the information about sales on page 116 to explain how a system of equations can be used to predict sales. Include an explanation of the meaning of the solution of the system of equations in the application at the beginning of the lesson. How reasonable would it be to use this system of equations to predict the company's online and in-store profits in 100 years? Explain your reasoning.

STANDARDIZED TEST PRACTICE

46. ACT/SAT Which of the following best describes the graph of the equations?

$4y = 3x + 8$
$-6x = -8y + 24$

A The lines are parallel.

B The lines have the same x-intercept.

C The lines are perpendicular.

D The lines have the same y-intercept.

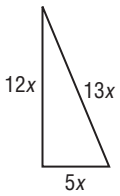
47. REVIEW Which set of dimensions corresponds to a triangle similar to the one shown below?

F 7 units, 11 units, 12 units

G 10 units, 23 units, 24 units

H 20 units, 48 units, 52 units

J 1 unit, 2 units, 3 units

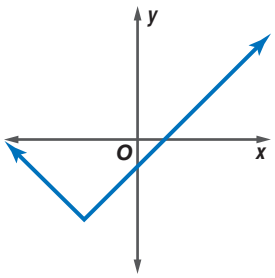


Spiral Review

48. CHORES Simon is putting up fence around his yard at a rate no faster than 15 feet per hour. Draw a graph that represents the length of fence that Simon has built. (Lesson 2-7)

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

49.



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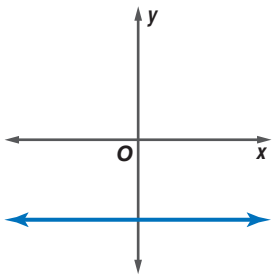
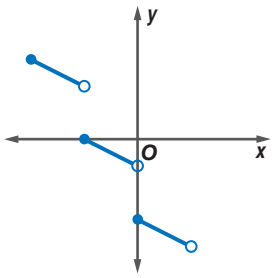


51.



GET READY for the Next Lesson

PREREQUISITE SKILL Simplify each expression. (Lesson 1-2)

52. $(3x + 5) - (2x + 3)$

53. $(3y - 11) + (6y + 12)$

54. $(5x - y) + (-8x + 7y)$

55. $6(2x + 3y - 1)$

56. $5(4x + 2y - x + 2)$

57. $3(x + 4y) - 2(x + 4y)$

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3-2 Solving Systems of Equations Algebraically

Main Ideas

- Solve systems of linear equations by using substitution.
- Solve systems of linear equations by using elimination.

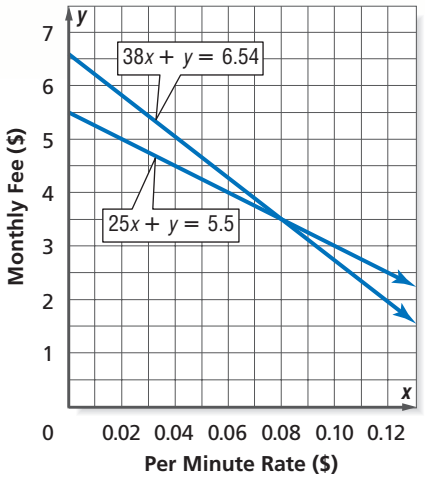
New Vocabulary

substitution method

elimination method

GET READY for the Lesson

In January, Yolanda's long-distance bill was \$5.50 for 25 minutes of calls. The bill was \$6.54 in February, when Yolanda made 38 minutes of calls. What are the rate per minute and flat fee the company charges?



Let x equal the rate per minute, and let y equal the monthly fee.

January bill: $25x + y = 5.5$
February bill: $38x + y = 6.54$

Sometimes it is difficult to determine the exact coordinates of the point where the lines intersect from the graph. For systems of equations like this one, it may be easier to solve the system by using algebraic methods.

Substitution One algebraic method is the **substitution method**. Using this method, one equation is solved for one variable in terms of the other. Then, this expression is substituted for the variable in the other equation.

EXAMPLE Solve by Using Substitution

1 **Use substitution to solve the system of equations.**

$x + 2y = 8$

$\frac{1}{2}x - y = 18$

Solve the first equation for x in terms of y.

$x + 2y = 8$ First equation

$x = 8 - 2y$ Subtract $2y$ from each side.

Substitute $8 - 2y$ for x in the second equation and solve for y.

$\frac{1}{2}x - y = 18$ Second equation

$\frac{1}{2}(8 - 2y) - y = 18$ Substitute $8 - 2y$ for x.

$4 - y - y = 18$ Distributive Property

$-2y = 14$ Subtract 4 from each side.

$y = -7$ Divide each side by -2.

(continued on the next page)

Study Tip

Coefficient of 1

It is easier to solve for the variable that has a coefficient of 1.

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Now, substitute the value for y in either original equation and solve for x.

$x + 2y = 8$ First equation

$x + 2(-7) = 8$ Replace y with -7.

$x - 14 = 8$ Simplify.

$x = 22$

The solution of the system is $(22, -7)$.

CHECK Your Progress

1A. $2x - 3y = 2$
$x + 2y = 15$

1B. $7y = 26 + 11x$
$x - 3y = 0$

STANDARDIZED TEST EXAMPLE **Solve by Substitution**

2 Matthew stopped for gasoline twice on a long car trip. The price of gasoline at the first station where he stopped was \$2.56 per gallon. At the second station, the price was \$2.65 per gallon. Matthew bought a total of 36.1 gallons of gasoline and spent \$94.00. How many gallons of gasoline did Matthew buy at the first gas station?

A 17.6 **B** 18.5 **C** 19.2 **D** 20.1

Read the Item

You are asked to find the number of gallons of gasoline that Matthew bought at the first gas station.

Solve the Item

Step 1 Define variables and write the system of equations. Let x represent the number of gallons bought at the first station and y represent the number of gallons bought at the second station.

$x + y = 36.1$ The total number of gallons was 36.1.

$2.56x + 2.65y = 94$ The total price was \$94.

Step 2 Solve one of the equations for one of the variables in terms of the other. Since the coefficient of y is 1 and you are asked to find the value of x, it makes sense to solve the first equation for y in terms of x.

$x + y = 36.1$ First equation

$y = 36.1 - x$ Subtract x from each side.

Test-Taking Tip

Even if the question does not ask you for both variables, it is still a good idea to find both so that you can check your answer.

Step 3 Substitute $36.1 - x$ for y in the second equation.

$2.56x + 2.65y = 94$ Second equation

$2.56x + 2.65(36.1 - x) = 94$ Substitute $36.1 - x$ for y.

$2.56x + 95.665 - 2.65x = 94$ Distributive Property

$-0.09x = -1.665$ Simplify.

$x = 18.5$ Divide each side by -0.09.

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Step 4 Matthew bought 18.5 gallons of gasoline at the first gas station. The answer is B.

CHECK Your Progress

2. COMIC BOOKS Dante spent $11.25 on 3 new and 4 old comic books, and Samantha spent $15.75 on 10 old and 3 new ones. If comics of one type are sold at the same price, what is the price in dollars of a new comic book?

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Elimination Another algebraic method is the **elimination method**. Using this method, you eliminate one of the variables by adding or subtracting the equations. When you add two true equations, the result is a new equation that is also true.

EXAMPLE Solve by Using Elimination

3 **Use the elimination method to solve the system of equations.**

$4a + 2b = 15$

$2a + 2b = 7$

In each equation, the coefficient of b is 2. If one equation is subtracted from the other, the variable b will be eliminated.

$$\begin{array}{rl} 4a + 2b = 15 & \\ (-)\ 2a + 2b = 7 & \\ \hline 2a = 8 & \text{Subtract the equations.} \\ a = 4 & \text{Divide each side by 2.} \end{array}$$

Now find b by substituting 4 for a in either original equation.

$$\begin{array}{rl} 2a + 2b = 7 & \text{Second equation} \\ 2(4) + 2b = 7 & \text{Replace } a \text{ with 4.} \\ 8 + 2b = 7 & \text{Multiply.} \\ 2b = -1 & \text{Subtract 8 from each side.} \\ b = -\frac{1}{2} & \text{Divide each side by 2.} \end{array}$$

The solution is $\left(4, -\frac{1}{2}\right)$.

Study Tip

Alternative Method

You may find it confusing to subtract equations. It may be helpful to multiply the second equation by -1 and then add the equations.

CHECK Your Progress

3A. $2x + y = 4$
$3x + y = 8$

3B. $5b = 20 + 2a$
$2a + 4b = 7$

Sometimes, adding or subtracting the two equations will not eliminate either variable. You may use multiplication to write an equivalent equation so that one of the variables has the same or opposite coefficient in both equations. When you multiply an equation by a nonzero number, the new equation is equivalent to the original equation.

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EXAMPLE Multiply, Then Use Elimination

4 **Use the elimination method to solve the system of equations.**

$3x - 7y = -14$
$5x + 2y = 45$

Study Tip

Alternative Method

You could also multiply the first equation by 5 and the second equation by 3. Then subtract to eliminate the x variable.

Multiply the first equation by 2 and the second equation by 7. Then add the equations to eliminate the y variable.

$3x - 7y = -14$ Multiply by 2. $\rightarrow$ $6x - 14y = -28$

$5x + 2y = 45$ Multiply by 7. $\rightarrow$ $(+)\ 35x + 14y = 315$

$41x \quad = 287$ Add the equations.

$x = 7$ Divide each side by 41.

Replace x with 7 and solve for y.

$3x - 7y = -14$ First equation

$3(7) - 7y = -14$ Replace x with 7.

$21 - 7y = -14$ Multiply.

$-7y = -35$ Subtract 21 from each side.

$y = 5$ Divide each side by -7.

The solution is (7, 5).

CHECK Your Progress

4A. $3x + 4y = 14$
$4x + 5y = 17$

4B. $2x - 4y = 28$
$4x = 17 - 5y$

If you add or subtract two equations in a system and the result is an equation that is never true, then the system is inconsistent. If the result when you add or subtract two equations in a system is an equation that is always true, then the system is dependent.

EXAMPLE Inconsistent System

5 **Use the elimination method to solve the system of equations.**

$8x + 2y = 17$
$-4x - y = 9$

Use multiplication to eliminate x.

$8x + 2y = 17$ $\qquad$ $8x + 2y = 17$

$-4x - y = 9$ Multiply by 2. $\rightarrow$ $-8x - 2y = 18$

$0 = 35$ Add the equations.

Since there are no values of x and y that will make the equation $0 = 35$ true, there are no solutions for this system of equations.

CHECK Your Progress

5A. $8y = 2x + 48$
$y - \frac{1}{4}x = 6$

5B. $x - 0.5y = -3$
$2x - y = 6$

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CHECK Your Understanding

Example 1 (pp. 123–124)

Solve each system of equations by using substitution.

1. $y = 3x - 4$
$y = 4 + x$

2. $4c + 2d = 10$
$c + 3d = 10$

3. $a - b = 2$
$-2a + 3b = 3$

4. $3g - 2h = -1$
$4g + h = 17$

Example 2 (pp. 124–125)

5. STANDARDIZED TEST PRACTICE Campus Rentals rents 2- and 3-bedroom apartments for \$700 and \$900 per month, respectively. Last month they had six vacant apartments and reported \$4600 in lost rent. How many 2-bedroom apartments were vacant?

A 2 **B** 3 **C** 4 **D** 5

Examples 3–5 (pp. 125–126)

Solve each system of equations by using elimination.

6. $2r - 3s = 11$
$2r + 2s = 6$

7. $5m + n = 10$
$4m + n = 4$

8. $2p + 4q = 18$
$3p - 6q = 3$

9. $\frac{1}{4}x + y = \frac{11}{4}$
$x - \frac{1}{2}y = 2$

10. $\frac{1}{6}y - 2 = \frac{1}{9}$
$12 = 18y$

11. $1.25x - y = -7$
$4y = 5x + 28$

Exercises

HOMEWORK HELP

For Exercises	See Examples
12–17	1, 2
18–21	3, 4
22, 23	5

Solve each system of equations by using substitution.

12. $2j - 3k = 3$
$j + k = 14$

13. $2r + s = 11$
$6r - 2s = -2$

14. $5a - b = 17$
$3a + 2b = 5$

15. $-w - z = -2$
$4w + 5z = 16$

16. $3s + 2t = -3$
$s + \frac{1}{3}t = -4$

17. $2x + 4y = 6$
$7x = 4 + 3y$

Solve each system of equations by using elimination.

18. $u + v = 7$
$2u + v = 11$

19. $m - n = 9$
$7m + n = 7$

20. $r + 4s = -8$
$3r + 2s = 6$

21. $4x - 5y = 17$
$3x + 4y = 5$

22. $2c + 6d = 14$
$-\frac{7}{3} + \frac{1}{3}c = -d$

23. $6d + 3f = 12$
$2d = 8 - f$

Cross-Curricular Project

A system of equations can be used to compare home loan options. Visit algebra2.com to continue work on your project.

SKIING **For Exercises 24 and 25, use the following information.**
All 28 members in Crestview High School's Ski Club went on a one-day ski trip. Members can rent skis for \$16 per day or snowboards for \$19 per day. The club paid a total of \$478 for rental equipment.

24. Write a system of equations that represents the number of members who rented the two types of equipment.

25. How many members rented skis and how many rented snowboards?

INVENTORY **For Exercises 26 and 27, use the following information.**
Beatriz is checking a shipment of technology equipment that contains laser printers that cost \$700 each and color monitors that cost \$200 each. She counts 30 boxes on the loading dock. The invoice states that the order totals \$15,000.

26. Write a system of two equations that represents the number of each item.

27. How many laser printers and how many color monitors were delivered?

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Real-World Career
Teacher

Besides the time they spend in a classroom, teachers spend additional time preparing lessons, grading papers, and assessing students' progress.

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Solve each system of equations by using either substitution or elimination.

28. $3p - 6q = 6$
$2p - 4q = 4$

29. $10m - 9n = 15$
$5m - 4n = 10$

30. $3c - 7d = -3$
$2c + 6d = -34$

31. $6g - 8h = 50$
$6h = 22 - 4g$

32. $2p = 7 + q$
$6p - 3q = 24$

33. $3x = -31 + 2y$
$5x + 6y = 23$

34. $3u + 5v = 6$
$2u - 4v = -7$

35. $3a = -3 + 2b$
$3a + b = 3$

36. $0.25x + 1.75y = 1.25$
$-0.5x + 2 = 2.5y$

37. $8 = 0.4m + 1.8n$
$1.2m + 3.4n = 16$

38. $s + 3t = 27$
$2t = 19 - \frac{1}{2}s$

39. $2f + 2g = 18$
$\frac{1}{6}f + \frac{1}{3}g = 1$

TEACHING **For Exercises 40–42, use the following information.**
Mr. Talbot is writing a science test. It will have true/false questions worth 2 points each and multiple-choice questions worth 4 points each for a total of 100 points. He wants to have twice as many multiple-choice questions as true/false.

40. Write a system of equations that represents the number of each type of question.

41. How many of each type of question will be on the test?

42. If most of his students can answer true/false questions within 1 minute and multiple-choice questions within $1\frac{1}{2}$ minutes, will they have enough time to finish the test in 45 minutes?

EXERCISE **For Exercises 43 and 44, use the following information.**
Megan exercises every morning for 40 minutes. She does a combination of step aerobics, which burns about 11 Calories per minute, and stretching, which burns about 4 Calories per minute. Her goal is to burn 335 Calories during her routine.

43. Write a system of equations that represents Megan's morning workout.

44. How long should she do each activity in order to burn 335 Calories?

EXTRA PRACTICE
See pages 895, 928.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

45. **OPEN ENDED** Give a system of equations that is more easily solved by substitution and a system of equations that is more easily solved by elimination.

46. **REASONING** Make a conjecture about the solution of a system of equations if the result of subtracting one equation from the other is $0 = 0$.

47. **FIND THE ERROR** Juanita and Jamal are solving the system $2x - y = 6$ and $2x + y = 10$. Who is correct? Explain your reasoning.

Juanita

$2x - y = 6$
$(-)2x + y = 10$
$0 = -4$

The statement $0 = -4$ is never true, so there is no solution.

Jamal

$2x - y = 6$
$(+)2x + y = 10$
$4x = 16$
$x = 4$

$2x - y = 6$
$2(4) - y = 6$
$8 - y = 6$
$y = 2$

The solution is (4, 2).

48. **CHALLENGE** Solve the system of equations.

$\frac{1}{x} + \frac{3}{y} = \frac{3}{4}$ $\left(\textit{Hint}\text{: Let } m = \frac{1}{x} \text{ and } n = \frac{1}{y}.\right)$

$\frac{3}{x} - \frac{2}{y} = \frac{5}{12}$

Bob Daemmrich/PhotoEdit
www.MathSchoolinternational.com

49. ***Writing in Math*** Use the information on page 123 to explain how a system of equations can be used to find a flat fee and a per-unit rate. Include a solution of the system of equations in the application at the beginning of the lesson.

STANDARDIZED TEST PRACTICE

50. ACT/SAT In order to practice at home, Tadeo purchased a basketball and a volleyball that cost a total of \$67, not including tax. If the price of the basketball b is \$4 more than twice the cost of the volleyball v which system of linear equations could be used to determine the price of each ball?

A $b + v = 67$
$b = 2v - 4$

B $b + v = 67$
$b = 2v + 4$

C $b + v = 4$
$b = 2v - 67$

D $b + v = 4$
$b = 2v + 67$

51. REVIEW The caterer at a brunch bought several pounds of chicken salad and several pounds of tuna salad. The chicken salad cost \$9 per pound, and the tuna salad cost \$6 per pound. He bought a total of 14 pounds of salad and paid a total of \$111. How much chicken salad did he buy?

F 6 pounds

G 7 pounds

H 8 pounds

J 9 pounds

Spiral Review

Graph each system of equations and describe it as *consistent and independent*, *consistent and dependent*, or *inconsistent*. (Lesson 3-1)

52. $y = x + 2$
$y = x - 1$

53. $4y - 2x = 4$
$y - \frac{1}{2}x = 1$

54. $3x + y = 1$
$y = 2x - 4$

Graph each inequality. (Lesson 2-7)

55. $x + y \leq 3$

56. $5y - 4x < -20$

57. $3x + 9y \geq -15$

Write each equation in standard form. Identify A, B, and C. (Lesson 2-2)

58. $y = 7x + 4$

59. $x = y$

60. $3x = 2 - 5y$

61. $6x = 3y - 9$

62. $y = \frac{1}{2}x - 3$

63. $\frac{2}{3}y - 6 = 1 - x$

64. ELECTRICITY Find the amount of current I (in amperes) produced if the electromotive force E is 1.5 volts, the circuit resistance R is 2.35 ohms, and the resistance r within a battery is 0.15 ohms, using the formula $I = \frac{E}{R + r}$. (Lesson 1-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Determine whether the given point satisfies each inequality.** (Lesson 2-7)

65. $3x + 2y \leq 10$; $(2, -1)$

66. $4x - 2y > 6$; $(3, 3)$

67. $7x + 4y \geq -15$; $(-4, 2)$

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3-3 Solving Systems of Inequalities by Graphing

Main Ideas

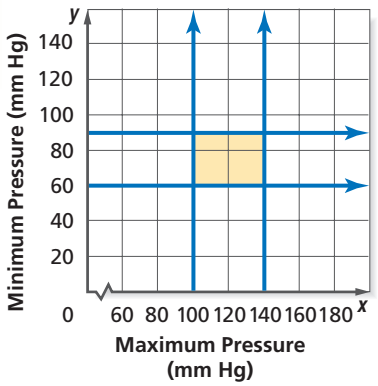
- Solve systems of inequalities by graphing.
- Determine the coordinates of the vertices of a region formed by the graph of a system of inequalities.

New Vocabulary

system of inequalities

GET READY for the Lesson

During one heartbeat, blood pressure reaches a maximum pressure and a minimum pressure, which are measured in millimeters of mercury (mm-Hg). It is expressed as the maximum over the minimum—for example, 120/80. Normal blood pressure for people under 40 ranges from 100 to 140 mm Hg for the maximum and from 60 to 90 mm Hg for the minimum. This can be represented by a system of inequalities.



Graph Systems of Inequalities To solve a **system of inequalities**, we need to find the ordered pairs that satisfy all of the inequalities in the system. The solution set is represented by the intersection of the graphs of the inequalities.

EXAMPLE 1 Intersecting Regions

Solve each system of inequalities.

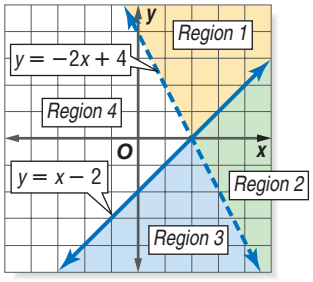
a. $y > -2x + 4$

$y \leq x - 2$

Solution of $y > -2x + 4 \rightarrow$ Regions 1 and 2

Solution of $y \leq x - 2 \rightarrow$ Regions 2 and 3

The region that provides a solution of both inequalities is the solution of the system. Region 2 is the solution of the system.

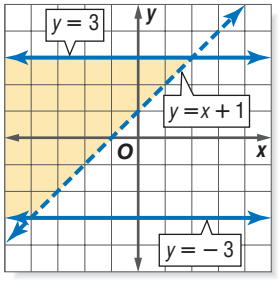


b. $y > x + 1$

$|y| \leq 3$

The inequality $|y| \leq 3$ can be written as $y \leq 3$ and $y \geq -3$.

Graph all of the inequalities on the same coordinate plane and shade the region or regions that are common to all.



Study Tip

Look Back

To review **graphing inequalities**, see Lesson 2-7.

Concepts in Motion

Animation algebra2.com

CHECK Your Progress

1A. $y \leq 3x - 4$

$y > -2x + 3$

1B. $|y| < 3$

$y \geq x - 1$

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Reading Math

Empty Set The empty set is also called the *null set.* It can be represented as ∅ or { }.

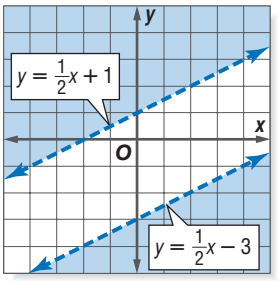
It is possible that two regions do *not* intersect. In such cases, we say the solution set is the empty set (∅) and no solution exists.

EXAMPLE 2 Separate Regions

Solve the system of inequalities by graphing.

$y > \frac{1}{2}x + 1$

$y < \frac{1}{2}x - 3$



Graph both inequalities. The graphs do not overlap, so the solution sets have no points in common. The solution set of the system is ∅.

CHECK Your Progress

2. $y > \frac{1}{4}x + 4$

$y < \frac{1}{4}x - 2$

Real-World EXAMPLE 3 Write and Use a System of Inequalities

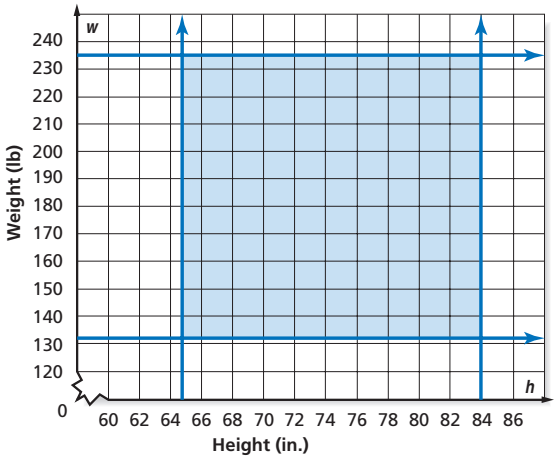
BASKETBALL The 2005–06 Denver Nuggets roster included players of varying weights and heights. Francisco Elson was the largest at 7′0″ and 235 pounds. The smallest player on the team was Earl Boykins at 5′5″ and 133 pounds. Write and graph a system of inequalities that represents the range of heights and weights for the members of the team.

Let h represent the height of a member of the Denver Nuggets. The possible heights for a member of the team are at least 65 inches, but no more than 84 inches. We can write two inequalities.

$h \geq 65$ and $h \leq 84$

Let w represent the weights of a player on the Denver Nuggets. The weights can be written as two inequalities.

$w \geq 133$ and $w \leq 235$



Graph all of the inequalities. Any ordered pair in the intersection of the graphs is a solution of the system. In this case, a solution of the system of inequalities is a potential height and weight combination for a member of the Denver Nuggets.

CHECK Your Progress

3. CATERING Classy Catering needs at least 15 food servers and 5 bussers to cater a large party. But in order to make a profit, they can have no more than 34 food servers and 7 bussers working at an event. Write and graph a system of inequalities that represents this information.

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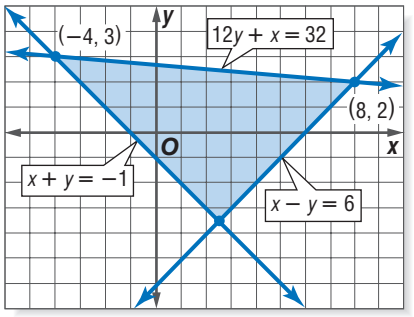
Find Vertices of a Polygonal Region Sometimes, the graph of a system of inequalities forms a polygonal region. To find the vertices of the region, determine the coordinates of the points at which the boundaries intersect.

EXAMPLE Find Vertices

4 **GEOMETRY** **Find the coordinates of the vertices of the figure formed by $x + y \geq -1$, $x - y \leq 6$, and $12y + x \leq 32$.**

Graph each inequality. The intersection of the graphs forms a triangle.



The coordinates $(-4, 3)$ and $(8, 2)$ can be determined from the graph. To find the third vertex, solve the system of equations $x + y = -1$ and $x - y = 6$.

Add the equations to eliminate y.

$$\begin{array}{rrl} x + y &= -1 & \\ (+)\ x - y &= \ \ 6 & \\ \hline 2x &= \ \ 5 & \text{Add the equations.} \\ x &= \frac{5}{2} & \text{Divide each side by 2.} \end{array}$$

Now find y by substituting $\frac{5}{2}$ for x in the first equation.

$$\begin{array}{rl} x + y = -1 & \text{First equation} \\ \frac{5}{2} + y = -1 & \text{Replace } x \text{ with } \frac{5}{2}. \\ y = -\frac{7}{2} & \text{Subtract } \frac{5}{2} \text{ from each side.} \end{array}$$

CHECK **Compare the coordinates to the coordinates on the graph.**

The x-coordinate of the third vertex is between 2 and 3, so $\frac{5}{2}$ is reasonable. The y-coordinate of the third vertex is between -3 and -4, so $-\frac{7}{2}$ is reasonable.

The vertices of the triangle are at $(-4, 3)$, $(8, 2)$, and $\left(\frac{5}{2}, -\frac{7}{2}\right)$.

CHECK Your Progress

4. Find the coordinates of the vertices of the figure formed by $x + y \leq 2$, $x - 2y \leq 8$, and $x + \left(-\frac{1}{3}y\right) \geq -\frac{2}{3}$.

Personal Tutor at algebra2.com

CHECK Your Understanding

Examples 1, 2 (pp. 130–131)

Solve each system of inequalities by graphing.

1. $x \leq 4$
$y > 2$

2. $y \leq -4x - 3$
$y > -4x + 1$

3. $|x - 1| \leq 2$
$x + y > 2$

4. $y \geq 3x + 3$
$y < 3x - 2$

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Example 3 (p. 131)

SHOPPING **For Exercises 5 and 6, use the following information.**
The most Jack can spend on bagels and muffins for the cross country team is $28. A package of 6 bagels costs $2.50. A package of muffins costs $3.50 and contains 8 muffins. He needs to buy at least 12 bagels and 24 muffins.

5. Graph the region that shows how many packages of each item he can purchase.

6. Give an example of three different purchases he can make.

Example 4 (p. 132)

Find the coordinates of the vertices of the figure formed by each system of inequalities.

7. $y \leq x$
$y \geq -3$
$3y + 5x \leq 16$

8. $y \geq x - 3$
$y \leq x + 7$
$x + y \leq 11$
$x + y \geq -1$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–17	1, 2
18, 19	3
20–23	4

Solve each system of inequalities by graphing.

9. $x \geq 2$
$y > 3$

10. $x \leq -1$
$y \geq -4$

11. $y < 2 - x$
$y > x + 4$

12. $x > 1$
$x \leq -1$

13. $3x + 2y \geq 6$
$4x - y \geq 2$

14. $4x - 3y < 7$
$2y - x < -6$

15. $3y \leq 2x - 8$
$y \geq \frac{2}{3}x - 1$

16. $y > x - 3$
$|y| \leq 2$

17. $2x + 5y \leq -15$
$y > \frac{-2}{5}x + 2$

18. **PART-TIME JOBS** Rondell makes $10 an hour cutting grass and $12 an hour for raking leaves. He cannot work more than 15 hours per week. Graph two inequalities that Rondell can use to determine how many hours he needs to work at each job if he wants to earn at least $120 per week.

19. **RECORDING** Jane's band wants to spend no more than $575 recording their first CD. The studio charges at least $35 an hour to record. Graph a system of inequalities to represent this situation.

Find the coordinates of the vertices of the figure formed by each system of inequalities.

20. $y \geq 0$
$x \geq 0$
$x + 2y \leq 8$

21. $y \geq -4$
$y \leq 2x + 2$
$2x + y \leq 6$

22. $x \leq 3$
$-x + 3y \leq 12$
$4x + 3y \geq 12$

23. $x + y \leq 9$
$x - 2y \leq 12$
$y \leq 2x + 3$

24. **GEOMETRY** Find the area of the region defined by the system of inequalities $y + x \leq 3$, $y - x \leq 3$, and $y \geq -1$.

25. **GEOMETRY** Find the area of the region defined by the system of inequalities $x \geq -3$, $y + x \leq 8$, and $y - x \geq -2$.

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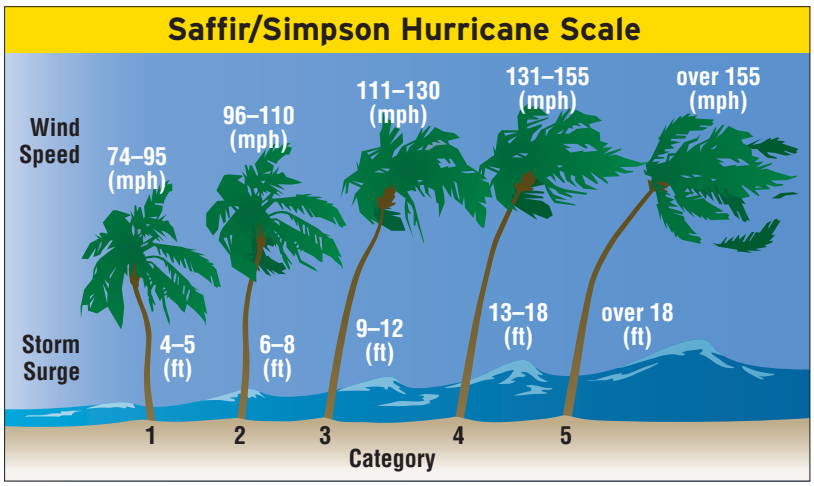
Real-World Career
Atmospheric Scientist
The best known use of atmospheric science is for weather forecasting. However, weather information is also studied for air-pollution control, agriculture, and transportation.

MathOnline
For more information, go to algebra2.com.

HURRICANES **For Exercises 26 and 27, use the following information.**
Hurricanes are divided into five categories according to their wind speed and storm surge. Category 5 is the most destructive type of hurricane.

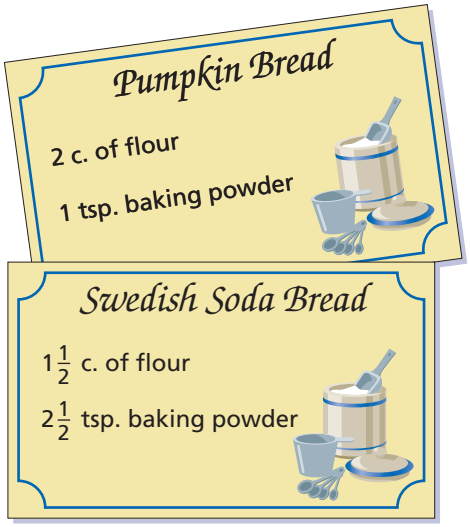


Source: National Oceanic and Atmospheric Administration

26. Write and graph the system of inequalities that represents the range of wind speeds s and storm surges h for a Category 3 hurricane.

27. On August 29, 2005, Hurricane Katrina hit the Gulf coasts of Louisiana and Mississippi. At its peak, Katrina had maximum sustained winds of 145 mph. Classify the strength of Hurricane Katrina and state the expected heights of its storm surges.

BAKING **For Exercises 28–30, use the recipes at the right.**
The Merry Bakers are baking pumpkin bread and Swedish soda bread for this week's specials. They have at most 24 cups of flour and at most 26 teaspoons of baking powder on hand.



28. Graph the inequalities that represent how many loaves of each type of bread the bakers can make.

29. List three different combinations of breads they can make.

30. Which combination uses all of the available flour and baking soda?

Solve each system of inequalities by graphing.

31. $y < 2x - 3$
$y \leq \frac{1}{2}x + 1$

32. $|x| \leq 3$
$|y| > 1$

33. $|x + 1| \leq 3$
$x + 3y \geq 6$

34. $y \geq 2x + 1$
$y \leq 2x - 2$
$3x + y \leq 9$

35. $x - 3y > 2$
$2x - y < 4$
$2x + 4y \geq -7$

36. $x \leq 1$
$y < 2x + 1$
$x + 2y \geq -3$

EXTRA PRACTICE
See pages 896, 928.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

37. OPEN ENDED Write a system of inequalities that has no solution.

38. REASONING Determine whether the following statement is *true* or *false*. If false, give a counterexample. *A system of two linear inequalities has either no points or infinitely many points in its solution.*

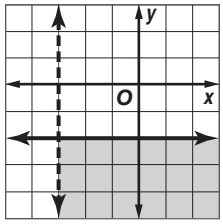
Doug Martin
www.MathSchoolinternational.com

39. CHALLENGE Find the area of the region defined by $|x| + |y| \leq 5$ and $|x| + |y| \geq 2$.

40. *Writing in Math* Using the information about blood pressure on page 130, explain how you can determine whether your blood pressure is in a normal range utilizing a graph of the system of inequalities.

STANDARDIZED TEST PRACTICE

41. ACT/SAT Choose the system of inequalities whose solution is represented by the graph.



A $y < -2$, $x < -3$

B $y \leq -2$, $x > -3$

C $x \leq -2$, $y > -3$

D $x < -3$, $y < -3$

42. REVIEW To be a member of the marching band, a student must have a GPA of at least 2.0 and must have attended at least five after-school practices. Choose the system of inequalities that best represents this situation.

F $x \geq 2$, $y \geq 5$

G $x \leq 2$, $y \leq 5$

H $x < 2$, $y < 5$

J $x > 2$, $y > 5$

Spiral Review

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

43. $4x - y = -20$, $x + 2y = 13$

44. $3x - 4y = -2$, $5x + 2y = 40$

45. $4x + 5y = 7$, $3x - 2y = 34$

Solve each system of equations by graphing. (Lesson 3-1)

46. $y = 2x + 1$, $y = -\frac{1}{2}x - 4$

47. $2x + y = -3$, $6x + 3y = -9$

48. $2x - y = 6$, $-x + 8y = 12$

49. RENTALS To rent an inflatable trampoline for parties, it costs \$75 an hour plus a set-up/tear-down fee of \$200. Write an equation that represents this situation in slope-intercept form. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each value if $f(x) = 4x + 3$ and $g(x) = 5x - 7$.** (Lesson 2-1)

50. $f(-2)$ **51.** $g(-1)$ **52.** $g(3)$ **53.** $g(-0.25)$

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EXTEND 3-3

Graphing Calculator Lab

Systems of Linear Inequalities

You can graph systems of linear inequalities with a TI–83/84 Plus graphing calculator using the Y= menu. You can choose different graphing styles to shade above or below a line.

EXAMPLE

Graph the system of inequalities in the standard viewing window.

$y \geq -2x + 3$

$y \leq x + 5$

Step 1

- Enter $-2x + 3$ as Y1. Since y is greater than or equal to $-2x + 3$, shade above the line.

  KEYSTROKES: -2 [X,T,θ,n] [+] 3

- Use the left arrow key to move your cursor as far left as possible. Highlight the graph style icon. Press [ENTER] until the shade above icon, ▀, appears.

Step 2

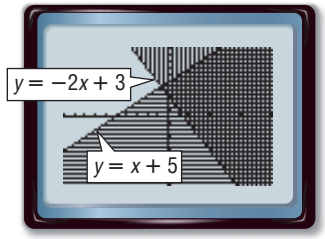
- Enter $x + 5$ as Y2. Since y is less than or equal to $x + 5$, shade below the line.

  KEYSTROKES: [X,T,θ,n] [+] 5

- Use the arrow and [ENTER] keys to choose the shade below icon, ▄.

Step 3

- Display the graphs by pressing [GRAPH]. Notice the shading pattern above the line $y = -2x + 3$ and the shading pattern below the line $y = x + 5$. The intersection of the graphs is the region where the patterns overlap. This region includes all the points that satisfy the system $y \geq -2x + 3$ and $y \leq x + 5$.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Exercises

Solve each system of inequalities. Sketch each graph on a sheet of paper.

1. $y \geq 4$
   $y \leq -x$

2. $y \geq -2x$
   $y \leq -3$

3. $y \geq 1 - x$
   $y \leq x + 5$

4. $y \geq x + 2$
   $y \leq -2x - 1$

5. $3y \geq 6x - 15$
   $2y \leq -x + 3$

6. $y + 3x \geq 6$
   $y - 2x \leq 9$

7. $6y + 4x \geq 12$
   $5y - 3x \leq -10$

8. $\frac{1}{4}y - x \geq -2$
   $\frac{1}{3}y + 2x \leq 4$

Other Calculator Keystrokes at algebra2.com

www.MathSchoolinternational.com

CHAPTER 3

Mid-Chapter Quiz

Lessons 3-1 through 3-3

Solve each system of equations by graphing. (Lesson 3-1)

1. $y = 3x + 10$
$y = -x + 6$

2. $2x + 3y = 12$
$2x - y = 4$

3. $x = y - 1$
$\frac{1}{3}y = x - 3$

4. $10 = -2x + y$
$-3x = -5y + 1$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

5. $y = x + 5$
$x + y = 9$

6. $2x + 6y = 2$
$3x + 2y = 10$

7. $\frac{3}{5}x + \frac{1}{12}y = 24$
$\frac{1}{9}x - \frac{2}{9}y = 13$

8. $-x = 16.95 - 7y$
$4x - 18.3 = -2y$

9. TRAVEL The busiest airport in the world is Atlanta's Hartsfield International Airport, and the second busiest airport is Chicago's O'Hare International Airport. Together they handled 160 million passengers in 2005. If Hartsfield handled 16 million more passengers than O'Hare, how many were handled by each airport? (Lesson 3-2)

10. MULTIPLE CHOICE Shenae spent \$42 on 2 cans of primer and 1 can of paint for her room. If the price of paint p is 150% of the price of primer r, which system of equations can be used to find the price of paint and primer? (Lesson 3-2)

A $p = r + \frac{1}{2}r$
$p + 2r = 42$

C $r = p + \frac{1}{2}p$
$p + 2p = 42$

B $p = r + 2r$
$p + \frac{1}{2}r = 42$

D $r = p + 2p$
$p + \frac{1}{2} = 42$

11. ART Marta can spend no more than \$225 on the art club's supply of brushes and paint. A box of brushes costs \$7.50 and contains 3 brushes. A box of paint costs \$21.45 and contains 10 tubes of paint. She needs at least 20 brushes and 56 tubes of paint. Graph the region that shows how many packages of each item can be purchased. (Lesson 3-3)

Solve each system of inequalities by graphing. (Lesson 3-3)

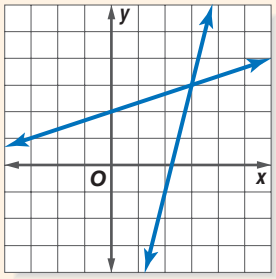
12. $y - x > 0$
$y + x < 4$

13. $y \geq 3x - 4$
$y \leq x + 3$

14. MULTIPLE CHOICE Which graph represents the following system of equations? (Lesson 3-3)

$\frac{1}{3}x + 2 = y$
$4x - 9 = y$

F



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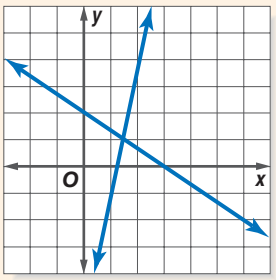

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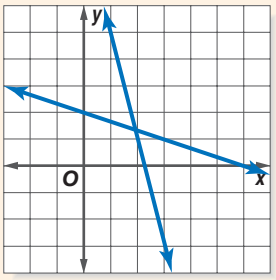

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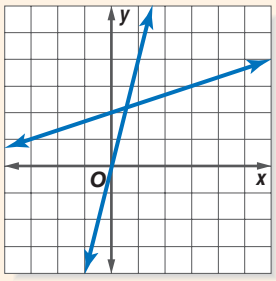

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3-4 Linear Programming

Main Ideas

- Find the maximum and minimum values of a function over a region.
- Solve real-world problems using linear programming.

New Vocabulary

constraints
feasible region
bounded
vertex
unbounded
linear programming

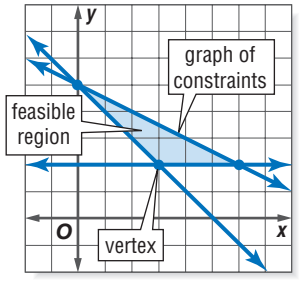
GET READY for the Lesson

The U.S. Coast Guard maintains the buoys that ships use to navigate. The ships that service buoys are called *buoy tenders.*

Suppose a buoy tender can carry up to 8 replacement buoys. The crew can repair a buoy in 1 hour and replace a buoy in $2\frac{1}{2}$ hours.

Maximum and Minimum Values The buoy tender captain can use a system of inequalities to represent the limits of time and replacements on the ship. If these inequalities are graphed, the points in the intersection are combinations of repairs and replacements that can be scheduled.

The inequalities are called the **constraints**. The intersection of the graphs is called the **feasible region**. When the graph of a system of constraints is a polygonal region like the one graphed at the right, we say that the region is **bounded**.

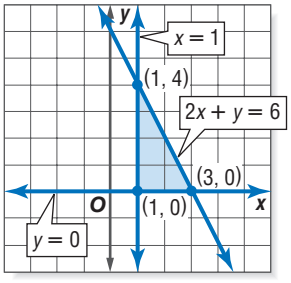


Since the buoy tender captain wants to service the maximum number of buoys, he will need to find the maximum value of the function for points in the feasible region. The maximum or minimum value of a related function *always* occurs at a **vertex** of the feasible region.

EXAMPLE 1 Bounded Region

Graph the following system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$x \geq 1$
$y \geq 0$
$2x + y \leq 6$
$f(x, y) = 3x + y$



Step 1 Graph the inequalities. The polygon formed is a triangle with vertices at (1, 4), (3, 0), and (1, 0).

Reading Math

Function Notation
The notation $f(x, y)$ is used to represent a function with two variables x and y. It is read *f of x and y*.

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Step 2 Use a table to find the maximum and minimum values of $f(x, y)$.

Substitute the coordinates of the vertices into the function.

(x, y)	$3x + y$	$f(x, y)$	
(1, 4)	3(1) + 4	7	
(3, 0)	3(3) + 0	9	← maximum
(1, 0)	3(1) + 0	3	← minimum

The maximum value is 9 at (3, 0). The minimum value is 3 at (1, 0).

Study Tip

Common Misconception

Do not assume that there is no minimum value if the feasible region is unbounded below the line, or that there is no maximum value if the feasible region is unbounded above the line.

CHECK Your Progress

1. $x \leq 2$

$3x - y \geq -2$

$y \geq x - 2$

$f(x, y) = 2x - 3y$

Sometimes a system of inequalities forms a region that is open. In this case, the region is said to be **unbounded**.

EXAMPLE 2 Unbounded Region

Graph the following system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the function for this region.

$2x + y \geq 3$

$3y - x \leq 9$

$2x + y \leq 10$

$f(x, y) = 5x + 4y$

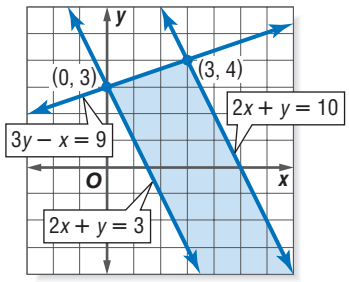
Review Vocabulary

Feasible

Everyday Use possible or likely

Math Use the area of a graph where it is possible to find a solution to a system of inequalities

Graph the system of inequalities. There are only two points of intersection, (0, 3) and (3, 4).



(x, y)	$5x + 4y$	$f(x, y)$
(0, 3)	5(0) + 4(3)	12
(3, 4)	5(3) + 4(4)	31

The maximum is 31 at (3, 4).

Although $f(0, 3)$ is 12, it is not the minimum value since there are other points in the solution that produce lesser values. For example, $f(3, -2) = 7$ and $f(20, -35) = -40$. It appears that because the region is unbounded, $f(x, y)$ has no minimum value.

CHECK Your Progress

2. $g \leq -3h + 4$

$g \geq -3h - 6$

$g \geq \frac{1}{3}h - 6$

$f(g, h) = 2g - 3h$

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Linear Programming The process of finding maximum or minimum values of a function for a region defined by inequalities is called **linear programming**.

KEY CONCEPT — *Linear Programming Procedure*

Step 1 Define the variables.

Step 2 Write a system of inequalities.

Step 3 Graph the system of inequalities.

Step 4 Find the coordinates of the vertices of the feasible region.

Step 5 Write a linear function to be maximized or minimized.

Step 6 Substitute the coordinates of the vertices into the function.

Step 7 Select the greatest or least result. Answer the problem.

Real-World Link

Animal surgeries are usually performed in the morning so that the animal can recover throughout the day while there is plenty of staff to monitor its progress.

Source: www.vetmedicine.miningco.com

Real-World EXAMPLE 3 Linear Programming

VETERINARY MEDICINE **As a receptionist for a veterinarian, one of Dolores Alvarez's tasks is to schedule appointments. She allots 20 minutes for a routine office visit and 40 minutes for a surgery. The veterinarian cannot do more than 6 surgeries per day. The office has 7 hours available for appointments. If an office visit costs $55 and most surgeries cost $125, how can she maximize the income for the day?**

Step 1 Define the variables.

v = the number of office visits
s = the number of surgeries

Step 2 Write a system of inequalities.

Since the number of appointments cannot be negative, v and s must be nonnegative numbers.

$v \geq 0$ and $s \geq 0$

An office visit is 20 minutes, and a surgery is 40 minutes. There are 7 hours available for appointments.

$20v + 40s \leq 420$ 7 hours = 420 minutes

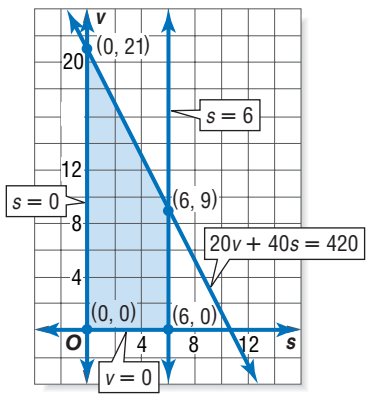
The veterinarian cannot do more than 6 surgeries per day.

$s \leq 6$

Step 3 Graph the system of inequalities.

Step 4 Find the coordinates of the vertices of the feasible region.

From the graph, the vertices of the feasible region are at (0, 0), (6, 0), (6, 9), and (0, 21). If the vertices could not be read from the graph easily, we could also solve a system of equations using the boundaries of the inequalities.



Concepts in Motion

Animation
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Caroline Penn/CORBIS

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Study Tip

Reasonableness

Check your solutions for reasonableness by thinking of the situation in context. Surgeries provide more income than office visits. So to maximize income, the veterinarian would do the most possible surgeries in a day.

Step 5 Write a function to be maximized or minimized.

The function that describes the income is $f(s, v) = 125s + 55v$. We wish to find the maximum value for this function.

Step 6 Substitute the coordinates of the vertices into the function.

(s, v)	$125s + 55v$	$f(s, v)$
(0, 0)	125(0) + 55(0)	0
(6, 0)	125(6) + 55(0)	750
(6, 9)	125(6) + 55(9)	1245
(0, 21)	125(0) + 55(21)	1155

Step 7 Select the greatest or least result. Answer the problem.

The maximum value of the function is 1245 at (6, 9). This means that the maximum income is \$1245 when Dolores schedules 6 surgeries and 9 office visits.

CHECK Your Progress

3. **BUSINESS** A landscaper balances his daily projects between small landscape jobs and mowing lawns. He allots 30 minutes per lawn and 90 minutes per small landscape job. He works at most ten hours per day. The landscaper earns \$35 per lawn and \$125 per landscape job. He cannot do more than 3 landscape jobs per day and get all of his mowing done. Find a combination of lawns mowed and completed landscape jobs per week that will maximize income.

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Example 1 (pp. 138–139)

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

1. $y \geq 2$
   $x \geq 1$
   $x + 2y \leq 9$
   $f(x, y) = 2x - 3y$

2. $y \leq 2x + 1$
   $1 \leq y \leq 3$
   $x + 2y \leq 12$
   $f(x, y) = 3x + y$

3. $x \leq 5$
   $y \geq -2$
   $y \leq x - 1$
   $f(x, y) = x - 2y$

4. $y \geq -x + 3$
   $1 \leq x \leq 4$
   $y \leq x + 4$
   $f(x, y) = -x + 4y$

Example 2 (p. 139)

5. $y \geq -x + 2$
   $2 \leq x \leq 7$
   $y \leq \frac{1}{2}x + 5$
   $f(x, y) = 8x + 3y$

6. $x + 2y \leq 6$
   $2x - y \leq 7$
   $x \geq -2, y \geq -3$
   $f(x, y) = x - y$

7. $x \geq -3$
   $y \leq 1$
   $3x + y \leq 6$
   $f(x, y) = 5x - 2y$

8. $y \leq x + 2$
   $y \leq 11 - 2x$
   $2x + y \geq -7$
   $f(x, y) = 4x - 3y$

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Example 3
(pp. 139–140)

MANUFACTURING **For Exercises 9–14, use the following information.**
The Future Homemakers Club is making canvas tote bags and leather tote bags for a fund-raiser. They will line both types of tote bags with canvas and use leather for the handles of both. For the canvas bags, they need 4 yards of canvas and 1 yard of leather. For the leather bags, they need 3 yards of leather and 2 yards of canvas. Their advisor purchased 56 yards of leather and 104 yards of canvas.

9. Let c represent the number of canvas bags and let ℓ represent the number of leather bags. Write a system of inequalities for the number of bags that can be made.

10. Draw the graph showing the feasible region.

11. List the coordinates of the vertices of the feasible region.

12. If the club plans to sell the canvas bags at a profit of $20 each and the leather bags at a profit of $35 each, write a function for the total profit on the bags.

13. How can the club make the maximum profit?

14. What is the maximum profit?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
15–20	1
21–27	2
28–33	3

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

15. $y \geq 1$
$x \leq 6$
$y \leq 2x + 1$
$f(x, y) = x + y$

16. $y \geq -4$
$x \leq 3$
$y \leq 3x - 4$
$f(x, y) = x - y$

17. $y \geq 2$
$1 \leq x \leq 5$
$y \leq x + 3$
$f(x, y) = 3x - 2y$

18. $y \geq 1$
$2 \leq x \leq 4$
$x - 2y \geq -4$
$f(x, y) = 3y + x$

19. $y \leq x + 6$
$y + 2x \geq 6$
$2 \leq x \leq 6$
$f(x, y) = -x + 3y$

20. $x - 3y \geq -7$
$5x + y \leq 13$
$x + 6y \geq -9$
$3x - 2y \geq -7$
$f(x, y) = x - y$

21. $x + y \geq 4$
$3x - 2y \leq 12$
$x - 4y \geq -16$
$f(x, y) = x - 2y$

22. $y \geq x - 3$
$y \leq 6 - 2x$
$2x + y \geq -3$
$f(x, y) = 3x + 4y$

23. $2x + 3y \geq 6$
$3x - 2y \geq -4$
$5x + y \geq 15$
$f(x, y) = x + 3y$

24. $2x + 2y \geq 4$
$2y \geq 3x - 6$
$4y \leq x + 8$
$f(x, y) = 3y + x$

25. $x \geq 0$
$y \geq 0$
$x + 2y \leq 6$
$2y - x \leq 2$
$x + y \leq 5$
$f(x, y) = 3x - 5y$

26. $x \geq 2$
$y \geq 1$
$x - 2y \geq -4$
$x + y \leq 8$
$2x - y \leq 7$
$f(x, y) = x - 4y$

27. **RESEARCH** Use the Internet or other reference to find an industry that uses linear programming. Describe the restrictions or constraints of the problem and explain how linear programming is used to help solve the problem.

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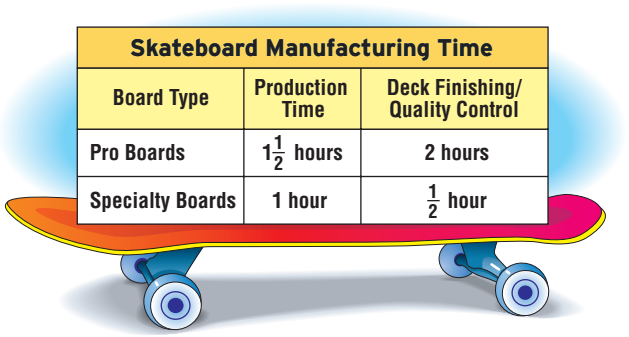
PRODUCTION **For Exercises 28–33, use the following information.**
The total number of workers' hours per day available for production in a skateboard factory is 85 hours. There are 40 workers' hours available for finishing decks and quality control each day. The table shows the number of hours needed in each department for two different types of skateboards.



Skateboard Manufacturing Time		
Board Type	Production Time	Deck Finishing/ Quality Control
Pro Boards	$1\frac{1}{2}$ hours	2 hours
Specialty Boards	1 hour	$\frac{1}{2}$ hour

28. Let g represent the number of pro boards and let c represent the number of specialty boards. Write a system of inequalities to represent the situation.

29. Draw the graph showing the feasible region.

30. List the coordinates of the vertices of the feasible region.

31. If the profit on a pro board is \$50 and the profit on a specialty board is \$65, write a function for the total profit on the skateboards.

32. Determine the number of each type of skateboard that needs to be made to have a maximum profit.

33. What is the maximum profit?

FARMING **For Exercises 34–37, use the following information.**
Dean Stadler has 20 days in which to plant corn and soybeans. The corn can be planted at a rate of 250 acres per day and the soybeans at a rate of 200 acres per day. He has 4500 acres available for planting these two crops.

34. Let c represent the number of acres of corn and let s represent the number of acres of soybeans. Write a system of inequalities to represent the possible ways Mr. Stadler can plant the available acres.

35. Draw the graph showing the feasible region and list the coordinates of the vertices of the feasible region.

36. If the profit is \$26 per acre on corn and \$30 per acre on soybeans, how much of each should Mr. Stadler plant? What is the maximum profit?

37. How much of each should Mr. Stadler plant if the profit on corn is \$29 per acre and the profit on soybeans is \$24 per acre? What is the maximum profit?

38. **MANUFACTURING** The Cookie Factory wants to sell chocolate chip and peanut butter cookies in combination packages of 6–12 cookies. At least three of each type of cookie should be in each package. The cost of making a chocolate chip cookie is 19¢, and the selling price is 44¢ each. The cost of making a peanut butter cookie is 13¢, and the selling price is 39¢. How many of each type of cookie should be in each package to maximize the profit?

EXTRA PRACTICE
See pages 896, 928.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. **OPEN ENDED** Create a system of inequalities that forms a bounded region.

40. **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true.
A function defined by a feasible region has a minimum and a maximum value.

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41. Which One Doesn't Belong? Given the following system of inequalities, which ordered pair does not belong? Explain your reasoning.

$$y \leq \frac{1}{2}x + 5 \qquad y < -3x + 7 \qquad y \geq -\frac{1}{3}x - 2$$

(0, 0)	(−2, 6)	(−3, 2)	(1, −1)

42. CHALLENGE The vertices of a feasible region are $A(1, 2)$, $B(5, 2)$, and $C(1, 4)$. Write a function where A is the maximum and B is the minimum.

43. *Writing in Math* Use the information about buoy tenders on page 138 to explain how linear programming can be used in scheduling work. Include a system of inequalities that represents the constraints that are used to schedule buoy repair and replacement and an explanation of the linear function that the buoy tender captain would wish to maximize.

STANDARDIZED TEST PRACTICE

44. ACT/SAT For a game she's playing, Liz must draw a card from a deck of 26 cards, one with each letter of the alphabet on it, and roll a six-sided die. What is the probability that Liz will roll an odd number and draw a letter in her name?

A $\frac{2}{3}$ **B** $\frac{1}{13}$ **C** $\frac{1}{26}$ **D** $\frac{3}{52}$

45. REVIEW Which of the following best describes the graphs of $y = 3x - 5$ and $4y = 12x + 16$?

F The lines have the same y-intercept.

G The lines have the same x-intercept.

H The lines are perpendicular.

J The lines are parallel.

Spiral Review

Solve each system of inequalities by graphing. (Lesson 3-3)

46. $2y + x \geq 4$
$y \geq x - 4$

47. $3x - 2y \leq -6$
$y \leq \frac{3}{2}x - 1$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

48. $4x + 5y = 20$
$5x + 4y = 7$

49. $6x + y = 15$
$x - 4y = -10$

50. $3x + 8y = 23$
$x - y = 4$

51. CARD COLLECTING Nathan has 50 baseball cards in his collection from the 1950's and 1960's. His goal is to buy 2 more cards each month. Write an equation that represents how many cards Nathan will have in his collection in x months if he meets his goal. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression if $x = -2$, $y = 6$, and $z = 5$.** (Lesson 1-1)

52. $x + y + z$

53. $2x - y + 3z$

54. $-x + 4y - 2z$

55. $5x + 2y - z$

56. $3x - y + 4z$

57. $-2x - 3y + 2z$

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3-5 Solving Systems of Equations in Three Variables

Main Ideas

- Solve systems of linear equations in three variables.
- Solve real-world problems using systems of linear equations in three variables.

New Vocabulary

ordered triple

GET READY for the Lesson

At the 2004 Summer Olympics in Athens, Greece, the United States won 103 medals. They won 6 more gold medals than bronze and 10 more silver medals than bronze.

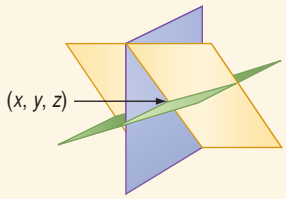
You can write and solve a system of three linear equations to determine how many of each type of medal the U.S. Olympians won. Let g represent the number of gold medals, let s represent the number of silver medals, and let b represent the number of bronze medals.

$g + s + b = 103$ U.S. Olympians won a total of 103 medals.
$g = b + 6$ They won 6 more gold medals than bronze.
$s = b + 10$ They won 10 more silver medals than bronze.

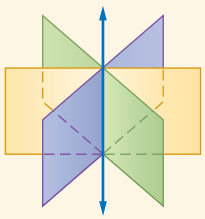
Systems in Three Variables The system of equations above has three variables. The graph of an equation in three variables, all to the first power, is a plane. The solution of a system of three equations in three variables can have one solution, infinitely many solutions, or no solution.

KEY CONCEPT *System of Equations in Three Variables*

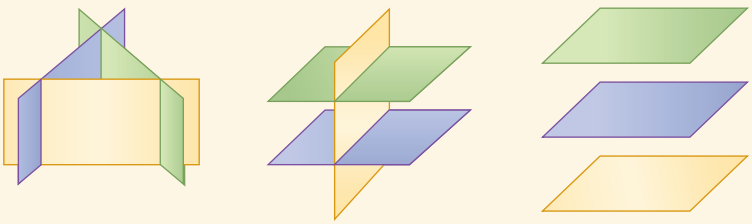
One Solution
- planes intersect in one point



Infinitely Many Solutions
- planes intersect in a line
- planes intersect in the same plane



No Solution
- planes have no point in common



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Solving systems of equations in three variables is similar to solving systems of equations in two variables. Use the strategies of substitution and elimination. The solution of a system of equations in three variables x, y, and z is called an **ordered triple** and is written as (x, y, z).

EXAMPLE One Solution

1 **Solve the system of equations.**

$x + 2y + z = 10$

$2x - y + 3z = -5$

$2x - 3y - 5z = 27$

Study Tip

Elimination

Remember that you can eliminate any of the three variables.

Step 1 Use elimination to make a system of two equations in two variables.

$$\begin{array}{rl} x + 2y + z = 10 & \xrightarrow{\text{Multiply by 2.}} \quad 2x + 4y + 2z = 20 \\ 2x - y + 3z = -5 & \underline{(-)\ 2x - y + 3z = -5} \\ & 5y - z = 25 \quad \text{Subtract to eliminate } x. \end{array}$$

$$\begin{array}{rl} 2x - y + 3z = -5 & \text{Second equation} \\ \underline{(-)\ 2x - 3y - 5z = 27} & \text{Third equation} \\ 2y + 8z = -32 & \text{Subtract to eliminate } x. \end{array}$$

Notice that the x terms in each equation have been eliminated. The result is two equations with the same two variables y and z.

Step 2 Solve the system of two equations.

$$\begin{array}{rl} 5y - z = 25 & \xrightarrow{\text{Multiply by 8.}} \quad 40y - 8z = 200 \\ 2y + 8z = -32 & \underline{(+)\ 2y + 8z = -32} \\ & 42y = 168 \quad \text{Add to eliminate } z. \\ & y = 4 \quad \text{Divide by 42.} \end{array}$$

Use one of the equations with two variables to solve for z.

$$\begin{array}{rl} 5y - z = 25 & \text{Equation with two variables} \\ 5(4) - z = 25 & \text{Replace } y \text{ with 4.} \\ 20 - z = 25 & \text{Multiply.} \\ z = -5 & \text{Simplify.} \end{array}$$

The result is $y = 4$ and $z = -5$.

Step 3 Solve for x using one of the original equations with three variables.

$$\begin{array}{rl} x + 2y + z = 10 & \text{Original equation with three variables} \\ x + 2(4) + (-5) = 10 & \text{Replace } y \text{ with 4 and } z \text{ with } -5. \\ x + 8 - 5 = 10 & \text{Multiply.} \\ x = 7 & \text{Simplify.} \end{array}$$

The solution is $(7, 4, -5)$. Check this solution in the other two original equations.

CHECK Your Progress

1A. $2x - y + 3z = -2$
$x + 4y - 2z = 16$
$5x + y - 1z = 14$

1B. $3x + y + z = 0$
$-x + 2y - 2z = -3$
$4x - y - 3z = 9$

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EXAMPLE Infinitely Many Solutions

2 **Solve the system of equations.**

$4x - 6y + 4z = 12$

$6x - 9y + 6z = 18$

$5x - 8y + 10z = 20$

Eliminate x in the first two equations.

$4x - 6y + 4z = 12$ — Multiply by 3. → $12x - 18y + 12z = 36$

$6x - 9y + 6z = 18$ — Multiply by -2. → $(+)\ -12x + 18y - 12z = -36$ Add the equations.

$0 = 0$

The equation $0 = 0$ is always true. This indicates that the first two equations represent the same plane. Check to see if this plane intersects the third plane.

$4x - 6y + 4z = 12$ — Multiply by 5. → $20x - 30y + 20z = 60$

$5x - 8y + 10z = 20$ — Multiply by -2. → $(+)\ -10x + 16y - 20z = -40$ Add the equations.

$10x - 14y = 20$ Divide by the GCF, 2.

$5x - 7y = 10$

The planes intersect in a line. So, there are an infinite number of solutions.

CHECK Your Progress

2A. $8x + 12y - 24z = -40$
$3x - 8y + 12z = 23$
$2x + 3y - 6z = -10$

2B. $3x - 2y + 4z = 8$
$-6x + 4y - 8z = -16$
$x + 2y - 4z = 4$

Study Tip

Common Misconception

Not every ordered triple is a solution of a system in three variables with an infinite number of solutions. The solution set contains an infinite number of ordered triples but not every ordered triple.

EXAMPLE No Solution

3 **Solve the system of equations.**

$6a + 12b - 8c = 24$

$9a + 18b - 12c = 30$

$4a + 8b - 7c = 26$

Eliminate a in the first two equations.

$6a + 12b - 8c = 24$ — Multiply by 3. → $18a + 36b - 24c = 72$

$9a + 18b - 12c = 30$ — Multiply by 2. → $(-)\ 18a + 36b - 24c = 60$ Subtract the equations.

$0 = 12$

The equation $0 = 12$ is never true. So, there is no solution of this system.

CHECK Your Progress

3A. $8x + 4y - 3z = 7$
$4x + 2y - 6z = -15$
$10x + 5y - 15z = -25$

3B. $4x - 3y - 2z = 8$
$x + 5y + 3z = 9$
$-8x + 6y + 4z = 2$

Real-World Problems When solving problems involving three variables, use the four-step plan to help organize the information.

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Real-World EXAMPLE Write and Solve a System of Equations

4 INVESTMENTS **Andrew Chang has $15,000 that he wants to invest in certificates of deposit (CDs). For tax purposes, he wants his total interest per year to be $800. He wants to put $1000 more in a 2-year CD than in a 1-year CD and invest the rest in a 3-year CD. How much should Mr. Chang invest in each type of CD?**

Number of Years	1	2	3
Rate	3.4%	5.0%	6.0%

Real-World Link

A certificate of deposit (CD) is a way to invest your money with a bank. The bank generally pays higher interest rates on CDs than savings accounts. However, you must invest your money for a specific time period, and there are penalties for early withdrawal.

Explore Read the problem and define the variables.

a = the amount of money invested in a 1-year certificate

b = the amount of money in a 2-year certificate

c = the amount of money in a 3-year certificate

Plan Mr. Chang has $15,000 to invest.

$$a + b + c = 15{,}000$$

The interest he earns should be $800. The interest equals the rate times the amount invested.

$$0.034a + 0.05b + 0.06c = 800$$

There is $1000 more in the 2-year certificate than in the 1-year certificate.

$$b = a + 1000$$

Solve Substitute $b = a + 1000$ in each of the first two equations.

$$\begin{aligned} a + (a + 1000) + c &= 15{,}000 && \text{Replace } b \text{ with } (a + 1000).\\ 2a + 1000 + c &= 15{,}000 && \text{Simplify.}\\ 2a + c &= 14{,}000 && \text{Subtract 1000 from each side.} \end{aligned}$$

$$\begin{aligned} 0.034a + 0.05(a + 1000) + 0.06c &= 800 && \text{Replace } b \text{ with } (a + 1000).\\ 0.034a + 0.05a + 50 + 0.06c &= 800 && \text{Distributive Property}\\ 0.084a + 0.06c &= 750 && \text{Simplify.} \end{aligned}$$

Now solve the system of two equations in two variables.

$$\begin{aligned} 2a + c &= 14{,}000 \\ 0.084a + 0.06c &= 750 \end{aligned} \quad \xrightarrow{\text{Multiply by 0.06.}} \quad \begin{aligned} 0.12a + 0.06c &= 840 \\ (-)\ 0.084a + 0.06c &= 750 \\ \hline 0.036a &= 90 \\ a &= 2500 \end{aligned}$$

Substitute 2500 for a in one of the original equations.

$$\begin{aligned} b &= a + 1000 && \text{Third equation}\\ &= 2500 + 1000 && a = 2500\\ &= 3500 && \text{Add.} \end{aligned}$$

M. Angelo/CORBIS

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Substitute 2500 for a and 3500 for b in one of the original equations.

$$\begin{aligned} a + b + c &= 15{,}000 && \text{First equation} \\ 2500 + 3500 + c &= 15{,}000 && a = 2500,\ b = 3500 \\ 6000 + c &= 15{,}000 && \text{Add.} \\ c &= 9000 && \text{Subtract 6000 from each side.} \end{aligned}$$

So, Mr. Chang should invest \$2500 in a 1-year certificate, \$3500 in a 2-year certificate, and \$9000 in a 3-year certificate.

Check Is the answer reasonable? Have all the criteria been met?

The total investment is \$15,000.

$2500 + 3500 + 9000 = 15{,}000$ ✓

The interest earned will be \$800.

$0.034(2500) + 0.05(3500) + 0.06(9000) = 800$

$85 + 175 + 540 = 800$ ✓

There is \$1000 more in the 2-year certificate than the 1-year certificate.

$3500 = 2500 + 1000$ ✓ The answer is reasonable.

CHECK Your Progress

4. BASKETBALL Macario knows that he has scored a total of 70 points so far this basketball season. His coach told him that he has scored 37 times, but Macario wants to know how many free throws, field goals, and three pointers he has made. The sum of his field goals and three pointers equal twice the number of free throws minus two. How many free throws, field goals, and three pointers has Macario made?

Concepts in Motion
Interactive Lab
algebra2.com

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Examples 1–3 (pp. 146–147)

Solve each system of equations.

1. $x + 2y = 12$
$3y - 4z = 25$
$x + 6y + z = 20$

2. $9a + 7b = -30$
$8b + 5c = 11$
$-3a + 10c = 73$

3. $r - 3s + t = 4$
$3r - 6s + 9t = 5$
$4r - 9s + 10t = 9$

4. $2r + 3s - 4t = 20$
$4r - s + 5t = 13$
$3r + 2s + 4t = 15$

5. $2x - y + z = 1$
$x + 2y - 4z = 3$
$4x + 3y - 7z = -8$

6. $x + y + z = 12$
$6x - 2y - z = 16$
$3x + 4y + 2z = 28$

Example 4 (pp. 148–149)

COOKING For Exercises 7 and 8, use the following information.
Jambalaya is a Cajun dish made from chicken, sausage, and rice. Simone is making a large pot of jambalaya for a party. Chicken costs \$6 per pound, sausage costs \$3 per pound, and rice costs \$1 per pound. She spends \$42 on 13.5 pounds of food. She buys twice as much rice as sausage.

7. Write a system of three equations that represents how much food Simone purchased.

8. How much chicken, sausage, and rice will she use in her dish?

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Exercises

HOMEWORK HELP

For Exercises	See Examples
9–19	1–3
20–23	4

Solve each system of equations.

9. $2x - y = 2$
$3z = 21$
$4x + z = 19$

10. $-4a = 8$
$5a + 2c = 0$
$7b + 3c = 22$

11. $5x + 2y = 4$
$3x + 4y + 2z = 6$
$7x + 3y + 4z = 29$

12. $8x - 6z = 38$
$2x - 5y + 3z = 5$
$x + 10y - 4z = 8$

13. $4a + 2b - 6c = 2$
$6a + 3b - 9c = 3$
$8a + 4b - 12c = 6$

14. $2r + s + t = 14$
$-r - 3s + 2t = -2$
$4r - 6s + 3t = -5$

15. $3x + y + z = 4$
$2x + 2y + 3z = 3$
$x + 3y + 2z = 5$

16. $4a - 2b + 8c = 30$
$a + 2b - 7c = -12$
$2a - b + 4c = 15$

17. $9x - 3y + 12z = 39$
$12x - 4y + 16z = 52$
$3x - 8y + 12z = 23$

18. The sum of three numbers is 20. The second number is 4 times the first, and the sum of the first and third is 8. Find the numbers.

19. The sum of three numbers is 12. The first number is twice the sum of the second and third. The third number is 5 less than the first. Find the numbers.

Real-World Link

In 2005, Katie Smith became the first person in the WNBA to score 5000 points.

Source: www.wnba.com

BASKETBALL **For Exercises 20 and 21, use the following information.**
In the 2004 season, Seattle's Lauren Jackson was ranked first in the WNBA for total points and points per game. She scored 634 points making 362 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 26 more 2-point field goals than free throws.

20. Write a system of equations that represents the number of goals she made.

21. Find the number of each type of goal she made.

FOOD **For Exercises 22 and 23, use the following information.**
Maka loves the lunch combinations at Rosita's Mexican Restaurant. Today however, she wants a different combination than the ones listed on the menu.

22. Assume that the price of a combo meal is the same price as purchasing each item separately. Find the price for an enchilada, a taco, and a burrito.

23. If Maka wants 2 burritos and 1 enchilada, how much should she plan to spend?

24. **TRAVEL** Jonathan and members of his Spanish Club are going to Costa Rica. He purchases 10 traveler's checks in denominations of $20, $50, and $100, totaling $370. He has twice as many $20 checks as $50 checks. How many of each denomination of traveler's checks does he have?

EXTRA PRACTICE
See pages 896, 928.

Math Online
Self-Check Quiz at algebra2.com

Solve each system of equations.

25. $6x + 2y + 4z = 2$
$3x + 4y - 8z = -3$
$-3x - 6y + 12z = 5$

26. $r + s + t = 5$
$2r - 7s - 3t = 13$
$\frac{1}{2}r - \frac{1}{3}s + \frac{2}{3}t = -1$

27. $2a - b + 3c = -7$
$4a + 5b + c = 29$
$a - \frac{2b}{3} + \frac{c}{4} = -10$

Andy Lyons/Allsport/Getty Images
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H.O.T. Problems

28. OPEN ENDED Write an example of a system of three equations in three variables that has $(-3, 5, 2)$ as a solution. Show that the ordered triple satisfies all three equations.

29. REASONING Compare and contrast solving a system of two equations in two variables to solving a system of equations of three equations in three variables.

30. FIND THE ERROR Melissa is solving the system of equations $r + 2s + t = 3$, $2r + 4s + 2t = 6$, and $3r + 6s + 3t = 12$. Is she correct? Explain.

$r + 2s + t = 3 \rightarrow 2r + 4s + 2t = 6$

$2r + 4s + 2t = 6 \rightarrow (-)2r + 4s + 2t = 6$

$0 = 0$

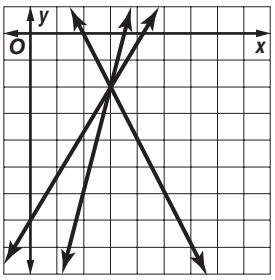
The second equation is a multiple of the first, so they are the same plane. There are infinitely many solutions.

31. CHALLENGE The general form of an equation for a parabola is $y = ax^2 + bx + c$, where (x, y) is a point on the parabola. If three points on the parabola are $(0, 3)$, $(-1, 4)$, and $(2, 9)$, determine the values of a, b, c. Write the equation of the parabola.

32. *Writing in Math* Use the information on page 145 to explain how you can determine the number and type of medals 2004 U.S. Olympians won in Athens. Demonstrate how to find the number of each type of medal won by the U.S. Olympians and describe another situation where you can use a system of three equations in three variables to solve a problem.

STANDARDIZED TEST PRACTICE

33. ACT/SAT The graph depicts which system of equations?



A $y + 14 = 4x$
$y = 4 - 2x$
$-7 = y - \frac{5}{3}x$

C $y - 14 = 4x$
$y = 4 + 2x$
$-7 = y + \frac{5}{3}$

B $y + 14x = 4$
$-2y = 4 + y$
$-7 = y - \frac{5}{3}x$

D $y - 14x = 4$
$2x = 4 + y$
$7 = y - \frac{5}{3}x$

34. REVIEW What is the solution to the system of equations shown below?

$$\begin{cases} x - y + z = 0 \\ -5x + 3y - 2z = -1 \\ 2x - y + 4z = 11 \end{cases}$$

F $(0, 3, 3)$

G $(2, 5, 3)$

H no solution

J infinitely many solutions

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Spiral Review

35. MILK The Yoder Family Dairy produces at most 200 gallons of skim and whole milk each day for delivery to large bakeries and restaurants. Regular customers require at least 15 gallons of skim and 21 gallons of whole milk each day. If the profit on a gallon of skim milk is \$0.82 and the profit on a gallon of whole milk is \$0.75, how many gallons of each type of milk should the dairy produce each day to maximize profits? (Lesson 3-4)

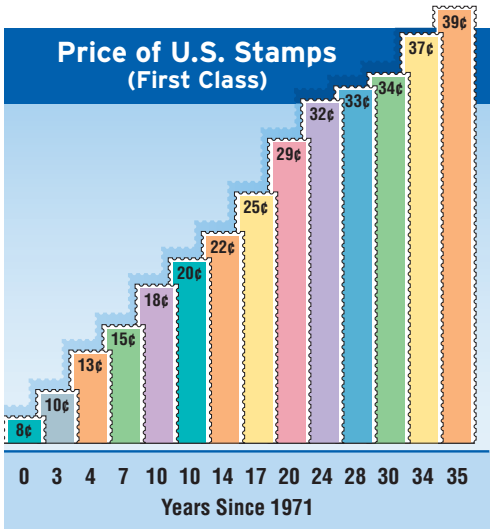
Solve each system of inequalities by graphing. (Lesson 3-3)

36. $y \leq x + 2$
$y \geq 7 - 2x$

37. $4y - 2x > 4$
$3x + y > 3$

38. $3x + y \geq 1$
$2y - x \leq -4$

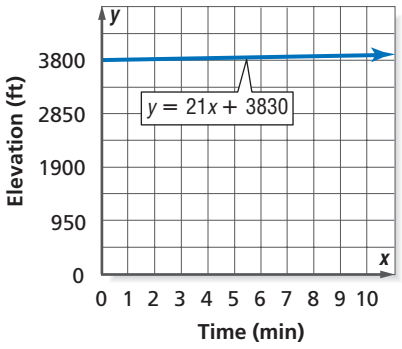
ANALYZE GRAPHS **For Exercises 39 and 40, use the following information.**
The table shows the price for first-class stamps since July 1, 1971. (Lesson 2-5)



39. Write a prediction equation for this relationship.

40. Predict the price for a first-class stamp issued in the year 2015.

41. HIKING Miguel is hiking on the Alum Cave Bluff Trail in the Great Smoky Mountains. The graph represents Miguel's elevation y at each time x. At what elevation did Miguel begin his climb? How is that represented in the equation? (Lesson 2-4)



Find each value if $f(x) = 6x + 2$ and $g(x) = 3x^2 - x$. (Lesson 2-1)

42. $f(-1)$

43. $f\left(\frac{1}{2}\right)$

44. $g(1)$

45. $g(-3)$

46. TIDES Ocean tides are caused by gravitational forces exerted by the Moon. Tides are also influenced by the size, boundaries, and depths of ocean basins and inlets. The highest tides on Earth occur in the Bay of Fundy in Nova Scotia, Canada. During the middle of the tidal range, the ocean shore is 30 meters from a rock bluff. The tide causes the shoreline to advance 8 meters and retreat 8 meters throughout the day. Write and solve an equation describing the maximum and minimum distances from the rock bluff to the ocean during high and low tide. (Lesson 1-4)

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CHAPTER 3

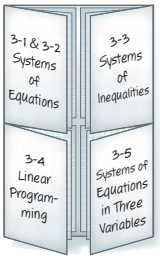
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Systems of Equations (Lessons 3-1 and 3-2)

- The solution of a system of equations can be found by graphing the two equations and determining at what point they intersect.
- In the substitution method, one equation is solved for a variable and substituted to find the value of another variable.
- In the elimination method, one variable is eliminated by adding or subtracting the equations.

Systems of Inequalities (Lesson 3-3)

- The solution of a system of inequalities is found by graphing the inequalities and determining the intersection of the graphs.

Linear Programming (Lesson 3-4)

- The maximum and minimum values of a function are determined by linear programming techniques.

Systems of Three Equations (Lesson 3-5)

- A system of equations in three variables can be solved algebraically by using the substitution method or the elimination method.

Key Vocabulary

bounded region (p. 138)
consistent system (p. 118)
constraints (p. 138)
dependent system (p. 118)
elimination method (p. 125)
feasible region (p. 138)
inconsistent system (p. 118)
independent system (p. 118)
linear programming (p. 140)
ordered triple (p. 146)
substitution method (p. 123)
system of equations (p. 116)
system of inequalities (p. 130)
unbounded region (p. 139)
vertex (p. 138)

Vocabulary Check

Choose the term from the list above that best matches each phrase.

1. the inequalities of a linear programming problem
2. a system of equations that has an infinite number of solutions
3. the region of a graph where every constraint is met
4. a method of solving equations in which one equation is solved for one variable in terms of the other variable
5. a system of equations that has at least one solution
6. a system of equations that has exactly one solution
7. a method of solving equations in which one variable is eliminated when the two equations are combined
8. the solution of a system of equations in three variables (x, y, z)
9. two or more equations with the same variables
10. two or more inequalities with the same variables

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Lesson-by-Lesson Review

3–1 Solving Systems of Equations by Graphing (pp. 116–122)

Solve each system of linear equations by graphing.

11. $3x + 2y = 12$
$x - 2y = 4$

12. $8x - 10y = 7$
$4x - 5y = 7$

13. $y - 2x = 8$
$y = \frac{1}{2}x - 4$

14. $20y + 13x = 10$
$0.65x + y = 0.5$

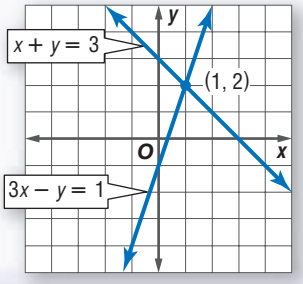
15. PLUMBING Two plumbers offer competitive services. The first charges a \$35 house-call fee and \$28 per hour. The second plumber charges a \$42 house-call fee and \$21 per hour. After how many hours do the two plumbers charge the same amount?

Example 1 **Solve the system of equations by graphing.**

$x + y = 3$
$3x - y = 1$

Graph both equations on the same coordinate plane.

The solution of the system is (1, 2).



3–2 Solving Systems of Equations Algebraically (pp. 123–129)

Solve each system of equations by using either substitution or elimination.

16. $x + y = 5$
$2x - y = 4$

17. $2x - 3y = 9$
$4x + 2y = -22$

18. $7y - 2x = 10$
$-3y + x = -3$

19. $x + y = 4$
$x - y = 8.5$

20. $-6y - 2x = 0$
$11y + 3x = 4$

21. $3x - 5y = -13$
$4x + 2y = 0$

22. CLOTHING Colleen bought 15 used and lightly used T-shirts at a thrift store. The used shirts cost \$0.70 less than the lightly used shirts. Her total, minus tax, was \$16.15. If Colleen bought 8 used shirts and paid \$0.70 less per shirt than for a lightly used shirt, how much does each type of shirt cost?

Example 2 **Solve the system of equations by using either substitution or elimination.**

$x = 4y + 7$
$y = -3 - x$

Substitute $-3 - x$ for y in the first equation.

$x = 4y + 7$	First equation
$x = 4(-3 - x) + 7$	Substitute $-3 - x$ for y.
$x = -12 - 4x + 7$	Distributive Property
$5x = -5$	Add $4x$ to each side.
$x = -1$	Divide each side by 5.

Now substitute the value for x in either original equation.

$y = -3 - x$	Second equation
$= -3 - (-1)$ or -2	Replace x with -1 and simplify.

The solution of the system is $(-1, -2)$.

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Mixed Problem Solving
For mixed problem-solving practice, see page 928.

3–3 Solving Systems of Inequalities by Graphing (pp. 130–135)

Solve each system of inequalities by graphing. Use a table to analyze the possible solutions.

23. $y \leq 4$
$y > -3$

24. $|y| > 3$
$x \leq 1$

25. $y < x + 1$
$x > 5$

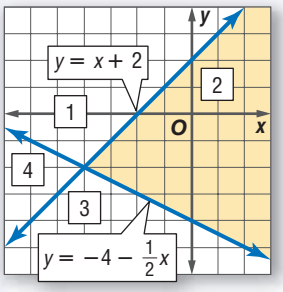
26. $y \leq x + 4$
$2y \geq x - 3$

27. JOBS Tamara spends no more than 5 hours working at a local manufacturing plant. It takes her 25 minutes to set up her equipment and at least 45 minutes for each unit she constructs. Draw a diagram that represents this information.

Example 3 **Solve the system of inequalities by graphing.**

$y \leq x + 2$

$y \geq -4 - \frac{1}{2}x$



The solution of the system is the region that satisfies both inequalities. The solution of this system is region 2.

3–4 Linear Programming (pp. 138–144)

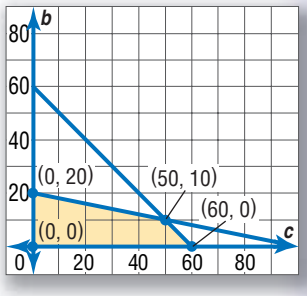
28. MANUFACTURING A toy manufacturer is introducing two new dolls to their customers: My First Baby, which talks, laughs, and cries, and My Real Baby, which simulates using a bottle and crawls. In one hour the company can produce 8 First Babies or 20 Real Babies. Because of the demand, the company must produce at least twice as many First Babies as Real Babies. The company spends no more than 48 hours per week making these two dolls. The profit on each First Baby is \$3.00 and the profit on each Real Baby is \$7.50. Find the number and type of dolls that should be produced to maximize the profit.

Example 4 **The area of a parking lot is 600 square meters. A car requires 6 square meters of space, and a bus requires 30 square meters of space. The attendant can handle no more than 60 vehicles. If a car is charged \$3 to park and a bus is charged \$8, how many of each should the attendant accept to maximize income?**

Let c = the number of cars and b = the number of buses.

$c \geq 0$, $b \geq 0$, $6c + 30b \leq 600$, and $c + b \leq 60$

Graph the inequalities. The vertices of the feasible region are (0, 0), (0, 20), (50, 10), and (60, 0).



The profit function is $f(c, b) = 3c + 8b$. The maximum value of \$230 occurs at (50, 10). So the attendant should accept 50 cars and 10 buses.

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3–5 Solving Systems of Equations in Three Variables (pp. 145–152)

Solve each system of equations.

29. $x + 4y - z = 6$
$3x + 2y + 3z = 16$
$2x - y + z = 3$

30. $2a + b - c = 5$
$a - b + 3c = 9$
$3a - 6c = 6$

31. $e + f = 4$
$2d + 4e - f = -3$
$3e = -3$

32. SUBS Ryan, Tyee, and Jaleel are ordering subs from a shop that lets them choose the number of meats, cheeses, and veggies that they want. Their sandwiches and how much they paid are displayed in the table. How much does each topping cost?

Name	Meat	Cheese	Veggie	Price
Ryan	1	2	5	$5.70
Tyee	3	2	2	$7.85
Jaleel	2	1	4	$6.15

Example 5 **Solve the system of equations.**
$x + 3y + 2z = 1$
$2x + y - z = 2$
$x + y + z = 2$

Use elimination to make a system of two equations in two variables.

$$\begin{array}{rl} 2x + 6y + 4z = 2 & \text{First equation} \times 2 \\ (-)\ 2x + y - z = 2 & \text{Second equation} \\ \hline 5y + 5z = 0 & \text{Subtract.} \end{array}$$

Do the same with the first and third equations to get $2y + z = -1$.

Solve the system of two equations.

$$\begin{array}{rl} 5y + 5z = 0 & \\ (-)10y + 5z = -5 & \\ \hline -5y = 5 & \text{Subtract to eliminate } z. \\ y = -1 & \text{Divide each side by } -5. \end{array}$$

Substitute -1 for y in one of the equations with two variables and solve for z.

Then, substitute -1 for y and the value you received for z into an equation from the original system to solve for x.

The solution is $(2, -1, 1)$.

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CHAPTER 3

Practice Test

Solve each system of equations.

1. $-4x + y = -5$
$2x + y = 7$

2. $x + y = -8$
$-3x + 2y = 9$

3. $3x + 2y = 18$
$y = 6x - 6$

4. $-6x + 3y = 33$
$-4x + y = 16$

5. $-7x + 6y = 42$
$3x + 4y = 28$

6. $2y = 5x - 1$
$x + y = -1$

Solve each system of inequalities by graphing.

7. $y \geq x - 3$
$y \geq -x + 1$

8. $x + 2y \geq 7$
$3x - 4y < 12$

9. $3x + y < -5$
$2x - 4y \geq 6$

10. $2x + y \geq 7$
$3y \leq 4x + 1$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and the minimum values of the given function.

11. $5 \geq y \geq -3$
$4x + y \leq 5$
$-2x + y \leq 5$
$f(x, y) = 4x - 3y$

12. $x \geq -10$
$1 \geq y \geq -6$
$3x + 4y \leq -8$
$2y \geq x - 10$
$f(x, y) = 2x + y$

13. **MULTIPLE CHOICE** Which statement best describes the graphs of the two equations?

$16x - 2y = 24$
$12x = 3y - 36$

A The lines are parallel.

B The lines are the same.

C The lines intersect in only one point.

D The lines intersect in more than one point, but are not the same.

Solve each system of equations.

14. $x + y + z = -1$
$2x + 4y + z = 1$
$x + 2y - 3z = -3$

15. $x + z = 7$
$2y - z = -3$
$-x - 3y + 2z = 11$

16. **MULTIPLE CHOICE** Carla, Meiko, and Kayla went shopping to get ready for college. Their purchases and total amounts spent are shown in the table below.

Person	Shirts	Pants	Shoes	Total Spent
Carla	3	4	2	$149.79
Meiko	5	3	3	$183.19
Kayla	6	5	1	$181.14

Assume that all of the shirts were the same price, all of the pants were the same price, and all of the shoes were the same price. What was the price of each item?

F shirt, $12.95; pants, $15.99; shoes, $23.49

G shirt, $15.99; pants, $12.95; shoes, $23.49

H shirt, $15.99; pants, $23.49; shoes, $12.95

J shirt, $23.49; pants, $15.99; shoes, $12.95

MANUFACTURING **For Exercises 17–19, use the following information.**
A sporting goods manufacturer makes a $5 profit on soccer balls and a $4 profit on volleyballs. Cutting requires 2 hours to make 75 soccer balls and 3 hours to make 60 volleyballs. Sewing needs 3 hours to make 75 soccer balls and 2 hours to make 60 volleyballs. The cutting department has 500 hours available, and the sewing department has 450 hours available.

17. How many soccer balls and volleyballs should be made to maximize the company's profit?

18. What is the maximum profit the company can make from these two products?

19. What would the maximum profit be if Cutting and Sewing got new equipment that allowed them to produce soccer balls at the same rate, but allowed Cutting to produce 75 volleyballs in 3 hours and Sewing to make 75 volleyballs in 2 hours?

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CHAPTER 3

Standardized Test Practice

Cumulative, Chapters 1–3

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. At the Gallatin Valley Cinema, the cost of 2 boxes of popcorn and 1 soda is \$11.50. The cost of 3 boxes of popcorn and 4 sodas is \$27.25. Which pair of equations can be used to determine p, the cost of a box of popcorn, and s, the cost of a soda?

A $2p + s = 27.25$
$3p + 4s = 11.50$

B $2p - s = 11.50$
$3p - 4s = 27.25$

C $2p + s = 11.50$
$3p + 4s = 27.25$

D $p + s = 11.50$
$p + 4 = 27.25$

2. What are the x-intercepts of the graph of the equation $y = x^2 - 2x - 15$?

F $x = -3, x = 5$

G $x = -1, x = 15$

H $x = -5, x = 3$

J $x = -5, x = -3$

3. GRIDDABLE What is the y-coordinate of the solution to the system of equations below?

$y = 4x - 7$

$y = -\frac{1}{2}x + 2$

4. As a fund-raiser, the student council sold T-shirts and sweatshirts. They sold a total of 105 T-shirts and sweatshirts and raised \$1170. If the cost of a T-shirt t was \$10 and the cost of a sweatshirt s was \$15, what was the number of sweatshirts sold?

A 24

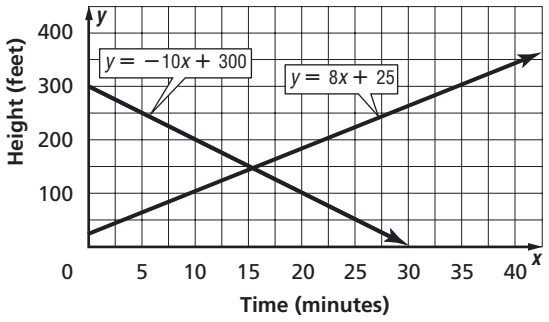
B 52

C 81

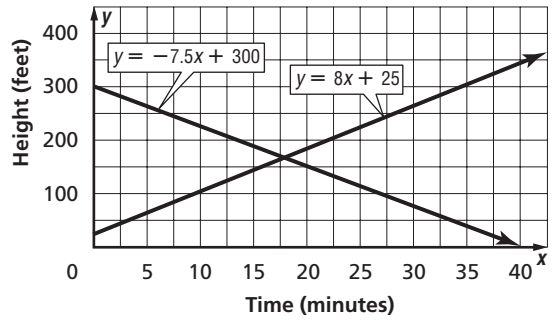
D 105

5. At the Carter County Fair, one hot air balloon is descending at a rate of 10 feet per minute from a height of 300 feet. At the same time, another hot air balloon is climbing from ground level at a rate of 8 feet per minute. Which graph shows when the two hot air balloons will be at the same altitude?

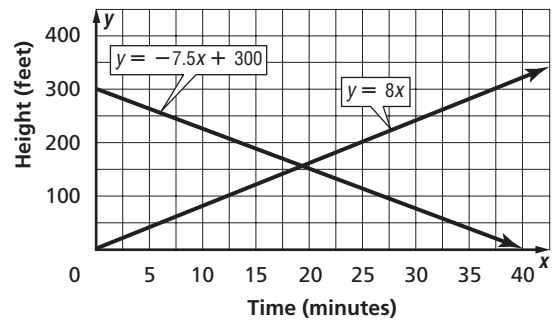
F



G



H



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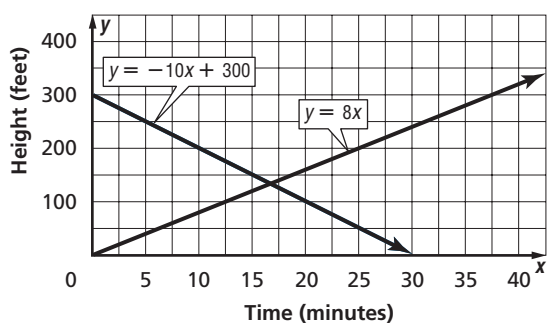

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6. Which of the following best describes the graph of the equations below?
$3y = 4x - 3$
$8y = -6x - 5$

A The lines have the same y-intercept.

B The lines have the same x-intercept.

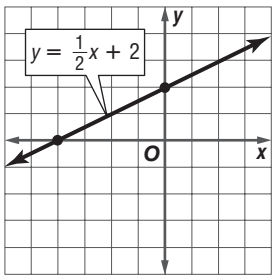
C The lines are perpendicular.

D The lines are parallel.

TEST-TAKING TIP

QUESTION 6 This problem does not include a drawing. Make one. It can help you quickly see how to solve the problem.

7. The graph of the equation $y = \frac{1}{2}x + 2$ is given below. Suppose you graph $y = x - 1$ on the grid.



What is the solution to the system of equations?

F (0, −1)

G (7, 6)

H (6, 5)

J no solution

8. The equations of two lines are $2x - y = 6$ and $4x - y = -2$. Which of the following describes their point of intersection?

A (2, −2)

B (−8, −38)

C (−4, −14)

D no intersection

9. Let p represent the price that Ella charges for a necklace. Let $f(x)$ represent the total amount of money that Ella makes for selling x necklaces. The function $f(x)$ is best represented by

F $x + p$

G xp^2

H px

J $x^2 + p$

10. GRIDDABLE Martha had some money saved for a week long vacation. The first day of the vacation she spent $125 on food and a hotel. On the second day, she was given $80 from her sister for expenses. Martha then had $635 left for the rest of the vacation. How much money, in dollars, did she begin the vacation with?

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Christine had one dress and three sweaters cleaned at the dry cleaner and the charge was $19.50. The next week, she had two dresses and two sweaters cleaned for a total charge of $23.00.

a. Let d represent the price of cleaning a dress and s represent the price of cleaning a sweater. Write a system of linear equations to represent the prices of cleaning each item.

b. Solve the system of equations using substitution or elimination. Explain your choice of method.

c. What will the charge be if Christine takes two dresses and four sweaters to be cleaned?

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson...	3-1	2-4	3-1	3-1	3-2	3-1	3-1	3-1	2-4	3-2	3-2

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CHAPTER 4

Matrices

BIG Ideas

- Organize data in matrices.
- Perform operations with matrices and determinants.
- Transform figures on a coordinate plane.
- Find the inverse of a matrix.
- Use matrices to solve systems of equations.

Key Vocabulary

determinant (p. 194)
identity matrix (p. 208)
inverse (p. 209)
matrix (p. 162)
scalar multiplication (p. 171)

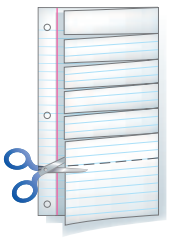
Real-World Link

Data Organization Matrices are often used to organize data. If the number of male and female students who participate in various sports are organized in separate matrices, the total number of participants can be found by adding the matrices.

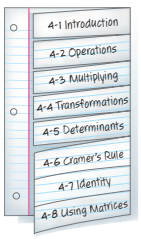
FOLDABLES Study Organizer

Matrices Make this Foldable to help you organize your notes. Begin with one sheet of notebook paper.

1 **Fold** lengthwise to the holes. Cut eight tabs in the top sheet.



2 **Label** each tab with a lesson number and title.



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GET READY for Chapter 4

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

QUICKCheck

Name the additive inverse and the multiplicative inverse for each number. (Lesson 1-2)

1. 3 **2.** -11 **3.** 8 **4.** -0.5

5. 1.25 **6.** $\frac{5}{9}$ **7.** $-\frac{8}{3}$ **8.** $-1\frac{1}{5}$

9. FOOTBALL After the quarterback from Central High takes a snap from the center, he drops back 4 yards. How many yards forward does Central High have to go to make it back to the line of scrimmage? (Lesson 1-2)

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

10. $x = y + 5$
$3x + y = 19$

11. $3x - 2y = 1$
$4x + 2y = 20$

12. $5x + 3y = 25$
$4x + 7y = -3$

13. $y = x - 7$
$2x - 8y = 2$

14. MONEY Last year the chess team paid \$7 per hat and \$15 per shirt for a total purchase of \$330. This year they spent \$360 to buy the same number of shirts and hats because the hats now cost \$8 and the shirts cost \$16. Write and solve a system of two equations that represents the number of hats and shirts bought each year. (Lesson 3-2)

QUICKReview

EXAMPLE 1

Name the additive inverse and the multiplicative inverse for $-\frac{1}{2}$.

The additive inverse of $-\frac{1}{2}$ is a number x such that $-\frac{1}{2} + x = 0$.

$x = \frac{1}{2}$ Add $\frac{1}{2}$ to each side.

The multiplicative inverse of $-\frac{1}{2}$ is a number x, such that $-\frac{1}{2}x = 1$.

$x = -2$ Multiply each side by -2.

EXAMPLE 2

Solve the following system of equations by using either substitution or elimination.

$2y = -x + 3$
$6x + 7y = 8$

Since x has a coefficient of -1 in the first equation, use the substitution method. First solve that equation for x.

$2y = -x + 3 \rightarrow x = -2y + 3$

$6(-2y + 3) + 7y = 8$ Substitute $-2y + 3$ for x.

$-12y + 18 + 7y = 8$ Distributive Property

$-5y = -10$ Combine like terms.

$y = 2$ Divide each side by -5.

To find x, use $y = 2$ in the first equation.

$2(2) = -x + 3$ Substitute 2 for y.

$4 = -x + 3$ Multiply.

$x = -1$ Subtract 4 from and add x to each side.

The solution is $(-1, 2)$.

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4-1 Introduction to Matrices

Main Ideas

- Organize data in matrices.
- Solve equations involving matrices.

New Vocabulary

matrix
element
dimension
row matrix
column matrix
square matrix
zero matrix
equal matrices

Reading Math

Matrices The plural of *matrix* is *matrices*.

GET READY for the Lesson

There are many types of sport-utility vehicles (SUVs) in many prices and styles. So, Oleta makes a list of qualities to consider for some top-rated models. She organizes the information in a matrix to easily compare the features of each vehicle.

	Base Price ($)	Horse-power	Exterior Length (in.)	Cargo Space (ft^3)	Fuel Economy (mpg)
Hybrid SUV	19,940	153	174.9	66.3	22
Standard SUV	31,710	275	208.4	108.8	15
Mid-Size SUV	27,350	255	188.0	90.3	17
Compact SUV	21,295	165	175.2	64.1	21

Source: cars.com

Organize Data A **matrix** is a rectangular array of variables or constants in horizonal rows and vertical columns, usually enclosed in brackets.

Real-World EXAMPLE Organize Data into a Matrix

1 **The prices for two cable companies are listed below. Use a matrix to organize the information. When is each company's service less expensive?**

Metro Cable	
Basic Service (26 channels)	$11.95
Standard Service (53 channels)	$30.75
Premium Channels (in addition to Standard Service)	
• One Premium	$10.00
• Two Premiums	$19.00
• Three Premiums	$25.00

Cable City	
Basic Service (26 channels)	$9.95
Standard Service (53 channels)	$31.95
Premium Channels (in addition to Standard Service)	
• One Premium	$8.95
• Two Premiums	$16.95
• Three Premiums	$22.95

Organize the costs into labeled columns and rows.

	Basic	Standard	Standard Plus One Premium	Standard Plus Two Premiums	Standard Plus Three Premiums
Metro Cable	11.95	30.75	40.75	49.75	55.75
Cable City	9.95	31.95	40.90	48.90	54.90

Metro Cable has the best price for standard service and standard plus one premium channel. Cable City has the best price for the other categories.

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CHECK Your Progress

1. Use a matrix to organize and compare the following information about some roller coasters.

Roller Coaster	Batman the Escape	Great White	Mr. Freeze
Speed (mph)	55	50	70
Height (feet)	90	108	218
Length (feet)	2300	2562	1300

Reading Math

Element The elements of a matrix can be represented using double subscript notation. The element a_{ij} is the element in row i column j.

In a matrix, numbers or data are organized so that each position in the matrix has a purpose. Each value in the matrix is called an **element**. A matrix is usually named using an uppercase letter.

$$A = \begin{bmatrix} 2 & 6 & 1 \\ 7 & 1 & 5 \\ 9 & 3 & 0 \\ 12 & 15 & 26 \end{bmatrix}$$

4 rows; 3 columns

The element 15 is in row 4, column 2.

A matrix can be described by its **dimensions**. A matrix with m rows and n columns is an $m \times n$ matrix (read "m by n"). Matrix A above is a 4×3 matrix since it has 4 rows and 3 columns.

EXAMPLE **Dimensions of a Matrix**

2 **State the dimensions of matrix B if $B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$.**

$$B = \begin{bmatrix} 1 & -3 \\ -5 & 18 \\ 0 & -2 \end{bmatrix}$$

3 rows; 2 columns

Since matrix B has 3 rows and 2 columns, the dimensions of matrix B are 3×2.

CHECK Your Progress

2. State the dimensions of matrix L if $L = \begin{bmatrix} -2 & 1 & 3 & -4 \\ 0 & 3 & 0 & 7 \end{bmatrix}$.

Certain matrices have special names. A matrix that has only one row is called a **row matrix**, while a matrix that has only one column is called a **column matrix**. A matrix that has the same number of rows and columns is called a **square matrix**. Another special type of matrix is the **zero matrix**, in which every element is 0. The zero matrix can have any dimension.

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Equations Involving Matrices Two matrices are considered **equal matrices** if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.

Example: $\begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 0 \\ 0 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ The matrices have the same dimensions and the corresponding elements are equal. The matrices are equal.

Non-example: $\begin{bmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 3 \end{bmatrix} \neq \begin{bmatrix} 6 & 0 & 1 \\ 3 & 9 & 3 \end{bmatrix}$ The matrices have different dimensions. They are not equal.

Non-example: $\begin{bmatrix} 1 & 2 \\ 8 & 5 \end{bmatrix} \neq \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix}$ Not all corresponding elements are equal. The matrices are not equal.

The definition of equal matrices can be used to find values when elements of equal matrices are algebraic expressions.

EXAMPLE Solve an Equation Involving Matrices

3 **Solve $\begin{bmatrix} y \\ 3x \end{bmatrix} = \begin{bmatrix} 6 - 2x \\ 31 + 4y \end{bmatrix}$ for x and y.**

Since the matrices are equal, the corresponding elements are equal. When you write the sentences to show this equality, two linear equations are formed.

$y = 6 - 2x$

$3x = 31 + 4y$

This system can be solved using substitution.

$3x = 31 + 4y$	Second equation
$3x = 31 + 4(6 - 2x)$	Substitute $6 - 2x$ for y.
$3x = 31 + 24 - 8x$	Distributive Property
$11x = 55$	Add $8x$ to each side.
$x = 5$	Divide each side by 11.

To find the value for y, substitute 5 for x in either equation.

$y = 6 - 2x$	First equation
$y = 6 - 2(5)$	Substitute 5 for x.
$y = -4$	Simplify.

The solution is $(5, -4)$.

CHECK Your Progress

3. Solve $\begin{bmatrix} 5x + 2 & y - 4 \\ 0 & 4z + 6 \end{bmatrix} = \begin{bmatrix} 12 & -8 \\ 0 & 2 \end{bmatrix}$.

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CHECK Your Understanding

Example 1 (pp. 162-163)

WEATHER For Exercises 1 and 2, use the table that shows a five-day forecast indicating high (H) and low (L) temperatures.

Fri	Sat	Sun	Mon	Tue
H 88	H 88	H 90	H 86	H 85
L 54	L 54	L 56	L 53	L 52

1. Organize the temperatures in a matrix.
2. Which day will be the warmest?

Example 2 (p. 163)

State the dimensions of each matrix.

3. $[3 \quad 4 \quad 5 \quad 6 \quad 7]$

4. $\begin{bmatrix} 10 & -6 & 18 & 0 \\ -7 & 5 & 2 & 4 \\ 3 & 11 & 9 & 7 \end{bmatrix}$

Example 3 (p. 164)

Solve each equation.

5. $\begin{bmatrix} x+4 \\ 2y \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

6. $[9 \quad 13] = [x+2y \quad 4x+1]$

Exercises

HOMEWORK HELP

For Exercises	See Examples
7–8	1
9–14	2
15–20	3

Organize the information in a matrix.

7.

Ocean	Area (mi^2)	Average Depth (ft)
Pacific	60,060,700	13,215
Atlantic	29,637,900	12,880
Indian	26,469,500	13,002
Southern	7,848,300	16,400
Arctic	5,427,000	3,953

Source: factmonster.com

8.

Top Hockey Goalies				
Goalie	Games	Wins	Losses	Ties
Roy	1029	551	315	131
Sawchuk	971	447	330	172
Plante	837	435	247	146
Esposito	886	423	306	152
Hall	906	407	326	163

Source: factmonster.com

State the dimensions of each matrix.

9. $\begin{bmatrix} 6 & -1 & 5 \\ -2 & 3 & -4 \end{bmatrix}$

10. $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 8 \\ 6 & 2 & 4 \\ 1 & 3 & 6 \\ 5 & 9 & 2 \end{bmatrix}$

12. $\begin{bmatrix} -3 & 17 & -22 \\ 9 & 31 & 16 \\ 20 & -15 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 17 & -2 & 8 & -9 & 6 \\ 5 & 11 & 20 & -1 & 4 \end{bmatrix}$

14. $\begin{bmatrix} 16 & 8 \\ 10 & 5 \\ 0 & 0 \end{bmatrix}$

Solve each equation.

15. $[4x \quad 3y] = [12 \quad -1]$

16. $[2x \quad 3 \quad 3z] = [5 \quad 3y \quad 9]$

17. $\begin{bmatrix} 4x \\ 5 \end{bmatrix} = \begin{bmatrix} 15+x \\ 2y-1 \end{bmatrix}$

18. $\begin{bmatrix} x+3y \\ 3x+y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

19. $\begin{bmatrix} 2x+y \\ x-3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

20. $\begin{bmatrix} 4x-3 & 3y \\ 7 & 13 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ 7 & 2z+1 \end{bmatrix}$

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Real-World Link

Adjusting for inflation, *Cleopatra* (1963) is the most expensive movie ever made. Its $44 million budget is equivalent to $306,867,120 today.

Source: *The Guiness Book of Records*

DINING OUT **For Exercises 21 and 22, use the following information.**
A newspaper rated several restaurants by cost, level of service, atmosphere, and location using a scale of ★ being low and ★★★★ being high.

Restaurant	Cost	Service	Atmosphere	Location
Catalina Grill	★★	★	★	★
Oyster Club	★★★	★★	★	★★
Casa di Pasta	★★★★	★★★	★★★	★★★
Mason's Steakhouse	★★	★★★★	★★★★	★★★

21. Write a 4 × 4 matrix to organize this information.

22. Which restaurant would you select based on this information, and why?

MOVIES **For Exercises 23 and 24, use the advertisement shown at the right.**

23. Write a matrix for the prices of movie tickets for adults, children, and seniors.

24. What are the dimensions of the matrix?

HOTELS **For Exercises 25 and 26, use the costs for an overnight stay at a hotel that are given below.**
Single Room: $60 weekday; $79 weekend
Double Room: $70 weekday; $89 weekend
Suite: $75 weekday; $95 weekend

25. Write a 3 × 2 matrix that represents the cost of each room.

26. Write a 2 × 3 matrix that represents the cost of each room.

EXTRA PRACTICE
See pages 897, 929.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

27. **RESEARCH** Use the Internet or other resource to find the meaning of the word *matrix*. How does the meaning of this word in other fields compare to its mathematical meaning?

28. **OPEN ENDED** Give examples of a row matrix, a column matrix, a square matrix, and a zero matrix. State the dimensions of each matrix.

CHALLENGE **For Exercises 29 and 30, use the matrix at the right.**

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & \dots \\ 2 & 5 & 9 & 14 & 20 & \dots \\ 4 & 8 & 13 & 19 & 26 & \dots \\ 7 & 12 & 18 & 25 & 33 & \dots \\ 11 & 17 & 24 & 32 & 41 & \dots \\ 16 & 23 & 31 & 40 & 50 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

29. Study the pattern of numbers. Complete the matrix for column 6 and row 7.

30. In which row and column will 100 occur?

31. ***Writing in Math*** Use the information about SUVs on page 162 to explain how a matrix can help Sabrina decide which SUV to buy.

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STANDARDIZED TEST PRACTICE

32. ACT/SAT The results of a recent poll are organized in the matrix.

$$\begin{array}{l} \\ \text{Proposition 1} \\ \text{Proposition 2} \\ \text{Proposition 3} \end{array} \begin{array}{c} \begin{array}{cc} \text{For} & \text{Against} \end{array} \\ \begin{bmatrix} 1553 & 771 \\ 689 & 1633 \\ 2088 & 229 \end{bmatrix} \end{array}$$

Based on these results, which conclusion is NOT valid?

A There were 771 votes cast against Proposition 1.

B More people voted against Proposition 1 than voted for Proposition 2.

C Proposition 2 has little chance of passing.

D More people voted for Proposition 1 than for Proposition 3.

33. REVIEW The chart shows an expression evaluated for four different values of x.

x	$x^2 + x + 1$
1	3
2	7
3	13
5	31

A student concludes that for all values of x, $x^2 + x + 1$ produces a prime number. Which value of x serves as a counterexample to prove this conclusion false?

F -4 **H** -2

G -3 **J** 4

Spiral Review

Solve each system of equations. (Lesson 3-5)

34. $3x - 3y = 6$
$-6y = -30$
$5z - 2x = 6$

35. $3a + 2b = 27$
$5a - 7b + c = 5$
$-2a + 10b + 5c = -29$

36. $3r - 15s + 4t = -57$
$9r + 45s - t = 26$
$-6r + 10s + 3t = -19$

37. BUSINESS A factory is making skirts and dresses from the same fabric. Each skirt requires 1 hour of cutting and 1 hour of sewing. Each dress requires 2 hours of cutting and 3 hours of sewing. The cutting department can cut up to 120 hours each week and the sewing department can sew up to 150 hours each week. If profits are \$12 for each skirt and \$18 for each dress, how many of each should the factory make for maximum profit? (Lesson 3-4)

38. Write an equation in slope-intercept form of the line that passes through the points indicated in the table. (Lesson 2–4)

39. Write an equation in standard form of the line that passes through the points indicated in the table. (Lesson 2–1)

x	y
-3	-1
2	$\frac{7}{3}$
3	3

Find each value if $f(x) = x^2 - 3x + 2$. (Lesson 2-1)

40. $f(3)$ **41.** $f(0)$ **42.** $f(2)$ **43.** $f(-3)$

GET READY for the Next Lesson

Find the value of each expression. (Lesson 1-2)

44. $8 + (-5)$ **45.** $6(-3)$ **46.** $\frac{1}{2}(34)$ **47.** $-5(3 - 18)$

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EXTEND 4-1

Spreadsheet Lab
Organizing Data

You can use a computer **spreadsheet** to organize and display data. Similar to a matrix, data in a spreadsheet are entered into rows and columns. Then you can use the data to create graphs or perform calculations.

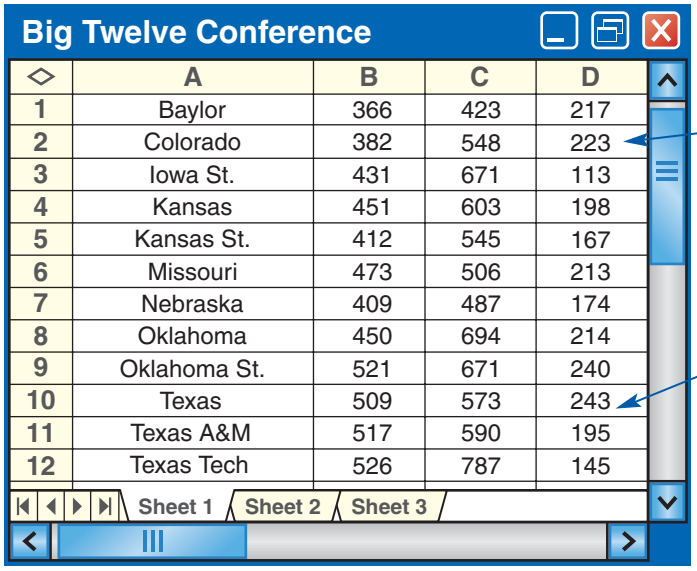
ACTIVITY

Enter the data on free throws (FT) and 2- and 3-point field goals (FG) in Big Twelve Conference Men's Basketball into a spreadsheet.

Big Twelve Conference 2004–2005 Men's Basketball							
Team	FT	2-PT FG	3-PT FG	Team	FT	2-PT FG	3-PT FG
Baylor	366	423	217	Nebraska	409	487	174
Colorado	382	548	223	Oklahoma	450	694	214
Iowa St.	431	671	113	Okahoma St.	521	671	240
Kansas	451	603	198	Texas	509	573	243
Kansas St.	412	545	167	Texas A&M	517	590	195
Missouri	473	506	213	Texas Tech	526	787	145

Source: SportsTicker

Use Column A for the team names, Column B for the numbers of free throws, Column C for the numbers of 2-point field goals, and Column D for the numbers of 3-point field goals.



Each **row** contains data for a different team. Row 2 represents Colorado.

Each **cell** of the spreadsheet contains one piece of data. Cell 10D contains the value 243, representing the number of 3-point field goals made by Texas.

Model and Analyze

1. Enter the data about sport-utility vehicles on page 162 into a spreadsheet.
2. Compare and contrast how data are organized in a spreadsheet and how they are organized in a matrix.

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4-2 Operations with Matrices

Main Ideas

- Add and subtract matrices.
- Multiply by a matrix scalar.

New Vocabulary

scalar

scalar multiplication

GET READY for the Lesson

Eneas, a hospital dietician, designs weekly menus for his patients and tracks nutrients for each daily diet. The table shows the Calories, protein, and fat in a patient's meals over a three-day period.

Day	Breakfast			Lunch			Dinner		
	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)	Calories	Protein (g)	Fat (g)
1	566	18	7	785	22	19	1257	40	26
2	482	12	17	622	23	20	987	32	45
3	530	10	11	710	26	12	1380	29	38

These data can be organized in three matrices representing breakfast, lunch, and dinner. The daily totals can then be found by adding the three matrices.

Add and Subtract Matrices Matrices can be added if and only if they have the same dimensions.

KEY CONCEPT *Addition and Subtraction of Matrices*

Words If A and B are two $m \times n$ matrices, then $A + B$ is an $m \times n$ matrix in which each element is the sum of the corresponding elements of A and B. Also, $A - B$ is an $m \times n$ matrix in which each element is the difference of the corresponding elements of A and B.

Symbols
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a+j & b+k & c+l \\ d+m & e+n & f+o \\ g+p & h+q & i+r \end{bmatrix}$$
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} a-j & b-k & c-l \\ d-m & e-n & f-o \\ g-p & h-q & i-r \end{bmatrix}$$

EXAMPLE 1 **Add Matrices**

a. Find $A + B$ if $A = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$.

$$A + B = \begin{bmatrix} 4 & -6 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 5 & -9 \end{bmatrix}$$ Definition of matrix addition

$$= \begin{bmatrix} 4 + (-3) & -6 + 7 \\ 2 + 5 & 3 + (-9) \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 7 & -6 \end{bmatrix}$$ Simplify.

(continued on the next page)

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b. Find $A + B$ if $A = \begin{bmatrix} 3 & -7 & 4 \\ 12 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 9 \\ 4 & -6 \end{bmatrix}$.

Since the dimensions of A are 2×3 and the dimensions of B are 2×2, you cannot add these matrices.

CHECK Your Progress

1. Find $A + B$ if $A = \begin{bmatrix} -5 & 7 \\ -1 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 3 \\ -4 & -5 \end{bmatrix}$.

EXAMPLE Subtract Matrices

2 Find $A - B$ if $A = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$.

$$A - B = \begin{bmatrix} 9 & 2 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 8 & -2 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 9-3 & 2-6 \\ -4-8 & 7-(-2) \end{bmatrix}$$ Subtract corresponding elements.

$$= \begin{bmatrix} 6 & -4 \\ -12 & 9 \end{bmatrix}$$ Simplify.

CHECK Your Progress

2. Find $A - B$ if $A = \begin{bmatrix} 12 & -4 \\ -5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ -3 & -2 \end{bmatrix}$.

Real-World Link

The rarest animal in the world today is a giant tortoise that lives in the Galapagos Islands. "Lonesome George" is the only remaining representative of his species (*Geochelone elephantopus abingdoni*). With virtually no hope of discovering another specimen, this species is now effectively extinct.

Source: ecoworld.com

Real-World EXAMPLE

3 **ANIMALS** **The table below shows the number of endangered and threatened species in the United States and in the world. How many more endangered and threatened species are there on the world list than on the U.S. list?**

Endangered and Threatened Species				
Type of Animal	United States		World	
	Endangered	Threatened	Endangered	Threatened
Mammals	68	10	319	27
Birds	77	13	252	19
Reptiles	14	22	78	37
Amphibians	11	10	19	11
Fish	71	43	82	44

Source: Fish and Wildlife Service, U.S. Department of Interior

The data in the table can be organized in two matrices. Find the difference of the matrix that represents species in the world and the matrix that represents species in the U.S.

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$$\begin{matrix} \text{World} \\ \begin{bmatrix} 319 & 27 \\ 252 & 19 \\ 78 & 37 \\ 19 & 11 \\ 82 & 44 \end{bmatrix} \end{matrix} - \begin{matrix} \text{U.S.} \\ \begin{bmatrix} 68 & 10 \\ 77 & 13 \\ 14 & 22 \\ 11 & 10 \\ 71 & 43 \end{bmatrix} \end{matrix} = \begin{matrix} \text{Endangered} & \text{Threatened} \\ \begin{bmatrix} 319-68 & 27-10 \\ 252-77 & 19-13 \\ 78-14 & 37-22 \\ 19-11 & 11-10 \\ 82-71 & 44-43 \end{bmatrix} \end{matrix}$$

Subtract corresponding elements.

$$= \begin{bmatrix} 251 & 17 \\ 175 & 6 \\ 64 & 15 \\ 8 & 1 \\ 11 & 1 \end{bmatrix}$$

The first column represents the difference in the number of endangered species on the world and U.S. lists. There are 251 mammals, 175 birds, 64 reptiles, 8 amphibians, and 11 fish species in this category.

The second column represents the difference in the number of threatened species on the world and U.S. lists. There are 17 mammals, 6 birds, 15 reptiles, 1 amphibian, and 1 fish species in this category.

CHECK Your Progress

3. Refer to the data on page 169 and use matrices to show the difference of Calories, protein, and fat between lunch and breakfast.

Online Personal Tutor at algebra2.com

Scalar Multiplication You can multiply any matrix by a constant called a **scalar.** This operation is called **scalar multiplication**.

KEY CONCEPT *Scalar Multiplication*

Words The product of a scalar k and an $m \times n$ matrix is an $m \times n$ matrix in which each element equals k times the corresponding elements of the original matrix.

Symbols $k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

EXAMPLE **Multiply a Matrix by a Scalar**

4 If $A = \begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$, find $3A$.

$$3A = 3\begin{bmatrix} 2 & 8 & -3 \\ 5 & -9 & 2 \end{bmatrix}$$

Substitution

$$= \begin{bmatrix} 3(2) & 3(8) & 3(-3) \\ 3(5) & 3(-9) & 3(2) \end{bmatrix} \text{ or } \begin{bmatrix} 6 & 24 & -9 \\ 15 & -27 & 6 \end{bmatrix}$$

Simplify.

CHECK Your Progress

4. If $A = \begin{bmatrix} 7 & -4 & 10 \\ -2 & 6 & -9 \end{bmatrix}$, find $-4A$.

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Many properties of real numbers also hold true for matrices.

CONCEPT SUMMARY *Properties of Matrix Operations*

For any matrices A, B, and C with the same dimensions and any scalar c, the following properties are true.

Commutative Property of Addition	$A + B = B + A$
Associative Property of Addition	$(A + B) + C = A + (B + C)$
Distributive Property	$c(A + B) = cA + cB$

EXAMPLE **Combination of Matrix Operations**

Study Tip

Additive Identity

The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called a *zero matrix*. It is the *additive identity matrix* for any 2×2 matrix. How is this similar to the additive identity for real numbers?

5 If $A = \begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$, find $5A - 2B$.

Perform the scalar multiplication first. Then subtract the matrices.

$$5A - 2B = 5\begin{bmatrix} 7 & 3 \\ -4 & -1 \end{bmatrix} - 2\begin{bmatrix} 9 & 6 \\ 3 & 10 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 5(7) & 5(3) \\ 5(-4) & 5(-1) \end{bmatrix} - \begin{bmatrix} 2(9) & 2(6) \\ 2(3) & 2(10) \end{bmatrix}$$ Multiply each element in the first matrix by 5 and multiply each element in the second matrix by 2.

$$= \begin{bmatrix} 35 & 15 \\ -20 & -5 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 6 & 20 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} 35 - 18 & 15 - 12 \\ -20 - 6 & -5 - 20 \end{bmatrix} \text{ or } \begin{bmatrix} 17 & 3 \\ -26 & -25 \end{bmatrix}$$ Subtract corresponding elements.

CHECK Your Progress

5. If $A = \begin{bmatrix} 4 & -2 \\ 5 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 2 \\ -1 & -3 \end{bmatrix}$, find $6A - 3B$.

GRAPHING CALCULATOR LAB

Matrix Operations

On the TI-83/84 Plus, 2nd [MATRX] accesses the matrix menu. Choose EDIT to define a matrix. Press 1 or ENTER and enter the dimensions of the matrix A using the ▶ key. Then enter each element by pressing ENTER after each entry. To display and use the matrix, exit the editing mode and choose the matrix under NAMES from the [MATRIX] menu.

Study Tip

Matrix Operations

The order of operations for matrices is similar to that of real numbers. Perform scalar multiplication before matrix addition and subtraction.

THINK AND DISCUSS

1. Enter $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$. What do the two numbers separated by a comma in the bottom left corner of the screen represent?

2. Enter $B = \begin{bmatrix} 1 & 9 & -3 \\ 8 & 6 & -5 \end{bmatrix}$. Find $A + B$. What is the result and why?

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CHECK Your Understanding

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

Example 1 (pp. 169–170)

1. $\begin{bmatrix} 5 & 8 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 12 & 6 \\ -8 & -3 \end{bmatrix} + \begin{bmatrix} 14 & -9 \\ 11 & -6 \end{bmatrix}$

Example 2 (p. 170)

3. $\begin{bmatrix} 3 & 7 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & -4 \end{bmatrix}$

4. $\begin{bmatrix} 4 & 12 \\ -3 & -7 \end{bmatrix} - \begin{bmatrix} 5 & 3 \\ -4 & -4 \end{bmatrix}$

Example 3 (pp. 170–171)

SPORTS **For Exercises 5–7, use the table below that shows high school participation in various sports.**

Sport	Males		Females	
	Schools	Participants	Schools	Participants
Basketball	17,389	544,811	17,061	457,986
Track and Field	15,221	504,801	15,089	418,322
Baseball/Softball	14,984	457,146	14,181	362,468
Soccer	10,219	349,785	9,490	309,032
Swimming and Diving	5,758	96,562	6,176	144,565

Source: National Federation of State High School Associations

5. Write two matrices that represent these data for males and females.

6. Find the total number of students that participate in each individual sport expressed as a matrix.

7. Could you add the two matrices to find the total number of schools that offer a particular sport? Why or why not?

Example 4 (p. 171)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

8. $3\begin{bmatrix} 6 & -1 & 5 & 2 \\ 7 & 3 & -2 & 8 \end{bmatrix}$

9. $-5\begin{bmatrix} 2 & -4 \\ -6 & 3 \\ -9 & -1 \end{bmatrix}$

Example 5 (p. 172)

Use matrices *A*, *B*, *C*, and *D* to find the following.

$A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 7 \\ 0 & -4 \end{bmatrix}$ $C = \begin{bmatrix} 9 & -4 \\ -6 & 5 \end{bmatrix}$ $D = \begin{bmatrix} 2 & -5 \end{bmatrix}$

10. $A + B + C$

11. $3B - 2C$

12. $4A + 2B - C$

13. $B + 2C + D$

Exercises

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

14. $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix}$

15. $\begin{bmatrix} -11 & 4 \\ -3 & 6 \end{bmatrix} + \begin{bmatrix} -2 & -5 \\ 5 & -3 \end{bmatrix}$

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HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18–21	2
22–24	3
25, 26	4
27, 28	5

Perform the indicated matrix operations. If the matrix does not exist, write impossible.

16. $\begin{bmatrix} -5 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

17. $\begin{bmatrix} 2 & 5 & 3 \\ -7 & -1 & 11 \\ 4 & -4 & 0 \end{bmatrix} + \begin{bmatrix} -9 & 2 & -5 \\ 1 & 6 & -3 \\ -9 & -12 & 8 \end{bmatrix}$

18. $\begin{bmatrix} -5 & 7 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & -2 \\ 9 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 12 & 0 & 8 \\ 9 & 15 & -11 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 4 \\ 9 & 2 & -6 \end{bmatrix}$

20. $\begin{bmatrix} 3 \\ -8 \\ -2 \end{bmatrix} - \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$

21. $\begin{bmatrix} -9 & 2 & -7 \\ 8 & 10 & 3 \\ -7 & 4 & 15 \end{bmatrix} - \begin{bmatrix} -1 & 3 & 6 \\ -7 & -3 & 5 \\ 2 & 11 & -4 \end{bmatrix}$

BUSINESS **For Exercises 22–24, use the following information.** An electronics store records each type of entertainment device sold at three of their branch stores so that they can monitor their purchases of supplies. Two weeks of sales are shown in the spreadsheets at the right.

	A	B	C	D	E
1	Week 1	Televisions	DVD players	Video game units	CD players
2	Store 1	325	215	147	276
3	Store 2	294	221	79	152
4	Store 3	175	191	100	146

	A	B	C	D	E
1	Week 2	Televisions	DVD players	Video game units	CD players
2	Store 1	306	162	145	257
3	Store 2	258	210	84	165
4	Store 3	188	176	99	112

22. Write a matrix for each week's sales.

23. Find the sum of the two weeks' sales expressed as a matrix.

24. Express the difference in sales from Week 1 to Week 2 as a matrix.

Perform the indicated matrix operation. If the matrix does not exist, write *impossible*.

25. $-2\begin{bmatrix} 2 & -4 & 1 \\ -3 & 5 & 8 \\ 7 & 6 & -2 \end{bmatrix}$

26. $3\begin{bmatrix} 5 & -3 \\ -10 & 8 \\ -1 & 7 \end{bmatrix}$

27. $5[0 \quad -1 \quad 7 \quad 2] + 3[5 \quad -8 \quad 10 \quad -4]$

28. $5\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix} + 6\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 8 \\ -4 \end{bmatrix}$

Use matrices A, B, C, and D to find the following.

$$A = \begin{bmatrix} 5 & 7 \\ -1 & 6 \\ 3 & -9 \end{bmatrix} \quad B = \begin{bmatrix} 8 & 3 \\ 5 & 1 \\ 4 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 4 \\ -2 & 5 \\ 7 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 2 \\ 9 & 0 \\ -3 & 0 \end{bmatrix}$$

29. $A + B$

30. $D - B$

31. $4C$

32. $6B - 2A$

33. $3C - 4A + B$

34. $C + \frac{1}{3}D$

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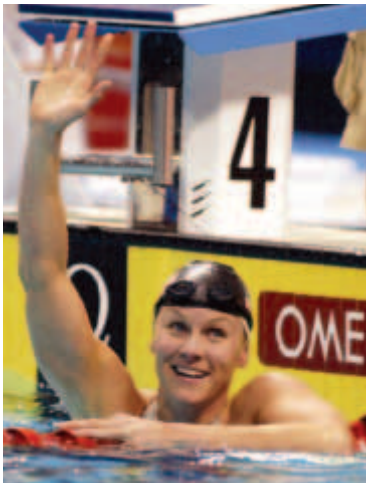

Real-World Link

Jenny Thompson won her record setting twelfth Olympic medal by winning the silver in the 4 × 100 Medley Relay at the 2004 Athens Olympics.

Source: athens2004.com

Perform the indicated matrix operation. If the matrix does not exist, write *impossible*.

35. $\begin{bmatrix} 1.35 & 5.80 \\ 1.24 & 14.32 \\ 6.10 & 35.26 \end{bmatrix} + \begin{bmatrix} 0.45 & 3.28 \\ 1.94 & 16.72 \\ 4.31 & 21.30 \end{bmatrix}$

36. $8\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix} - 2\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 1.5 \end{bmatrix}$

37. $\frac{1}{2}\begin{bmatrix} 4 & 6 \\ 3 & 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 9 & 27 \\ 0 & 3 \end{bmatrix}$

38. $5\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 2 & \frac{1}{3} & -1 \end{bmatrix} + 4\begin{bmatrix} -2 & \frac{3}{4} & 1 \\ \frac{1}{6} & 0 & \frac{5}{8} \end{bmatrix}$

SWIMMING **For Exercises 39–41, use the table that shows some of the world, Olympic, and U.S. women's freestyle swimming records.**

Distance (meters)	World	Olympic	U.S.
50	24.13 s	24.13 s	24.63 s
100	53.52 s	53.52 s	53.99 s
200	1:56.54 min	1:57.65 min	1:57.41 min
800	8:16.22 min	8:19.67 min	8:16.22 min

Source: hickoksports.com

39. Find the difference between U.S. and World records expressed as a column matrix.

40. Write a matrix that compares the total time of all four events for World, Olympic, and U.S. record holders.

41. In which events were the fastest times set at the Olympics?

RECREATION **For Exercises 42 and 43, use the following price list for one-day admissions to the community pool.**

42. Write the matrix that represents the additional cost for nonresidents.

43. Write a matrix that represents the difference in cost if a child or adult goes to the pool after 6:00 P.M.

Daily Admission Fees

Residents

Time of day	Child	Adult
Before 6:00 P.M.	$3.00	$4.50
After 6:00 P.M.	$2.00	$3.50

Nonresidents

Time of day	Child	Adult
Before 6:00 P.M.	$4.50	$6.75
After 6:00 P.M.	$3.00	$5.25

EXTRA PRACTICE
See pages 897, 929.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

44. CHALLENGE Determine values for each variable if $d = 1$, $e = 4d$, $z + d = e$, $f = \frac{x}{5}$, $ay = 1.5$, $x = \frac{d}{2}$, and $y = x + \frac{x}{2}$.

$$a\begin{bmatrix} x & y & z \\ d & e & f \end{bmatrix} = \begin{bmatrix} ax & ay & az \\ ad & ae & af \end{bmatrix}$$

45. OPEN ENDED Give an example of two matrices whose sum is a zero matrix.

46. CHALLENGE For matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the *transpose* of A is $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Write a matrix B that is equal to its transpose B^T.

47. *Writing in Math* Use the data on nutrition on page 169 to explain how matrices can be used to calculate daily dietary needs. Include three matrices that represent breakfast, lunch, and dinner over the three-day period, and a matrix that represents the total Calories, protein, and fat consumed each day.

Brent Smith/Reuters/CORBIS
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STANDARDIZED TEST PRACTICE

48. ACT/SAT Solve for x and y in the matrix equation $\begin{bmatrix} x \\ 7 \end{bmatrix} + \begin{bmatrix} 3y \\ -x \end{bmatrix} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$.

A $x = -5, y = 7$

B $x = 7, y = 3$

C $x = 7, y = 5$

D $x = 5, y = 7$

49. REVIEW What is the equation of the line that has a slope of 3 and passes through the point $(2, -9)$?

F $y = 3x + 11$

G $y = 3x - 11$

H $y = 3x + 15$

J $y = 3x - 15$

Spiral Review

State the dimensions of each matrix. (Lesson 4-1)

50. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 0 & 3 & 0 \end{bmatrix}$

52. $\begin{bmatrix} 5 & 1 & -6 & 2 \\ -38 & 5 & 7 & 3 \end{bmatrix}$

53. $\begin{bmatrix} 7 & -3 & 5 \\ 0 & 2 & -9 \\ 6 & 5 & 1 \end{bmatrix}$

54. $\begin{bmatrix} 8 & 6 \\ 5 & 2 \\ -4 & -1 \end{bmatrix}$

55. $\begin{bmatrix} 7 & 5 & 0 \\ -8 & 3 & 8 \\ 9 & -1 & 15 \\ 4 & 2 & 11 \end{bmatrix}$

Solve each system of equations. (Lesson 3-5)

56. $2a + b = 2$
$5a = 15$
$a + b + c = -1$

57. $r + s + t = 15$
$r + t = 12$
$s + t = 10$

58. $6x - 2y - 3z = -10$
$-6x + y + 9z = 3$
$8x - 3y = -16$

Solve each system by using substitution or elimination. (Lesson 3-2)

59. $2s + 7t = 39$
$5s - t = 5$

60. $3p + 6q = -3$
$2p - 3q = -9$

61. $a + 5b = 1$
$7a - 2b = 44$

SCRAPBOOKS For Exercises 62 and 63, use the following information. (Lesson 2-7)
Ian has \$6.00, and he wants to buy paper for his scrapbook. A sheet of printed paper costs 30¢, and a sheet of solid color paper costs 15¢.

62. Write and graph an inequality that describes this situation.

63. Does Ian have enough money to buy 14 pieces of each type of paper? Explain.

GET READY for the Next Lesson

Name the property illustrated by each equation. (Lesson 1-2)

64. $\frac{7}{9} \cdot \frac{9}{7} = 1$

65. $7 + (w + 5) = (7 + w) + 5$

66. $3(x + 12) = 3x + 3(12)$

67. $6(9a) = 9a(6)$

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4-3 Multiplying Matrices

Main Ideas

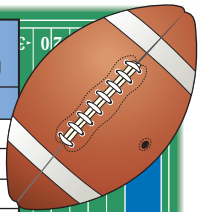
- Multiply matrices.
- Use the properties of matrix multiplication

GET READY for the Lesson

The table shows the scoring summary of the Carolina Panthers for the 2005 season. The team's record can be summarized in the record matrix R. The values for each type of score can be organized in the point values matrix P.

Carolina Panthers Regular Season Scoring

Type	Number
Touchdown	45
Extra Point	43
Field Goal	26
2-Point Conversion	1
Safety	0



Source: National Football League

Record

$$R = \begin{bmatrix} 45 \\ 43 \\ 26 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \text{touchdown} \\ \text{extra point} \\ \text{field goal} \\ \text{2–point conversion} \\ \text{safety} \end{matrix}$$

Point Values

$$P = \begin{bmatrix} 6 & 1 & 3 & 2 & 2 \end{bmatrix}$$

touchdown, extra point, field goal, 2-point conversion, safety

You can use matrix multiplication to find the total points scored.

Multiply Matrices You can multiply two matrices if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. When you multiply two matrices $A_{m \times n}$ and $B_{n \times r}$, the resulting matrix AB is an $m \times r$ matrix.

EXAMPLE 1 Dimensions of Matrix Products

Determine whether each matrix product is defined. If so, state the dimensions of the product.

a. $A_{2 \times 5}$ and $B_{5 \times 4}$

$A \cdot B = AB$

$2 \times 5 \quad 5 \times 4 \quad 2 \times 4$

The inner dimensions are equal, so the product is defined. Its dimensions are 2×4.

b. $A_{1 \times 3}$ and $B_{4 \times 3}$

$A \cdot B$

$1 \times 3 \quad 4 \times 3$

The inner dimensions are not equal, so the matrix product is not defined.

CHECK Your Progress

1A. $A_{4 \times 6}$ and $B_{6 \times 2}$

1B. $A_{3 \times 2}$ and $B_{3 \times 2}$

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The product of two matrices is found by multiplying corresponding columns and rows.

KEY CONCEPT — ***Multiplying Matrices***

Words The element a_{ij} of AB is the sum of the products of the corresponding elements in row i of A and column j of B.

Symbols
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} a_1x_1 + b_1x_2 & a_1y_1 + b_1y_2 \\ a_2x_1 + b_2x_2 & a_2y_1 + b_2y_2 \end{bmatrix}$$

EXAMPLE 2 Multiply Square Matrices

Find RS if $R = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $S = \begin{bmatrix} 3 & -9 \\ 5 & 7 \end{bmatrix}$.

$$RS = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 \\ 4 & 7 \end{bmatrix}$$

Step 1 Multiply the numbers in the first row of R by the numbers in the first column of S, add the products, and put the result in the first row, first column of RS.

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{3} & -9 \\ \mathbf{5} & 7 \end{bmatrix} = \begin{bmatrix} \mathbf{2(3) + (-1)(5)} & \\ & \end{bmatrix}$$

Step 2 Follow the same procedure as in Step 1 using the first row and second column numbers. Write the result in the first row, second column.

$$\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & \mathbf{-9} \\ 5 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & \mathbf{2(-9) + (-1)(7)} \\ & \end{bmatrix}$$

Step 3 Follow the same procedure with the second row and first column numbers. Write the result in the second row, first column.

$$\begin{bmatrix} 2 & -1 \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{3} & -9 \\ \mathbf{5} & 7 \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ \mathbf{3(3) + 4(5)} & \end{bmatrix}$$

Step 4 The procedure is the same for the numbers in the second row, second column.

$$\begin{bmatrix} 2 & -1 \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \cdot \begin{bmatrix} 3 & \mathbf{-9} \\ 5 & \mathbf{7} \end{bmatrix} = \begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & \mathbf{3(-9) + 4(7)} \end{bmatrix}$$

Step 5 Simplify the product matrix.

$$\begin{bmatrix} 2(3) + (-1)(5) & 2(-9) + (-1)(7) \\ 3(3) + 4(5) & 3(-9) + 4(7) \end{bmatrix} = \begin{bmatrix} 1 & -25 \\ 29 & 1 \end{bmatrix}$$

Study Tip

Multiplying Matrices

To avoid any miscalculations, find the product of the matrices in order as shown in Example 2. It may also help to cover rows or columns not being multiplied as you find elements of the product matrix.

Concepts in Motion

Animation
algebra2.com

CHECK Your Progress

2. Find UV if $U = \begin{bmatrix} 5 & 9 \\ -3 & -2 \end{bmatrix}$ and $V = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$.

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Real-World EXAMPLE

3 **SWIM MEET** **At a particular swim meet, 7 points were awarded for each first-place finish, 4 points for each second, and 2 points for each third. Which school won the meet?**

School	First Place	Second Place	Third Place
Central	4	7	3
Franklin	8	9	1
Hayes	10	5	3
Lincoln	3	3	6

Real-World Link

Swim meets consist of racing and diving competitions.

There are more than 241,000 high schools that participate each year.

Source: NFHS

Explore The final scores can be found by multiplying the swim results for each school by the points awarded for each first-, second-, and third-place finish.

Plan Write the results of the races and the points awarded in matrix form. Set up the matrices so that the number of rows in the points matrix equals the number of columns in the results matrix.

Results

$$R = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix}$$

Points

$$P = \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$

Solve Multiply the matrices.

$$RP = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 9 & 1 \\ 10 & 5 & 3 \\ 3 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 4(7) + 7(4) + 3(2) \\ 8(7) + 9(4) + 1(2) \\ 10(7) + 5(4) + 3(2) \\ 3(7) + 3(4) + 6(2) \end{bmatrix}$$ Multiply columns by rows.

$$= \begin{bmatrix} 62 \\ 94 \\ 96 \\ 45 \end{bmatrix}$$ Simplify.

The product matrix shows the scores for Central, Franklin, Hayes, and Lincoln in order. Hayes won the swim meet with a total of 96 points.

Check R is a 4×3 matrix and P is a 3×1 matrix; so their product should be a 4×1 matrix. *Why?*

CHECK Your Progress

3. Refer to the data in Exercises 22–24 on page 174. If the cost of televisions was \$250, DVD players was \$225, video game units was \$149, and CD players was \$75, use matrices to find the total sales for week 1.

Jean-Yves Ruszniewski/CORBIS

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Multiplicative Properties Recall that the same properties for real numbers also held true for matrix addition. However, some of these properties do *not* always hold true for matrix multiplication.

EXAMPLE **Commutative Property**

4 **Find each product if** $P = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$ **and** $Q = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$.

a. PQ

$$PQ = \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{bmatrix}$$

b. QP

$$QP = \begin{bmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 72 + 6 + 0 & -63 - 12 + 6 \\ 48 + 2 + 0 & -42 - 4 - 15 \end{bmatrix} \text{ or } \begin{bmatrix} 78 & -69 \\ 50 & -61 \end{bmatrix}$$

CHECK Your Progress

4. Use $A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 6 \\ -4 & 5 \end{bmatrix}$ to determine whether $AB = BA$ is true for the given matrices.

In Example 4, notice that $PQ \neq QP$. This demonstrates that the Commutative Property of Multiplication does not hold for matrix multiplication. The order in which you multiply matrices is very important.

EXAMPLE **Distributive Property**

5 **Find each product if** $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix}$, **and** $C = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$.

a. $A(B + C)$

$$A(B + C) = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \right)$$ Substitution

$$= \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 1 & 10 \end{bmatrix}$$ Add corresponding elements.

$$= \begin{bmatrix} 3(-1) + 2(1) & 3(6) + 2(10) \\ -1(-1) + 4(1) & -1(6) + 4(10) \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix}$$ Multiply columns by rows.

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b. $AB + AC$

$$AB + AC = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix}$$ Substitution

$$= \begin{bmatrix} 3(-2) + 2(6) & 3(5) + 2(7) \\ -1(-2) + 4(6) & -1(5) + 4(7) \end{bmatrix} + \begin{bmatrix} 3(1) + 2(-5) & 3(1) + 2(3) \\ -1(1) + 4(-5) & -1(1) + 4(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 29 \\ 26 & 23 \end{bmatrix} + \begin{bmatrix} -7 & 9 \\ -21 & 11 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} -1 & 38 \\ 5 & 34 \end{bmatrix}$$ Add corresponding elements.

CHECK Your Progress

5. Use the matrices $R = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $S = \begin{bmatrix} 4 & 6 \\ -2 & 5 \end{bmatrix}$, and $T = \begin{bmatrix} -3 & 7 \\ -4 & 8 \end{bmatrix}$ to determine if $(S + T)\,R = SR + TR$.

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Notice that in Example 5, $A(B + C) = AB + AC$. This and other examples suggest that the Distributive Property is true for matrix multiplication. Some properties of matrix multiplication are shown below.

KEY CONCEPT ***Properties of Matrix Multiplication***

For any matrices A, B, and C for which the matrix products are defined, and any scalar c, the following properties are true.

Associative Property of Matrix Multiplication	$(AB)C = A(BC)$
Associative Property of Scalar Multiplication	$c(AB) = (cA)B = A(cB)$
Left Distributive Property	$C(A + B) = CA + CB$
Right Distributive Property	$(A + B)C = AC + BC$

To show that a property is true for all cases, you must show it is true for the general case. To show that a property is *not* always true, you only need to find one counterexample.

CHECK Your Understanding

Example 1 (p. 177)

Determine whether each matrix product is defined. If so, state the dimensions of the product.

1. $A_{3 \times 5} \cdot B_{5 \times 2}$ **2.** $X_{2 \times 3} \cdot Y_{2 \times 3}$ **3.** $R_{3 \times 2} S_{2 \times 22}$

Find each product, if possible.

Example 2 (p. 178)

4. $\begin{bmatrix} 2 & 1 \\ 7 & -5 \end{bmatrix} \cdot \begin{bmatrix} -6 & 3 \\ -2 & -4 \end{bmatrix}$ **5.** $\begin{bmatrix} 10 & -2 \\ -7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$

Example 3 (p. 179)

6. $\begin{bmatrix} 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 5 \\ -2 & 0 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

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SPORTS **For Exercises 10 and 11, use the table below that shows the number of kids registered for baseball and softball.**

The Westfall Youth Baseball and Softball League charges the following registration fees: ages 7–8, $45; ages 9–10, $55; and ages 11–14, $65.

Team Members		
Age	Baseball	Softball
7–8	350	280
9–10	320	165
11–14	180	120

10. Write a matrix for the registration fees and a matrix for the number of players.

11. Find the total amount of money the league received from baseball and softball registrations.

Examples 4, 5 (pp. 180–181)

Use $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ to determine whether the following equations are true for the given matrices.

12. $AB = BA$

13. $A(BC) = (AB)C$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–19	1
20–27	2, 3
28–30	3
31, 32	4
33, 34	5

Determine whether each matrix product is defined. If so, state the dimensions of the product.

14. $A_{4 \times 3} \cdot B_{3 \times 2}$

15. $X_{2 \times 2} \cdot Y_{2 \times 2}$

16. $P_{1 \times 3} \cdot Q_{4 \times 1}$

17. $R_{1 \times 4} \cdot S_{4 \times 5}$

18. $M_{4 \times 3} \cdot N_{4 \times 3}$

19. $A_{3 \times 1} \cdot B_{1 \times 5}$

Find each product, if possible.

20. $\begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

21. $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -7 \end{bmatrix}$

22. $\begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$

23. $\begin{bmatrix} -1 & 0 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \\ 7 & -2 \end{bmatrix}$

24. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

25. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{bmatrix}$

27. $\begin{bmatrix} -4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 \end{bmatrix}$

BUSINESS **For Exercises 28–30, use the table and the following information.**

Solada Fox sells fruit from her three farms. Apples are $22 a case, peaches are $25 a case, and apricots are $18 a case.

Number of Cases in Stock of Each Type of Fruit			
Farm	Apples	Peaches	Apricots
1	290	165	210
2	175	240	190
3	110	75	0

28. Write an inventory matrix for the number of cases for each type of fruit for each farm and a cost matrix for the price per case for each type of fruit.

29. Find the total income of the three fruit farms expressed as a matrix.

30. What is the total income from all three fruit farms combined?

Use $A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$, and scalar $c = 3$ to determine whether the following equations are true for the given matrices.

31. $c(AB) = A(cB)$

32. $(AB)C = (CB)A$

33. $AC + BC = (A + B)C$

34. $C(A + B) = AC + BC$

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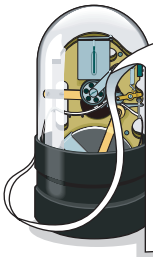
FUND-RAISING **For Exercises 35 and 36, use the following information.**
Lawrence High School sold wrapping paper and boxed cards for their fund-raising event. The school gets $1.00 for each roll of wrapping paper sold and $0.50 for each box of cards sold.

Total Amounts for Each Class		
Class	Wrapping Paper	Cards
Freshmen	72	49
Sophomores	68	63
Juniors	90	56
Seniors	86	62

35. Use a matrix to determine which class earned the most money.

36. What is the total amount of money the school made from the fund-raiser?

FINANCE **For Exercises 37–39, use the table below that shows the purchase price and selling price of stock for three companies.**
For a class project, Taini "bought" shares of stock in three companies. She bought 150 shares of a utility company, 100 shares of a computer company, and 200 shares of a food company. At the end of the project she "sold" all of her stock.



Company	Purchase Price (per share)	Selling Price (per share)
Utility	$54.00	$55.20
Computer	$48.00	$58.60
Food	$60.00	$61.10

EXTRA PRACTICE
See pages 897, 929.
MathOnline
Self-Check Quiz at algebra2.com

37. Organize the data in two matrices and use matrix multiplication to find the total amount she spent for the stock.

38. Write two matrices and use matrix multiplication to find the total amount she received for selling the stock.

39. Use matrix operations to find how much money Taini "made" or "lost" in her project.

H.O.T. Problems

40. **OPEN ENDED** Give an example of two matrices whose product is a 3×2 matrix.

41. **REASONING** Determine whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

For any matrix $A_{m \times n}$ for $m \neq n$, A^2 is defined.

42. **CHALLENGE** Give an example of two matrices A and B for which multiplication is commutative so that $AB = BA$. Explain how you found A and B.

43. **CHALLENGE** Find the values of a, b, c, and d to make the statement $\begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 7 \end{bmatrix}$ true. If matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ was multiplied by any other two-column matrix, what do you think the result would be?

44. ***Writing in Math*** Use the data on the Carolina Panthers found on page 177 to explain how matrices can be used in sports statistics. Describe a matrix that represents the total number of points scored in the 2005 season, and an example of another sport where different point values are used in scoring.

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STANDARDIZED TEST PRACTICE

45. ACT/SAT What are the dimensions of the matrix that results from the multiplication shown?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$$

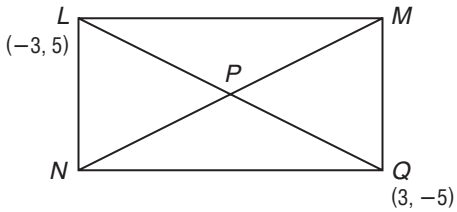
A 1×4

B 3×3

C 4×1

D 4×3

46. REVIEW Rectangle $LMNQ$ has diagonals that intersect at point P.



Which of the following represents point P?

F (1, 1)

G (2, 2)

H (0, 0)

J (−1, −1)

Spiral Review

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

47. $3\begin{bmatrix} 4 & -2 \\ -1 & 7 \end{bmatrix}$

48. $\begin{bmatrix} 3 & 5 & 9 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$

49. $2\begin{bmatrix} 6 & 3 \\ -8 & -2 \end{bmatrix} - 4\begin{bmatrix} 8 & 1 \\ 3 & -4 \end{bmatrix}$

Solve each equation. (Lesson 4-1)

50. $\begin{bmatrix} 3x + 2 \\ 15 \end{bmatrix} = \begin{bmatrix} 23 \\ -4y - 1 \end{bmatrix}$

51. $\begin{bmatrix} x + 3y \\ 2x - y \end{bmatrix} = \begin{bmatrix} -22 \\ 19 \end{bmatrix}$

52. $\begin{bmatrix} x + 3z \\ -2x + y - z \\ 5y - 7z \end{bmatrix} = \begin{bmatrix} -19 \\ -2 \\ 24 \end{bmatrix}$

53. VACATIONS Mrs. Franklin is planning a family vacation. She bought 8 rolls of film and 2 camera batteries for $23. The next day, her daughter went back and bought 6 more rolls of film and 2 batteries for her camera. This bill was $18. What are the prices of a roll of film and a camera battery? (Lesson 3-2)

Find the x-intercept and the y-intercept of the graph of each equation. Then graph the equation. (Lesson 2-2)

54. $y = 3 - 2x$

55. $x - \frac{1}{2}y = 8$

56. $5x - 2y = 10$

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each set of ordered pairs on a coordinate plane.** (Lesson 2-1)

57. {(2, 4), (−1, 3), (0, −2)}

58. {(−3, 5), (−2, −4), (3, −2)}

59. {(−1, 2), (2, 4), (3, −3), (4, −1)}

60. {(−3, 3), (1, 3), (4, 2), (−1, −5)}

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4-4 Transformations with Matrices

Main Ideas

- Use matrices to determine the coordinates of a translated or dilated figure.
- Use matrix multiplication to find the coordinates of a reflected or rotated figure.

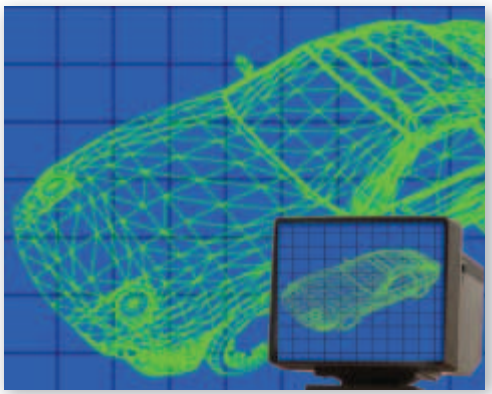
New Vocabulary

vertex matrix
transformation
preimage
image
translation
dilation
reflection
rotation

GET READY for the Lesson

Computer animation creates the illusion of motion by using a succession of computer-generated still images. Computer animation is used to create movie special effects and to simulate images that would be impossible to show otherwise.

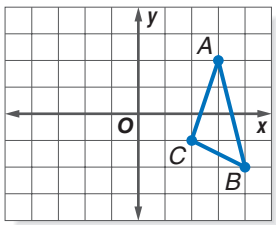
Complex geometric figures can be broken into simple triangles and then moved to other parts of the screen using matrices.



Translations and Dilations Points on a coordinate plane can be represented by matrices. The ordered pair (x, y) can be represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. Likewise, polygons can be represented by placing all of the column matrices of the coordinates of the vertices into one matrix, called a **vertex matrix**.

Triangle ABC with vertices $A(3, 2)$, $B(4, -2)$, and $C(2, -1)$ can be represented by the following vertex matrix.

$$\triangle ABC = \begin{matrix} A & B & C \\ \begin{bmatrix} 3 & 4 & 2 \\ 2 & -2 & -1 \end{bmatrix} & \begin{matrix} \leftarrow x\text{-coordinates} \\ \leftarrow y\text{-coordinates} \end{matrix} \end{matrix}$$



Notice that the triangle has 3 vertices and the vertex matrix has 3 columns. In general, the vertex matrix for a polygon with n vertices will have dimensions of $2 \times n$.

Matrices can be used to perform transformations. **Transformations** are functions that map points of a **preimage** onto its **image.**

One type of transformation is a translation. A **translation** occurs when a figure is moved from one location to another without changing its size, shape, or orientation. You can use matrix addition and a *translation matrix* to find the coordinates of a translated figure. The dimensions of a translation matrix should be the same as the dimensions of the vertex matrix.

Reading Math

Coordinate Matrix
A matrix containing coordinates of a geometric figure is also called a *coordinate matrix.*

Dennis Hallinan/Alamy Images
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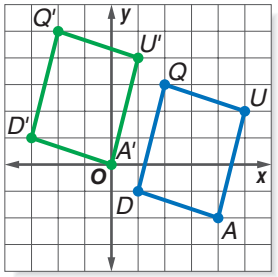
EXAMPLE 1 Translate a Figure

Find the coordinates of the vertices of the image of quadrilateral *QUAD* with *Q*(2, 3), *U*(5, 2), *A*(4, −2), and *D*(1, −1) if it is moved 4 units to the left and 2 units up. Then graph *QUAD* and its image *Q′U′A′D′*.

Write the vertex matrix for quadrilateral *QUAD*. $\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$

To translate the quadrilateral 4 units to the left, add −4 to each *x*-coordinate. To translate the figure 2 units up, add 2 to each *y*-coordinate. This can be done by adding the translation matrix $\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ to the vertex matrix of *QUAD*.

Vertex Matrix of *QUAD*		Translation Matrix		Vertex Matrix of *Q′U′A′D′*
$\begin{bmatrix} 2 & 5 & 4 & 1 \\ 3 & 2 & -2 & -1 \end{bmatrix}$	+	$\begin{bmatrix} -4 & -4 & -4 & -4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$	=	$\begin{bmatrix} -2 & 1 & 0 & -3 \\ 5 & 4 & 0 & 1 \end{bmatrix}$



The vertices of *Q′U′A′D′* are *Q′*(−2, 5), *U′*(1, 4), *A′*(0, 0), and *D′*(−3, 1). *QUAD* and *Q′U′A′D′* have the same size and shape.

CHECK Your Progress

1. Find the coordinates of the vertices of the image of triangle *RST* with *R*(−1, 5), *S*(2, 1), and *T*(−3, 2) if it is moved 3 units to the right and 4 units up. Then graph *RST* and its image *R′S′T′*.

STANDARDIZED TEST EXAMPLE 2 Find a Translation Matrix

Rectangle *A′B′C′D′* is the result of a translation of rectangle *ABCD*. A table of the vertices of each rectangle is shown. Find the coordinates of *D′*.

Rectangle *ABCD*	Rectangle *A′B′C′D′*
A(−4, 5)	*A′*(−1, 1)
B(1, 5)	*B′*(4, 1)
C(1, −2)	*C′*(4, −6)
D(−4, −2)	*D′*

A (−7, 2) **B** (−7, −6) **C** (−1, −6) **D** (−1, 2)

Test-Taking Tip

Sometimes you need to solve for unknown value(s) before you can solve for the value(s) requested in the question.

Read the Test Item

You are given the coordinates of the preimage and image of points *A*, *B*, and *C*. Use this information to find the translation matrix. Then you can use the translation matrix to find the coordinates of *D*.

Solve the Test Item

Step 1 Write a matrix equation. Let (*c*, *d*) represent the coordinates of *D*.

$$\begin{bmatrix} -4 & 1 & 1 & -4 \\ 5 & 5 & -2 & -2 \end{bmatrix} + \begin{bmatrix} x & x & x & x \\ y & y & y & y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

$$\begin{bmatrix} -4+x & 1+x & 1+x & -4+x \\ 5+y & 5+y & -2+y & -2+y \end{bmatrix} = \begin{bmatrix} -1 & 4 & 4 & c \\ 1 & 1 & -6 & d \end{bmatrix}$$

Extra Examples at algebra2.com

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Step 2 The matrices are equal, so corresponding elements are equal.

$-4 + x = -1$ Solve for x. $\qquad$ $5 + y = 1$ Solve for y.

$x = 3$ $\qquad$ $y = -4$

Step 3 Use the values for x and y to find the values for $D'(c, d)$.

$-4 + 3 = c$ $\qquad$ $-2 + (-4) = d$

$-1 = c$ $\qquad$ $-6 = d$

So the coordinates for D are $(-1, -6)$, and the answer is C.

CHECK Your Progress

2. Triangle $X'Y'Z'$ is the result of a translation of triangle XYZ. Find the coordinates of Z' using the information shown in the table.

Triangle XYZ	Triangle $X'Y'Z'$
$X(3, -1)$	$X'(1, 0)$
$Y(-4, 2)$	$Y'(-6, 3)$
$Z(5, 1)$	Z'

F $(3, 2)$ **G** $(7, 2)$ **H** $(7, 0)$ **J** $(3, 0)$

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Study Tip

Dilations

In a dilation, all linear measures of the image change in the same ratio. The image is similar to the preimage.

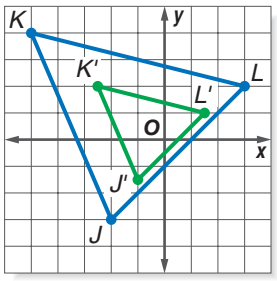
When a figure is enlarged or reduced, the transformation is called a **dilation**. A dilation is performed relative to its center. Unless otherwise specified, the center is the origin. You can use scalar multiplication to perform dilations.

EXAMPLE Dilation

3 **Dilate $\triangle JKL$ with $J(-2, -3)$, $K(-5, 4)$, and $L(3, 2)$ so that its perimeter is half the original perimeter. Find the coordinates of the vertices of $\triangle J'K'L'$.**

If the perimeter of a figure is half the original perimeter, then the lengths of the sides of the figure will be one-half the measure of the original lengths. Multiply the vertex matrix by the scale factor of $\frac{1}{2}$.

$$\frac{1}{2}\begin{bmatrix} -2 & -5 & 3 \\ -3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & 2 & 1 \end{bmatrix}$$



The coordinates of the vertices of $\triangle J'K'L'$ are $J'\left(-1, -\frac{3}{2}\right)$, $K'\left(-\frac{5}{2}, 2\right)$, and $L'\left(\frac{3}{2}, 1\right)$.

CHECK Your Progress

3. Dilate rectangle $MNPQ$ with $M(4, 4)$, $N(4, 12)$, $P(8, 4)$, and $Q(8, 12)$ so that its perimeter is one fourth the original perimeter. Find the coordinates of the vertices of rectangle $M'N'P'Q'$.

Reflections and Rotations A **reflection** maps every point of a figure to an image across a line of symmetry using a *reflection matrix*.

CONCEPT SUMMARY *Reflection Matrices*

For a reflection over the:	x-axis	y-axis	line $y = x$
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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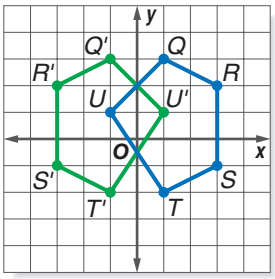
EXAMPLE Reflection

4 **Find the coordinates of the vertices of the image of pentagon $QRSTU$ with $Q(1, 3)$, $R(3, 2)$, $S(3, -1)$, $T(1, -2)$, and $U(-1, 1)$ after a reflection across the y-axis.**

Write the ordered pairs as a vertex matrix. Then multiply the vertex matrix by the reflection matrix for the y-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 3 & 1 & -1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 & -1 & 1 \\ 3 & 2 & -1 & -2 & 1 \end{bmatrix}$$

Notice that the preimage and image are congruent. Both figures have the same size and shape.



CHECK Your Progress

4. Find the coordinates of the vertices of the image of pentagon $QRSTU$ after a reflection across the x-axis.

A **rotation** occurs when a figure is moved around a center point, usually the origin. To determine the vertices of a figure's image by rotation, multiply its vertex matrix by a *rotation matrix*.

CONCEPT SUMMARY Rotation Matrices

For a counterclockwise rotation about the origin of:	90°	180°	270°
Multiply the vertex matrix on the left by:	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

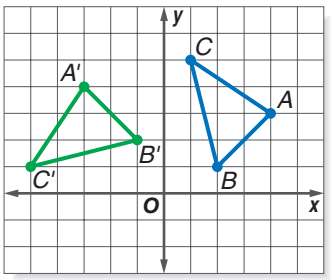
EXAMPLE Rotation

5 **Find the coordinates of the vertices of the image $\triangle ABC$ with $A(4, 3)$, $B(2, 1)$, and $C(1, 5)$ after it is rotated 90° counterclockwise about the origin.**

Write the ordered pairs in a vertex matrix. Then mutiply the vertex matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ 4 & 2 & 1 \end{bmatrix}$$

The coordinates of the vertices of $\triangle A'B'C'$ are $A'(-3, 4)$, $B'(-1, 2)$, and $C'(-5, 1)$. The image is congruent to the preimage.



Concepts in Motion
Interactive Lab
algebra2.com

CHECK Your Progress

5. Find the coordinates of the vertices of the image of $\triangle XYZ$ with $X(-5, -6)$, $Y(-1, -3)$, and $Z(-2, -4)$ after it is rotated 180° counterclockwise about the origin.

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CHECK Your Understanding

Example 1 (pp. 185–186)

Triangle ABC with vertices $A(1, 4)$, $B(2, -5)$, and $C(-6, -6)$ is translated 3 units right and 1 unit down.

1. Write the translation matrix.

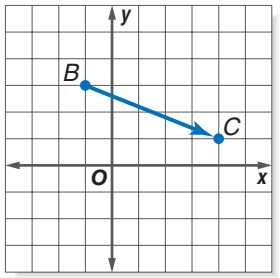
2. Find the coordinates of $\triangle A'B'C'$.

3. Graph the preimage and the image.

Example 2 (pp. 186–187)

4. STANDARDIZED TEST PRACTICE A point is translated from B to C as shown at the right. If a point at $(-4, 3)$ is translated in the same way, what will be its new coordinates?

A $(3, 4)$ **B** $(1, 1)$ **C** $(-8, 8)$ **D** $(1, 6)$



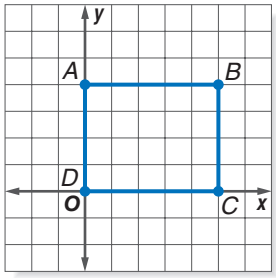
Example 3 (p. 187)

For Exercises 5–11, use the rectangle at the right.

5. Write the coordinates in a vertex matrix.

6. Find the coordinates of the image after a dilation by a scale factor of 3.

7. Find the coordinates of the image after a dilation by a scale factor of $\frac{1}{2}$.



Example 4 (p. 188)

8. Find the coordinates of the image after a reflection over the x-axis.

9. Find the coordinates of the image after a reflection over the y-axis.

10. Find the coordinates of the image after a rotation of 180°.

Example 5 (p. 188)

11. Find the coordinates of the image after a rotation of 270°.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12, 13	1
14, 15	2
16, 17	3
18, 19	4
20, 21	5

Write the translation matrix for each figure. Then find the coordinates of the image after the translation. Graph the preimage and the image on a coordinate plane.

12. $\triangle DEF$ with $D(1, 4)$, $E(2, -5)$, and $F(-6, -6)$, translated 4 units left and 2 units up

13. $\triangle MNO$ with $M(-7, 6)$, $N(1, 7)$, and $O(-3, 1)$, translated 2 units right and 6 units down

14. Rectangle $RSUT$ with vertices $R(-3, 2)$, $S(1, 2)$, $U(1, -1)$, $T(-3, -1)$ is translated so that T' is at $(-4, 1)$. Find the coordinates of R' and U'.

15. Triangle DEF with vertices $D(-2, 2)$, $E(3, 5)$, and $F(5, -2)$ is translated so that D' is at $(1, -5)$. Find the coordinates of E' and F'.

Write the vertex matrix for each figure. Then find the coordinates of the image after the dilation. Graph the preimage and the image on a coordinate plane.

16. $\triangle ABC$ with $A(0, 2)$, $B(1.5, -1.5)$, and $C(-2.5, 0)$ is dilated so that its perimeter is three times the original perimeter.

17. $\triangle XYZ$ with $X(-6, 2)$, $Y(4, 8)$, and $Z(2, -6)$ is dilated so that its perimeter is one half times the original perimeter.

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Write the vertex matrix and the reflection matrix for each figure. Then find the coordinates of the image after the reflection. Graph the preimage and the image on a coordinate plane.

18. The vertices of $\triangle XYZ$ are $X(1, -1)$, $Y(2, -4)$, and $Z(7, -1)$. The triangle is reflected over the line $y = x$.

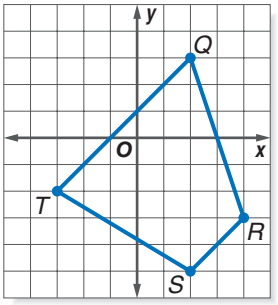
19. The vertices of rectangle $ABDC$ are $A(-3, 5)$, $B(5, 5)$, $D(5, -1)$, and $C(-3, -1)$. The rectangle is reflected over the x-axis.

Write the vertex matrix and the rotation matrix for each figure. Then find the coordinates of the image after the rotation. Graph the preimage and the image on a coordinate plane.

20. Parallelogram $DEFG$ with $D(2, 4)$, $E(5, 4)$, $F(4, 1)$, and $G(1, 1)$ is rotated 270° counterclockwise about the origin.

21. $\triangle MNO$ with $M(-2, -6)$, $N(1, 4)$, and $O(3, -4)$ is rotated 180° counterclockwise about the origin.

For Exercises 22–24, refer to the quadrilateral $QRST$ shown at the right.



22. Write the vertex matrix. Multiply the vertex matrix by -1.

23. Graph the preimage and image.

24. What type of transformation does the graph represent?

25. A triangle is rotated 90° counterclockwise about the origin. The coordinates of the vertices are $J'(-3, -5)$, $K'(-2, 7)$, and $L'(1, 4)$. What were the coordinates of the triangle in its original position?

26. A triangle is rotated 90° clockwise about the origin. The coordinates of the vertices are $F'(2, -3)$, $G'(-1, -2)$, and $H'(3, -2)$. What were the coordinates of the triangle in its original position?

27. A quadrilateral is reflected across the y-axis. The coordinates of the vertices are $P'(-2, 2)$, $Q'(4, 1)$, $R'(-1, -5)$, and $S'(-3, -4)$. What were the coordinates of the quadrilateral in its original position?

For Exercises 28–31, use rectangle $ABCD$ with vertices $A(-4, 4)$, $B(4, 4)$, $C(4, -4)$, and $D(-4, -4)$.

28. Find the coordinates of the image in matrix form after a reflection over the x-axis followed by a reflection over the y-axis.

29. Find the coordinates of the image in matrix form after a 180° rotation about the origin.

30. Find the coordinates of the image in matrix form after a reflection over the line $y = x$.

31. What do you observe about these three matrices? Explain.

Real-World Link

Douglas Engelbart invented the "X-Y position indicator for a display system" in 1964. He nicknamed this invention "the mouse" because a tail came out the end.

Source: about.com

TECHNOLOGY **For Exercises 32 and 33, use the following information.**

As you move the mouse for your computer, a corresponding arrow is translated on the screen. Suppose the position of the cursor on the screen is given in inches with the origin at the bottom left-hand corner of the screen.

32. Write a translation matrix that can be used to move the cursor 3 inches to the right and 4 inches up.

33. If the cursor is currently at (3.5, 2.25), what are the coordinates of the position after the translation?

Michael Denora/Getty Images

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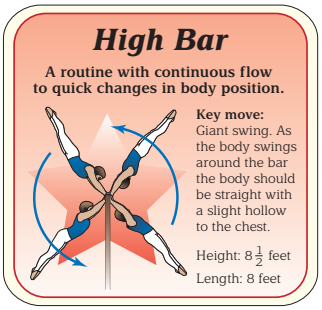
LANDSCAPING **For Exercises 34 and 35, use the following information.**
A garden design is plotted on a coordinate grid. The original plan shows a fountain with vertices at (−2, −2), (−6, −2), (−8, −5), and (−4, −5). Changes to the plan now require that the fountain's perimeter be three-fourths that of the original.

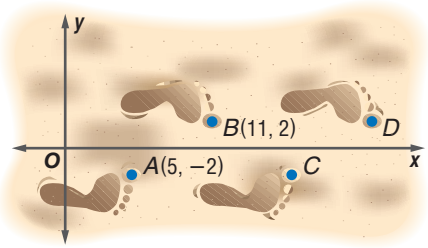
34. Determine the coordinates for the vertices of the fountain.

35. The center of the fountain was at (−5, −3.5). What will be the coordinates of the center after the changes in the plan have been made?

36. GYMNASTICS The drawing at the right shows four positions of a man performing the giant swing in the high bar event. Suppose this drawing is placed on a coordinate grid with the hand grips at $H(0, 0)$ and the toe of the figure in the upper right corner at $T(7, 8)$. Find the coordinates of the toes of the other three figures, if each successive figure has been rotated 90° counterclockwise about the origin.



FOOTPRINTS **For Exercises 37–40, use the following information.**
The combination of a reflection and a translation is called a *glide reflection*. An example is a set of footprints.



37. Describe the reflection and transformation combination shown at the right.

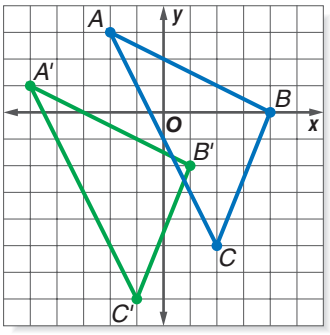
38. Write two matrix operations that can be used to find the coordinates of point C.

39. Does it matter which operation you do first? Explain.

40. What are the coordinates of the next two footprints?

EXTRA PRACTICE
See pages 898, 929.
MathOnline
Self-Check Quiz at algebra2.com

41. Write the translation matrix for $\triangle ABC$ and its image $\triangle A'B'C'$ shown at the right.



42. Compare and contrast the size and shape of the preimage and image for each type of transformation. For which types of transformations are the images congruent to the preimage?

H.O.T. Problems

43. OPEN ENDED Write a translation matrix that moves $\triangle DEF$ up and left.

44. CHALLENGE Do you think a matrix exists that would represent a reflection over the line $x = 3$? If so, make a conjecture and verify it.

45. REASONING Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

The image of a dilation is congruent to its preimage.

46. *Writing in Math* Use the information about computer animation on page 185 to explain how matrices can be used with transformations in computer animation. Include an example of how a figure with 5 points (coordinates) changes as a result of repeated dilations.

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STANDARDIZED TEST PRACTICE

47. ACT/SAT Triangle ABC has vertices with coordinates $A(-4, 2)$, $B(-4, -3)$, and $C(3, -2)$. After a dilation, triangle $A'B'C'$ has coordinates $A'(-12, 6)$, $B'(-12, -9)$, and $C'(9, -6)$. How many times as great is the perimeter of $\triangle A'B'C'$ as that of $\triangle ABC$?

A 3

B 6

C 12

D $\frac{1}{3}$

48. REVIEW Melanie wanted to find 5 consecutive whole numbers that add up to 95. She wrote the equation $(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 98$. What does the variable n represent in the equation?

F The least of the 5 whole numbers

G The middle of the 5 whole numbers

H The greatest of the 5 whole numbers

J The difference between the least and the greatest of the 5 whole numbers.

Spiral Review

Determine whether each matrix product is defined. If so, state the dimensions of the product. (Lesson 4-3)

49. $A_{2 \times 3} \cdot B_{3 \times 2}$ **50.** $A_{4 \times 1} \cdot B_{2 \times 1}$ **51.** $A_{2 \times 5} \cdot B_{5 \times 5}$

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*. (Lesson 4-2)

52. $2\begin{bmatrix} 4 & 9 & -8 \\ 6 & -11 & -2 \\ 12 & -10 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

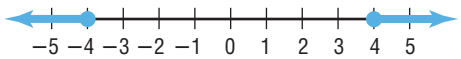
53. $4\begin{bmatrix} 3 & 4 & -7 \\ 6 & -9 & -2 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 6 & -4 \\ -7 & 10 & 1 \\ -2 & 1 & 5 \end{bmatrix}$

Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function. (Lesson 2-1)

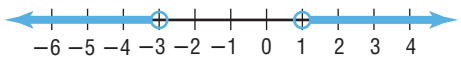
54. (3, 5), (4, 6), (5, −4) **55.** $x = -5y + 2$ **56.** $x = y^2$

Write an absolute value inequality for each graph. (Lesson 1-6)

57.



58.



59. BUSINESS Reliable Rentals rents cars for \$12.95 per day plus 15¢ per mile. Luis Romero works for a company that limits expenses for car rentals to \$90 per day. How many miles can Mr. Romero drive each day? (Lesson 1-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Use cross products to solve each proportion.**

60. $\frac{x}{8} = \frac{3}{4}$ **61.** $\frac{4}{20} = \frac{1}{m}$ **62.** $\frac{2}{3} = \frac{a}{42}$

63. $\frac{2}{y} = \frac{8}{9}$ **64.** $\frac{4}{n} = \frac{6}{2n - 3}$ **65.** $\frac{x}{5} = \frac{x + 1}{8}$

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CHAPTER 4

Mid-Chapter Quiz

Lessons 4-1 through 4-4

Solve each equation. (Lesson 4-1)

1. $\begin{bmatrix} 3x+1 \\ 7y \end{bmatrix} = \begin{bmatrix} 19 \\ 21 \end{bmatrix}$

2. $\begin{bmatrix} 2x+y \\ 4x-3y \end{bmatrix} = \begin{bmatrix} 9 \\ 23 \end{bmatrix}$

BUSINESS **For Exercises 3 and 4, use the table and the following information.**

The manager of The Best Bagel Shop keeps records of the types of bagels sold each day at their two stores. Two days of sales are shown below.

Day	Store	Type of Bagel			
		Sesame	Poppy	Wheat	Plain
Monday	East	120	80	64	75
	West	65	105	77	53
Tuesday	East	112	79	56	74
	West	69	95	82	50

3. Write a matrix for each day's sales. (Lesson 4-1)

4. Find the sum of the two days' sales using matrix addition. (Lesson 4-2)

Perform the indicated matrix operations. (Lesson 4-2)

5. $\begin{bmatrix} 3 & 0 \\ 7 & 12 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 4 & -1 \end{bmatrix}$

6. $5\begin{bmatrix} -2 & 4 & 5 \\ 0 & -4 & 7 \end{bmatrix}$

7. **MULTIPLE CHOICE** Solve for x and y in the matrix equation $\begin{bmatrix} 4x \\ -y \end{bmatrix} + \begin{bmatrix} -3y \\ 4 \end{bmatrix} = \begin{bmatrix} 22 \\ 2 \end{bmatrix}$. (Lesson 4-2)

A $x = 7, y = 2$ **C** $x = -7, y = 2$

B $x = -7, y = -2$ **D** $x = 7, y = -2$

Find each product, if possible. (Lesson 4-3)

8. $\begin{bmatrix} 4 & 0 & -8 \\ 7 & -2 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 6 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & -2 \\ -3 & 5 & 4 \end{bmatrix}$

RESTAURANTS **For Exercises 10–13, use the table and the following information.** (Lesson 4-3)

At Joe's Diner, the employees get paid weekly. The diner is closed on Mondays and Tuesdays. The servers make $20 per day (plus tips), cooks make $64 per day, and managers make $96 per day.

Number of Staff			
Day	Servers	Cooks	Managers
Wed.	8	3	2
Thur.	11	4	2
Fri.	17	6	5
Sat.	18	6	5
Sun.	14	5	3

10. Write a matrix for the number of staff needed for each day at the diner.

11. Write a cost matrix for the cost per type of employee.

12. Find the total cost of the wages for each day expressed as a matrix.

13. What is the total cost of wages for the week?

14. **MULTIPLE CHOICE** What is the product of $[5 \ -2 \ 3]$ and $\begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$? (Lesson 4-3)

F $\begin{bmatrix} 11 \\ -1 \end{bmatrix}$

G $[11 \ -1]$

H $\begin{bmatrix} 5 & -10 \\ 0 & -6 \\ 6 & -15 \end{bmatrix}$

J undefined

For Exercises 15 and 16, reflect square $ABCD$ with vertices $A(1, 2)$, $B(4, -1)$, $C(1, -4)$, and $D(-2, -1)$ over the y-axis. (Lesson 4-4)

15. Write the coordinates in a vertex matrix.

16. Find the coordinates of $A'B'C'D'$. Then graph $ABCD$ and $A'B'C'D'$.

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4-5 Determinants

Main Ideas

- Evaluate the determinant of a 2×2 matrix.
- Evaluate the determinant of a 3×3 matrix.

New Vocabulary

determinant
second-order determinant
third-order determinant
expansion by minors
minor

GET READY for the Lesson

The "Bermuda Triangle" is an area located off the southeastern Atlantic coast of the United States that is noted for a high incidence of unexplained losses of ships, small boats, and aircraft. Using the coordinates of the vertices of this triangle, you can find the value of a determinant to approximate the area of the triangle.

Determinants of 2×2 Matrices Every square matrix has a number associated with it called its **determinant.** The determinant of $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ can be represented by $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ or $\det \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$. The determinant of a 2×2 matrix is called a **second-order determinant.**

KEY CONCEPT *Second-Order Determinant*

Words The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.

Symbols $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 3(5) - (-1)(2) = 17$

EXAMPLE **Second-Order Determinant**

1 Find the value of the determinant $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix}$.

$\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - 5(6)$ Definition of determinant

$= -16 - 30$ or -46 Multiply.

CHECK Your Progress

Find the value of each determinant.

1A. $\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix}$

1B. $\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$

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Study Tip

Determinants

Note that only square matrices have determinants.

Determinants of 3 × 3 Matrices Determinants of 3 × 3 matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & f \\ g & h & i \end{vmatrix}$ The minor of a is $\begin{vmatrix} e & f \\ h & i \end{vmatrix}$.

$\begin{vmatrix} a & \cancel{b} & c \\ d & e & f \\ g & \cancel{h} & i \end{vmatrix}$ The minor of b is $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$.

$\begin{vmatrix} a & b & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix}$ The minor of c is $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

KEY CONCEPT — ***Third-Order Determinant***

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

EXAMPLE **Expansion by Minors**

2 **Evaluate** $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$ **using expansion by minors.**

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix}$$ Expansion by minors

$$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30)$$ Evaluate determinants.

$$= 2(36) - 7(24) - 3(-39)$$

$$= 72 - 168 + 117 \text{ or } 21$$ Multiply.

CHECK Your Progress

2. Evaluate $\begin{vmatrix} -2 & 3 & -1 \\ 5 & -3 & 8 \\ 4 & -6 & -5 \end{vmatrix}$ using expansion by minors.

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Another method for evaluating a third-order determinant is by using diagonals.

Step 1 Begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix}$$

Step 2 Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix} \qquad aei \quad bfg \quad cdh$$

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \begin{vmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{vmatrix} \qquad gec \quad hfa \quad idb$$

Step 3 To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is $aei + bfg + cdh - gec - hfa - idb$.

EXAMPLE Use Diagonals

3 **Evaluate** $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$ **using diagonals.**

Step 1 Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{vmatrix}$$

Step 2 Find the products of the elements of the diagonals.

$$\begin{vmatrix} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{vmatrix} \qquad 4 \quad 0 \quad 60$$

$$\begin{vmatrix} -1 & 3 & -3 & -1 & 3 \\ 4 & -2 & -1 & 4 & -2 \\ 0 & -5 & 2 & 0 & -5 \end{vmatrix} \qquad 0 \quad -5 \quad 24$$

Step 3 Add the bottom products and subtract the top products.
$4 + 0 + 60 - 0 - (-5) - 24 = 45$

The value of the determinant is 45.

CHECK Your Progress

3. Evaluate $\begin{vmatrix} 1 & -5 & 3 \\ 0 & 2 & -7 \\ 5 & -1 & -2 \end{vmatrix}$ using diagonals.

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One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.

Study Tip

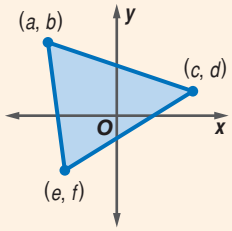
Area Formula

Notice that it is necessary to use the absolute value of A to guarantee a nonnegative value for the area.

KEY CONCEPT — **Area of a Triangle**

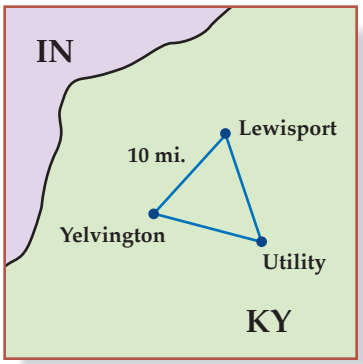
The area of a triangle having vertices at (a, b), (c, d), and (e, f) is $|A|$, where

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$



Real-World EXAMPLE

4 **RADIO** **A local radio station in Kentucky wants to place a tower that is strong enough to cover the cities of Yelvington, Utility, and Lewisport. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Kentucky with Yelvington at the origin, the coordinates of the three cities are (0, 0), (3, 0), and (1, 2). Use a determinant to estimate the area the signal must cover.**



$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$ Area Formula

$$= \frac{1}{2}\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$ $(a, b) = (3, 0)$, $(c, d) = (0, 2)$, $(e, f) = (0, 0)$

$$= \frac{1}{2}\left[3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}\right]$$ Expansion by minors

$$= \frac{1}{2}[3(2 - 0) - 0(1 - 0) + 1(0 - 0)]$$ Evaluate 2×2 determinants.

$$= \frac{1}{2}(6 - 0 - 0)$$ Multiply.

$$= \frac{1}{2}(6) \text{ or } 3$$ Simplify.

Remember that 1 unit equals 10 miles, so 1 square unit = 10×10 or 100 square miles. Thus, the area is 3×100 or 300 square miles.

CHECK Your Progress

4. Find the area of the triangle whose vertices are located at (2, 3), (−4, −3), and (1, −2).

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CHECK Your Understanding

Example 1 (p. 194)

Find the value of each determinant.

1. $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$

2. $\begin{vmatrix} -3 & -6 \\ 4 & 8 \end{vmatrix}$

Example 2 (p. 195)

Evaluate each determinant using expansion by minors.

3. $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$

Example 3 (p. 196)

Evaluate each determinant using diagonals.

5. $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

6. $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$

Example 4 (p. 197)

7. GEOMETRY What is the area of $\triangle ABC$ with $A(5, 4)$, $B(3, -4)$, and $C(-3, -2)$?

8. Find the area of the triangle whose vertices are located at $(2, -1)$, $(1, 2)$, and $(-1, 0)$.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–16	1
17–22	2
23–25	3
26–29	4

Find the value of each determinant.

9. $\begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix}$

10. $\begin{vmatrix} 8 & 5 \\ 6 & 1 \end{vmatrix}$

11. $\begin{vmatrix} -7 & 3 \\ -9 & 7 \end{vmatrix}$

12. $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix}$

13. $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$

14. $\begin{vmatrix} -9 & 0 \\ -12 & -7 \end{vmatrix}$

15. $\begin{vmatrix} 7 & 5.2 \\ -4 & 1.6 \end{vmatrix}$

16. $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$

17. $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 6 & 4 \\ 2 & 5 & 1 \end{vmatrix}$

18. $\begin{vmatrix} 7 & 3 & -4 \\ -2 & 9 & 6 \\ 0 & 0 & 0 \end{vmatrix}$

19. $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 5 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

20. $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

21. $\begin{vmatrix} 1 & 5 & -4 \\ -7 & 3 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

22. $\begin{vmatrix} 3 & 7 & 6 \\ -1 & 6 & 2 \\ 8 & -3 & -5 \end{vmatrix}$

23. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 9 & 5 \\ 8 & 7 & 4 \end{vmatrix}$

24. $\begin{vmatrix} 1 & 5 & 2 \\ -6 & -7 & 8 \\ 5 & 9 & -3 \end{vmatrix}$

25. $\begin{vmatrix} 8 & -9 & 0 \\ 1 & 5 & 4 \\ 6 & -2 & 3 \end{vmatrix}$

26. GEOGRAPHY Mr. Cardona is a regional sales manager for a company in Florida. Tampa, Orlando, and Ocala outline his region. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Florida with Tampa at the origin, the coordinates of the three cities are (0, 0), (7, 5), and (2.5, 10). Estimate the area of his sales territory.

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Real-World Career

Archaeologist

Archaeologists attempt to reconstruct past ways of life by examining preserved bones, the ruins of buildings, and artifacts such as tools, pottery, and jewelry.

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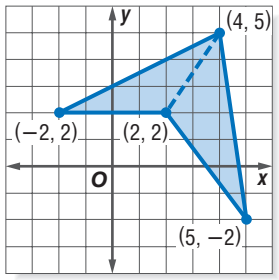
H.O.T. Problems

27. ARCHAEOLOGY During an archaeological dig, a coordinate grid is laid over the site to identify the location of artifacts as they are excavated. Suppose three corners of a building have been unearthed at $(-1, 6)$, $(4, 5)$, and $(-1, -2)$. If each square on the grid measures one square foot, estimate the area of the floor of the building, assuming that it is triangular.

28. GEOMETRY Find the area of a triangle whose vertices are located at $(4, 1)$, $(2, -1)$, and $(0, 2)$.

29. GEOMETRY Find the area of the polygon shown at the right.



30. Solve for x if $\det \begin{bmatrix} 2 & x \\ 5 & -3 \end{bmatrix} = 24$.

31. Solve $\det \begin{bmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{bmatrix} = -3$ for x.

32. GEOMETRY Find the value of x such that the area of a triangle whose vertices have coordinates $(6, 5)$, $(8, 2)$, and $(x, 11)$ is 15 square units.

33. GEOMETRY The area of a triangle ABC is 2 square units. The vertices of the triangle are $A(-1, 5)$, $B(3, 1)$, and $C(-1, y)$. What are the possible values of y?

MATRIX FUNCTION You can use a TI-83/84 Plus to find determinants of square matrices using the MATRIX functions. Enter the matrix under the EDIT menu. Then from the home screen choose det(, which is option 1 on the MATH menu, followed by the matrix name to calculate the determinant.

Use a graphing calculator to find the value of each determinant.

34. $\begin{bmatrix} 3 & -6.5 \\ 8 & 3.75 \end{bmatrix}$

35. $\begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{bmatrix}$

36. $\begin{bmatrix} 10 & 12 & 4 \\ -3 & 18 & -9 \\ 16 & -2 & -1 \end{bmatrix}$

37. OPEN ENDED Write a matrix whose determinant is zero.

38. FIND THE ERROR Khalid and Erica are finding the determinant of $\begin{bmatrix} 8 & 3 \\ -5 & 2 \end{bmatrix}$. Who is correct? Explain your reasoning.

Khalid

$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15)$

$= 31$

Erica

$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15$

$= 1$

39. REASONING Find a counterexample to disprove the following statement.
Two different matrices can never have the same determinant.

40. CHALLENGE Find a third-order determinant in which no element is 0, but for which the determinant is 0.

41. *Writing in Math* Use the information about the "Bermuda Triangle" on page 194 to explain how matrices can be used to find the area covered in this triangle. Then use your method to find the area.

EXTRA PRACTICE

See pages 898, 929.

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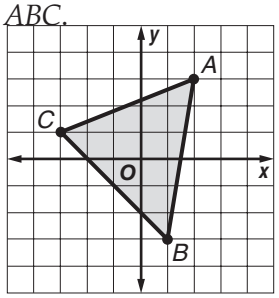
STANDARDIZED TEST PRACTICE

42. ACT/SAT Find the area of triangle ABC.



A 10 units^2

B 12 units^2

C 14 units^2

D 16 units^2

43. REVIEW Use the table to determine the expression that best represents the number of faces of any prism having a base with n sides.

Base	Sides of Base	Faces of Prisms
Triangle	3	5
Quadrilateral	4	6
Pentagon	5	7
Hexagon	6	8
Heptagon	7	9
Octagon	8	10

F $2(n - 1)$

G $2(n + 1)$

H $n + 2$

J $2n$

Spiral Review

For Exercises 44 and 45, use the following information. (Lesson 4-4)
The vertices of $\triangle ABC$ are $A(-2, 1)$, $B(1, 2)$ and $C(2, -3)$. The triangle is dilated so that its perimeter is $2\frac{1}{2}$ times the original perimeter.

44. Write the coordinates of $\triangle ABC$ in a vertex matrix.

45. Find the coordinates of $\triangle A'B'C'$. Then graph $\triangle ABC$ and $\triangle A'B'C'$.

Find each product, if possible. (Lesson 4-3)

46. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & 2 \end{bmatrix}$

47. $\begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -4 & 2 \end{bmatrix}$

48. $\begin{bmatrix} 7 & -5 & 4 \\ 6 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -8 \\ 1 & 2 \end{bmatrix}$

49. MARATHONS The length of a marathon was determined in the 1908 Olympic Games in London, England. The race began at Windsor Castle and ended in front of the royal box at London's Olympic Stadium, which was a distance of 26 miles 385 yards. Determine how many feet the marathon covers using the formula $f(m, y) = 5280m + 3y$, where m is the number of miles and y is the number of yards. (Lesson 3-4)

Write an equation in slope-intercept form for the line that satisfies each set of conditions. (Lesson 2-4)

50. slope 1, passes through (5, 3)

51. slope $-\frac{4}{3}$, passes through (6, −8)

52. passes through (3, 7) and (−2, −3)

53. passes through (0, 5) and (10, 10)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each system of equations.** (Lesson 3-2)

54. $x + y = -3$
$3x + 4y = -12$

55. $x + y = 10$
$2x + y = 11$

56. $2x + y = 5$
$4x + y = 9$

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4-6 Cramer's Rule

Main Ideas

- Solve systems of two linear equations by using Cramer's Rule.
- Solve systems of three linear equations by using Cramer's Rule.

New Vocabulary

Cramer's Rule

Study Tip

Look Back

To review **solving systems of equations**, see Lesson 3-2.

GET READY for the Lesson

Two sides of a triangle are contained in lines whose equations are $1.4x + 3.8y = 3.4$ and $2.5x - 1.7y = -10.9$. To find the coordinates of the vertex of the triangle between these two sides, you must solve the system of equations. One method for solving systems of equations is Cramer's Rule.

Systems of Two Linear Equations

Cramer's Rule uses determinants to solve systems of equations. Consider the following system.

$$ax + by = e \quad \text{a, b, c, d, e, and f represent constants, \textit{not} variables.}$$
$$cx + dy = f$$

Solve for x by using elimination.

$$\begin{array}{rll} adx + bdy = de & & \text{Multiply the first equation by } d. \\ (-)\ bcx + bdy = bf & & \text{Multiply the second equation by } b. \\ \hline adx - bcx = de - bf & & \text{Subtract.} \\ (ad - bc)x = de - bf & & \text{Factor.} \\ x = \dfrac{de - bf}{ad - bc} & & \text{Divide. Notice that } ad - bc \text{ must not be zero.} \end{array}$$

Solving for y in the same way produces the following expression.

$$y = \frac{af - ce}{ad - bc}$$

So the solution of the system of equations is $\left(\frac{de - bf}{ad - bc}, \frac{af - ce}{ad - bc}\right)$.

The fractions have a common denominator. It can be written using a determinant. The numerators can also be written as determinants.

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad de - bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad af - ce = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

KEY CONCEPT — ***Cramer's Rule for Two Variables***

The solution of the system of linear equations

$$ax + by = e$$
$$cx + dy = f$$

is (x, y), where $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

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EXAMPLE System of Two Equations

1 **Use Cramer's Rule to solve the system of equations.**

$5x + 7y = 13$

$2x - 5y = 13$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 13 & 7 \\ 13 & -5 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}} \qquad a = 5, b = 7, c = 2, d = -5, e = 13, \text{ and } f = 13 \qquad = \frac{\begin{vmatrix} 5 & 13 \\ 2 & 13 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 2 & -5 \end{vmatrix}}$$

$$= \frac{13(-5) - 13(7)}{5(-5) - 2(7)} \qquad \text{Evaluate each determinant.} \qquad = \frac{5(13) - 2(13)}{5(-5) - 2(7)}$$

$$= \frac{-156}{-39} \text{ or } 4 \qquad \text{Simplify.} \qquad = \frac{39}{-39} \text{ or } -1$$

The solution is $(4, -1)$.

CHECK Your Progress

Use Cramer's Rule to solve the systems of equations.

1A. $4x - 2y = -2$
$-x + 3y = 13$

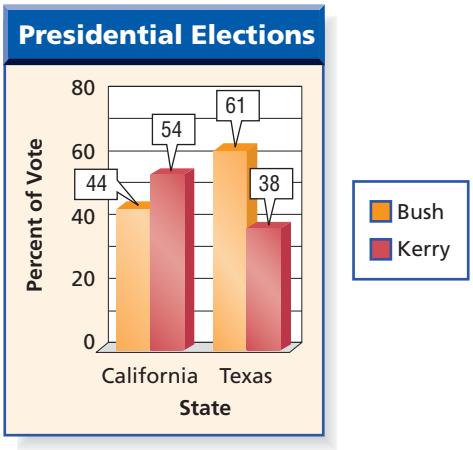
1B. $2x - 3y = 12$
$-6x + y = -20$

Real-World Link

In 2000, George W. Bush became the first son of a former president to win the presidency since John Quincy Adams did it in 1825.

Real-World EXAMPLE

2 **ELECTIONS** **In the 2004 presidential election, George W. Bush received about 10,000,000 votes in California and Texas, while John Kerry received about 9,500,000 votes in those states. The graph shows the percent of the popular vote that each candidate received in those states.**



a. **Write a system of equations that represents the total number of votes cast for each candidate in these two states.**

Words George W. Bush received 44% and 61% of the **votes in California** and **Texas**, respectively, for a total of 10,000,000 votes.

John Kerry received 54% and 38% of the **votes in California** and **Texas**, respectively, for a total of 9,500,000 votes.

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Extra Examples at algebra2.com

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You know the total votes for each candidate in Texas and California and the percent of the votes cast for each. You need to know the number of votes for each candidate in each state.

Variables Let x represent the total number of votes in California.
Let y represent the total number of votes in Texas.

Equations $0.44x + 0.61y = 10{,}000{,}000$ Votes for Bush
$0.54x + 0.38y = 9{,}500{,}000$ Votes for Kerry

b. Find the total number of popular votes cast in California and Texas.
Use Cramer's Rule to solve the system of equations.
Let $a = 0.44$, $b = 0.61$, $c = 0.54$, $d = 0.38$, $e = 10{,}000{,}000$, and $f = 9{,}500{,}000$.

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad \text{Cramer's Rule} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 10{,}000{,}000 & 0.61 \\ 9{,}500{,}000 & 0.38 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 0.44 & 10{,}000{,}000 \\ 0.54 & 9{,}500{,}000 \end{vmatrix}}{\begin{vmatrix} 0.44 & 0.61 \\ 0.54 & 0.38 \end{vmatrix}}$$

$$= \frac{10{,}000{,}000(0.38) - 9{,}500{,}000(0.61)}{0.44(0.38) - 0.54(0.61)} \qquad = \frac{0.44(9{,}500{,}000) - 0.54(10{,}000{,}000)}{0.44(0.38) - 0.54(0.61)}$$

$$= \frac{-1995000}{-0.1622} \qquad = \frac{-1220000}{-0.1622}$$

$$\approx 12{,}299{,}630 \qquad \approx 7{,}521{,}578$$

The solution of the system is about (12,299,630, 7,521,578).

So, there were about 12,300,000 popular votes cast in California and about 7,500,000 popular votes cast in Texas.

CHECK If you add the votes that Bush and Kerry received, the result is 10,000,000 + 9,500,000 or 19,500,000. If you add the popular votes in California and Texas, the result is 12,300,000 + 7,500,000 or 19,800,000. The difference of 300,000 votes is reasonable considering there were over 19 million total votes.

CHECK Your Progress

At the game on Friday, the Athletic Boosters sold chips C for \$0.50 and candy bars B for \$0.50 and made \$27. At Saturday's game, they raised the prices of chips to \$0.75 and candy bars to \$1.00. They made \$48 for the same amount of chips and candy bars sold.

2A. Write a system of equations that represents the total number of chips and candy bars sold at the games on Friday and Saturday.

2B. Find the total number of chips and candy bars that were sold on each day.

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Systems of Three Linear Equations You can also use Cramer's Rule to solve a system of three equations in three variables.

KEY CONCEPT *Cramer's Rule for Three Variables*

The solution of the system whose equations are

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = \ell$$

is (x, y, z), where $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \neq 0$.

EXAMPLE System of Three Equations

3 **Use Cramer's Rule to solve the system of equations.**

$3x + y + z = -1$

$-6x + 5y + 3z = -9$

$9x - 2y - z = 5$

Cross-Curricular Project

Math online You can use Cramer's Rule to compare home loans. Visit algebra2.com to continue work on your project.

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ \ell & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & \ell & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & \ell \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -1 & 1 & 1 \\ -9 & 5 & 3 \\ 5 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & -1 & 1 \\ -6 & -9 & 3 \\ 9 & 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}} \qquad = \frac{\begin{vmatrix} 3 & 1 & -1 \\ -6 & 5 & -9 \\ 9 & -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ -6 & 5 & 3 \\ 9 & -2 & -1 \end{vmatrix}}$$

Use a calculator to evaluate each determinant.

$x = \frac{-2}{-9}$ or $\frac{2}{9}$ $\qquad y = \frac{12}{-9}$ or $-\frac{4}{3}$ $\qquad z = \frac{3}{-9}$ or $-\frac{1}{3}$

The solution is $\left(\frac{2}{9}, -\frac{4}{3}, -\frac{1}{3}\right)$.

CHECK Your Progress

3. $2x + y - z = -2$

$-x + 2y + z = -0.5$

$x + y + 2z = 3.5$

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CHECK Your Understanding

Example 1 (p. 202)

Use Cramer's Rule to solve each system of equations.

1. $x - 4y = 1$
$2x + 3y = 13$

2. $0.2a = 0.3b$
$0.4a - 0.2b = 0.2$

Example 2 (pp. 202–203)

INVESTING For Exercises 3 and 4, use the following information.
Jarrod Wright has a total of \$5000 in his savings account and in a certificate of deposit. His savings account earns 3.5% interest annually. The certificate of deposit pays 5% interest annually if the money is invested for one year. He calculates that his interest earnings for the year will be \$227.50.

3. Write a system of equations for the amount of money in each investment.

4. How much money is in his savings account and in the certificate of deposit?

Example 3 (p. 204)

Use Cramer's Rule to solve each system of equations.

5. $2x - y + 3z = 5$
$3x + 2y - 5z = 4$
$x - 4y + 11z = 3$

6. $a + 9b - 2c = 2$
$-a - 3b + 4c = 1$
$2a + 3b - 6c = -5$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–12	1
13–17	2
18–21	3

Use Cramer's Rule to solve each system of equations.

7. $5x + 2y = 8$
$2x - 3y = 7$

8. $2m + 7n = 4$
$m - 2n = -20$

9. $2r - s = 1$
$3r + 2s = 19$

10. $3a + 5b = 33$
$5a + 7b = 51$

11. $2m - 4n = -1$
$3n - 4m = -5$

12. $4x + 3y = 6$
$8x - y = -9$

13. GEOMETRY The two sides of an angle are contained in lines whose equations are $4x + y = -4$ and $2x - 3y = -9$. Find the coordinates of the vertex of the angle.

14. GEOMETRY Two sides of a parallelogram are contained in the lines whose equations are $2.3x + 1.2y = 2.1$ and $4.1x - 0.5y = 14.3$. Find the coordinates of a vertex of the parallelogram.

STATE FAIR For Exercises 15 and 16, use the following information.
Jackson and Drew each purchased some game and ride tickets.

Person	Ticket Type	Tickets	Total
Jackson	game ride	6 15	\$93
Drew	game ride	7 12	\$81

15. Write a system of two equations using the given information.

16. Find the price for each type of ticket.

17. RINGTONES Ella's cell phone provider sells standard and premium ringtones. One month, Ella bought 2 standard and 2 premium ringtones for \$8.96. The next month Ella paid \$9.46 for 1 standard and 3 premium ringtones. What are the prices for standard and premium ringtones?

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Use Cramer's Rule to solve each system of equations.

18. $x + y + z = 6$
$2x + y - 4z = -15$
$5x - 3y + z = -10$

19. $a - 2b + c = 7$
$6a + 2b - 2c = 4$
$4a + 6b + 4c = 14$

20. $r - 2s - 5t = -1$
$r + 2s - 2t = 5$
$4r + s + t = -1$

21. $3a + c = 23$
$4a + 7b - 2c = -22$
$8a - b - c = 34$

22. $4x + 2y - 3z = -32$
$-x - 3y + z = 54$
$2y + 8z = 78$

23. $2r + 25s = 40$
$10r + 12s + 6t = -2$
$36r - 25s + 50t = -10$

24. $0.5r - s = -1$
$0.75r + 0.5s = -0.25$

25. $1.5m - 0.7n = 0.5$
$2.2m - 0.6n = -7.4$

26. $\frac{1}{3}r + \frac{2}{5}s = 5$
$\frac{2}{3}r - \frac{1}{2}s = -3$

27. $\frac{3}{4}x + \frac{1}{2}y = \frac{11}{12}$
$\frac{1}{2}x - \frac{1}{4}y = \frac{1}{8}$

Real-World Link
Video games are becoming increasingly popular among adults. In fact, more than 5% of adults play video games 2 or more times per week.

Source: U.S. Census Bureau

28. ARCADE GAMES Marcus and Cody purchased game cards to play virtual games at the arcade. Marcus used 47 points from his game card to drive the race car simulator and the snowboard simulator four times each. Cody used 48.25 points from his game card to drive the race car five times and the snowboard three times. How many points does each game charge per play?

29. PRICING The Harvest Nut Company sells made-to-order trail mixes. Sam's favorite mix contains peanuts, raisins, and carob-coated pretzels. Peanuts sell for \$3.20 per pound, raisins are \$2.40 per pound, and the carob-coated pretzels are \$4.00 per pound. Sam bought a 5-pound mixture for \$16.80 that contained twice as many pounds of carob-coated pretzels as raisins. How many pounds of peanuts, raisins, and carob-coated pretzels did Sam buy?

EXTRA PRACTICE
See pages 898, 929.
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H.O.T. Problems

30. OPEN ENDED Write a system of equations that *cannot* be solved using Cramer's Rule.

31. REASONING Write a system of equations whose solution is

$$x = \frac{\begin{vmatrix} -6 & 5 \\ 30 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}, \; y = \frac{\begin{vmatrix} 3 & -6 \\ 4 & 30 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 4 & -2 \end{vmatrix}}.$$

32. CHALLENGE In Cramer's Rule, if the value of the determinant is zero, what must be true of the graph of the system of equations represented by the determinant? Give examples to support your answer.

33. *Writing in Math* Use the information about two sides of the triangle on page 201 to explain how Cramer's Rule can be used to solve systems of equations. Include an explanation of how Cramer's rule uses determinants, and a situation where Cramer's rule would be easier to use to solve a system of equations than substitution or elimination.

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STANDARDIZED TEST PRACTICE

34. ACT/SAT Each year at Capital High School the students vote to choose the theme of that year's homecoming dance. The theme "A Night Under the Stars" received 225 votes, and "The Time of My Life" received 480 votes. If 40% of girls voted for "A Night Under the Stars", 75% of boys voted for "The Time of My Life", and all of the students voted, how many girls and boys are there at Capital High School?

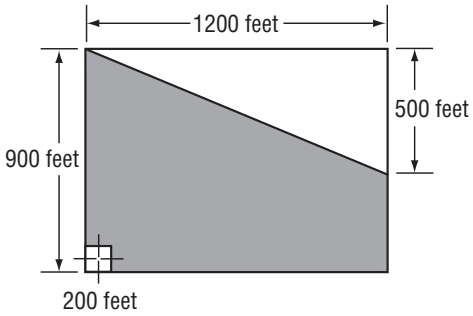
A 854 boys and 176 girls

B 705 boys and 325 girls

C 395 boys and 310 girls

D 380 boys and 325 girls

35. REVIEW What is the area of the shaded part of the rectangle below?



F 440,000 ft^2

G 540,000 ft^2

H 640,000 ft^2

J 740,000 ft^2

Spiral Review

Find the value of each determinant. (Lesson 4-5)

36. $\begin{vmatrix} 3 & 2 \\ -2 & 4 \end{vmatrix}$

37. $\begin{vmatrix} 8 & 6 \\ 4 & 8 \end{vmatrix}$

38. $\begin{vmatrix} -5 & 2 \\ 4 & 9 \end{vmatrix}$

For Exercises 39 and 40, use the following information. (Lesson 4-4)
Triangle ABC with vertices $A(0, 2)$, $B(-3, -1)$, and $C(-2, -4)$ is translated 1 unit right and 3 units up.

39. Write the translation matrix.

40. Find the coordinates of $\triangle A'B'C'$. Then graph the preimage and the image.

Solve each system of equations by graphing. (Lesson 3-1)

41. $y = 3x + 5$
$y = -2x - 5$

42. $x + y = 7$
$\frac{1}{2}x - y = -1$

43. $x - 2y = 10$
$2x - 4y = 12$

44. BUSINESS The Friendly Fix-It Company charges a base fee of \$45 for any in-home repair. In addition, the technician charges \$30 per hour. Write an equation for the cost c of an in-home repair of h hours. (Lesson 1-3)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each product, if possible.** (Lesson 4-3)

45. $\begin{bmatrix} 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix}$

46. $\begin{bmatrix} 0 & 9 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -6 \\ 8 & 1 \end{bmatrix}$

47. $\begin{bmatrix} 5 & -4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

48. $\begin{bmatrix} 7 & 11 & -5 \\ 3 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$

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4-7 Identity and Inverse Matrices

Main Ideas

- Determine whether two matrices are inverses.
- Find the inverse of a 2×2 matrix.

New Vocabulary

identity matrix

inverse

GET READY for the Lesson

With the rise of Internet shopping, ensuring the privacy of the user's personal information has become an important priority. Companies protect their computers by using codes. Cryptography is a method of preparing coded messages that can only be deciphered by using a "key."

The following technique is a simplified version of how cryptography works.

- First, assign a number to each letter of the alphabet.
- Convert your message into a matrix and multiply it by the coding matrix. The message is now unreadable to anyone who does not have the key to the code.
- To decode the message, the recipient of the coded message must multiply by the inverse of the coding matrix.

Code																			
_	0	A	1	B	2	C	3	D	4	E	5	F	6	G	7	H	8
I	9	J	10	K	11	L	12	M	13	N	14	O	15	P	16	Q	17
R	18	S	19	T	20	U	21	V	22	W	23	X	24	Y	25	Z	26

Identity and Inverse Matrices Recall that for real numbers, the multiplicative identity is 1. For matrices, the **identity matrix** is a square matrix that, when multiplied by another matrix, equals that same matrix.

2×2 Identity Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3×3 Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

KEY CONCEPT *Identity Matrix for Multiplication*

Word The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.

Symbols If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Michael Keller/CORBIS

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Two $n \times n$ matrices are **inverses** of each other if their product is the identity matrix. If matrix A has an inverse symbolized by A^{-1}, then $A \cdot A^{-1} = A^{-1} \cdot A = I$.

EXAMPLE Verify Inverse Matrices

1 **Determine whether each pair of matrices are inverses of each other.**

a. $X = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$

If X and Y are inverses, then $X \cdot Y = Y \cdot X = I$.

$$X \cdot Y = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{4} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 1-2 & 1+\frac{1}{2} \\ -\frac{1}{2}+(-4) & -\frac{1}{2}+1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 1\frac{1}{2} \\ -4\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ Matrix multiplication

Since $X \cdot Y \neq I$, they are *not* inverses.

b. $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

If P and Q are inverses, then $P \cdot Q = Q \cdot P = I$.

$$P \cdot Q = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

$$Q \cdot P = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 3-2 & 4-4 \\ -\frac{3}{2}+\frac{3}{2} & -2+3 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Matrix multiplication

Since $P \cdot Q = Q \cdot P = I$, P and Q are inverses.

Study Tip

Verifying Inverses

Since multiplication of matrices is not commutative, it is necessary to check the product in both orders.

CHECK Your Progress

1. $X = \begin{bmatrix} 4 & -1 \\ 2 & -2 \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

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Find Inverse Matrices Some matrices do not have an inverse. You can determine whether a matrix has an inverse by using the determinant.

KEY CONCEPT *Inverse of a 2 × 2 Matrix*

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Notice that $ad - bc$ is the value of det A. Therefore, if the value of the determinant of a matrix is 0, the matrix cannot have an inverse.

EXAMPLE **Find the Inverse of a Matrix**

2 **Find the inverse of each matrix, if it exists.**

a. $R = \begin{bmatrix} -4 & -3 \\ 8 & 6 \end{bmatrix}$

First find the determinant to see if the matrix has an inverse.

$$\begin{vmatrix} -4 & -3 \\ 8 & 6 \end{vmatrix} = -24 - (-24) = 0$$

Since the determinant equals 0, R^{-1} does not exist.

b. $P = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

Find the determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 6 - 5 \text{ or } 1$$

Since the determinant does not equal 0, P^{-1} exists.

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{Definition of inverse}$$

$$= \frac{1}{3(2) - 1(5)}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad a = 3, b = 1, c = 5, d = 2$$

$$= 1\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

$$= \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad \text{Simplify.}$$

CHECK Find the product of the matrices. If the product is I, then they are inverses.

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 & 2 - 2 \\ -15 + 15 & -5 + 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

CHECK Your Progress

2A. $\begin{bmatrix} -3 & 7 \\ 1 & -4 \end{bmatrix}$ **2B.** $\begin{bmatrix} 2 & 1 \\ -4 & 3 \end{bmatrix}$

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Real-World Link

The Enigma was a German coding machine used in World War II. Its code was considered to be unbreakable. However, the code was eventually solved by a group of Polish mathematicians.

Source: bletchleypark.org.uk

Matrices can be used to code messages by placing the message in a $n \times 2$ matrix.

Real-World EXAMPLE

3 **a. CRYPTOGRAPHY** **Use the table at the beginning of the lesson to assign a number to each letter in the message GO_TONIGHT. Then code the message with the matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.**

Convert the message to numbers using the table.

G O _ T O N I G H T

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

Write the message in matrix form. Arrange the numbers in a matrix with 2 columns and as many rows as are needed. Then multiply the message matrix B by the coding matrix A.

$$BA = \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 14 + 60 & 7 + 45 \\ 0 + 80 & 0 + 60 \\ 30 + 56 & 15 + 42 \\ 18 + 28 & 9 + 21 \\ 16 + 80 & 8 + 60 \end{bmatrix}$$ Multiply the matrices.

$$= \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix}$$ Write an equation.

The coded message is 74 | 52 | 80 | 60 | 86 | 57 | 46 | 30 | 96 | 68.

Study Tip

Messages

If there is an odd number of letters to be coded, add a 0 at the end of the message.

b. Use the inverse matrix A^{-1} to decode the message in Example 3a.

First find the inverse matrix of $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{2(3) - (1)(4)}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$ $a = 2, b = 1, c = 4, d = 3$

$$= \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$$ Simplify.

$$= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$ Simplify.

(continued on the next page)

Volker Steger/Science Photo Library/Ph

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Next, decode the message by multiplying the coded matrix C by A^{-1}.

$$CA^{-1} = \begin{bmatrix} 74 & 52 \\ 80 & 60 \\ 86 & 57 \\ 46 & 30 \\ 96 & 68 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{bmatrix}$$ Write an equation.

$$= \begin{bmatrix} 111 - 104 & -37 + 52 \\ 120 - 120 & -40 + 60 \\ 129 - 114 & -43 + 57 \\ 69 - 60 & -23 + 30 \\ 144 - 136 & -48 + 68 \end{bmatrix}$$ Multiply the matrices.

$$= \begin{bmatrix} 7 & 15 \\ 0 & 20 \\ 15 & 14 \\ 9 & 7 \\ 8 & 20 \end{bmatrix}$$ Simplify.

Use the table again to convert the numbers to letters. You can now read the message.

7 | 15 | 0 | 20 | 15 | 14 | 9 | 7 | 8 | 20

G O _ T O N I G H T

CHECK Your Progress

3. Use the table at the beginning of the lesson to assign a number to each letter in the message SECRET_CODE. Then code the message with the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Use the inverse matrix A^{-1} to decode the message.

CHECK Your Understanding

Example 1 (p. 209)

Determine whether each pair of matrices are inverses of each other.

1. $A = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$

2. $X = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, Y = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

3. $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

4. $F = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}, G = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$

Example 2 (p. 210)

Find the inverse of each matrix, if it exists.

5. $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & -8 \\ -1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$

Example 3 (pp. 211–212)

8. CRYPTOGRAPHY Code a message using your own coding matrix. Give your message and the matrix to a friend to decode. (*Hint*: Use a coding matrix whose determinant is 1 and that has all positive elements.)

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–12	1
13–21	2
22–24	3

Determine whether each pair of matrices are inverses of each other.

9. $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

10. $R = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, S = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$

11. $A = \begin{bmatrix} 6 & 2 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -\frac{5}{2} & -3 \end{bmatrix}$

12. $X = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}, Y = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

13. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$

16. $\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 7 \\ 2 & -6 \end{bmatrix}$

19. $\begin{bmatrix} 4 & -3 \\ 2 & 7 \end{bmatrix}$

20. $\begin{bmatrix} -2 & 0 \\ 5 & 6 \end{bmatrix}$

21. $\begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}$

CRYPTOGRAPHY For Exercises 22–24, use the alphabet table at the right.

Your friend sent you messages that were coded with the coding matrix $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Use the inverse of matrix C to decode each message.

CODE		
A 26	J 17	S 8
B 25	K 16	T 7
C 24	L 15	U 6
D 23	M 14	V 5
E 22	N 13	W 4
F 21	O 12	X 3
G 20	P 11	Y 2
H 19	Q 10	Z 1
I 18	R 9	_ 0

22. 50 | 36 | 51 | 29 | 18 | 18 | 26 | 13 | 33 | 26 | 44 | 22 | 48 | 33 | 59 | 34 | 61 | 35 | 4 | 2

23. 59 | 33 | 8 | 8 | 39 | 21 | 7 | 7 | 56 | 37 | 25 | 16 | 4 | 2

24. 59 | 34 | 49 | 31 | 40 | 20 | 16 | 14 | 21 | 15 | 25 | 25 | 36 | 24 | 32 | 16

25. RESEARCH Use the Internet or other reference to find examples of codes used throughout history. Explain how messages were coded.

Determine whether each statement is *true* or *false*.

26. Only square matrices have multiplicative identities.

27. Only square matrices have multiplicative inverses.

28. Some square matrices do not have multiplicative inverses.

29. Some square matrices do not have multiplicative identities.

Determine whether each pair of matrices are inverses of each other.

30. $C = \begin{bmatrix} 1 & 5 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} \frac{2}{7} & \frac{5}{7} \\ \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$

31. $J = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}, K = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} & \frac{7}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$

EXTRA PRACTICE
See pages 899, 929.

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Find the inverse of each matrix, if it exists.

32. $\begin{bmatrix} 2 & -5 \\ 6 & 1 \end{bmatrix}$

33. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{6} & \frac{1}{4} \end{bmatrix}$

34. $\begin{bmatrix} \frac{3}{10} & \frac{5}{8} \\ \frac{1}{5} & \frac{3}{4} \end{bmatrix}$

35. GEOMETRY Compare the matrix used to reflect a figure over the x-axis to the matrix used to reflect a figure over the y-axis.

a. Are they inverses?

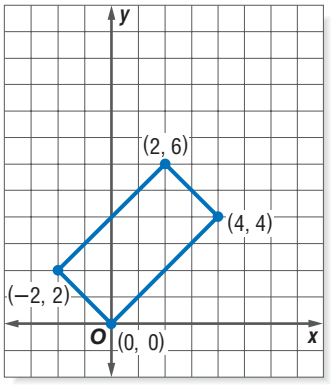
b. Does your answer make sense based on the geometry? Use a drawing to support your answer.

36. GEOMETRY The matrix used to rotate a figure 270° counterclockwise about the origin is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Compare this matrix with the matrix used to rotate a figure 90° counterclockwise about the origin.

a. Are they inverses?

b. Does your answer make sense? Use a drawing to support your answer.

GEOMETRY For Exercises 37–41, use the figure at the right.



37. Write the vertex matrix A for the rectangle.

38. Use matrix multiplication to find BA if $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

39. Graph the vertices of the transformed rectangle. Describe the transformation.

40. Make a conjecture about what transformation B^{-1} describes on a coordinate plane.

41. Find B^{-1} and multiply it by BA. Make a drawing to verify your conjecture.

Graphing Calculator

INVERSE FUNCTION The x^{-1} key on a TI-83/84 Plus graphing calculator is used to find the inverse of a matrix. If you get a SINGULAR MATRIX error on the screen, then the matrix has no inverse. Find the inverse of each matrix.

42. $\begin{bmatrix} -11 & 9 \\ 6 & -5 \end{bmatrix}$

43. $\begin{bmatrix} 12 & 4 \\ 15 & 5 \end{bmatrix}$

44. $\begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 5 & 2 \end{bmatrix}$

H.O.T. Problems

45. REASONING Explain how to find the inverse of a 2 × 2 matrix.

46. OPEN ENDED Create a square matrix that does not have an inverse. Explain how you know it has no inverse.

47. CHALLENGE For which values of a, b, c, and d will $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^{-1}$?

48. *Writing in Math* Use the information about cryptography on page 208 to explain how inverse matrices are used in cryptography. Explain why the inverse matrix works in decoding a message, and describe the conditions you must consider when writing a message in matrix form.

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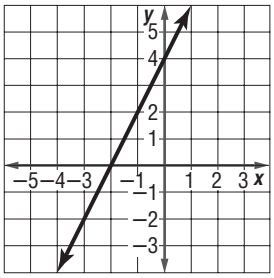
STANDARDIZED TEST PRACTICE

49. ACT/SAT The message **MEET_ME_TOMORROW** is converted into numbers (0 = space, A = 1, B = 2, etc.) and encoded using a numeric key. After the message is encoded it becomes 31 | −11 | 30 | 50 | 13 | 39 | 10 | −10 | 55 | 5 | 41 | 19 | 54 | 18 | 53 | 39. Which key was used to encode this message?

A $\begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$ C $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$

B $\begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$ D $\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$

50. REVIEW Line q is shown below. Which equation best represents a line parallel to line q?



F $y = x + 2$ H $y = 2x - 3$

G $y = x + 5$ J $y = -2x + 2$

Spiral Review

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

51. $3x + 2y = -2$
$x - 3y = 14$

52. $2x + 5y = 35$
$7x - 4y = -28$

53. $4x - 3z = -23$
$-2x - 5y + z = -9$
$y - z = 3$

Evaluate each determinant. (Lesson 4-5)

54. $\begin{vmatrix} 2 & 8 & -6 \\ 4 & 5 & 2 \\ -3 & -6 & -1 \end{vmatrix}$

55. $\begin{vmatrix} -3 & -3 & 1 \\ -9 & -2 & 3 \\ 5 & -2 & -1 \end{vmatrix}$

56. $\begin{vmatrix} 5 & -7 & 3 \\ -1 & -2 & -9 \\ 5 & -7 & 3 \end{vmatrix}$

Find each product, if possible. (Lesson 4-3)

57. $[5 \quad 2] \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

58. $\begin{bmatrix} 7 & 4 \\ -1 & 2 \\ -3 & 5 \end{bmatrix} \cdot [3 \quad 5]$

59. $[4 \quad 2 \quad 0] \cdot \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$

Solve each system of equations. (Lesson 3-2)

60. $3x + 5y = 2$
$2x - y = -3$

61. $6x + 2y = 22$
$3x + 7y = 41$

62. $3x - 2y = -2$
$4x + 7y = 65$

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

63. (2, 5), (6, 9)

64. (1, 0), (−2, 9)

65. (−5, 4), (−3, −6)

66. (−2, 2), (−5, 1)

67. (0, 3), (−2, −2)

68. (−8, 9), (0, 6)

69. OCEANOGRAPHY The bottom of the Mariana Trench in the Pacific Ocean is 6.8 miles below sea level. Water pressure in the ocean is represented by the function $f(x) = 1.15x$, where x is the depth in miles and $f(x)$ is the pressure in tons per square inch. Find the pressure in the Mariana Trench. (Lesson 2-1)

GET READY for the Next Lesson

Solve each equation. (Lesson 1-3)

70. $3k + 8 = 5$

71. $12 = -5h + 2$

72. $7z - 4 = 5z + 8$

73. $\frac{x}{2} + 5 = 7$

74. $\frac{3 + n}{6} = -4$

75. $6 = \frac{s - 8}{-7}$

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4-8 Using Matrices to Solve Systems of Equations

Main Ideas

- Write matrix equations for systems of equations.
- Solve systems of equations using matrix equations.

New Vocabulary

matrix equation

GET READY for the Lesson

An ecologist is studying two species of birds that compete for food and territory. He estimates that a particular region with an area of 14.25 acres (approximately 69,000 square yards) can supply 20,000 pounds of food for the birds.

Species A needs 140 pounds of food and has a territory of 500 square yards per nesting pair. Species B needs 120 pounds of food and has a territory of 400 square yards per nesting pair. The biologist can use this information to find the number of birds of each species that the area can support.

Write Matrix Equations The situation above can be represented using a system of equations that can be solved using matrices. Let's examine a similar situation. Consider the system of equations below. You can write this system with matrices by using the left and right sides of the equations.

$$\begin{matrix} 5x + 7y = 11 \\ 3x + 8y = 18 \end{matrix} \rightarrow \begin{bmatrix} 5x + 7y \\ 3x + 8y \end{bmatrix} = \begin{bmatrix} 11 \\ 18 \end{bmatrix}$$

Write the matrix on the left as the product of the coefficient matrix and the variable matrix.

$$\begin{matrix} A & & X & & B \\ \begin{bmatrix} 5 & 7 \\ 3 & 8 \end{bmatrix} & \cdot & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} 11 \\ 18 \end{bmatrix} \\ \text{coefficient matrix} & & \text{variable matrix} & & \text{constant matrix} \end{matrix}$$

The system of equations is now expressed as a **matrix equation**.

EXAMPLE **Two-Variable Matrix Equation**

1 **Write a matrix equation for the system of equations.**

$5x - 6y = -47$

$3x + 2y = -17$

Determine the coefficient, variable, and constant matrices.

$$\begin{matrix} 5x - 6y = -47 \\ 3x + 2y = -17 \end{matrix} \rightarrow \begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

SuperStock/Alamy Images

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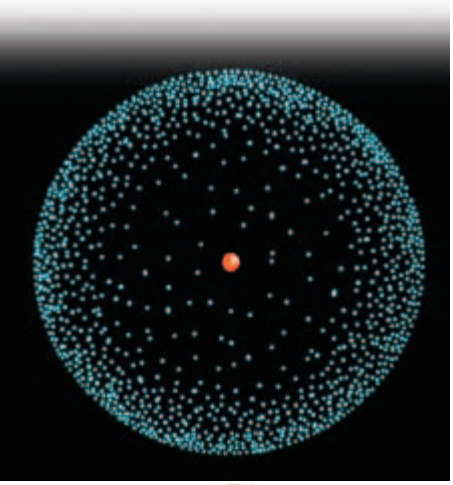
Write the matrix equation.

$$\begin{matrix} A & & X & = & B \end{matrix}$$
$$\begin{bmatrix} 5 & -6 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -47 \\ -17 \end{bmatrix}$$

CHECK Your Progress

1. $2x + 4y = 7$
$3x - y = 6$



Real-World Link

Atomic mass units (amu) are relative units of weight because they were compared to the weight of a hydrogen atom. So a molecule of nitrogen, whose weight is 14.0 amu, weighs 14 times as much as a hydrogen atom.

Source: www.sizes.com

Real-World EXAMPLE

2 **CHEMISTRY** **The molecular formula for glucose is $C_6H_{12}O_6$, which represents that a molecule of glucose has 6 carbon (C) atoms, 12 hydrogen (H) atoms, and 6 oxygen (O) atoms. One molecule of glucose weighs 180 atomic mass units (amu), and one oxygen atom weighs 16 amu. The formulas and weights for glucose and sucrose are listed below.**

Sugar	Formula	Atomic Weight (amu)
glucose	$C_6H_{12}O_6$	180
sucrose	$C_{12}H_{22}O_{11}$	342

a. Write a system of equations that represents the weight of each atom.

Let c represent the weight of a carbon atom.

Let h represent the weight of a hydrogen atom.

Glucose:
$$6c + 12h + 6(16) = 180 \quad \text{Equation for glucose}$$
$$6c + 12h + 96 = 180 \quad \text{Simplify.}$$
$$6c + 12h = 84 \quad \text{Subtract 96 from each side.}$$

Sucrose:
$$12c + 22h + 11(16) = 342 \quad \text{Equation for sucrose}$$
$$12c + 22h + 176 = 342 \quad \text{Simplify.}$$
$$12c + 22h = 166 \quad \text{Subtract 176 from each side.}$$

b. Write a matrix equation for the system of equations.

Determine the coefficient, variable, and constant matrices. Then write the matrix equation.

$$\begin{matrix} 6c + 12h = 84 \\ 12c + 22h = 166 \end{matrix} \rightarrow \begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

$$\begin{matrix} A & & X & = & B \end{matrix}$$
$$\begin{bmatrix} 6 & 12 \\ 12 & 22 \end{bmatrix} \cdot \begin{bmatrix} c \\ h \end{bmatrix} = \begin{bmatrix} 84 \\ 166 \end{bmatrix}$$

You will solve this matrix equation in Exercise 3.

CHECK Your Progress

2. The formula for propane is C_3H_8, and its atomic weight is 44 amu. Butane is C_4H_{10}, and its atomic weight is 58 amu. Write a system of equations for the weight of each. Then write a matrix equation for the system of equations.

Ken Eward/S.S./Photo Researchers

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Study Tip

Solving Using Inverses

Notice that A^{-1} is on the left on both sides of the equation. It is important to multiply both sides of the matrix equation with the inverse in the same order since matrix multiplication is not commutative.

Solve Systems of Equations A matrix equation in the form $AX = B$, where A is a coefficient matrix, X is a variable matrix, and B is a constant matrix, can be solved in a similar manner as a linear equation of the form $ax = b$.

$ax = b$	Write the equation.	$AX = B$
$\left(\frac{1}{a}\right)ax = \left(\frac{1}{a}\right)b$	Multiply each side by the inverse of the coefficient, if it exists.	$A^{-1}AX = A^{-1}B$
$1x = \left(\frac{1}{a}\right)b$	$\left(\frac{1}{a}\right)a = 1, A^{-1}A = I$	$IX = A^{-1}B$
$x = \left(\frac{1}{a}\right)b$	$1x = x, IX = X$	$X = A^{-1}B$

Notice that the solution of the matrix equation is the product of the inverse of the coefficient matrix and the constant matrix.

EXAMPLE 3 Solve Systems of Equations

Use a matrix equation to solve each system of equations.

a. $6x + 2y = 11$
$3x - 8y = 1$

The matrix equation is $\begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & 2 \\ 3 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-48 - 6}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \text{ or } -\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix}$$

Step 2 Multiply each side of the matrix equation by the inverse matrix.

$$-\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & -2 \\ 3 & -8 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -8 & -2 \\ -3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$ Multiply each side by A^{-1}.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{54}\begin{bmatrix} -90 \\ -27 \end{bmatrix}$$ Multiply matrices.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{2} \end{bmatrix}$$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

The solution is $\left(\frac{5}{3}, \frac{1}{2}\right)$. Check this solution in the original equation.

Study Tip

Identity Matrix

The identity matrix on the left verifies that the inverse matrix has been calculated correctly.

b. $6a - 9b = -18$
$8a - 12b = 24$

The matrix equation is $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$, when $A = \begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \end{bmatrix}$, and $B = \begin{bmatrix} -18 \\ 24 \end{bmatrix}$.

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Review Vocabulary

Inconsistent System of Equations: a system of equations that does not have a solution (Lesson 3-1)

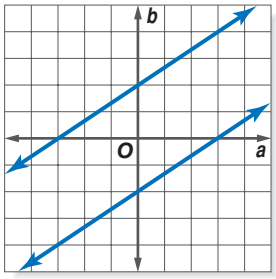
Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{-72 + 72}\begin{bmatrix} -12 & 9 \\ -8 & 6 \end{bmatrix}$$

The determinant of the coefficient matrix $\begin{bmatrix} 6 & -9 \\ 8 & -12 \end{bmatrix}$ is 0, so A^{-1} does not exist.

There is no unique solution of this system.

Graph the system of equations. Since the lines are parallel, this system has no solution. Therefore, the system is inconsistent.



CHECK Your Progress

3A. $-2x + 3y = -7$
$4x - 8y = 16$

3B. $2x - 4y = -24$
$3x - 6y = -12$

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To solve a system of equations with three variables, you can use the 3×3 identity matrix. However, finding the inverse of a 3×3 matrix may be tedious. Graphing calculators and computers offer fast and accurate calculations.

GRAPHING CALCULATOR LAB

Systems of Three Equations in Three Variables

You can use a graphing calculator and a matrix equation to solve systems of equations. Consider the system of equations below.

$3x - 2y + z = 0$
$2x + 3y - z = 17$
$5x - y + 4z = -7$

THINK AND DISCUSS

1. Write a matrix equation for the system of equations.
2. Enter the coefficient matrix as matrix A and the constant matrix as matrix B. Find the product of A^{-1} and B. Recall that the x^{-1} key is used to find A^{-1}.
3. How is the result related to the solution?

CHECK Your Understanding

Example 1 (pp. 216–217)

Write a matrix equation for each system of equations.

1. $x - y = -3$
$x + 3y = 5$

2. $2g + 3h = 8$
$-4g - 7h = -5$

Example 2 (p. 217)

3. CHEMISTRY Refer to Example 2 on page 217. Solve the system of equations to find the weight of a carbon, hydrogen, and oxygen atom.

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Example 3 (pp. 218–219)

Use a matrix equation to solve each system of equations.

4. $5x - 3y = -30$
$8x + 5y = 1$

5. $5s + 4t = 12$
$4s - 3t = -1.25$

6. $3x + 6y = 11$
$2x + 4y = 7$

7. $3x + 4y = 3$
$6x + 8y = 5$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–11	1
12, 13	2
14–23	3, 4

Write a matrix equation for each system of equations.

8. $3x - y = 0$
$x + 2y = -21$

9. $4x - 7y = 2$
$3x + 5y = 9$

10. $5a - 6b = -47$
$3a + 2b = -17$

11. $3m - 7n = -43$
$6m + 5n = -10$

12. MONEY Mykia had 25 quarters and dimes. The total value of all the coins was \$4.00. How many quarters and dimes did Mykia have?

13. PILOT TRAINING Flight instruction costs \$105 per hour, and the simulator costs \$45 per hour. Hai-Ling spent 4 more hours in airplane training than in the simulator. If Hai-Ling spent \$3870, how much time did he spend training in an airplane and in a simulator?

Use a matrix equation to solve each system of equations.

14. $p - 2q = 1$
$p + 5q = 22$

15. $3x - 9y = 12$
$-2x + 6y = 9$

16. $-2x + 4y = 3$
$2x - 4y = 5$

17. $6r + s = 9$
$3r = -2s$

18. $5a + 9b = -28$
$2a - b = -2$

19. $6x - 10y = 7$
$3x - 5y = 8$

20. $4m - 7n = -63$
$3m + 2n = 18$

21. $8x - 3y = 19.5$
$2.5x + 7y = 18$

22. $x + 2y = 8$
$3x + 2y = 6$

23. $4x - 3y = 5$
$2x + 9y = 6$

24. NUMBER THEORY Find two numbers whose sum is 75 and the second number is 15 less than twice the first.

25. CHEMISTRY Refer to Check Your Progress 2 on page 217. Solve the system of equations to find the weights of a carbon and a hydrogen atom.

26. SPORTING GOODS Use three rows from the table of sporting goods sales and write a matrix. Then use the matrix to find the cost of each type of ball.

Day	Baseballs	Basketballs	Footballs	Sales (\$)
Monday	10	3	6	97
Tuesday	13	1	4	83
Wednesday	8	5	2	79
Thursday	15	2	7	116
Friday	9	0	8	84

EXTRA PRACTICE
See pages 899, 929.
MathOnline
Self-Check Quiz at algebra2.com

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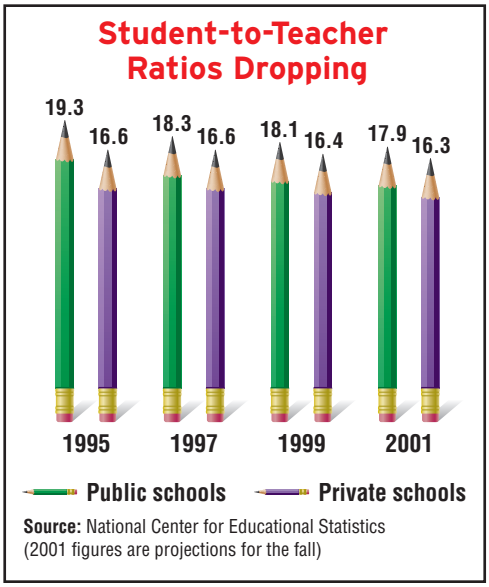
27. SCHOOLS The graphic shows that student-to-teacher ratios are dropping in both public and private schools. If these rates of change remain constant, predict when the student-to-teacher ratios for private and public schools will be the same.



28. CHEMISTRY Cara is preparing an acid solution. She needs 200 milliliters of 48% concentration solution. Cara has 60% and 40% concentration solutions in her lab. How many milliliters of 40% acid solution should be mixed with 60% acid solution to make the required amount of 48% acid solution?

Graphing Calculator

Use a graphing calculator to solve each system of equations using inverse matrices.

29. $2a - b + 4c = 6$
$a + 5b - 2c = -6$
$3a - 2b + 6c = 8$

30. $3x - 5y + 2z = 22$
$2x + 3y - z = -9$
$4x + 3y + 3z = 1$

31. $2q + r + s = 2$
$-q - r + 2s = 7$
$-3q + 2r + 3s = 7$

H.O.T. Problems

32. REASONING Write the matrix equation $\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ as a system of linear equations.

33. OPEN ENDED Write a system of equations that does not have a unique solution.

34. FIND THE ERROR Tommy and Laura are solving a system of equations.

They find that $A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Who is correct? Explain your reasoning.

Tommy

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -9 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Laura

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 31 \end{bmatrix}$$

35. CHALLENGE What can you conclude about the solution set of a system of equations if the coefficient matrix does not have an inverse?

36. *Writing in Math* Use the information about ecology found on page 216 to explain how matrices can be used to find the number of species of birds that an area can support. Demonstrate a system of equations that can be used to find the number of each species the region can support, and a solution of the problem using matrices.

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STANDARDIZED TEST PRACTICE

37. ACT/SAT The Yogurt Shoppe sells cones in three sizes: small, \$0.89; medium, \$1.19; and large, \$1.39. One day Scott sold 52 cones. He sold seven more medium cones than small cones. If he sold \$58.98 in cones, how many medium cones did he sell?

A 11 **C** 24

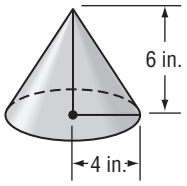
B 17 **D** 36

38. ACT/SAT What is the solution to the system of equations $6a + 8b = 5$ and $10a - 12b = 2$?

F $\left(\frac{3}{4}, \frac{1}{2}\right)$ **H** $\left(\frac{1}{2}, \frac{3}{4}\right)$

G $\left(\frac{1}{2}, -\frac{1}{2}\right)$ **J** $\left(\frac{1}{2}, \frac{1}{4}\right)$

39. REVIEW A right circular cone has radius 4 inches and height 6 inches.



What is the lateral area of the cone? (Lateral area of cone $= \pi r\ell$, where ℓ = slant height)?

A 24π sq in.

B $2\sqrt{13}\pi$ sq in.

C $2\sqrt{52}\pi$ sq in.

D $8\sqrt{13}\pi$ sq in.

Spiral Review

Find the inverse of each matrix, if it exists. (Lesson 4-7)

40. $\begin{bmatrix} 4 & 4 \\ 2 & 3 \end{bmatrix}$ **41.** $\begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$ **42.** $\begin{bmatrix} -3 & -6 \\ 5 & 10 \end{bmatrix}$

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

43. $6x + 7y = 10$
$3x - 4y = 20$

44. $6a + 7b = -10.15$
$9.2a - 6b = 69.944$

45. $\frac{x}{2} - \frac{2y}{3} = 2\frac{1}{3}$
$3x + 4y = -50$

46. ECOLOGY If you recycle a $3\frac{1}{2}$-foot stack of newspapers, one less 20-foot loblolly pine tree will be needed for paper. Use a prediction equation to determine how many feet of loblolly pine trees will *not* be needed for paper if you recycle a pile of newspapers 20 feet tall. (Lesson 2-5)

Cross-Curricular Project

Algebra and Consumer Science

What Does it Take to Buy a House? It is time to complete your project. Use the information and data you have gathered about home buying and selling to prepare a portfolio or Web page. Be sure to include your tables, graphs, and calculations in the presentation. You may also wish to include additional data, information, or pictures.

MathOnline **Cross-Curricular Project at algebra2.com**

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EXTEND 4-8

Graphing Calculator Lab

Augmented Matrices

Using a TI-83/84 Plus, you can solve a system of linear equations using the MATRIX function. An **augmented matrix** contains the coefficient matrix with an extra column containing the constant terms. The reduced row echelon function of a graphing calculator reduces the augmented matrix so that the solution of the system of equations can be easily determined.

ACTIVITY

Write an augmented matrix for the system of equations. Then solve the system by using the reduced row echelon form on the graphing calculator.

$3x + y + 3z = 2$
$2x + y + 2z = 1$
$4x + 2y + 5z = 5$

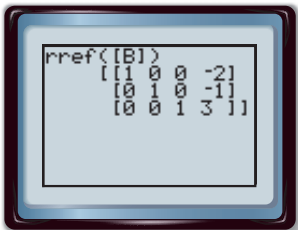
Step 1 Write the augmented matrix and enter it into a calculator.

The augmented matrix $B = \left[\begin{array}{ccc:c} 3 & 1 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 2 & 5 & 5 \end{array}\right]$.

KEYSTROKES: *Review matrices on page 172.*

Step 2 Find the reduced row echelon form (rref) using the graphing calculator.

KEYSTROKES: 2nd [MATRIX] ▶ ALPHA [B] 2nd [MATRIX] 2) ENTER



Study the reduced echelon matrix. The first three columns are the same as a 3×3 identity matrix. The first row represents $x = -2$, the second row represents $y = -1$, and the third row represents $z = 3$. The solution is $(-2, -1, 3)$.

EXERCISES

Write an augmented matrix for each system of equations. Then solve with a graphing calculator. Round to the nearest hundredth.

1. $x - 3y = 5$
$2x + y = 1$

2. $15x + 11y = 36$
$4x - 3y = -26$

3. $2x - y = 5$
$2x - 3y = 1$

4. $-x + 3y = 10$
$4x + 2y = 16$

5. $8x - 7y = 45.1$
$2x + 5y = -8.3$

6. $0.5x + 0.7y = 5.5$
$3x - 2.5y = -0.5$

7. $3x - y = 0$
$2x - 3y = 1$

8. $3x - 2y + z = -2$
$x - y + 3z = 5$
$-x + y + z = -1$

9. $x - y + z = 2$
$x - z = 1$
$y + 2z = 0$

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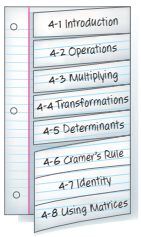
CHAPTER 4 Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Matrices (Lesson 4-1)

- A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns.
- Equal matrices have the same dimensions and corresponding elements are equal.

Operations (Lessons 4-2, 4-3)

- Matrices can be added or subtracted if they have the same dimensions. Add or subtract corresponding elements.
- To multiply a matrix by a scalar k, multiply each element in the matrix by k.
- Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.
- Use matrix addition and a translation matrix to find the coordinates of a translated figure.
- Use scalar multiplication to perform dilations.

Transformations (Lesson 4-4)

- To rotate a figure counterclockwise about the origin, multiply the vertex matrix on the left by a rotation matrix.

Identity and Inverse Matrices (Lesson 4-7)

- An identity matrix is a square matrix with ones on the diagonal and zeros in the other positions.
- Two matrices are inverses of each other if their product is the identity matrix.

Matrix Equations (Lesson 4-8)

- To solve a matrix equation, find the inverse of the coefficient matrix. Then multiply each side of the equation by the inverse matrix.

Key Vocabulary

Cramer's Rule (p. 201)	inverse (p. 209)
determinant (p. 194)	matrix (p. 162)
dilation (p. 187)	matrix equation (p. 216)
dimension (p. 163)	reflection (p. 188)
element (p. 163)	rotation (p. 188)
equal matrices (p. 164)	scalar multiplication (p. 171)
identity matrix (p. 208)	translation (p. 185)

Vocabulary Check

Choose the correct term from the list above to complete each sentence.

1. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a(n) _____ for multiplication.
2. _____ is the process of multiplying a matrix by a constant.
3. A(n) _____ is when a figure is moved around a center point.
4. The _____ of $\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$ is -1.
5. A(n)_____ is the product of the coefficient matrix and the variable matrix equal to the constant matrix.
6. The _____ of a matrix tell how many rows and columns are in the matrix.
7. A(n) _____ is a rectangular array of constants or variables.
8. Each value in a matrix is called an _____.
9. If the product of two matrices is the identity matrix, they are _____.
10. _____ can be used to solve a system of equations.
11. (A)n _____ is when a geometric figure is enlarged or reduced.
12. A(n) _____ occurs when a figure is slid from one location to another on the coordinate plane.

Vocabulary Review at algebra2.com

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Lesson-by-Lesson Review

4-1 Introduction to Matrices (pp. 162–167)

Solve each equation.

13. $\begin{bmatrix} 2y - x \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ 4y - 1 \end{bmatrix}$

14. $\begin{bmatrix} 7x \\ x + y \end{bmatrix} = \begin{bmatrix} 5 + 2y \\ 11 \end{bmatrix}$

15. $\begin{bmatrix} 3x + y \\ x - 3y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

16. $\begin{bmatrix} 2x - y \\ 6x - y \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix}$

17. FAMILY Three sisters, Tionna, Diana, and Caroline each have 3 children. Tionna's children are 17, 20, and 23 years old. Diana's children are 12, 19, and 22 years old. Caroline's children are 6, 7, and 11 years old. Write a matrix of the children's ages. Which element represents the youngest child?

Example 1 Solve $\begin{bmatrix} 2x \\ y \end{bmatrix} = \begin{bmatrix} 32 + 6y \\ 7 - x \end{bmatrix}$.

Write two linear equations.

$2x = 32 + 6y$
$y = 7 - x$

Solve the system of equations.

$2x = 32 + 6y$	First equation
$2x = 32 + 6(7 - x)$	Substitute $7 - x$ for y.
$2x = 32 + 42 - 6x$	Distributive Property
$8x = 74$	Add $6x$ to each side.
$x = 9.25$	Divide each side by 8.

To find the value of y, substitute 9.25 for x in either equation.

$y = 7 - x$	Second equation
$= 7 - 9.25$	Substitute 9.25 for x.
$= -2.25$	Simplify.

The solution is (9.25, −2.25).

4-2 Operations with Matrices (pp. 169–176)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

18. $\begin{bmatrix} -4 & 3 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 3 & -8 \end{bmatrix}$

19. $[0.2 \quad 1.3 \quad -0.4] - [2 \quad 1.7 \quad 2.6]$

20. $\begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 & 4 \\ -16 & 8 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & -3 \\ 4 & -5 & 2 \end{bmatrix} - 2\begin{bmatrix} -2 & 3 & 5 \\ -3 & -1 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 90 & 70 & 85 \\ 72 & 53 & 97 \\ 84 & 61 & 79 \end{bmatrix} - \begin{bmatrix} 93 & 77 & 91 \\ 83 & 52 & 92 \\ 83 & 64 & 89 \end{bmatrix}$

Example 2 Find $A - B$ if $A = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$.

$A - B = \begin{bmatrix} 3 & 8 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} -4 & 6 \\ 1 & 9 \end{bmatrix}$ Matrix subtraction

$= \begin{bmatrix} 3 - (-4) & 8 - 6 \\ -5 - 1 & 2 - 9 \end{bmatrix}$ Subtract.

$= \begin{bmatrix} 7 & 2 \\ -6 & -7 \end{bmatrix}$ Simplify.

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CHAPTER 4

Study Guide and Review

4-3 Multiplying Matrices (pp. 177–184)

Find each product, if possible.

23. $[2 \quad 7] \cdot \begin{bmatrix} 5 \\ -4 \end{bmatrix}$

24. $\begin{bmatrix} 8 & -3 \\ 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & -5 \end{bmatrix}$

25. $\begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 4 & 5 \\ 3 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

26. SHOPPING Mark went shopping and bought two shirts, three pairs of pants, one belt, and two pairs of shoes. The following matrix shows the prices for each item respectively.

[\$20.15 \$32 \$15 \$25.99]

Use matrix multiplication to find the total amount of money Mark spent while shopping.

Example 3 Find XY if $X = [6 \quad 4]$ and $Y = \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}$.

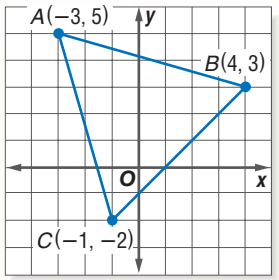
$XY = [6 \quad 4] \cdot \begin{bmatrix} 2 & 5 \\ -3 & 0 \end{bmatrix}$ Write an equation.

$= [6(2) + 4(-3) \quad 6(5) + 4(0)]$ Multiply columns by rows.

$= [0 \quad 30]$ Simplify.

4-4 Transformations with Matrices (pp. 185–192)

For Exercises 27–30, use the figure to find the coordinates of the image after each transformation.



27. translation 4 units right and 5 units down

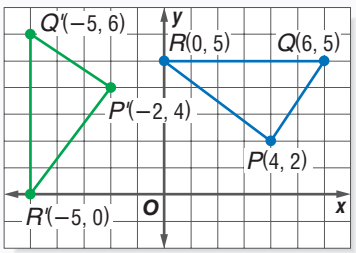
28. dilation by a scale factor of 2

29. reflection over the y-axis

30. rotation of 180°

31. MAPS Kala is drawing a map of her neighborhood. Her house is represented by quadrilateral $ABCD$ with $A(2, 2)$, $B(6, 2)$, $C(6, 6)$, and $D(2, 6)$. Kala wants to use the same coordinates to make a map one half the size. What will the new coordinates of her house be?

Example 4 **Find the coordinates of the vertices of the image of $\triangle PQR$ with $P(4, 2)$, $Q(6, 5)$, and $R(0, 5)$ after a rotation of 90° counterclockwise about the origin.**



Write the ordered pairs in a vertex matrix. Then multiply by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 6 & 0 \\ 2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -5 \\ 4 & 6 & 0 \end{bmatrix}$$

The coordinates of the vertices of $\triangle P'Q'R'$ are $P'(-2, 4)$, $Q'(-5, 6)$, and $R'(-5, 0)$.

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Mixed Problem Solving
For mixed problem-solving practice, see page 929.

4-5 Determinants (pp. 194–200)

Find the value of each determinant.

32. $\begin{vmatrix} 4 & 11 \\ -7 & 8 \end{vmatrix}$

33. $\begin{vmatrix} 6 & -7 \\ 5 & 3 \end{vmatrix}$

34. $\begin{vmatrix} 12 & 8 \\ 9 & 6 \end{vmatrix}$

35. $\begin{vmatrix} 2 & -3 & 1 \\ 0 & 7 & 8 \\ 2 & 1 & 3 \end{vmatrix}$

36. $\begin{vmatrix} 7 & -4 & 5 \\ 1 & 3 & -6 \\ 5 & -1 & -2 \end{vmatrix}$

37. $\begin{vmatrix} 6 & 3 & -2 \\ -4 & 2 & 5 \\ -3 & -1 & 0 \end{vmatrix}$

38. GEOMETRY Alex wants to find the area of a triangle. He draws the triangle on a coordinate plane and finds that it has vertices at (2, 1), (3, 4) and (1, 4). Find the area of the triangle.

Example 5 Evaluate $\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix}$.

$\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix} = 3(2) - (-4)(6)$ Definition of determinant

$= 6 - (-24)$ or 30 Simplify.

Example 6 Evaluate $\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix}$ **using expansion by minors.**

$$\begin{vmatrix} 3 & 1 & 5 \\ 1 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 5\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix}$$

$$= 3(-4 - (-1)) - 1(2 - 0) + 5(-1 - 0)$$

$$= -9 - 2 - 5 \text{ or } -16$$

4-6 Cramer's Rule (pp. 201–207)

Use Cramer's Rule to solve each system of equations.

39. $9a - b = 1$
$3a + 2b = 12$

40. $x + 5y = 14$
$-2x + 6y = 4$

41. $4f + 5g = -2$
$-3f - 7g = 8$

42. $-6m + n = -13$
$11m - 6n = 3$

43. $6x - 7z = 13$
$8y + 2z = 14$
$7x + z = 6$

44. $2a - b - 3c = -20$
$4a + 2b + c = 6$
$2a + b - c = -6$

45. ENTERTAINMENT Selena paid \$25.25 to play three games of miniature golf and two rides on go-karts. Selena paid \$25.75 for four games of miniature golf and one ride on the go-karts. Use Cramer's Rule to find out how much each activity costs.

Example 7 **Use Cramer's Rule to solve $5a - 3b = 7$ *and* $3a + 9b = -3$.**

$$a = \frac{\begin{vmatrix} 7 & -3 \\ -3 & 9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}} \quad \text{Cramer's Rule} \quad b = \frac{\begin{vmatrix} 5 & 7 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 3 & 9 \end{vmatrix}}$$

$$= \frac{63 - 9}{45 + 9} \quad \text{Evaluate each determinant.} \quad = \frac{-15 - 21}{45 + 9}$$

$$= \frac{54}{54} \text{ or } 1 \quad \text{Simplify.} \quad = \frac{-36}{54} \text{ or } -\frac{2}{3}$$

The solution is $\left(1, -\frac{2}{3}\right)$.

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CHAPTER 4

Study Guide and Review

4-7 Identity and Inverse Matrices (pp. 208–215)

Find the inverse of each matrix, if it exists.

46. $\begin{bmatrix} 3 & 2 \\ 4 & -2 \end{bmatrix}$

47. $\begin{bmatrix} 8 & 6 \\ 9 & 7 \end{bmatrix}$

48. $\begin{bmatrix} 0 & 2 \\ 5 & -4 \end{bmatrix}$

49. $\begin{bmatrix} 6 & -1 & 0 \\ 5 & 8 & -2 \end{bmatrix}$

50. CRYPTOGRAPHY Martin wrote a coded message to his friend using a coding matrix, $C = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. What is Martin's message if the matrix he gave his friend was $\begin{bmatrix} 26 & 12 \\ 80 & 80 \\ 75 & 25 \\ 24 & 38 \\ 94 & 98 \\ 32 & 24 \\ 53 & 101 \end{bmatrix}$? (*Hint*: Assume that the letters are labeled 1–26 with A = 1 and _ = 0.)

Example 8 Find the inverse of $S = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$.

First evaluate the determinant.

$\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix} = 3 - (-8)$ or 11

Then use the formula for the inverse matrix.

$S^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$

4-8 Using Matrices to Solve Systems of Equations (pp. 216–222)

Solve each matrix equation or system of equations by using inverse matrices.

51. $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$

52. $\begin{bmatrix} 4 & 1 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

53. $3x + 8 = -y$
$4x - 2y = -14$

54. $3x - 5y = -13$
$4x + 3y = 2$

55. SHOES Joan is preparing a dye solution for her shoes. For the right color she needs 1500 milliliters of a 63% concentration solution. The store has only 75% and 50% concentration solutions. How many milliliters of 50% dye solution should be mixed with 75% dye solution to make the necessary amount of 63% dye solution?

Example 9 Solve $\begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \end{bmatrix}$.

Step 1 Find the inverse of the coefficient matrix.

$A^{-1} = \frac{1}{-12-16}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix}$ or $-\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix}$

Step 2 Multiply each side by the inverse matrix.

$$-\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 8 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= -\frac{1}{28}\begin{bmatrix} -3 & -8 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{28}\begin{bmatrix} -140 \\ 28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

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CHAPTER 4

Practice Test

Solve each equation.

1. $\begin{bmatrix} 3x+1 \\ 2y \end{bmatrix} = \begin{bmatrix} 10 \\ 4+y \end{bmatrix}$

2. $\begin{bmatrix} 2x & y+1 \\ 13 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -7 \\ 13 & z-8 \end{bmatrix}$

Perform the indicated operations. If the matrix does not exist, write *impossible*.

3. $\begin{bmatrix} 2 & -4 & 1 \\ 3 & 8 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ -4 & 3 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 6 & 7 \\ 1 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -4 & 3 \\ -1 & -2 \\ 2 & 5 \end{bmatrix}$

Find the value of each determinant.

6. $\begin{vmatrix} -1 & 4 \\ -6 & 3 \end{vmatrix}$

7. $\begin{vmatrix} -2 & 0 & 5 \\ -3 & 4 & 0 \\ 1 & 3 & -1 \end{vmatrix}$

Find the inverse of each matrix, if it exists.

8. $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$

9. $\begin{bmatrix} -6 & -3 \\ 8 & 4 \end{bmatrix}$

Solve each matrix equation or system of equations by using inverse matrices.

10. $\begin{bmatrix} 5 & 7 \\ -9 & 3 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 41 \\ -105 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 3 \\ 11 & -7 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \end{bmatrix}$

12. $5a + 2b = -49$
$2a + 9b = 5$

13. $4c + 9d = 6$
$13c - 11d = -61$

14. ACCOUNTING A small business' bank account is charged a service fee for each electronic credit and electronic debit transaction. Their transactions and charges for two recent months are listed in the table.

Month	Electronic Credits	Electronic Debits	Cost
January	28	18	$7.22
February	25	31	$7.79

Use a system of equations to find the fee for each electronic credit and electronic debit transaction.

For Exercises 15–17, use $\triangle ABC$ whose vertices have coordinates $A(6, 3)$, $B(1, 5)$, and $C(-1, 4)$.

15. Use a determinant to find the area of $\triangle ABC$.

16. Translate $\triangle ABC$ so that the coordinates of B' are $(3, 1)$. What are the coordinates of A' and C'?

17. Find the coordinates of the vertices of a triangle that is a dilation of $\triangle ABC$ with a perimeter five times that of $\triangle ABC$.

18. MULTIPLE CHOICE Lupe is preparing boxes of assorted chocolates. Chocolate-covered peanuts cost $7 per pound. Chocolate-covered caramels cost $6.50 per pound. The boxes of assorted candies contain five more pounds of peanut candies than caramel candies. If the total amount sold was $575, how many pounds of each candy were needed to make the boxes?

A 40 lb peanut, 45 lb caramel

B 40 lb caramel, 45 lb peanut

C 40 lb peanut, 35 lb caramel

D 40 lb caramel, 35 lb peanut

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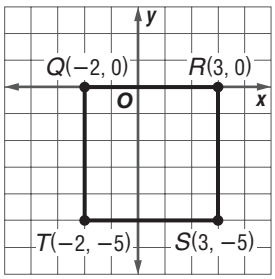
CHAPTER 4

Standardized Test Practice

Cumulative, Chapters 1–4

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Figure $QRST$ is shown on the coordinate plane.



Which transformation creates an image with a vertex at the origin?

A Reflect figure $QRST$ across the line $y = -1$.

B Reflect figure $QRST$ across the line $x = -3$.

C Rotate figure $QRST$ 180 degrees around R.

D Translate figure $QRST$ to the left 3 units and up 5 units.

2. The algebraic form of a linear function is $d = 35t$, where d is the distance in miles and t is the time in hours. Which one of the following choices identifies the same linear function?

F For every 6 hours that a car is driven, it travels about 4 miles.

G For every 6 hours that a car is driven, it travels about 210 miles.

H

t	d
0	0
2	17.5
4	8.75
6	5.83

J

t	d
0	0
70	2
140	4
210	6

3. GRIDDABLE What is the value of a in the matrix equation below?

$$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 9 \end{bmatrix}$$

TEST-TAKING TIP

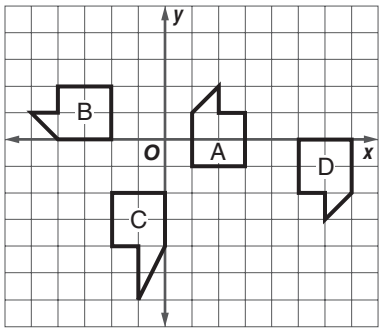
Question 3 When answering questions, read carefully and make sure that you know exactly what the question is asking you to find. For example, if you find the value of b in question 3, you have not solved the problem. You need to find the value of a.

4. Pedro is creating a scale drawing of a car. He finds that the height of the car in the drawing is $\frac{1}{32}$ of the actual height of the car x. Which equation best represents this relationship?

A $y = x - \frac{1}{32}$ **C** $y = \frac{1}{32}x$

B $y = -\frac{1}{32}x$ **D** $y = x + \frac{1}{32}$

5. Which pair of polygons is congruent?



F Polygon A and Polygon B

G Polygon B and Polygon C

H Polygon A and Polygon C

J Polygon C and Polygon D

6. For Marla's vacation, it will cost her \$100 to drive her car plus between \$0.50 to \$0.75 per mile. If she will drive her car for 400 miles, what is a reasonable conclusion about c, the total cost to drive her car on the vacation?

A $300 < c < 400$ **C** $100 < c < 400$

B $300 < c \leq 400$ **D** $300 \leq c \leq 400$

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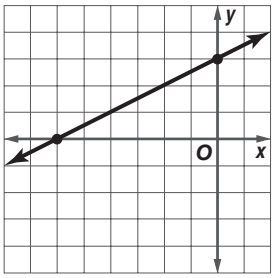
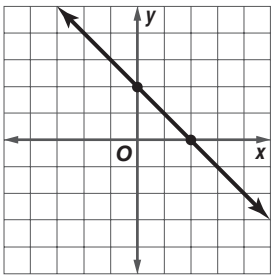
7. What are the slope and y-intercept of the equation of the line graphed below?



F $m = 4; b = \frac{2}{3}$ **H** $m = \frac{1}{2}; b = 3$

G $m = 4; b = \frac{3}{2}$ **J** $m = \frac{1}{2}; b = 4$

8. The graph of a line is shown below.



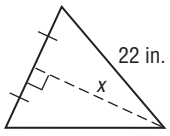
If the slope of this line is multiplied by 2 and the y-intercept increases by 1 unit, which linear equation represents these changes?

A $y = -\frac{1}{2}x + 1$ **C** $y = -4x + 3$

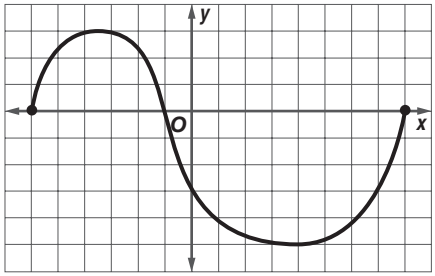
B $y = -2x + 1$ **D** $y = -2x + 3$

9. Given the equilateral triangle below, what is the approximate measure of x?

F 19.1 in.
G 22.0 in.
H 24.6 in.
J 31.1 in.



10. What is the domain of the function shown on the graph?



A $\{x \mid -5 \le x \le 3\}$ **C** $\{x \mid -5 < x < 3\}$

B $\{x \mid -6 \le x \le 8\}$ **D** $\{x \mid -6 < x < 8\}$

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. The Colonial High School Yearbook Staff is selling yearbooks and chrome picture frames engraved with the year. The number of yearbooks and frames sold to members of each grade is shown in the table.

Sales for Each Class		
Grade	**Yearbooks**	**Frames**
9th	423	256
10th	464	278
11th	546	344
12th	575	497

a. Find the difference in the sales of yearbooks and frames made to the 10^{th} and 11^{th} grade classes.

b. Find the total numbers of yearbooks and frames sold.

c. A yearbook costs \$48, and a frame costs \$18. Find the sales of yearbooks and frames for each class.

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson or Page...	4-4	2-4	4-8	4-2	4-5	3-3	2-3	2-3	879	2-1	1-3

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UNIT 2

Quadratic, Polynomial, and Radical Equations and Inequalities

Focus

Use functions and equations as means for analyzing and understanding a broad variety of relationships.

CHAPTER 5

Quadratic Functions and Inequalities

BIG Idea Formulate equations and inequalities based on quadratic functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

BIG Idea Interpret and describe the effects of changes in the parameters of quadratic functions.

CHAPTER 6

Polynomial Functions

BIG Idea Use properties and attributes of polynomial functions and apply functions to problem situations.

CHAPTER 7

Radical Equations and Inequalities

BIG Idea Formulate equations and inequalities based on square root functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

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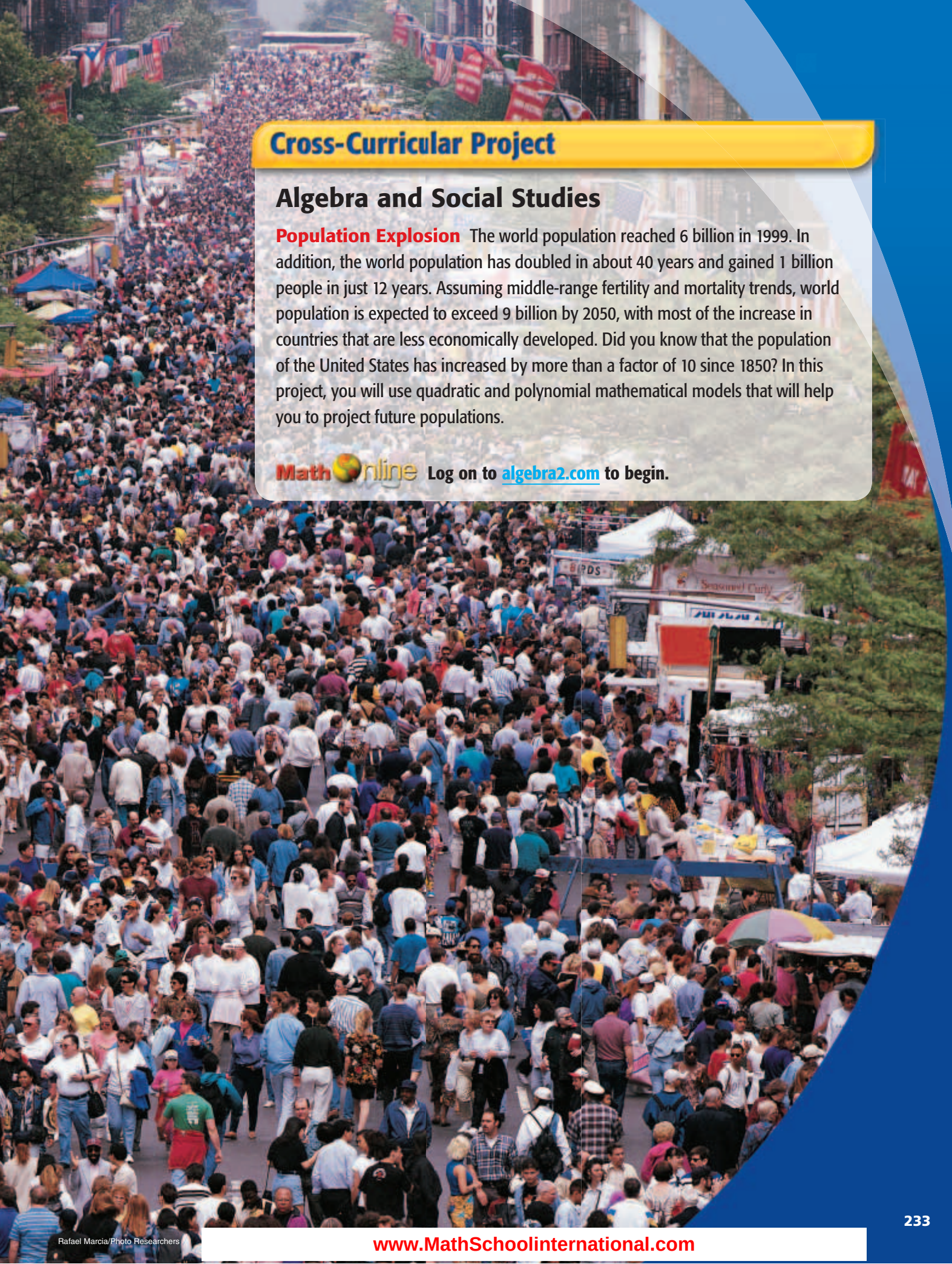

Cross-Curricular Project

Algebra and Social Studies

Population Explosion The world population reached 6 billion in 1999. In addition, the world population has doubled in about 40 years and gained 1 billion people in just 12 years. Assuming middle-range fertility and mortality trends, world population is expected to exceed 9 billion by 2050, with most of the increase in countries that are less economically developed. Did you know that the population of the United States has increased by more than a factor of 10 since 1850? In this project, you will use quadratic and polynomial mathematical models that will help you to project future populations.

MathOnline **Log on to algebra2.com to begin.**

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CHAPTER 5

Quadratic Functions and Inequalities

BIG Ideas

- Graph quadratic functions.
- Solve quadratic equations.
- Perform operations with complex numbers.
- Graph and solve quadratic inequalities.

Key Vocabulary

discriminant (p. 279)

imaginary unit (p. 260)

root (p. 246)

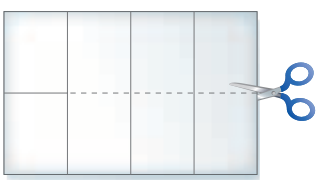
Real-World Link

Suspension Bridges Quadratic functions can be used to model real-world phenomena like the motion of a falling object. They can also be used to model the shape of architectural structures such as the supporting cables of the Mackinac Suspension Bridge in Michigan.

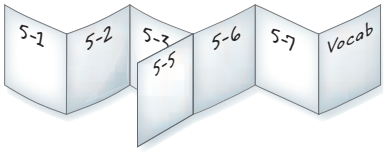
FOLDABLES Study Organizer

Quadratic Functions and Inequalities Make this Foldable to help you organize your notes. Begin with one sheet of 11″ by 17″ paper.

1 **Fold** in half lengthwise. Then fold in fourths crosswise. Cut along the middle fold from the edge to the last crease as shown.



2 **Refold** along the lengthwise fold and staple the uncut section at the top. Label each section with a lesson number and close to form a booklet.



Jon Arnold Images/Alamy Images

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GET READY for Chapter 5

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICK Check

Given $f(x) = 2x^2 - 6$ and $g(x) = -x^2 + 4x - 4$, find each value. (Lesson 2-1)

1. $f(1)$
2. $f(4)$
3. $f(0)$
4. $f(-2)$
5. $g(0)$
6. $g(-1)$
7. $g(2)$
8. $g(0.5)$

FISH For Exercises 9 and 10, use the following information.

Tuna swim at a steady rate of 9 miles per hour until they die, and they never stop moving. (Lesson 2–1)

9. Write a function that is a model for the situation.

10. Evaluate the function to estimate how far a 2-year-old tuna has traveled.

Factor completely. If the polynomial is not factorable, write *prime*. (Prerequisite Skills, p. 877)

11. $x^2 + 11x + 30$
12. $x^2 - 13x + 36$
13. $x^2 - x - 56$
14. $x^2 - 5x - 14$
15. $x^2 + x + 2$
16. $x^2 + 10x + 25$
17. $x^2 - 22x + 121$
18. $x^2 - 9$

19. FLOOR PLAN A living room has a floor space of $x^2 + 11x + 28$ square feet. If the width of the room is $(x + 4)$ feet, what is the length? (Prerequisite Skills, p. 877)

QUICK Review

EXAMPLE 1 **Given $f(x) = 3x^2 + 2$ and $g(x) = 0.5x^2 + 2x - 1$, find each value.**

a. $f(3)$

$f(x) = 3x^2 + 2$ Original function

$f(3) = 3(3)^2 + 2$ Substitute 3 for x.

$= 27 + 2$ or 29 Simplify.

b. $g(-4)$

$g(x) = 0.5x^2 + 2x - 1$ Original function

$g(-4) = 0.5(-4)^2 + 2(-4) - 1$ Substitute -4 for x.

$= 8 + (-8) - 1$ Multiply.

$= -1$ Simplify.

EXAMPLE 2 **Factor $x^2 - x - 2$ completely. If the polynomial is not factorable, write *prime*.**

To find the coefficients of the x-terms, you must find two numbers whose product is $(1)(-2)$ or -2, and whose sum is -1. The two coefficients must be 1 and -2 since $(1)(-2) = -2$ and $1 + (-2) = -1$. Rewrite the expression and factor by grouping.

$x^2 - x - 2$

$= x^2 + x - 2x - 2$ Substitute $x - 2x$ for $-x$.

$= (x^2 + x) + (-2x - 2)$ Associative Property

$= x(x + 1) - 2(x + 1)$ Factor out the GCF.

$= (x + 1)(x - 2)$ Distributive Property

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

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5-1 Graphing Quadratic Functions

Main Ideas

- Graph quadratic functions.
- Find and interpret the maximum and minimum values of a quadratic function.

New Vocabulary

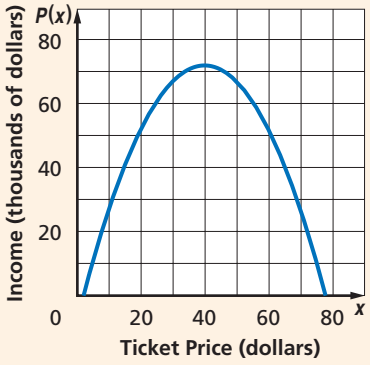
quadratic function
quadratic term
linear term
constant term
parabola
axis of symmetry
vertex
maximum value
minimum value

GET READY for the Lesson

Rock music managers handle publicity and other business issues for the artists they manage. One group's manager has found that based on past concerts, the predicted income for a performance is $P(x) = -50x^2 + 4000x - 7500$, where x is the price per ticket in dollars.

The graph of this quadratic function is shown at the right. At first the income increases as the price per ticket increases, but as the price continues to increase, the income declines.

Rock Concert Income



Graph Quadratic Functions A **quadratic function** is described by an equation of the following form.

quadratic term, linear term, constant term

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$

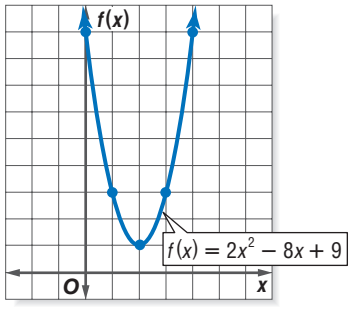
The graph of any quadratic function is called a **parabola**. To graph a quadratic function, graph ordered pairs that satisfy the function.

EXAMPLE 1 Graph a Quadratic Function

Graph $f(x) = 2x^2 - 8x + 9$ by making a table of values.

Choose integer values for x and evaluate the function for each value. Graph the resulting coordinate pairs and connect the points with a smooth curve.

x	$2x^2 - 8x + 9$	$f(x)$	$(x, f(x))$
0	$2(0)^2 - 8(0) + 9$	9	(0, 9)
1	$2(1)^2 - 8(1) + 9$	3	(1, 3)
2	$2(2)^2 - 8(2) + 9$	1	(2, 1)
3	$2(3)^2 - 8(3) + 9$	3	(3, 3)
4	$2(4)^2 - 8(4) + 9$	9	(4, 9)



CHECK Your Progress

Graph each function by making a table of values.

1A. $g(x) = -x^2 + 2x - 6$ **1B.** $f(x) = x^2 - 8x + 15$

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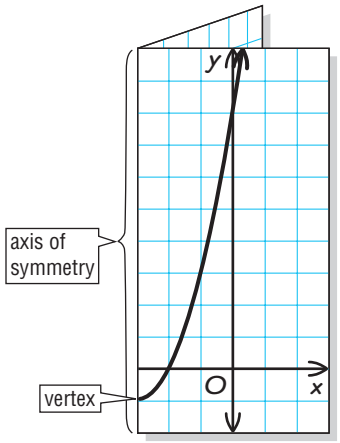
All parabolas have an **axis of symmetry**. If you were to fold a parabola along its axis of symmetry, the portions of the parabola on either side of this line would match.

The point at which the axis of symmetry intersects a parabola is called the **vertex**. The y-intercept of a quadratic function, the equation of the axis of symmetry, and the x-coordinate of the vertex are related to the equation of the function as shown below.



Study Tip

Graphing Quadratic Functions

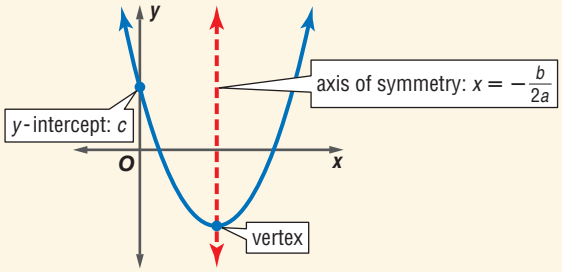
Knowing the location of the axis of symmetry, y-intercept, and vertex can help you graph a quadratic function.

KEY CONCEPT — **Graph of a Quadratic Equation**

Words Consider the graph of $y = ax^2 + bx + c$, where $a \neq 0$.

- The y-intercept is $a(0)^2 + b(0) + c$ or c.
- The equation of the axis of symmetry is $x = -\frac{b}{2a}$.
- The x-coordinate of the vertex is $-\frac{b}{2a}$.

Model



EXAMPLE 2 Axis of Symmetry, y-Intercept, and Vertex

Consider the quadratic function $f(x) = x^2 + 9 + 8x$.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

Begin by rearranging the terms of the function so that the quadratic term is first, the linear term is second, and the constant term is last. Then identify a, b, and c.

$$f(x) = ax^2 + bx + c$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$f(x) = x^2 + 9 + 8x \rightarrow f(x) = 1x^2 + 8x + 9 \rightarrow a = 1, b = 8, \text{ and } c = 9$$

The y-intercept is 9. Use a and b to find the equation of the axis of symmetry.

$x = -\frac{b}{2a}$ Equation of the axis of symmetry

$= -\frac{8}{2(1)}$ $a = 1, b = 8$

$= -4$ Simplify.

The equation of the axis of symmetry is $x = -4$. Therefore, the x-coordinate of the vertex is -4.

(continued on the next page)

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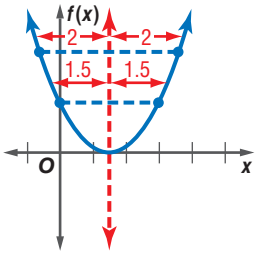
Study Tip

Symmetry

Sometimes it is convenient to use symmetry to help find other points on the graph of a parabola. Each point on a parabola has a mirror image located the same distance from the axis of symmetry on the other side of the parabola.



b. Make a table of values that includes the vertex.

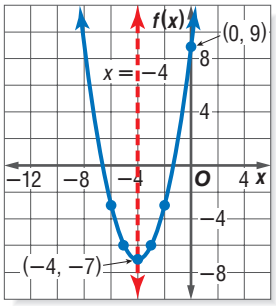
Choose some values for x that are less than -4 and some that are greater than -4. This ensures that points on each side of the axis of symmetry are graphed.

x	$x^2 + 8x + 9$	$f(x)$	$(x, f(x))$
-6	$(-6)^2 + 8(-6) + 9$	-3	$(-6, -3)$
-5	$(-5)^2 + 8(-5) + 9$	-6	$(-5, -6)$
-4	$(-4)^2 + 8(-4) + 9$	-7	$(-4, -7)$ ← Vertex
-3	$(-3)^2 + 8(-3) + 9$	-6	$(-3, -6)$
-2	$(-2)^2 + 8(-2) + 9$	-3	$(-2, -3)$

c. Use this information to graph the function.

Graph the vertex and y-intercept. Then graph the points from your table, connecting them and the y-intercept with a smooth curve.

As a check, draw the axis of symmetry, $x = -4$, as a dashed line. The graph of the function should be symmetrical about this line.



CHECK Your Progress

Consider the quadratic function $g(x) = 3 - 6x + x^2$.

2A. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

2B. Make a table of values that includes the vertex.

2C. Use this information to graph the function.

Maximum and Minimum Values The y-coordinate of the vertex of a quadratic function is the **maximum value** or **minimum value** attained by the function.

Study Tip

Domain

The domain of a quadratic function is all real numbers.

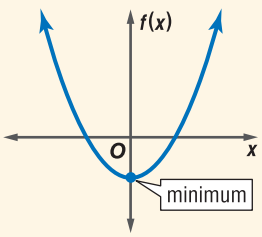
KEY CONCEPT *Maximum and Minimum Value*

Words The graph of $f(x) = ax^2 + bx + c$, where $a \neq 0$,

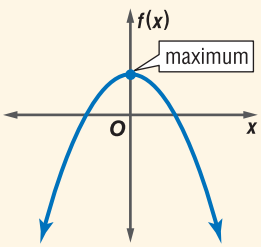
- opens up and has a minimum value when $a > 0$, and
- opens down and has a maximum value when $a < 0$.
- The range of a quadratic function is all real numbers greater than or equal to the minimum, or all real numbers less than or equal to the maximum.

Models

***a* is positive.**



***a* is negative.**



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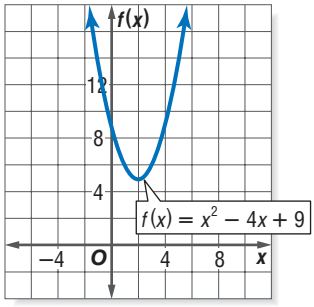
EXAMPLE Maximum or Minimum Value

3 **Consider the function $f(x) = x^2 - 4x + 9$.**

a. Determine whether the function has a maximum or a minimum value.

For this function, $a = 1$, $b = -4$, and $c = 9$. Since $a > 0$, the graph opens up and the function has a minimum value.

b. State the maximum or minimum value of the function.



The minimum value of the function is the y-coordinate of the vertex.

The x-coordinate of the vertex is $-\frac{-4}{2(1)}$ or 2.

Find the y-coordinate of the vertex by evaluating the function for $x = 2$.

$f(x) = x^2 - 4x + 9$ Original function

$f(2) = (2)^2 - 4(2) + 9$ or 5 $x = 2$

Therefore, the minimum value of the function is 5.

c. State the domain and range of the function.

The domain is all real numbers. The range is all reals greater than or equal to the minimum value. That is, $\{f(x) \mid f(x) \geq 5\}$.

Study Tip

Common Misconception

The terms *minimum point* and *minimum value* are not interchangeable. The minimum point is the set of coordinates that describe the location of the vertex. The minimum value of a function is the y-coordinate of the minimum point. It is the least value obtained when $f(x)$ is evaluated for all values of x.

CHECK Your Progress

Consider $g(x) = 2x^2 - 4x - 3$.

3A. Determine whether the function has a maximum or minimum value.

3B. State the maximum or minimum value of the function.

3C. What are the domain and range of the function?

When quadratic functions are used to model real-world situations, their maximum or minimum values can have real-world meaning.

Real-World EXAMPLE

4 **TOURISM A tour bus in Boston serves 400 customers a day. The charge is \$5 per person. The owner of the bus service estimates that the company would lose 10 passengers a day for each \$0.50 fare increase.**

a. How much should the fare be in order to maximize the income for the company?

Words The income is the number of passengers multiplied by the price per ticket.

Variables Let x = the number of \$0.50 fare increases.
Then $5 + 0.50x$ = the price per passenger and
$400 - 10x$ = the number of passengers.

Let $I(x)$ = income as a function of x.

(continued on the next page)

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	The income	is	the number of passengers	multiplied by	the price per passenger.
Equation	$I(x)$	$=$	$(400 - 10x)$	$\cdot$	$(5 + 0.50x)$

$$= 400(5) + 400(0.50x) - 10x(5) - 10x(0.50x)$$

$= 2000 + 200x - 50x - 5x^2$ Multiply.

$= 2000 + 150x - 5x^2$ Simplify.

$= -5x^2 + 150x + 2000$ Rewrite in $ax^2 + bx + c$ form.

$I(x)$ is a quadratic function with $a = -5$, $b = 150$, and $c = 2000$. Since $a < 0$, the function has a maximum value at the vertex of the graph. Use the formula to find the x-coordinate of the vertex.

x-coordinate of the vertex $= -\frac{b}{2a}$ Formula for the x-coordinate of the vertex

$= -\frac{150}{2(-5)}$ $a = -5, b = 150$

$= 15$ Simplify.

This means the company should make 15 fare increases of \$0.50 to maximize its income. Thus, the ticket price should be $5 + 0.50(15)$ or \$12.50.

The domain of the function is all real numbers, but negative values of x would correspond to a decreased fare. Therefore, a value of 15 fare increases is reasonable.

Real-World Link

Known as "Beantown," Boston is the largest city and unofficial capital of New England.

Source: boston-online.com

b. What is the maximum income the company can expect to make?

To determine maximum income, find the maximum value of the function by evaluating $I(x)$ for $x = 15$.

$I(x) = -5x^2 + 150x + 2000$ Income function

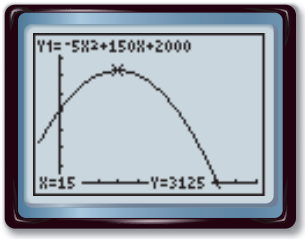
$I(15) = -5(15)^2 + 150(15) + 2000$ $x = 15$

$= 3125$ Use a calculator.

Thus, the maximum income the company can expect is \$3125. The increased fare would produce greater income. The income from the lower fare was \$5(400), or \$2000. So an answer of \$3125 is reasonable.

CHECK Graph this function on a graphing calculator and use the CALC menu to confirm this solution.

KEYSTROKES: 2nd [CALC] 4 0 ENTER 25 ENTER ENTER



[−5, 50] scl: 5 by [−100, 4000] scl: 500

At the bottom of the display are the coordinates of the maximum point on the graph. The y-value is the maximum value of the function, or 3125. The graph shows the range of the function as all reals less than or equal to 3125. ✓

CHECK Your Progress

4. Suppose that for each \$0.50 increase in the fare, the company will lose 8 passengers. Determine how much the fare should be in order to maximize the income, and then determine the maximum income.

Online Personal Tutor at algebra2.com

Donovan Reese/Getty Images

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CHECK Your Understanding

Examples 1, 2 (pp. 236–238)

Complete parts a–c for each quadratic function.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

b. Make a table of values that includes the vertex.

c. Use this information to graph the function.

1. $f(x) = -4x^2$

2. $f(x) = x^2 + 2x$

3. $f(x) = -x^2 + 4x - 1$

4. $f(x) = x^2 + 8x + 3$

5. $f(x) = 2x^2 - 4x + 1$

6. $f(x) = 3x^2 + 10x$

Example 3 (p. 239)

Determine whether each function has a maximum or a minimum value and find the maximum or minimum value. Then state the domain and range of the function.

7. $f(x) = -x^2 + 7$

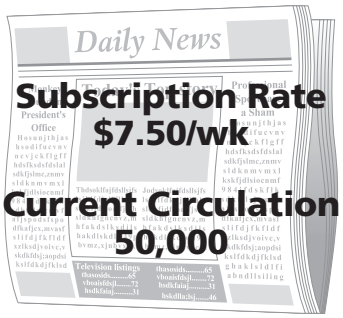
8. $f(x) = x^2 - x - 6$

9. $f(x) = 4x^2 + 12x + 9$

10. $f(x) = -x^2 - 4x + 1$

Example 4 (pp. 239–240)

11. NEWSPAPERS Due to increased production costs, the *Daily News* must increase its subscription rate. According to a recent survey, the number of subscriptions will decrease by about 1250 for each 25¢ increase in the subscription rate. What weekly subscription rate will maximize the newspaper's income from subscriptions?



Exercises

HOMEWORK HELP

For Exercises	See Examples
12–21	1, 2
22–31	3
32–36	4

Complete parts a–c for each quadratic function.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

b. Make a table of values that includes the vertex.

c. Use this information to graph the function.

12. $f(x) = 2x^2$

13. $f(x) = -5x^2$

14. $f(x) = x^2 + 4$

15. $f(x) = x^2 - 9$

16. $f(x) = 2x^2 - 4$

17. $f(x) = 3x^2 + 1$

18. $f(x) = x^2 - 4x + 4$

19. $f(x) = x^2 - 9x + 9$

20. $f(x) = x^2 - 4x - 5$

21. $f(x) = x^2 + 12x + 36$

Determine whether each function has a maximum or a minimum value and find the maximum or minimum value. Then state the domain and range of the function.

22. $f(x) = 3x^2$

23. $f(x) = -x^2 - 9$

24. $f(x) = x^2 - 8x + 2$

25. $f(x) = x^2 + 6x - 2$

26. $f(x) = 4x - x^2 + 1$

27. $f(x) = 3 - x^2 - 6x$

28. $f(x) = x^2 - 10x - 1$

29. $f(x) = x^2 + 8x + 15$

30. $f(x) = -x^2 + 12x - 28$

31. $f(x) = -14x - x^2 - 109$

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Real-World Link

The Exchange House in London, England, is supported by two interior and two exterior steel arches. V-shaped braces add stability to the structure.

Source: Council on Tall Buildings and Urban Habitat

ARCHITECTURE **For Exercises 32 and 33, use the following information.**
The shape of each arch supporting the Exchange House can be modeled by $h(x) = -0.025x^2 + 2x$, where $h(x)$ represents the height of the arch and x represents the horizontal distance from one end of the base in meters.

32. Write the equation of the axis of symmetry and find the coordinates of the vertex of the graph of $h(x)$.

33. According to this model, what is the maximum height of the arch?

PHYSICS **For Exercises 34–36, use the following information.**
An object is fired straight up from the top of a 200-foot tower at a velocity of 80 feet per second. The height $h(t)$ of the object t seconds after firing is given by $h(t) = -16t^2 + 80t + 200$.

34. What are the domain and range of the function? What domain and range values are reasonable in the given situation?

35. Find the maximum height reached by the object and the time that the height is reached.

36. Interpret the meaning of the y-intercept in the context of this problem.

Complete parts a–c for each quadratic function.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

b. Make a table of values that includes the vertex.

c. Use this information to graph the function.

37. $f(x) = 3x^2 + 6x - 1$

38. $f(x) = -2x^2 + 8x - 3$

39. $f(x) = -3x^2 - 4x$

40. $f(x) = 2x^2 + 5x$

41. $f(x) = 0.5x^2 - 1$

42. $f(x) = -0.25x^2 - 3x$

43. $f(x) = \frac{1}{2}x^2 + 3x + \frac{9}{2}$

44. $f(x) = x^2 - \frac{2}{3}x - \frac{8}{9}$

Determine whether each function has a maximum or a minimum value and find the maximum or minimum value. Then state the domain and range of the function.

45. $f(x) = 2x + 2x^2 + 5$

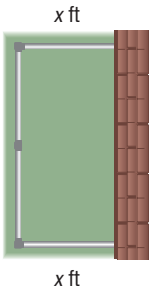
46. $f(x) = x - 2x^2 - 1$

47. $f(x) = -7 - 3x^2 + 12x$

48. $f(x) = -20x + 5x^2 + 9$

49. $f(x) = -\frac{1}{2}x^2 - 2x + 3$

50. $f(x) = \frac{3}{4}x^2 - 5x - 2$



CONSTRUCTION **For Exercises 51–54, use the following information.**
Jaime has 120 feet of fence to make a rectangular kennel for his dogs. He will use his house as one side.

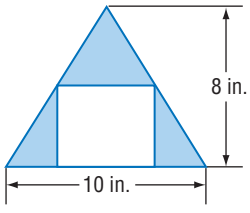
51. Write an algebraic expression for the kennel's length.

52. What are reasonable values for the domain of the area function?

53. What dimensions produce a kennel with the greatest area?

54. Find the maximum area of the kennel.

55. **GEOMETRY** A rectangle is inscribed in an isosceles triangle as shown. Find the dimensions of the inscribed rectangle with maximum area. (*Hint:* Use similar triangles.)



Aidan O'Rourke
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FUND-RAISING **For Exercises 56 and 57, use the following information.** Last year, 300 people attended the Sunnybrook High School Drama Club's winter play. The ticket price was $8. The advisor estimates that 20 fewer people would attend for each $1 increase in ticket price.

56. What ticket price would give the most income for the Drama Club?

57. If the Drama Club raised its tickets to this price, how much income should it expect to bring in?

Graphing Calculator

MAXIMA AND MINIMA You can use the MINIMUM or MAXIMUM feature on a graphing calculator to find the minimum or maximum of a quadratic function. This involves defining an interval that includes the vertex of the parabola. A lower bound is an x-value left of the vertex, and an upper bound is an x-value right of the vertex.

Step 1 Graph the function so that the vertex of the parabola is visible.

Step 2 Select 3:minimum or 4:maximum from the CALC menu.

Step 3 Using the arrow keys, locate a left bound and press ENTER.

Step 4 Locate a right bound and press ENTER twice. The cursor appears on the maximum or minimum of the function. The maximum or minimum value is the y-coordinate of that point.

EXTRA PRACTICE
See page 899, 930.
Math Online
Self-Check Quiz at
algebra2.com

Find the value of the maximum or minimum of each quadratic function to the nearest hundredth.

58. $f(x) = 3x^2 - 7x + 2$

59. $f(x) = -5x^2 + 8x$

60. $f(x) = 2x^2 - 3x + 2$

61. $f(x) = -6x^2 + 9x$

62. $f(x) = 7x^2 + 4x + 1$

63. $f(x) = -4x^2 + 5x$

H.O.T. Problems

64. OPEN ENDED Give an example of a quadratic function that has a domain of all real numbers and a range of all real numbers less than a maximum value. State the maximum value and sketch the graph of the function.

65. CHALLENGE Write an expression for the minimum value of a function of the form $y = ax^2 + c$, where $a > 0$. Explain your reasoning. Then use this function to find the minimum value of $y = 8.6x^2 - 12.5$.

66. ***Writing in Math*** Use the information on page 236 to explain how income from a rock concert can be maximized. Include an explanation of how to algebraically and graphically determine what ticket price should be charged to achieve maximum income.

STANDARDIZED TEST PRACTICE

67. ACT/SAT The graph of which of the following equations is symmetrical about the y-axis?

A $y = x^2 + 3x - 1$

B $y = -x^2 + x$

C $y = 6x^2 + 9$

D $y = 3x^2 - 3x + 1$

68. REVIEW In which equation does every real number x correspond to a nonnegative real number y?

F $y = -x^2$

G $y = -x$

H $y = x$

J $y = x^2$

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Spiral Review

Solve each system of equations by using inverse matrices. (Lesson 4-8)

69. $2x + 3y = 8$
$x - 2y = -3$

70. $x + 4y = 9$
$3x + 2y = -3$

Find the inverse of each matrix, if it exists. (Lesson 4-7)

71. $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$

72. $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$

Perform the indicated operation, if possible. (Lesson 4-5)

73. $\begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -3 & 2 \\ 1 & 4 \end{bmatrix}$

74. $\begin{bmatrix} 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 & 1 \\ -3 & 2 & 0 \end{bmatrix}$

Perform the indicated operations. (Lesson 4-2)

75. $[4 \quad 1 \quad -3] + [6 \quad -5 \quad 8]$

76. $[2 \quad -5 \quad 7] - [-3 \quad 8 \quad -1]$

77. $4\begin{bmatrix} -7 & 5 & -11 \\ 2 & -4 & 9 \end{bmatrix}$

78. $-2\begin{bmatrix} -3 & 0 & 12 \\ -7 & \frac{1}{3} & 4 \end{bmatrix}$

79. CONCERTS The price of two lawn seats and a pavilion seat at an outdoor amphitheater is \$75. The price of three lawn seats and two pavilion seats is \$130. How much do lawn and pavilion seats cost? (Lesson 3-2)

Solve each system of equations. (Lesson 3-2)

80. $4a - 3b = -4$
$3a - 2b = -4$

81. $2r + s = 1$
$r - s = 8$

82. $3x - 2y = -3$
$3x + y = 3$

83. Graph the system of equations $y = -3x$ and $y - x = 4$. State the solution. Is the system of equations *consistent and independent*, *consistent and dependent*, or *inconsistent*? (Lesson 3-1)

Find the slope of the line that passes through each pair of points. (Lesson 2-3)

84. $(6, 7), (0, -5)$

85. $(-3, -2), (-1, -4)$

86. $(-3, 2), (5, 6)$

87. $(-2, 8), (1, -7)$

88. $(3, 8), (7, 22)$

89. $(4, 21), (9, 12)$

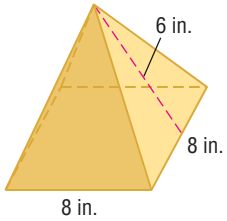
Solve each equation. Check your solutions. (Lesson 1-4)

90. $|x - 3| = 7$

91. $-4|d + 2| = -12$

92. $5|k - 4| = k + 8$

93. GEOMETRY The formula for the surface area of a regular pyramid is $S = \frac{1}{2}P\ell + B$ where P is the perimeter of the base, ℓ is the slant height of the pyramid, and B is the area of the base. Find the surface area of the pyramid shown. (Lesson 1-1)



GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each function for the given value.** (Lesson 2-1)

94. $f(x) = x^2 + 2x - 3, x = 2$

95. $f(x) = -x^2 - 4x + 5, x = -3$

96. $f(x) = 3x^2 + 7x, x = -2$

97. $f(x) = \frac{2}{3}x^2 + 2x - 1, x = -3$

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READING MATH

Roots of Equations and Zeros of Functions

The *solution* of an equation is called the *root* of the equation.

Example **Find the root of $0 = 3x - 12$.**

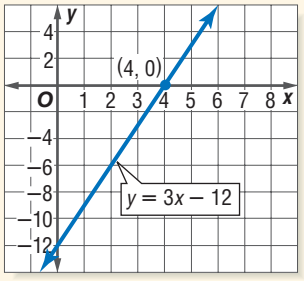
$0 = 3x - 12$ Original equation

$12 = 3x$ Add 12 to each side.

$4 = x$ Divide each side by 4.

The root of the equation is 4.

You can also find the root of an equation by finding the *zero* of its related function. Values of x for which $f(x) = 0$ are called *zeros* of the function f.



Linear Equation	Related Linear Function
$0 = 3x - 12$	$f(x) = 3x - 12$ or $y = 3x - 12$

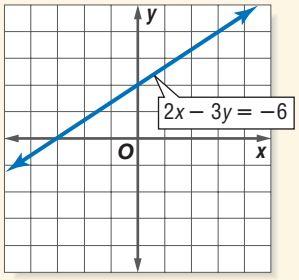
The zero of a function is the *x-intercept* of its graph. Since the graph of $y = 3x - 12$ intercepts the x-axis at 4, the zero of the function is 4.

You will learn about roots of quadratic equations and zeros of quadratic functions in Lesson 5-2.

Reading to Learn

1. Use $0 = 2x - 9$ and $f(x) = 2x - 9$ to distinguish among roots, solutions, and zeros.
2. Relate x-intercepts of graphs and solutions of equations.

Determine whether each statement is *true* or *false*. Explain your reasoning.



3. The function graphed at the right has two zeros, −3 and 2.
4. The root of $4x + 7 = 0$ is −1.75.
5. $f(0)$ is a zero of the function $f(x) = -\frac{1}{2}x + 5$.
6. **PONDS** The function $y = 24 - 2x$ represents the inches of water in a pond y after it is drained for x minutes. Find the zero and describe what it means in the context of this situation. Make a connection between the zero of the function and the root of $0 = 24 - 2x$.

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5-2 Solving Quadratic Equations by Graphing

Main Ideas

- Solve quadratic equations by graphing.
- Estimate solutions of quadratic equations by graphing.

New Vocabulary

quadratic equation
standard form
root
zero

GET READY for the Lesson

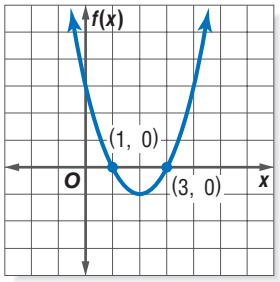
As you speed to the top of a free-fall ride, you are pressed against your seat so that you feel like you're being pushed downward. Then as you free-fall, you fall at the same rate as your seat. Without the force of your seat pressing on you, you *feel* weightless. The height above the ground (in feet) of an object in free-fall can be determined by the quadratic function $h(t) = -16t^2 + h_0$, where t is the time in seconds and the initial height is h_0 feet.

Solve Quadratic Equations When a quadratic function is set equal to a value, the result is a quadratic equation. A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$. When a quadratic equation is written in this way, and a, b, and c are all integers, it is in **standard form**.

Reading Math

Roots, Zeros, Intercepts In general, equations have roots, functions have zeros, and graphs of functions have x-intercepts.

The solutions of a quadratic equation are called the **roots** of the equation. One method for finding the roots of a quadratic equation is to find the **zeros** of the related quadratic function. The zeros of the function are the x-intercepts of its graph. These are the solutions of the related equation because $f(x) = 0$ at those points. The zeros of the function graphed at the right are 1 and 3.



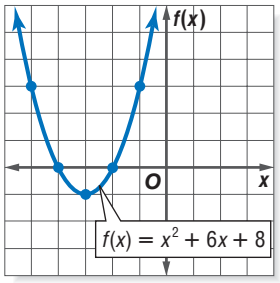
EXAMPLE **Two Real Solutions**

1 **Solve $x^2 + 6x + 8 = 0$ by graphing.**

Graph the related quadratic function $f(x) = x^2 + 6x + 8$. The equation of the axis of symmetry is $x = -\frac{6}{2(1)}$ or -3. Make a table using x values around -3. Then, graph each point.

x	-5	-4	-3	-2	-1
$f(x)$	3	0	-1	0	3

We can see that the zeros of the function are -4 and -2. Therefore, the solutions of the equation are -4 and -2.



CHECK Your Progress **Solve each equation by graphing.**

1A. $x^2 - x - 6 = 0$ **1B.** $x^2 + x = 2$

There are three possible outcomes when solving a quadratic equation.

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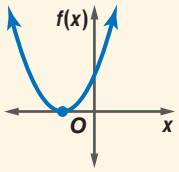
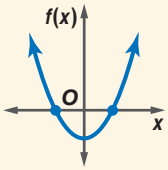
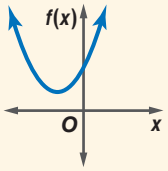
KEY CONCEPT — Solutions of a Quadratic Equation

Words A quadratic equation can have one real solution, two real solutions, or no real solution.

Models

One Real Solution	Two Real Solutions	No Real Solution
		

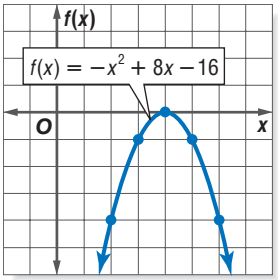
EXAMPLE 2 One Real Solution

Solve $8x - x^2 = 16$ by graphing.

$8x - x^2 = 16 \rightarrow -x^2 + 8x - 16 = 0$ Subtract 16 from each side.

Graph the related quadratic function $f(x) = -x^2 + 8x - 16$.

x	2	3	4	5	6
$f(x)$	−4	−1	0	−1	−4



Notice that the graph has only one x-intercept, 4. Thus, the equation's only solution is 4.

Study Tip

One Real Solution

When a quadratic equation has one real solution, it really has two solutions that are the same number.

CHECK Your Progress **Solve each equation by graphing.**

2A. $10x = -25 - x^2$ **2B.** $-x^2 - 2x = 1$

EXAMPLE 3 No Real Solution

NUMBER THEORY Find two real numbers with a sum of 6 and a product of 10 or show that no such numbers exist.

Explore Let x = one of the numbers. Then $6 - x$ = the other number.

Plan

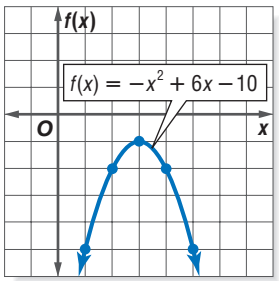
$x(6 - x) = 10$ The product is 10.

$6x - x^2 = 10$ Distributive Property

$-x^2 + 6x - 10 = 0$ Subtract 10 from each side.

Solve Graph the related function.

The graph has no x-intercepts. This means the original equation has no real solution. Thus, it is *not* possible for two numbers to have a sum of 6 and a product of 10.



Check Try finding the product of several pairs of numbers with sums of 6. Is each product less than 10 as the graph suggests?

CHECK Your Progress

3. Find two real numbers with a sum of 8 and a product of 12 or show that no such numbers exist.

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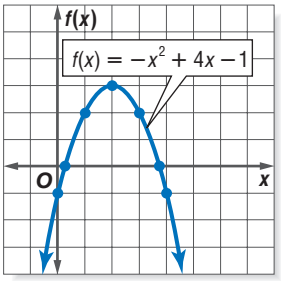
Estimate Solutions Often exact roots cannot be found by graphing. You can estimate solutions by stating the integers between which the roots are located.

EXAMPLE Estimate Roots

4 **Solve $-x^2 + 4x - 1 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.**



x	0	1	2	3	4
$f(x)$	−1	2	3	2	−1

The x-intercepts of the graph indicate that one solution is between 0 and 1, and the other is between 3 and 4.

Study Tip

Location of Roots

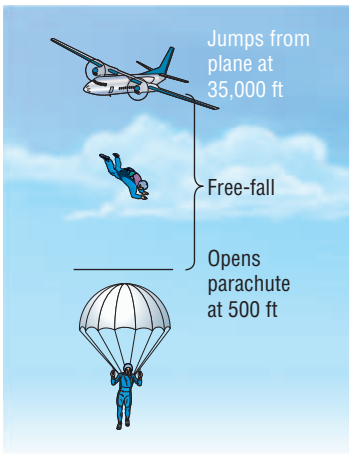
Notice in the table of values that the value of the function changes from negative to positive between the x-values of 0 and 1, and 3 and 4.

CHECK Your Progress

4. Solve $x^2 + 5x - 2 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

Real-World EXAMPLE

5 **EXTREME SPORTS** **In 1999, Adrian Nicholas broke the world record for the longest human flight. He flew 10 miles from a drop point in 4 minutes 55 seconds using an aerodynamic suit. Using the information at the right and ignoring air resistance, how long would he have been in free-fall had he not used this suit? Use the formula $h(t) = -16t^2 + h_0$, where the time t is in seconds and the initial height h_0 is in feet.**



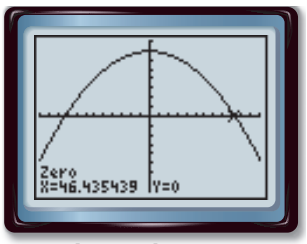
We need to find t when $h_0 = 35{,}000$ and $h(t) = 500$. Solve $500 = -16t^2 + 35{,}000$.

$500 = -16t^2 + 35{,}000$ Original equation

$0 = -16t^2 + 34{,}500$ Subtract 500 from each side.

Graph the related function $y = -16t^2 + 34{,}500$ on a graphing calculator.

Use the **Zero** feature, 2nd [CALC], to find the positive zero of the function, since time cannot be negative. Use the arrow keys to locate a left bound and press ENTER. Then, locate a right bound and press ENTER twice. The positive zero of the function is approximately 46.4. Mr. Nicholas would have been in free-fall for about 46 seconds.



[−60, 60] scl: 5 by
[−40000, 40000] scl: 5000

CHECK Your Progress

5. If Mr. Nicholas had jumped from the plane at 40,000 feet, how long would he have been in free-fall had he not used his special suit?

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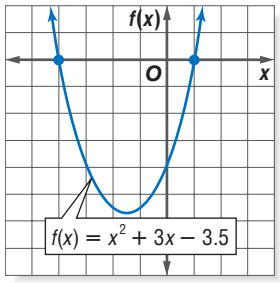
CHECK Your Understanding

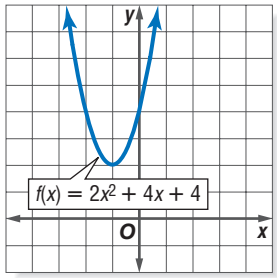
Examples 1–3 (pp. 246–247)

Use the related graph of each equation to determine its solutions.

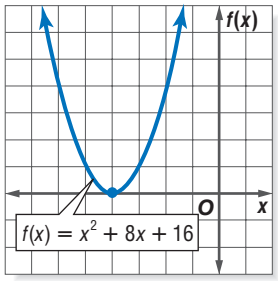
1. $x^2 + 3x - 3.5 = 0$



2. $2x^2 + 4x + 4 = 0$



3. $x^2 + 8x + 16 = 0$



Examples 1–4 (pp. 246–248)

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

4. $-x^2 - 7x = 0$

5. $x^2 - 2x - 24 = 0$

6. $25 + x^2 + 10x = 0$

7. $-14x + x^2 + 49 = 0$

8. $x^2 + 16x + 64 = -6$

9. $x^2 - 12x = -37$

10. $4x^2 - 7x - 15 = 0$

11. $2x^2 - 2x - 3 = 0$

Examples 1, 3 (pp. 246, 247)

12. NUMBER THEORY Use a quadratic equation to find two real numbers with a sum of 5 and a product of −14, or show that no such numbers exist.

Example 5 (p. 248)

13. ARCHERY An arrow is shot upward with a velocity of 64 feet per second. Ignoring the height of the archer, how long after the arrow is released does it hit the ground? Use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of an object in feet, v_0 is the object's initial velocity in feet per second, and t is the time in seconds.

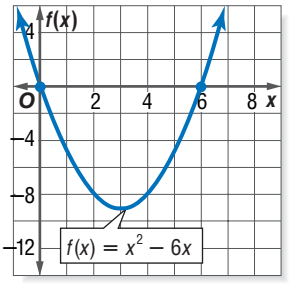
Exercises

HOMEWORK HELP

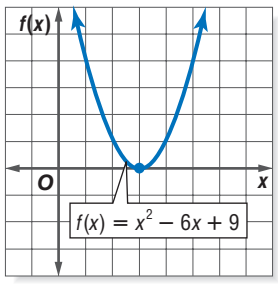
For Exercises	See Examples
14–19	1–3
20–29	1–4
30, 31	5

Use the related graph of each equation to determine its solutions.

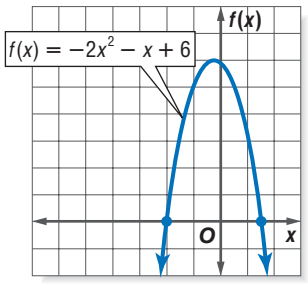
14. $x^2 - 6x = 0$



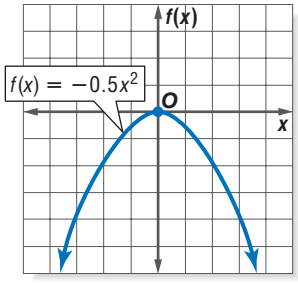
15. $x^2 - 6x + 9 = 0$



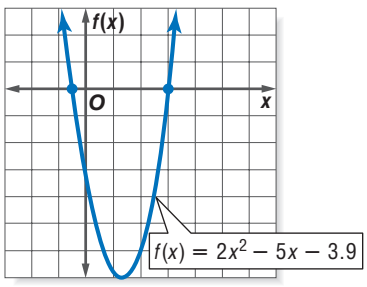
16. $-2x^2 - x + 6 = 0$



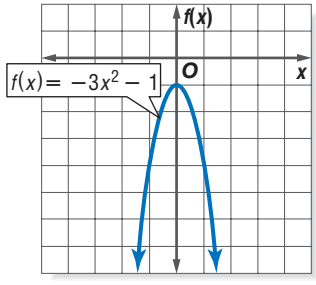
17. $-0.5x^2 = 0$



18. $2x^2 - 5x - 3.9 = 0$



19. $-3x^2 - 1 = 0$



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Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

20. $x^2 - 3x = 0$
21. $-x^2 + 4x = 0$
22. $-x^2 + x = -20$
23. $x^2 - 9x = -18$
24. $14x + x^2 + 49 = 0$
25. $-12x + x^2 = -36$
26. $x^2 + 2x + 5 = 0$
27. $-x^2 + 4x - 6 = 0$
28. $x^2 + 4x - 4 = 0$
29. $x^2 - 2x - 1 = 0$

For Exercises 30 and 31, use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of an object in feet, v_0 is the object's initial velocity in feet per second, and t is the time in seconds.

30. TENNIS A tennis ball is hit upward with a velocity of 48 feet per second. Ignoring the height of the tennis player, how long does it take for the ball to fall to the ground?

31. BOATING A boat in distress launches a flare straight up with a velocity of 190 feet per second. Ignoring the height of the boat, how many seconds will it take for the flare to hit the water?

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

32. $2x^2 - 3x = 9$
33. $4x^2 - 8x = 5$
34. $2x^2 = -5x + 12$
35. $2x^2 = x + 15$
36. $x^2 + 3x - 2 = 0$
37. $x^2 - 4x + 2 = 0$
38. $-2x^2 + 3x + 3 = 0$
39. $0.5x^2 - 3 = 0$

NUMBER THEORY **Use a quadratic equation to find two real numbers that satisfy each situation, or show that no such numbers exist.**

40. Their sum is -17 and their product is 72.
41. Their sum is 7 and their product is 14.
42. Their sum is -9 and their product is 24.
43. Their sum is 12 and their product is -28.

44. LAW ENFORCEMENT Police officers can use the length of skid marks to help determine the speed of a vehicle before the brakes were applied. If the skid marks are on dry concrete, the formula $\frac{s^2}{24} = d$ can be used. In the formula, s represents the speed in miles per hour and d represents the length of the skid marks in feet. If the length of the skid marks on dry concrete are 50 feet, how fast was the car traveling?

45. PHYSICS Suppose you could drop a small object from the Observatory of the Empire State Building. How long would it take for the object to reach the ground, assuming there is no air resistance? Use the information at the left and the formula $h(t) = -16t^2 + h_0$, where t is the time in seconds and the initial height h_0 is in feet.

Real-World Link

Located on the 86th floor, 1050 feet (320 meters) above the streets of New York City, the Observatory offers panoramic views from within a glass-enclosed pavilion and from the surrounding open-air promenade.

Source: www.esbnyc.com

EXTRA PRACTICE
See pages 900, 930.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

46. OPEN ENDED Give an example of a quadratic equation with a double root, and state the relationship between the double root and the graph of the related function.

47. REASONING Explain how you can estimate the solutions of a quadratic equation by examining the graph of its related function.

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www.MathSchoolinternational.com

48. CHALLENGE A quadratic function has values $f(-4) = -11$, $f(-2) = 9$, and $f(0) = 5$. Between which two x-values must $f(x)$ have a zero? Explain your reasoning.

49. *Writing in Math* Use the information on page 246 to explain how a quadratic function models a free-fall ride. Include a graph showing the height at any given time of a free-fall ride that lifts riders to a height of 185 feet and an explanation of how to use this graph to estimate how long the riders would be in free-fall if the ride were allowed to hit the ground before stopping.

STANDARDIZED TEST PRACTICE

50. ACT/SAT If one of the roots of the equation $x^2 + kx - 12 = 0$ is 4, what is the value of k?

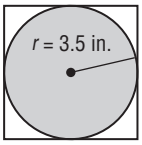
A -1

B 0

C 1

D 3

51. REVIEW What is the area of the square in square inches?



F 49

G 51

H 53

J 55

Spiral Review

Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex for each quadratic function. Then graph the function by making a table of values. (Lesson 5-1)

52. $f(x) = x^2 - 6x + 4$ **53.** $f(x) = -4x^2 + 8x - 1$ **54.** $f(x) = \frac{1}{4}x^2 + 3x + 4$

55. Solve the system $4x - y = 0$, $2x + 3y = 14$ by using inverse matrices. (Lesson 4-8)

Evaluate the determinant of each matrix. (Lesson 4-3)

56. $\begin{bmatrix} 6 & 4 \\ -3 & 2 \end{bmatrix}$ **57.** $\begin{bmatrix} 2 & -1 & -6 \\ 5 & 0 & 3 \\ -3 & 2 & 11 \end{bmatrix}$ **58.** $\begin{bmatrix} 6 & 5 & 2 \\ -3 & 0 & -6 \\ 1 & 4 & 2 \end{bmatrix}$

59. COMMUNITY SERVICE A drug awareness program is being presented at a theater that seats 300 people. Proceeds will be donated to a local drug information center. If every two adults must bring at least one student, what is the maximum amount of money that can be raised? (Lesson 3-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Factor completely.** (p. 753)

60. $x^2 + 5x$ **61.** $x^2 - 100$

62. $x^2 - 11x + 28$ **63.** $x^2 - 18x + 81$

64. $3x^2 + 8x + 4$ **65.** $6x^2 - 14x - 12$

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EXTEND 5-2

Graphing Calculator Lab

Modeling Using Quadratic Functions

ACTIVITY

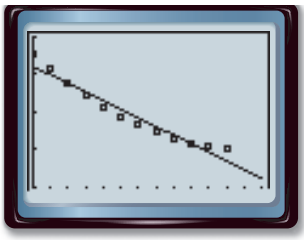
FALLING WATER **Water drains from a hole made in a 2-liter bottle. The table shows the level of the water y measured in centimeters from the bottom of the bottle after x seconds. Find and graph a linear regression equation and a quadratic regression equation. Determine which equation is a better fit for the data.**

Time (s)	0	20	40	60	80	100	120	140	160	180	200	220
Water level (cm)	42.6	40.7	38.9	37.2	35.8	34.3	33.3	32.3	31.5	30.8	30.4	30.1

Step 1 Find a linear regression equation.

- Enter the times in L1 and the water levels in L2. Then find a linear regression equation. Graph a scatter plot and the equation.

KEYSTROKES: *Review lists and finding and graphing a linear regression equation on page 92.*



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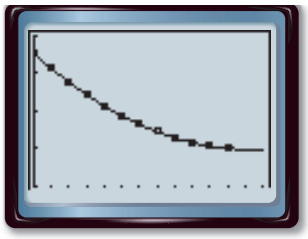
Step 2 Find a quadratic regression equation.

- Find the quadratic regression equation. Then copy the equation to the Y= list and graph.

KEYSTROKES: STAT ▶ 5 ENTER Y= VARS 5 ▶ ▶ ENTER GRAPH



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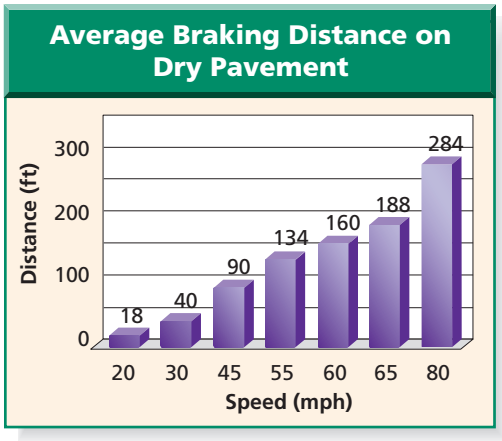
The graph of the linear regression equation appears to pass through just two data points. However, the graph of the quadratic regression equation fits the data very well.

Exercises

For Exercises 1–4, use the graph of the braking distances for dry pavement.

1. Find and graph a linear regression equation and a quadratic regression equation for the data. Determine which equation is a better fit for the data.
2. Use the CALC menu with each regression equation to estimate the braking distance at speeds of 100 and 150 miles per hour.
3. How do the estimates found in Exercise 2 compare?
4. How might choosing a regression equation that does not fit the data well affect predictions made by using the equation?



Source: Missouri Department of Revenue

Other Calculator Keystrokes at algebra2.com

www.MathSchoolinternational.com

5-3 Solving Quadratic Equations by Factoring

Main Ideas

- Write quadratic equations in intercept form.
- Solve quadratic equations by factoring.

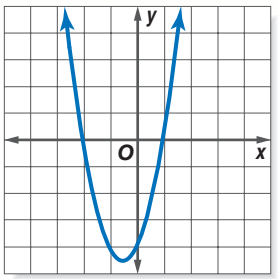
New Vocabulary

intercept form

FOIL method

GET READY for the Lesson

The **intercept form** of a quadratic equation is $y = a(x - p)(x - q)$. In the equation, p and q represent the x-intercepts of the graph corresponding to the equation. The intercept form of the equation shown in the graph is $y = 2(x - 1)(x + 2)$. The x-intercepts of the graph are 1 and -2. The standard form of the equation is $y = 2x^2 + 2x - 4$.



Intercept Form Changing a quadratic equation in intercept form to standard form requires the use of the FOIL method. The **FOIL method** uses the Distributive Property to multiply binomials.

KEY CONCEPT *FOIL Method for Multiplying Binomials*

The product of two binomials is the sum of the products of **F** the *first* terms, **O** the *outer* terms, **I** the *inner* terms, and **L** the *last* terms.

To change $y = 2(x - 1)(x + 2)$ to standard form, use the FOIL method to find the product of $(x - 1)$ and $(x + 2)$, $x^2 + x - 2$, and then multiply by 2. The standard form of the equation is $y = 2x^2 + 2x - 4$.

You have seen that a quadratic equation of the form $(x - p)(x - q) = 0$ has roots p and q. You can use this pattern to find a quadratic equation for a given pair of roots.

EXAMPLE 1 Write an Equation Given Roots

Write a quadratic equation with $\frac{1}{2}$ and -5 as its roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

$$(x - p)(x - q) = 0 \quad \text{Write the pattern.}$$

$$\left(x - \tfrac{1}{2}\right)[x - (-5)] = 0 \quad \text{Replace } p \text{ with } \tfrac{1}{2} \text{ and } q \text{ with } -5.$$

$$\left(x - \tfrac{1}{2}\right)(x + 5) = 0 \quad \text{Simplify.}$$

$$x^2 + \tfrac{9}{2}x - \tfrac{5}{2} = 0 \quad \text{Use FOIL.}$$

$$2x^2 + 9x - 5 = 0 \quad \text{Multiply each side by 2 so that } b \text{ and } c \text{ are integers.}$$

Study Tip

Writing an Equation

The pattern $(x - p)(x - q) = 0$ produces one equation with roots p and q.

In fact, there are an infinite number of equations that have these same roots.

CHECK Your Progress

1. Write a quadratic equation with $-\frac{1}{3}$ and 4 as its roots. Write the equation in standard form.

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Solve Equations by Factoring In the last lesson, you learned to solve a quadratic equation by graphing. Another way to solve a quadratic equation is by factoring an equation in standard form. When an equation in standard form is factored and written in intercept form $y = a(x - p)(x - q)$, the solutions of the equation are p and q.

The following factoring techniques, or patterns, will help you factor polynomials. Then you can use the Zero Product Property to solve equations.

CONCEPT SUMMARY *Factoring Techniques*

Factoring Technique	General Case
Greatest Common Factor (GCF)	$a^3b^2 - 3ab^2 = ab^2(a^2 - 3)$
Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
General Trinomials	$acx^2 + (ad + bc)x + bd =$ $(ax + b)(cx + d)$

The FOIL method can help you factor a polynomial into the product of two binomials. Study the following example.

$$(ax + b)(cx + d) = \overbrace{ax \cdot cx}^{\text{F}} + \overbrace{ax \cdot d}^{\text{O}} + \overbrace{b \cdot cx}^{\text{I}} + \overbrace{b \cdot d}^{\text{L}}$$
$$= acx^2 + (ad + bc)x + bd$$

Notice that the product of the coefficient of x^2 and the constant term is $abcd$. The product of the two terms in the coefficient of x is also $abcd$.

EXAMPLE Two or Three Terms

2 **Factor each polynomial.**

a. $5x^2 - 13x + 6$

To find the coefficients of the x-terms, you must find two numbers with a product of $5 \cdot 6$ or 30, and a sum of -13. The two coefficients must be -10 and -3 since $(-10)(-3) = 30$ and $-10 + (-3) = -13$.

Rewrite the expression using $-10x$ and $-3x$ in place of $-13x$ and factor by grouping.

$5x^2 - 13x + 6 = 5x^2 - 10x - 3x + 6$ Substitute $-10x - 3x$ for $-13x$.

$= (5x^2 - 10x) + (-3x + 6)$ Associative Property

$= 5x(x - 2) - 3(x - 2)$ Factor out the GCF of each group.

$= (5x - 3)(x - 2)$ Distributive Property

b. $m^6 - n^6$

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ Difference of two squares

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$ Sum and difference of two cubes

Study Tip

The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

CHECK Your Progress

2A. $3xy^2 - 48x$ **2B.** $c^3d^3 + 27$

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Solving quadratic equations by factoring is an application of the **Zero Product Property**.

KEY CONCEPT — **Zero Product Property**

Words For any real numbers a and b, if $ab = 0$, then either $a = 0$, $b = 0$, or both a and b equal zero.

Example If $(x + 5)(x - 7) = 0$, then $x + 5 = 0$ or $x - 7 = 0$.

EXAMPLE 3 Two Roots

Solve $x^2 = 6x$ by factoring. Then graph.

$x^2 = 6x$ Original equation

$x^2 - 6x = 0$ Subtract $6x$ from each side.

$x(x - 6) = 0$ Factor the binomial.

$x = 0$ or $x - 6 = 0$ Zero Product Property

$x = 6$ Solve the second equation.

The solution set is $\{0, 6\}$.

To complete the graph, find the vertex. Use the equation for the axis of symmetry.

$x = -\frac{b}{2a}$ Equation of the axis of symmmetry

$= -\frac{-6}{2}(1)$ $a = 1, b = -6$

$= 3$ Simplify.

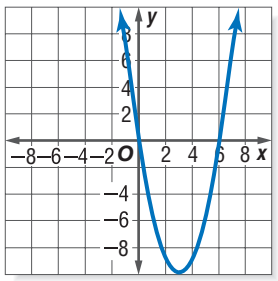
Therefore, the x-coordinate of the vertex is 3. Substitute 3 into the equation to find the y-value.

$y = x^2 - 6x$ Original equation

$= 3^2 - 6(3)$ $x = 3$

$= 9 - 18$ Simplify.

$= -9$ Subtract.



The vertex is at $(3, -9)$. Graph the x-intercepts $(0, 0)$ and $(6, 0)$ and the vertex $(3, -9)$, connecting them with a smooth curve.

CHECK Your Progress **Solve each equation by factoring. Then graph.**

3A. $3x^2 = 9x$ **3B.** $6x^2 = 1 - x$

Study Tip

Double Roots

The application of the Zero Product Property produced two identical equations, $x - 8 = 0$, both of which have a root of 8. For this reason, 8 is called the *double root* of the equation.

Personal Tutor at algebra2.com

EXAMPLE 4 Double Root

Solve $x^2 - 16x + 64 = 0$ by factoring.

$x^2 - 16x + 64 = 0$ Original equation

$(x - 8)(x - 8) = 0$ Factor.

$x - 8 = 0$ or $x - 8 = 0$ Zero Product Property

$x = 8$ $x = 8$ Solve each equation.

The solution set is $\{8\}$.

(continued on the next page)

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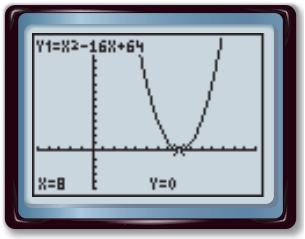
CHECK The graph of the related function, $f(x) = x^2 - 16x + 64$, intersects the x-axis only once. Since the zero of the function is 8, the solution of the related equation is 8.



CHECK Your Progress

Solve each equation by factoring.

4A. $x^2 + 12x + 36 = 0$ **4B.** $x^2 - 25 = 0$

CHECK Your Understanding

Example 1 (p. 253)

Write a quadratic equation with the given root(s). Write the equation in standard form.

1. $-4, 7$ **2.** $\frac{1}{2}, \frac{4}{3}$ **3.** $-\frac{3}{5}, -\frac{1}{3}$

Example 2 (p. 254)

Factor each polynomial.

4. $x^3 - 27$ **5.** $4xy^2 - 16x$ **6.** $3x^2 + 8x + 5$

Examples 3, 4 (pp. 255–256)

Solve each equation by factoring. Then graph.

7. $x^2 - 11x = 0$ **8.** $x^2 + 6x - 16 = 0$ **9.** $4x^2 - 13x = 12$

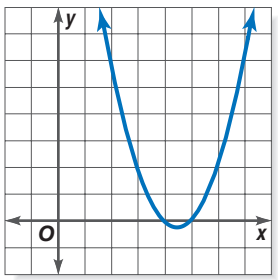
10. $x^2 - 14x = -49$ **11.** $x^2 + 9 = 6x$ **12.** $x^2 - 3x = -\frac{9}{4}$

Exercises

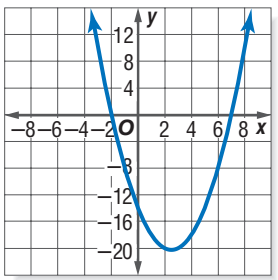
HOMEWORK HELP	
For Exercises	See Examples
13–16	1
17–20	2
21–32	3, 4

Write a quadratic equation in standard form for each graph.

13.



14.



Write a quadratic equation in standard form with the given roots.

15. $4, -5$ **16.** $-6, -8$

Factor each polynomial.

17. $x^2 - 7x + 6$ **18.** $x^2 + 8x - 9$

19. $3x^2 + 12x - 63$ **20.** $5x^2 - 80$

Solve each equation by factoring. Then graph.

21. $x^2 + 5x - 24 = 0$ **22.** $x^2 - 3x - 28 = 0$

23. $x^2 = 25$ **24.** $x^2 = 81$

25. $x^2 + 3x = 18$ **26.** $x^2 - 4x = 21$

27. $-2x^2 + 12x - 16 = 0$ **28.** $-3x^2 - 6x + 9 = 0$

29. $x^2 + 36 = 12x$ **30.** $x^2 + 64 = 16x$

31. NUMBER THEORY Find two consecutive even integers with a product of 224.

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32. PHOTOGRAPHY A rectangular photograph is 8 centimeters wide and 12 centimeters long. The photograph is enlarged by increasing the length and width by an equal amount in order to double its area. What are the dimensions of the new photograph?

Solve each equation by factoring.

33. $3x^2 = 5x$

34. $4x^2 = -3x$

35. $4x^2 + 7x = 2$

36. $4x^2 - 17x = -4$

37. $4x^2 + 8x = -3$

38. $6x^2 + 6 = -13x$

39. $9x^2 + 30x = -16$

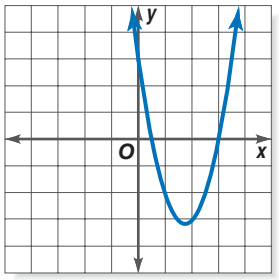
40. $16x^2 - 48x = -27$

41. Find the roots of $x(x + 6)(x - 5) = 0$.

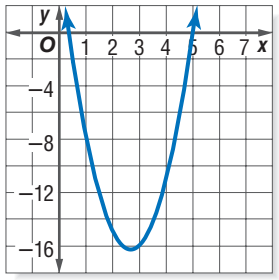
42. Solve $x^3 = 9x$ by factoring.

Write a quadratic equation with the given graph or roots.

43.



44.



45. $-\frac{2}{3}, \frac{3}{4}$

46. $-\frac{3}{2}, -\frac{4}{5}$

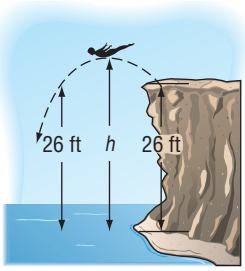
Matthew McVay/Stock Boston

Real-World Link

A board foot is a measure of lumber volume. One piece of lumber 1 foot long by 1 foot wide by 1 inch thick measures one board foot.

Source: www.wood-worker.com

47. DIVING To avoid hitting any rocks below, a cliff diver jumps up and out. The equation $h = -16t^2 + 4t + 26$ describes her height h in feet t seconds after jumping. Find the time at which she returns to a height of 26 feet.



FORESTRY **For Exercises 48 and 49, use the following information.**

Lumber companies need to be able to estimate the number of board feet that a given log will yield. One of the most commonly used formulas for estimating board feet is the *Doyle Log Rule*, $B = \frac{L}{16}(D^2 - 8D + 16)$ where B is the number of board feet, D is the diameter in inches, and L is the length of the log in feet.

48. Rewrite Doyle's formula for logs that are 16 feet long.

49. Find the root(s) of the quadratic equation you wrote in Exercise 48. What do the root(s) tell you about the kinds of logs for which Doyle's rule makes sense?

EXTRA PRACTICE

See pages 900, 930.

MathOnline

Self-Check Quiz at algebra2.com

H.O.T. Problems

50. FIND THE ERROR Lina and Kristin are solving $x^2 + 2x = 8$. Who is correct? Explain your reasoning.

Lina

$x^2 + 2x = 8$

$x(x + 2) = 8$

$x = 8$ or $x + 2 = 8$

$x = 6$

Kristin

$x^2 + 2x = 8$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x + 4 = 0$ or $x - 2 = 0$

$x = -4$ $x = 2$

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51. OPEN ENDED Choose two integers. Then write an equation with those roots in standard form. How would the equation change if the signs of the two roots were switched?

52. CHALLENGE For a quadratic equation of the form $(x - p)(x - q) = 0$, show that the axis of symmetry of the related quadratic function is located halfway between the x-intercepts p and q.

53. *Writing in Math* Use the information on page 253 to explain how to solve a quadratic equation using the Zero Product Property. Explain why you cannot solve $x(x + 5) = 24$ by solving $x = 24$ and $x + 5 = 24$.

STANDARDIZED TEST PRACTICE

54. ACT/SAT Which quadratic equation has roots $\frac{1}{2}$ and $\frac{1}{3}$?

A $5x^2 - 5x - 2 = 0$

B $5x^2 - 5x + 1 = 0$

C $6x^2 + 5x - 1 = 0$

D $6x^2 - 5x + 1 = 0$

55. REVIEW What is the solution set for the equation $3(4x + 1)^2 = 48$?

F $\left\{\frac{5}{4}, -\frac{3}{4}\right\}$

G $\left\{-\frac{5}{4}, \frac{3}{4}\right\}$

H $\left\{\frac{15}{4}, -\frac{17}{4}\right\}$

J $\left\{\frac{1}{3}, -\frac{4}{3}\right\}$

Spiral Review

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

56. $0 = -x^2 - 4x + 5$

57. $0 = 4x^2 + 4x + 1$

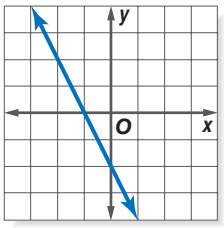
58. $0 = 3x^2 - 10x - 4$

59. Determine whether $f(x) = 3x^2 - 12x - 7$ has a maximum or a minimum value. Then find the maximum or minimum value. (Lesson 5-1)

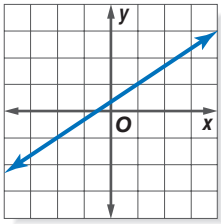
60. CAR MAINTENANCE Vince needs 12 quarts of a 60% anti-freeze solution. He will combine an amount of 100% anti-freeze with an amount of a 50% anti-freeze solution. How many quarts of each solution should be mixed to make the required amount of the 60% anti-freeze solution? (Lesson 4-8)

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

61.



62.



GET READY for the Next Lesson

PREREQUISITE SKILL Name the property illustrated by each equation. (Lesson 1-2)

63. $2x + 4y + 3z = 2x + 3z + 4y$

64. $3(6x - 7y) = 3(6x) + 3(-7y)$

65. $(3 + 4) + x = 3 + (4 + x)$

66. $(5x)(-3y)(6) = (-3y)(6)(5x)$

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5-4 Complex Numbers

Main Ideas

- Find square roots and perform operations with pure imaginary numbers.
- Perform operations with complex numbers.

New Vocabulary

square root
imaginary unit
pure imaginary number
Square Root Property
complex number
complex conjugates

GET READY for the Lesson

Concepts in MOtion
Interactive Lab algebra2.com

Consider $2x^2 + 2 = 0$. One step in the solution of this equation is $x^2 = -1$. Since there is no real number that has a square of -1, there are no real solutions. French mathematician René Descartes (1596–1650) proposed that a number i be defined such that $i^2 = -1$.

Square Roots and Pure Imaginary Numbers A **square root** of a number n is a number with a square of n. For example, 7 is a square root of 49 because $7^2 = 49$. Since $(-7)^2 = 49$, -7 is also a square root of 49. Two properties will help you simplify expressions that contain square roots.

KEY CONCEPT *Product and Quotient Properties of Square Roots*

Words For nonnegative real numbers a and b,
$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, and
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $b \neq 0$.

Examples $\sqrt{3 \cdot 2} = \sqrt{3} \cdot \sqrt{2}$
$\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}}$

Simplified square root expressions do not have radicals in the denominator, and any number remaining under the square root has no perfect square factor other than 1.

EXAMPLE Properties of Square Roots

1 Simplify.

a. $\sqrt{50}$

$\sqrt{50} = \sqrt{25 \cdot 2}$
$= \sqrt{25} \cdot \sqrt{2}$
$= 5\sqrt{2}$

b. $\sqrt{\frac{11}{49}}$

$\sqrt{\frac{11}{49}} = \frac{\sqrt{11}}{\sqrt{49}}$
$= \frac{\sqrt{11}}{7}$

CHECK Your Progress

1A. $\sqrt{45}$

1B. $\sqrt{\frac{32}{81}}$

Since i is defined to have the property that $i^2 = -1$, the number i is the principal square root of -1; that is, $i = \sqrt{(-1)}$. i is called the **imaginary unit**. Numbers of the form $3i$, $-5i$, and $i\sqrt{2}$ are called **pure imaginary numbers**. Pure imaginary numbers are square roots of negative real numbers. For any positive real number b, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$ or bi.

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Reading Math

Imaginary Unit i is usually written before radical symbols to make it clear that it is not under the radical.

EXAMPLE 2 Square Roots of Negative Numbers

Simplify.

a. $\sqrt{-18}$

$$\begin{aligned}\sqrt{-18} &= \sqrt{-1 \cdot 3^2 \cdot 2}\\ &= \sqrt{-1} \cdot \sqrt{3^2} \cdot \sqrt{2}\\ &= i \cdot 3 \cdot \sqrt{2} \text{ or } 3i\sqrt{2}\end{aligned}$$

b. $\sqrt{-125x^5}$

$$\begin{aligned}\sqrt{-125x^5} &= \sqrt{-1 \cdot 5^2 \cdot x^4 \cdot 5x}\\ &= \sqrt{-1} \cdot \sqrt{5^2} \cdot \sqrt{x^4} \cdot \sqrt{5x}\\ &= i \cdot 5 \cdot x^2 \cdot \sqrt{5x} \text{ or } 5ix^2\sqrt{5x}\end{aligned}$$

CHECK Your Progress

2A. $\sqrt{-27}$ **2B.** $\sqrt{-216y^4}$

The Commutative and Associative Properties of Multiplication hold true for pure imaginary numbers.

EXAMPLE 3 Products of Pure Imaginary Numbers

Simplify.

a. $-2i \cdot 7i$

$$\begin{aligned}-2i \cdot 7i &= -14i^2\\ &= -14(-1) \quad i^2 = -1\\ &= 14\end{aligned}$$

b. $\sqrt{-10} \cdot \sqrt{-15}$

$$\begin{aligned}\sqrt{-10} \cdot \sqrt{-15} &= i\sqrt{10} \cdot i\sqrt{15}\\ &= i^2\sqrt{150}\\ &= -1 \cdot \sqrt{25} \cdot \sqrt{6}\\ &= -5\sqrt{6}\end{aligned}$$

c. i^{45}

$$\begin{aligned}i^{45} &= i \cdot i^{44} && \text{Multiplying powers}\\ &= i \cdot (i^2)^{22} && \text{Power of a Power}\\ &= i \cdot (-1)^{22} && i^2 = -1\\ &= i \cdot 1 \text{ or } i && (-1)^{22} = 1\end{aligned}$$

CHECK Your Progress

3A. $3i \cdot 4i$ **3B.** $\sqrt{-20} \cdot \sqrt{-12}$ **3C.** i^{31}

You can solve some quadratic equations by using the **Square Root Property.**

Reading Math

Plus or Minus $\pm\sqrt{n}$ is read *plus or minus the square root of n.*

KEY CONCEPT *Square Root Property*

For any real number n, if $x^2 = n$, then $x = \pm\sqrt{n}$.

EXAMPLE 4 Equation with Pure Imaginary Solutions

Solve $3x^2 + 48 = 0$.

$$\begin{aligned}3x^2 + 48 &= 0 && \text{Original equation}\\ 3x^2 &= -48 && \text{Subtract 48 from each side.}\\ x^2 &= -16 && \text{Divide each side by 3.}\\ x &= \pm\sqrt{-16} && \text{Square Root Property}\\ x &= \pm 4i && \sqrt{-16} = \sqrt{16} \cdot \sqrt{-1}\end{aligned}$$

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CHECK Your Progress **Solve each equation.**

4A. $4x^2 + 100 = 0$ **4B.** $x^2 + 4 = 0$

Operations with Complex Numbers Consider $5 + 2i$. Since 5 is a real number and $2i$ is a pure imaginary number, the terms are not like terms and cannot be combined. This type of expression is called a **complex number**.

KEY CONCEPT ***Complex Numbers***

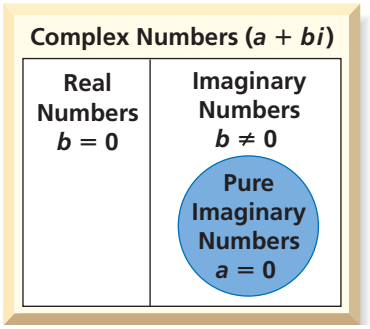
Words A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit. a is called the real part, and b is called the imaginary part.

Examples $7 + 4i$ and $2 - 6i = 2 + (-6)i$

The Venn diagram shows the complex numbers.

- If $b = 0$, the complex number is a real number.
- If $b \neq 0$, the complex number is imaginary.
- If $a = 0$, the complex number is a pure imaginary number.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.



Reading Math

Complex Numbers The form $a + bi$ is sometimes called the *standard form* of a complex number.

EXAMPLE **Equate Complex Numbers**

5 **Find the values of x and y that make the equation $2x - 3 + (y - 4)i = 3 + 2i$ true.**

Set the real parts equal to each other and the imaginary parts equal to each other.

$2x - 3 = 3$ Real parts

$2x = 6$ Add 3 to each side.

$x = 3$ Divide each side by 2.

$y - 4 = 2$ Imaginary parts

$y = 6$ Add 4 to each side.

CHECK Your Progress

5. Find the values of x and y that make the equation $5x + 1 + (3 + 2y)i = 2x - 2 + (y - 6)i$ true.

To add or subtract complex numbers, combine like terms. That is, combine the real parts and combine the imaginary parts.

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EXAMPLE 6 Add and Subtract Complex Numbers

Simplify.

a. $(6 - 4i) + (1 + 3i)$

$(6 - 4i) + (1 + 3i) = (6 + 1) + (-4 + 3)i$ Commutative and Associative Properties

$= 7 - i$ Simplify.

b. $(3 - 2i) - (5 - 4i)$

$(3 - 2i) - (5 - 4i) = (3 - 5) + [-2 - (-4)]i$ Commutative and Associative Properties

$= -2 + 2i$ Simplify.

CHECK Your Progress

6A. $(-2 + 5i) + (1 - 7i)$ **6B.** $(4 + 6i) - (-1 + 2i)$

Study Tip

Complex Numbers

While all real numbers are also complex, the term *Complex Numbers* usually refers to a number that is not real.

One difference between real and complex numbers is that complex numbers cannot be represented by lines on a coordinate plane. However, complex numbers can be graphed on a *complex plane*. A complex plane is similar to a coordinate plane, except that the horizontal axis represents the real part a of the complex number, and the vertical axis represents the imaginary part b of the complex number.

You can also use a complex plane to model the addition of complex numbers.

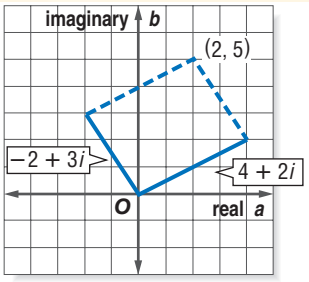
ALGEBRA LAB

Adding Complex Numbers Graphically

Use a complex plane to find $(4 + 2i) + (-2 + 3i)$.

- Graph $4 + 2i$ by drawing a segment from the origin to (4, 2) on the complex plane.
- Graph $-2 + 3i$ by drawing a segment from the origin to (−2 , 3) on the complex plane.
- Given three vertices of a parallelogram, complete the parallelogram.
- The fourth vertex at (2, 5) represents the complex number $2 + 5i$.

So, $(4 + 2i) + (-2 + 3i) = 2 + 5i$.



MODEL AND ANALYZE

1. Model $(-3 + 2i) + (4 - i)$ on a complex plane.
2. Describe how you could model the difference $(-3 + 2i) - (4 - i)$ on a complex plane.

Complex numbers are used with electricity. In a circuit with alternating current, the voltage, current, and impedance, or hindrance to current, can be represented by complex numbers. To multiply these numbers, use the FOIL method.

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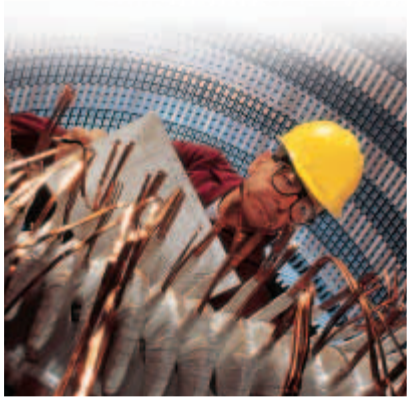
Study Tip

Electrical engineers use j as the imaginary unit to avoid confusion with the I for current.



Real-World Career

Electrical Engineer

The chips and circuits in computers are designed by electrical engineers.

Math Online

For more information, go to algebra2.com.

Real-World EXAMPLE

7 **ELECTRICITY** **In an AC circuit, the voltage E, current I, and impedance Z are related by the formula $E = I \cdot Z$. Find the voltage in a circuit with current $1 + 3j$ amps and impedance $7 - 5j$ ohms.**

$E = I \cdot Z$ Electricity formula

$= (1 + 3j) \cdot (7 - 5j)$ $I = 1 + 3j, Z = 7 - 5j$

$= 1(7) + 1(-5j) + (3j)7 + 3j(-5j)$ FOIL

$= 7 - 5j + 21j - 15j^2$ Multiply.

$= 7 + 16j - 15(-1)$ $j^2 = -1$

$= 22 + 16j$ Add.

The voltage is $22 + 16j$ volts.

CHECK Your Progress

7. Find the voltage in a circuit with current $2 - 4j$ amps and impedance $3 - 2j$ ohms.

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Two complex numbers of the form $a + bi$ and $a - bi$ are called **complex conjugates**. The product of complex conjugates is always a real number. You can use this fact to simplify the quotient of two complex numbers.

EXAMPLE **Divide Complex Numbers**

8 **Simplify.**

a. $\dfrac{3i}{2 + 4i}$

$\dfrac{3i}{2 + 4i} = \dfrac{3i}{2 + 4i} \cdot \dfrac{2 - 4i}{2 - 4i}$ $2 + 4i$ and $2 + 4i$ are conjugates.

$= \dfrac{6i - 12i^2}{4 - 16i^2}$ Multiply.

$= \dfrac{6i + 12}{20}$ $i^2 = -1$

$= \dfrac{3}{5} + \dfrac{3}{10}i$ Standard form

b. $\dfrac{5 + i}{2i}$

$\dfrac{5 + i}{2i} = \dfrac{5 + i}{2i} \cdot \dfrac{i}{i}$ Why multiply by $\frac{i}{i}$ instead of $\frac{-2i}{-2i}$?

$= \dfrac{5i + i^2}{2i^2}$ Multiply.

$= \dfrac{5i - 1}{-2}$ $i^2 = -1$

$= \dfrac{1}{2} - \dfrac{5}{2}i$ Standard form

CHECK Your Progress

8A. $\dfrac{-2i}{3 + 5i}$ **8B.** $\dfrac{2 + i}{1 - i}$

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CHECK Your Understanding

Examples 1–3 (pp. 259–260)

Simplify.

1. $\sqrt{56}$
2. $\sqrt{80}$
3. $\sqrt{\frac{48}{49}}$
4. $\sqrt{\frac{120}{9}}$
5. $\sqrt{-36}$
6. $\sqrt{-50x^2y^2}$
7. $(6i)(-2i)$
8. $5\sqrt{-24} \cdot 3\sqrt{-18}$
9. i^{29}
10. i^{80}

Example 4 (p. 260)

Solve each equation.

11. $2x^2 + 18 = 0$
12. $-5x^2 - 25 = 0$

Example 5 (p. 261)

Find the values of m and n that make each equation true.

13. $2m + (3n + 1)i = 6 - 8i$
14. $(2n - 5) + (-m - 2)i = 3 - 7i$

Example 6 (p. 262)

15. **ELECTRICITY** The current in one part of a series circuit is $4 - j$ amps. The current in another part of the circuit is $6 + 4j$ amps. Add these complex numbers to find the total current in the circuit.

Examples 7, 8 (p. 263)

Simplify.

16. $(-2 + 7i) + (-4 - 5i)$
17. $(8 + 6i) - (2 + 3i)$
18. $(3 - 5i)(4 + 6i)$
19. $(1 + 2i)(-1 + 4i)$
20. $\frac{2 - i}{5 + 2i}$
21. $\frac{3 + i}{1 + 4i}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
22–25	1
26–29	2
30–33	3
34–37	6
38, 39, 50	7
40, 41, 51	8
42–45	4
46–49	5

Simplify.

22. $\sqrt{125}$
23. $\sqrt{147}$
24. $\sqrt{\frac{192}{121}}$
25. $\sqrt{\frac{350}{81}}$
26. $\sqrt{-144}$
27. $\sqrt{-81}$
28. $\sqrt{-64x^4}$
29. $\sqrt{-100a^4b^2}$
30. $(-2i)(-6i)(4i)$
31. $3i(-5i)^2$
32. i^{13}
33. i^{24}
34. $(5 - 2i) + (4 + 4i)$
35. $(-2 + i) + (-1 - i)$
36. $(15 + 3i) - (9 - 3i)$
37. $(3 - 4i) - (1 - 4i)$
38. $(3 + 4i)(3 - 4i)$
39. $(1 - 4i)(2 + i)$
40. $\frac{4i}{3 + i}$
41. $\frac{4}{5 + 3i}$

Solve each equation.

42. $5x^2 + 5 = 0$
43. $4x^2 + 64 = 0$
44. $2x^2 + 12 = 0$
45. $6x^2 + 72 = 0$

Find the values of m and n that make each equation true.

46. $8 + 15i = 2m + 3ni$
47. $(m + 1) + 3ni = 5 - 9i$
48. $(2m + 5) + (1 - n)i = -2 + 4i$
49. $(4 + n) + (3m - 7)i = 8 - 2i$

ELECTRICITY For Exercises 50 and 51, use the formula $E = I \cdot Z$.

50. The current in a circuit is $2 + 5j$ amps, and the impedance is $4 - j$ ohms. What is the voltage?

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51. The voltage in a circuit is $14 - 8j$ volts, and the impedance is $2 - 3j$ ohms. What is the current?

52. Find the sum of $ix^2 - (2 + 3i)x + 2$ and $4x^2 + (5 + 2i)x - 4i$.

53. Simplify $[(3 + i)x^2 - ix + 4 + i] - [(-2 + 3i)x^2 + (1 - 2i)x - 3]$.

Simplify.

54. $\sqrt{-13} \cdot \sqrt{-26}$

55. $(4i)\left(\frac{1}{2}i\right)^2(-2i)^2$

56. i^{38}

57. $(3 - 5i) + (3 + 5i)$

58. $(7 - 4i) - (3 + i)$

59. $(-3 - i)(2 - 2i)$

60. $\frac{(10 + i)^2}{4 - i}$

61. $\frac{2 - i}{3 - 4i}$

62. $(-5 + 2i)(6 - i)(4 + 3i)$

63. $(2 + i)(1 + 2i)(3 - 4i)$

64. $\frac{5 - i\sqrt{3}}{5 + i\sqrt{3}}$

65. $\frac{1 - i\sqrt{2}}{1 + i\sqrt{2}}$

Solve each equation, and locate the complex solutions in the complex plane.

66. $-3x^2 - 9 = 0$

67. $-2x^2 - 80 = 0$

68. $\frac{2}{3}x^2 + 30 = 0$

69. $\frac{4}{5}x^2 + 1 = 0$

Find the values of m and n that make each equation true.

70. $(m + 2n) + (2m - n)i = 5 + 5i$

71. $(2m - 3n)i + (m + 4n) = 13 + 7i$

EXTRA PRACTICE
See pages 900, 930.
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Self-Check Quiz at algebra2.com

72. ELECTRICITY The impedance in one part of a series circuit is $3 + 4j$ ohms, and the impedance in another part of the circuit is $2 - 6j$. Add these complex numbers to find the total impedance in the circuit.

H.O.T. Problems

73. OPEN ENDED Write two complex numbers with a product of 10.

74. CHALLENGE Copy and complete the table. Explain how to use the exponent to determine the simplified form of any power of i.

Power of i	Simplified Expression
i^6	?
i^7	?
i^8	?
i^9	?
i^{10}	?
i^{11}	?
i^{12}	?
i^{13}	?

75. Which One Doesn't Belong? Identify the expression that does not belong with the other three. Explain your reasoning.

$(3i)^2$ | $(2i)(3i)(4i)$ | $(6 + 2i) - (4 + 2i)$ | $(2i)^4$

76. REASONING Determine if each statement is *true* or *false*. If false, find a counterexample.

a. Every real number is a complex number.

b. Every imaginary number is a complex number.

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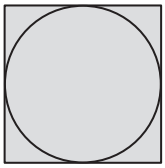
77. ***Writing in Math*** Use the information on page 261 to explain how complex numbers are related to quadratic equations. Explain how the a and c must be related if the equation $ax^2 + c = 0$ has complex solutions and give the solutions of the equation $2x^2 + 2 = 0$.

STANDARDIZED TEST PRACTICE

78. **ACT/SAT** The area of the square is 16 square units. What is the area of the circle?



A 2π units^2

B 12 units^2

C 4π units^2

D 16π units^2

79. If $i^2 = -1$, then what is the value of i^{71}?

F -1

G 0

H $-i$

J i

Spiral Review

Write a quadratic equation with the given root(s). Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. (Lesson 5-3)

80. $-3, 9$

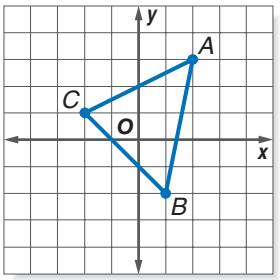
81. $-\frac{1}{3}, -\frac{3}{4}$

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

82. $3x^2 = 4 - 8x$

83. $2x^2 + 11x = -12$

Triangle ABC is reflected over the x-axis. (Lesson 4-4)



84. Write a vertex matrix for the triangle.

85. Write the reflection matrix.

86. Write the vertex matrix for $\triangle A'B'C'$.

87. Graph $\triangle A'B'C'$.

88. **FURNITURE** A new sofa, love seat, and coffee table cost \$2050. The sofa costs twice as much as the love seat. The sofa and the coffee table together cost \$1450. How much does each piece of furniture cost? (Lesson 3-5)

89. **DECORATION** Samantha is going to use more than 75 but less than 100 bricks to make a patio off her back porch. If each brick costs \$2.75, write and solve a compound inequality to determine the amount she will spend on bricks. (Lesson 1-6)

GET READY for the Next Lesson

Determine whether each polynomial is a perfect square trinomial. (Lesson 5-3)

90. $x^2 - 10x + 16$

91. $x^2 + 18x + 81$

92. $x^2 - 9$

93. $x^2 - 12x - 36$

94. $x^2 - x + \frac{1}{4}$

95. $2x^2 - 15x + 25$

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CHAPTER 5

Mid-Chapter Quiz

Lessons 5-1 through 5-4

1. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex for $f(x) = 3x^2 - 12x + 4$. Then graph the function by making a table of values. (Lesson 5-1)

2. **MULTIPLE CHOICE** For which function is the x-coordinate of the vertex at 4? (Lesson 5-1)

   A $f(x) = x^2 - 8x + 15$

   B $f(x) = -x^2 - 4x + 12$

   C $f(x) = x^2 + 6x + 8$

   D $f(x) = -x^2 - 2x + 2$

3. Determine whether $f(x) = 3 - x^2 + 5x$ has a maximum or minimum value. Then find this maximum or minimum value and state the domain and range of the function. (Lesson 5-1)

4. **BASEBALL** From 2 feet above home plate, Grady hits a baseball upward with a velocity of 36 feet per second. The height $h(t)$ of the baseball t seconds after Grady hits it is given by $h(t) = -16t^2 + 36t + 2$. Find the maximum height reached by the baseball and the time that this height is reached. (Lesson 5-1)

5. Solve $2x^2 - 11x + 12 = 0$ by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

NUMBER THEORY **Use a quadratic equation to find two real numbers that satisfy each situation, or show that no such numbers exist.** (Lesson 5-2)

6. Their sum is 12, and their product is 20.
7. Their sum is 5 and their product is 9.

8. **MULTIPLE CHOICE** For what value of x does $f(x) = x^2 + 5x + 6$ reach its minimum value? (Lesson 5-2)

   F -5 H $-\frac{5}{2}$

   G -3 J -2

9. **FOOTBALL** A place kicker kicks a ball upward with a velocity of 32 feet per second. Ignoring the height of the kicking tee, how long after the football is kicked does it hit the ground? Use the formula $h(t) = v_0 t - 16t^2$ where $h(t)$ is the height of an object in feet, v_0 is the object's initial velocity in feet per second, and t is the time in seconds. (Lesson 5-2)

Solve each equation by factoring. (Lesson 5-3)

10. $2x^2 - 5x - 3 = 0$
11. $6x^2 + 4x - 2 = 0$
12. $3x^2 - 6x - 24 = 0$
13. $x^2 + 12x + 20 = 0$

REMODELING **For Exercises 14 and 15, use the following information.** (Lesson 5-3)

Sandy'closet was supposed to be 10 feet by 12 feet. The architect decided that this would not work and reduced the dimensions by the same amount x on each side. The area of the new closet is 63 square feet.

14. Write a quadratic equation that represents the area of Sandy's closet now.
15. Find the new dimensions of her closet.

16. Write a quadratic equation in standard form with roots -4 and $\frac{1}{3}$. (Lesson 5-3)

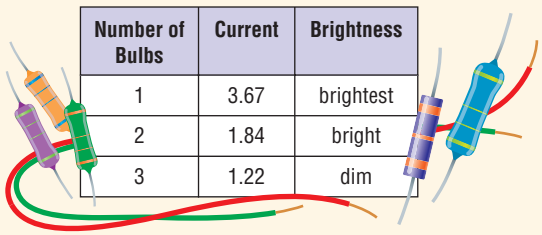
Simplify. (Lesson 5-4)

17. $\sqrt{-49}$
18. $\sqrt{-36a^3b^4}$
19. $(28 - 4i) - (10 - 30i)$
20. i^{89}
21. $(6 - 4i)(6 + 4i)$
22. $\frac{2 - 4i}{1 + 3i}$

23. **ELECTRICITY** The impedance in one part of a series circuit is $2 + 5j$ ohms and the impedance in another part of the circuit is $7 - 3j$ ohms. Add these complex numbers to find the total impedance in the circuit. (Lesson 5-4)

Series Circuit

Number of Bulbs	Current	Brightness
1	3.67	brightest
2	1.84	bright
3	1.22	dim



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5-5 Completing the Square

Main Ideas

- Solve quadratic equations by using the Square Root Property.
- Solve quadratic equations by completing the square.

New Vocabulary

completing the square

GET READY for the Lesson

Under a yellow caution flag, race car drivers slow to a speed of 60 miles per hour. When the green flag is waved, the drivers can increase their speed.

Suppose the driver of one car is 500 feet from the finish line. If the driver accelerates at a constant rate of 8 feet per second squared, the equation $t^2 + 22t + 121 = 246$ represents the time t it takes the driver to reach this line. To solve this equation, you can use the Square Root Property.

Square Root Property You have solved equations like $x^2 - 25 = 0$ by factoring. You can also use the Square Root Property to solve such an equation. This method is useful with equations like the one above that describes the race car's speed. In this case, the quadratic equation contains a perfect square trinomial set equal to a constant.

EXAMPLE Equation with Rational Roots

1 **Solve $x^2 + 10x + 25 = 49$ by using the Square Root Property.**

$x^2 + 10x + 25 = 49$	Original equation
$(x + 5)^2 = 49$	Factor the perfect square trinomial.
$x + 5 = \pm\sqrt{49}$	Square Root Property
$x + 5 = \pm 7$	$\sqrt{49} = 7$
$x = -5 \pm 7$	Add -5 to each side.
$x = -5 + 7$ or $x = -5 - 7$	Write as two equations.
$x = 2$ $\quad x = -12$	Solve each equation.

The solution set is $\{2, -12\}$. You can check this result by using factoring to solve the original equation.

CHECK Your Progress

Solve each equation by using the Square Root Property.

1A. $x^2 - 12x + 36 = 25$ **1B.** $x^2 - 16x + 64 = 49$

Roots that are irrational numbers may be written as exact answers in radical form or as *approximate* answers in decimal form when a calculator is used.

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EXAMPLE Equation with Irrational Roots

2 **Solve $x^2 - 6x + 9 = 32$ by using the Square Root Property.**

$x^2 - 6x + 9 = 32$	Original equation
$(x - 3)^2 = 32$	Factor the perfect square trinomial.
$x - 3 = \pm\sqrt{32}$	Square Root Property
$x = 3 \pm 4\sqrt{2}$	Add 3 to each side; $-\sqrt{32} = 4\sqrt{2}$
$x = 3 + 4\sqrt{2}$ or $x = 3 - 4\sqrt{2}$	Write as two equations.
$x \approx 8.7$ $\quad x \approx -2.7$	Use a calculator.

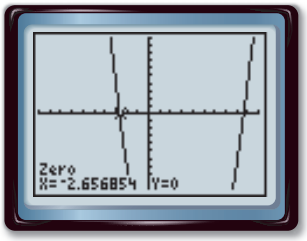
Study Tip

Plus or Minus

When using the Square Root Property, remember to put a $\pm$ sign before the radical.

The exact solutions of this equation are $3 - 4\sqrt{2}$ and $3 + 4\sqrt{2}$. The approximate solutions are -2.7 and 8.7. Check these results by finding and graphing the related quadratic function.

$x^2 - 6x + 9 = 32$	Original equation
$x^2 - 6x - 23 = 0$	Subtract 32 from each side.
$y = x^2 - 6x - 23$	Related quadratic function



CHECK Use the ZERO function of a graphing calculator. The approximate zeros of the related function are -2.7 and 8.7.

CHECK Your Progress

Solve each equation by using the Square Root Property.

2A. $x^2 + 8x + 16 = 20$ **2B.** $x^2 - 6x + 9 = 32$

Complete the Square The Square Root Property can only be used to solve quadratic equations when the quadratic expression is a perfect square. However, few quadratic expressions are perfect squares. To make a quadratic expression a perfect square, a method called **completing the square** may be used.

In a perfect square trinomial, there is a relationship between the coefficient of the linear term and the constant term. Consider the following pattern.

$$(x + 7)^2 = x^2 + 2(7)x + 7^2 \quad \text{Square of a sum pattern}$$

$$= x^2 + 14x + 49 \quad \text{Simplify.}$$

$$\left(\frac{14}{2}\right)^2 \rightarrow 7^2 \quad \text{Notice that 49 is } 7^2 \text{ and 7 is one half of 14.}$$

Use this pattern of coefficients to complete the square of a quadratic expression.

KEY CONCEPT *Completing the Square*

Words To complete the square for any quadratic expression of the form $x^2 + bx$, follow the steps below.

Step 1 Find one half of b, the coefficient of x.

Step 2 Square the result in Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$.

Symbols $x^2 + bx + \left(\frac{b}{2}\right)^2 = x + \left(\frac{b}{2}\right)^2$

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EXAMPLE Complete the Square

3 **Find the value of c that makes $x^2 + 12x + c$ a perfect square. Then write the trinomial as a perfect square.**

Step 1 Find one half of 12. $\frac{12}{2} = 6$

Step 2 Square the result of Step 1. $6^2 = 36$

Step 3 Add the result of Step 2 to $x^2 + 12x$. $x^2 + 12x + 36$

The trinomial $x^2 + 12x + 36$ can be written as $(x + 6)^2$.

CHECK Your Progress

3. Find the value of c that makes $x^2 - 14x + c$ a perfect square. Then write the trinomial as a perfect square.

Concepts in Motion
Animation
algebra2.com

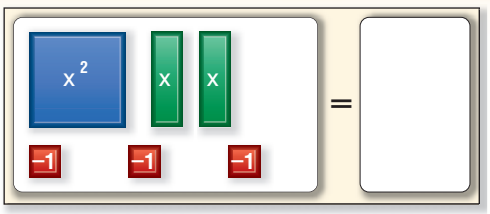
You can solve any quadratic equation by completing the square. Because you are solving an equation, add the value you use to complete the square to each side.

ALGEBRA LAB

Completing the Square

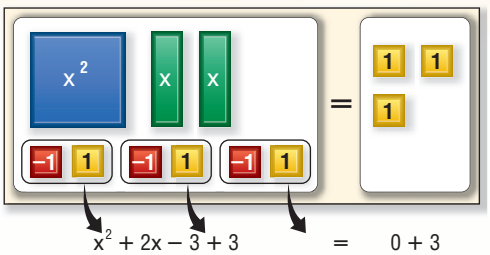
Use algebra tiles to complete the square for the equation $x^2 + 2x - 3 = 0$.

Step 1 Represent $x^2 + 2x - 3 = 0$ on an equation mat.



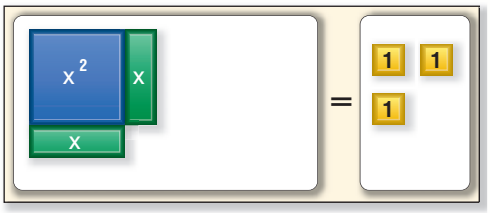
$x^2 + 2x - 3 = 0$

Step 2 Add 3 to each side of the mat. Remove the zero pairs.



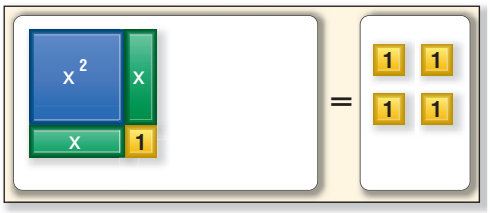
$x^2 + 2x - 3 + 3 = 0 + 3$

Step 3 Begin to arrange the x^2- and x-tiles into a square.



$x^2 + 2x = 3$

Step 4 To complete the square, add 1 yellow 1-tile to each side. The completed equation is $x^2 + 2x + 1 = 4$ or $(x + 1)^2 = 4$.



$x^2 + 2x + 1 = 3 + 1$

MODEL

Use algebra tiles to complete the square for each equation.

1. $x^2 + 2x - 4 = 0$

2. $x^2 + 4x + 1 = 0$

3. $x^2 - 6x = -5$

4. $x^2 - 2x = -1$

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EXAMPLE Solve an Equation by Completing the Square

4 **Solve $x^2 + 8x - 20 = 0$ by completing the square.**

$x^2 + 8x - 20 = 0$ Notice that $x^2 + 8x - 20$ is not a perfect square.

$x^2 + 8x = 20$ Rewrite so the left side is of the form $x^2 + bx$.

$x^2 + 8x + 16 = 20 + 16$ Since $\left(\frac{8}{2}\right)^2 = 16$, add 16 to each side.

$(x + 4)^2 = 36$ Write the left side as a perfect square by factoring.

$x + 4 = \pm 6$ Square Root Property

$x = -4 \pm 6$ Add -4 to each side.

$x = -4 + 6$ or $x = -4 - 6$ Write as two equations.

$x = 2$ $x = -10$ The solution set is $\{-10, 2\}$.

You can check this result by using factoring to solve the original equation.

Study Tip

Common Misconception

When solving equations by completing the square, don't forget to add $\left(\frac{b}{2}\right)^2$ to *each* side of the equation.

CHECK Your Progress

Solve each equation by completing the square.

4A. $x^2 - 10x + 24 = 0$ **4B.** $x^2 + 10x + 9 = 0$

When the coefficient of the quadratic term is not 1, you must divide the equation by that coefficient before completing the square.

EXAMPLE Equation with $a \neq 1$

5 **Solve $2x^2 - 5x + 3 = 0$ by completing the square.**

$2x^2 - 5x + 3 = 0$ Notice that $2x^2 - 5x + 3$ is not a perfect square.

$x^2 - \frac{5}{2}x + \frac{3}{2} = 0$ Divide by the coefficient of the quadratic term, 2.

$x^2 - \frac{5}{2}x = -\frac{3}{2}$ Subtract $\frac{3}{2}$ from each side.

$x^2 - \frac{5}{2}x + \frac{25}{16} = -\frac{3}{2} + \frac{25}{16}$ Since $\left(-\frac{5}{2} \div 2\right)^2 = \frac{25}{16}$, add $\frac{25}{16}$ to each side.

$\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$ Write the left side as a perfect square by factoring. Simplify the right side.

$x - \frac{5}{4} = \pm\frac{1}{4}$ Square Root Property

$x = \frac{5}{4} \pm \frac{1}{4}$ Add $\frac{5}{4}$ to each side.

$x = \frac{5}{4} + \frac{1}{4}$ or $x = \frac{5}{4} - \frac{1}{4}$ Write as two equations.

$x = \frac{3}{2}$ $x = 1$ The solution set is $\left\{1, \frac{3}{2}\right\}$.

Study Tip

Mental Math

Use mental math to find a number to add to each side to complete the square.

$\left(-\frac{5}{2} \div 2\right)^2 = \frac{25}{16}$

CHECK Your Progress

Solve each equation by completing the square.

5A. $3x^2 + 10x - 8 = 0$ **5B.** $3x^2 - 14x + 16 = 0$

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Not all solutions of quadratic equations are real numbers. In some cases, the solutions are complex numbers of the form $a + bi$, where $b \neq 0$.

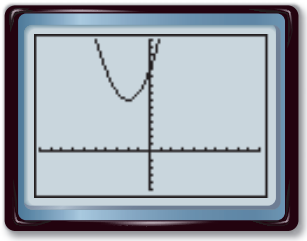
EXAMPLE Equation with Complex Solutions

6 **Solve $x^2 + 4x + 11 = 0$ by completing the square.**

$x^2 + 4x + 11 = 0$	Notice that $x^2 + 4x + 11$ is not a perfect square.
$x^2 + 4x = -11$	Rewrite so the left side is of the form $x^2 + bx$.
$x^2 + 4x + 4 = -11 + 4$	Since $\left(\frac{4}{2}\right)^2 = 4$, add 4 to each side.
$(x + 2)^2 = -7$	Write the left side as a perfect square by factoring.
$x + 2 = \pm\sqrt{-7}$	Square Root Property
$x + 2 = \pm i\sqrt{7}$	$\sqrt{-1} = i$
$x = -2 \pm i\sqrt{7}$	Subtract 2 from each side.

The solution set is $\{-2 + i\sqrt{7}, -2 - i\sqrt{7}\}$. Notice that these are imaginary solutions.

CHECK A graph of the related function shows that the equation has no real solutions since the graph has no x-intercepts. Imaginary solutions must be checked algebraically by substituting them in the original equation.



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CHECK Your Progress

Solve each equation by completing the square.

6A. $x^2 + 2x + 2 = 0$ **6B.** $x^2 - 6x + 25 = 0$

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CHECK Your Understanding

Examples 1 and 2 (pp. 268–269)

Solve each equation by using the Square Root Property.

1. $x^2 + 14x + 49 = 9$ **2.** $x^2 - 12x + 36 = 25$

3. $x^2 + 16x + 64 = 7$ **4.** $9x^2 - 24x + 16 = 2$

Example 2 (p. 269)

ASTRONOMY For Exercises 5–7, use the following information.
The height h of an object t seconds after it is dropped is given by $h = -\frac{1}{2}gt^2 + h_0$, where h_0 is the initial height and g is the acceleration due to gravity. The acceleration due to gravity near Earth's surface is 9.8 m/s^2, while on Jupiter it is 23.1 m/s^2. Suppose an object is dropped from an initial height of 100 meters from the surface of each planet.

5. On which planet should the object reach the ground first?

6. Find the time it takes for the object to reach the ground on each planet to the nearest tenth of a second.

7. Do the times to reach the ground seem reasonable? Explain.

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Example 3 (p. 270)

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

8. $x^2 - 12x + c$ **9.** $x^2 - 3x + c$

Examples 4–6 (pp. 271–272)

Solve each equation by completing the square.

10. $x^2 + 3x - 18 = 0$ **11.** $x^2 - 8x + 11 = 0$

12. $2x^2 - 3x - 3 = 0$ **13.** $3x^2 + 12x - 18 = 0$

14. $x^2 + 2x + 6 = 0$ **15.** $x^2 - 6x + 12 = 0$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–19, 40, 41	1
20–23	2
24–27	3
28–31	4
32–35	5
36–39	6

Solve each equation by using the Square Root Property.

16. $x^2 + 4x + 4 = 25$ **17.** $x^2 - 10x + 25 = 49$

18. $x^2 - 9x + \frac{81}{4} = \frac{1}{4}$ **19.** $x^2 + 7x + \frac{49}{4} = 4$

20. $x^2 + 8x + 16 = 7$ **21.** $x^2 - 6x + 9 = 8$

22. $x^2 + 12x + 36 = 5$ **23.** $x^2 - 3x + \frac{9}{4} = 6$

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

24. $x^2 + 16x + c$ **25.** $x^2 - 18x + c$

26. $x^2 - 15x + c$ **27.** $x^2 + 7x + c$

Solve each equation by completing the square.

28. $x^2 - 8x + 15 = 0$ **29.** $x^2 + 2x - 120 = 0$ **30.** $x^2 + 2x - 6 = 0$

31. $x^2 - 4x + 1 = 0$ **32.** $2x^2 + 3x - 5 = 0$ **33.** $2x^2 - 3x + 1 = 0$

34. $2x^2 + 7x + 6 = 0$ **35.** $9x^2 - 6x - 4 = 0$ **36.** $x^2 - 4x + 5 = 0$

37. $x^2 + 6x + 13 = 0$ **38.** $x^2 - 10x + 28 = 0$ **39.** $x^2 + 8x + 9 = -9$

40. MOVIE SCREENS The area A in square feet of a projected picture on a movie screen is given by $A = 0.16d^2$, where d is the distance from the projector to the screen in feet. At what distance will the projected picture have an area of 100 square feet?

41. FRAMING A picture has a square frame that is 2 inches wide. The area of the picture is one third of the total area of the picture and frame. What are the dimensions of the picture to the nearest quarter of an inch?

Solve each equation by using the Square Root Property.

42. $x^2 + x + \frac{1}{4} = \frac{9}{16}$ **43.** $x^2 + 1.4x + 0.49 = 0.81$

44. $4x^2 - 28x + 49 = 5$ **45.** $9x^2 + 30x + 25 = 11$

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

46. $x^2 + 0.6x + c$ **47.** $x^2 - 2.4x + c$

48. $x^2 - \frac{8}{3}x + c$ **49.** $x^2 + \frac{5}{2}x + c$

Solve each equation by completing the square.

50. $x^2 + 1.4x = 1.2$ **51.** $x^2 - 4.7x = -2.8$

52. $x^2 - \frac{2}{3}x - \frac{26}{9} = 0$ **53.** $x^2 - \frac{3}{2}x - \frac{23}{16} = 0$

54. $3x^2 - 4x = 2$ **55.** $2x^2 - 7x = -12$

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Real-World Link

Reverse ballistic testing—accelerating a target on a sled to impact a stationary test item at the end of the track—was pioneered at the Sandia National Laboratories' Rocket Sled Track Facility in Albuquerque, New Mexico. This facility provides a 10,000-foot track for testing items at very high speeds.

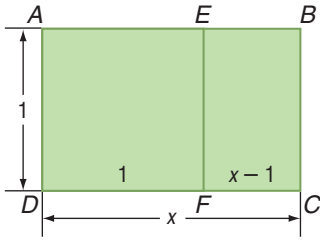
Source: sandia.gov

56. ENGINEERING In an engineering test, a rocket sled is propelled into a target. The sled's distance d in meters from the target is given by the formula $d = -1.5t^2 + 120$, where t is the number of seconds after rocket ignition. How many seconds have passed since rocket ignition when the sled is 10 meters from the target?

GOLDEN RECTANGLE For Exercises 57–59, use the following information.
A *golden rectangle* is one that can be divided into a square and a second rectangle that is geometrically similar to the original rectangle. The ratio of the length of the longer side to the shorter side of a golden rectangle is called the *golden ratio*.

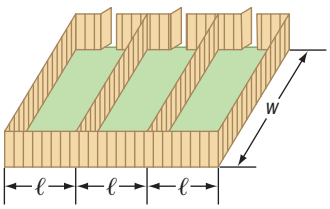
57. Find the ratio of the length of the longer side to the length of the shorter side for rectangle $ABCD$ and for rectangle $EBCF$.

58. Find the exact value of the golden ratio by setting the two ratios in Exercise 57 equal and solving for x. (*Hint:* The golden ratio is a positive value.)



59. RESEARCH Use the Internet or other reference to find examples of the golden rectangle in architecture. What applications does the golden ratio have in music?

60. KENNEL A kennel owner has 164 feet of fencing with which to enclose a rectangular region. He wants to subdivide this region into three smaller rectangles of equal length, as shown. If the total area to be enclosed is 576 square feet, find the dimensions of the enclosed region. (*Hint:* Write an expression for ℓ in terms of w.)



EXTRA PRACTICE
See pages 901, 930.

MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

61. OPEN ENDED Write a perfect square trinomial equation in which the linear coefficient is negative and the constant term is a fraction. Then solve the equation.

62. FIND THE ERROR Rashid and Tia are solving $2x^2 - 8x + 10 = 0$ by completing the square. Who is correct? Explain your reasoning.

Rashid

$2x^2 - 8x + 10 = 0$
$2x^2 - 8x = -10$
$2x^2 - 8x + 16 = -10 + 16$
$(x - 4)^2 = 6$
$x - 4 = \pm\sqrt{6}$
$x = 4 \pm \sqrt{6}$

Tia

$2x^2 - 8x + 10 = 0$
$x^2 - 4x = 0 - 5$
$x^2 - 4x + 4 = -5 + 4$
$(x - 2)^2 = -1$
$x - 2 = \pm i$
$x = 2 \pm i$

63. REASONING Determine whether the value of c that makes $ax^2 + bx + c$ a perfect square trinomial is *sometimes*, *always*, or *never* negative. Explain your reasoning.

CORBIS
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64. CHALLENGE Find all values of n such that $x^2 + bx + \left(\frac{b}{2}\right)^2 = n$ has

a. one real root. **b.** two real roots. **c.** two imaginary roots.

65. ***Writing in Math*** Use the information on page 268 to explain how you can find the time it takes an accelerating car to reach the finish line. Include an explanation of why $t^2 + 22t + 121 = 246$ cannot be solved by factoring and a description of the steps you would take to solve the equation.

STANDARDIZED TEST PRACTICE

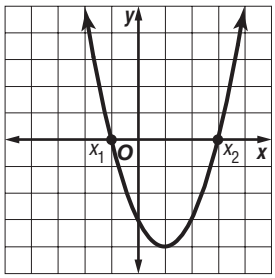
66. ACT/SAT The two zeros of a quadratic function are labeled x_1 and x_2 on the graph. Which expression has the greatest value?

A $2x_1$

B x_2

C $x_2 - x_1$

D $x_2 + x_1$



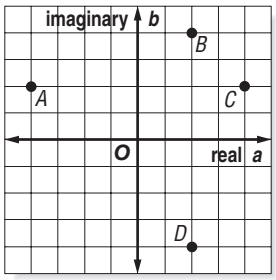
67. REVIEW If $i = \sqrt{-1}$ which point shows the location of $2 - 4i$ on the plane?

F point A

G point B

H point C

J point D



Spiral Review

Simplify. (Lesson 5-4)

68. i^{14}

69. $(4 - 3i) - (5 - 6i)$

70. $(7 + 2i)(1 - i)$

Solve each equation by factoring. (Lesson 5-3)

71. $4x^2 + 8x = 0$

72. $x^2 - 5x = 14$

73. $3x^2 + 10 = 17x$

Solve each system of equations by using inverse matrices. (Lesson 4-8)

74. $5x + 3y = -5$
$7x + 5y = -11$

75. $6x + 5y = 8$
$3x - y = 7$

CHEMISTRY **For Exercises 76 and 77, use the following information.**
For hydrogen to be a liquid, its temperature must be within 2°C of −257°C. (Lesson 1-4)

76. Write an equation to determine the least and greatest temperatures for this substance.

77. Solve the equation.

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $b^2 - 4ac$ for the given values of a, b, and c.** (Lesson 1-1)

78. $a = 1, b = 7, c = 3$

79. $a = 1, b = 2, c = 5$

80. $a = 2, b = -9, c = -5$

81. $a = 4, b = -12, c = 9$

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5-6 The Quadratic Formula and the Discriminant

Main Ideas

- Solve quadratic equations by using the Quadratic Formula.
- Use the discriminant to determine the number and type of roots of a quadratic equation.

New Vocabulary

Quadratic Formula

discriminant

GET READY for the Lesson

Competitors in the 10-meter platform diving competition jump upward and outward before diving into the pool below. The height h of a diver in meters above the pool after t seconds can be approximated by the equation $h = -4.9t^2 + 3t + 10$.

Dimitri Iundt/TempSport/CORBIS

Quadratic Formula You have seen that exact solutions to some quadratic equations can be found by graphing, by factoring, or by using the Square Root Property. While completing the square can be used to solve any quadratic equation, the process can be tedious if the equation contains fractions or decimals. Fortunately, a formula exists that can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$. This formula can be derived by solving the general form of a quadratic equation.

$ax^2 + bx + c = 0$	General quadratic equation
$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	Divide each side by a.
$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Subtract $\frac{c}{a}$ from each side.
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$	Complete the square.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Factor the left side. Simplify the right side.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Square Root Property
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from each side.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Simplify.

This equation is known as the **Quadratic Formula**.

Reading Math

Quadratic Formula The Quadratic Formula is read *x equals the opposite of b, plus or minus the square root of b squared minus 4ac, all divided by 2a.*

KEY CONCEPT *Quadratic Formula*

The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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EXAMPLE Two Rational Roots

1 **Solve $x^2 - 12x = 28$ by using the Quadratic Formula.**

First, write the equation in the form $ax^2 + bx + c = 0$ and identify a, b, and c.

$$\begin{array}{rcl} & a x^2 + & b x + \quad c = 0 \\ & \downarrow \quad \downarrow & \downarrow \\ x^2 - 12x = 28 \quad \rightarrow & 1x^2 - 12x - 28 = 0 & \end{array}$$

Then, substitute these values into the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(-28)}}{2(1)} \qquad \text{Replace } a \text{ with 1, } b \text{ with } -12, \text{ and } c \text{ with } -28.$$

$$= \frac{12 \pm \sqrt{144 + 112}}{2} \qquad \text{Simplify.}$$

$$= \frac{12 \pm \sqrt{256}}{2} \qquad \text{Simplify.}$$

$$= \frac{12 \pm 16}{2} \qquad \sqrt{256} = 16$$

$$x = \frac{12 + 16}{2} \text{ or } x = \frac{12 - 16}{2} \qquad \text{Write as two equations.}$$

$$= 14 \qquad = -2 \qquad \text{Simplify.}$$

The solutions are -2 and 14. Check by substituting each of these values into the original equation.

Study Tip

Quadratic Formula

Although factoring may be an easier method to solve the equations in Examples 1 and 2, the Quadratic Formula can be used to solve any quadratic equation.

CHECK Your Progress

Solve each equation by using the Quadratic Formula.

1A. $x^2 + 6x = 16$ **1B.** $2x^2 + 25x + 33 = 0$

When the value of the radicand in the Quadratic Formula is 0, the quadratic equation has exactly one rational root.

EXAMPLE One Rational Root

2 **Solve $x^2 + 22x + 121 = 0$ by using the Quadratic Formula.**

Identify a, b, and c. Then, substitute these values into the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

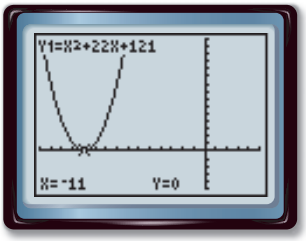
$$= \frac{-22 \pm \sqrt{(22)^2 - 4(1)(121)}}{2(1)} \qquad \text{Replace } a \text{ with 1, } b \text{ with 22, and } c \text{ with 121.}$$

$$= \frac{-22 \pm \sqrt{0}}{2} \qquad \text{Simplify.}$$

$$= \frac{-22}{2} \text{ or } -11 \qquad \sqrt{0} = 0$$

The solution is -11.

CHECK A graph of the related function shows that there is one solution at $x = -11$.



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Study Tip

Constants

The constants a, b, and c are not limited to being integers. They can be irrational or complex.

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CHECK Your Progress

Solve each equation by using the Quadratic Formula.

2A. $x^2 - 16x + 64 = 0$ **2B.** $x^2 + 34x + 289 = 0$

You can express irrational roots exactly by writing them in radical form.

EXAMPLE Irrational Roots

3 **Solve $2x^2 + 4x - 5 = 0$ by using the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

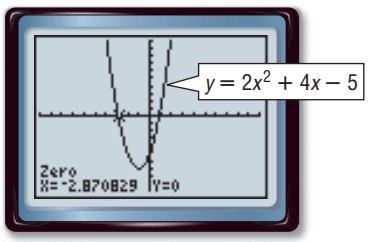
$$= \frac{-4 \pm \sqrt{(4)^2 - 4(2)(-5)}}{2(2)}$$ Replace a with 2, b with 4, and c with -5.

$$= \frac{-4 \pm \sqrt{56}}{4}$$ Simplify.

$$= \frac{-4 \pm 2\sqrt{14}}{4} \text{ or } \frac{-2 \pm \sqrt{14}}{2}$$ $\sqrt{56} = \sqrt{4 \cdot 14}$ or $2\sqrt{14}$

The approximate solutions are -2.9 and 0.9.

CHECK Check these results by graphing the related quadratic function, $y = 2x^2 + 4x - 5$. Using the ZERO function of a graphing calculator, the approximate zeros of the related function are -2.9 and 0.9.



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CHECK Your Progress

Solve each equation by using the Quadratic Formula.

3A. $3x^2 + 5x + 1 = 0$ **3B.** $x^2 - 8x + 9 = 0$

When using the Quadratic Formula, if the radical contains a negative value, the solutions will be complex. Complex solutions of quadratic equations with real coefficients always appear in conjugate pairs.

EXAMPLE Complex Roots

4 **Solve $x^2 - 4x = -13$ by using the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$ Replace a with 1, b with -4, and c with 13.

$$= \frac{4 \pm \sqrt{-36}}{2}$$ Simplify.

$$= \frac{4 \pm 6i}{2}$$ $\sqrt{-36} = \sqrt{36(-1)}$ or $6i$

$$= 2 \pm 3i$$ Simplify.

The solutions are the complex numbers $2 + 3i$ and $2 - 3i$.

Study Tip

Using the Quadratic Formula

Remember that to correctly identify a, b, and c for use in the Quadratic Formula, the equation must be written in the form $ax^2 + bx + c = 0$.

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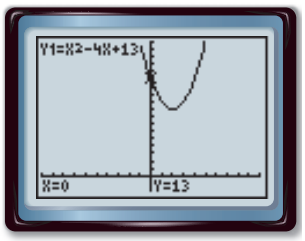
A graph of the related function shows that the solutions are complex, but it cannot help you find them.



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CHECK The check for $2 + 3i$ is shown below.

$x^2 - 4x = -13$ Original equation

$(2 + 3i)^2 - 4(2 + 3i) \stackrel{?}{=} -13$ $x = 2 + 3i$

$4 + 12i + 9i^2 - 8 - 12i \stackrel{?}{=} -13$ Square of a sum; Distributive Property

$-4 + 9i^2 \stackrel{?}{=} -13$ Simplify.

$-4 - 9 = -13$ ✓ $i^2 = -1$

CHECK Your Progress

Solve each equation by using the Quadratic Formula.

4A. $3x^2 + 5x + 4 = 0$ **4B.** $x^2 - 6x + 10 = 0$

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Reading Math

Roots Remember that the solutions of an equation are called *roots*.

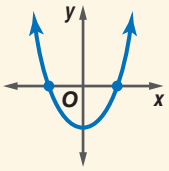
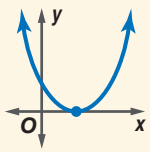
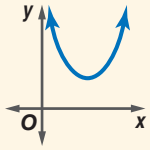
Roots and the Discriminant In Examples 1, 2, 3, and 4, observe the relationship between the value of the expression under the radical and the roots of the quadratic equation. The expression $b^2 - 4ac$ is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ ← discriminant

The value of the discriminant can be used to determine the number and type of roots of a quadratic equation. The following table summarizes the possible types of roots.

KEY CONCEPT *Discriminant*

Consider $ax^2 + bx + c = 0$, where a, b, and c are rational numbers.

Value of Discriminant	Type and Number of Roots	Example of Graph of Related Function
$b^2 - 4ac > 0$; $b^2 - 4ac$ is a perfect square.	2 real, rational roots	
$b^2 - 4ac > 0$; $b^2 - 4ac$ is *not* a perfect square.	2 real, irrational roots	
$b^2 - 4ac = 0$	1 real, rational root	
$b^2 - 4ac < 0$	2 complex roots	

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The discriminant can help you check the solutions of a quadratic equation. Your solutions must match in number and in type to those determined by the discriminant.

EXAMPLE **Describe Roots**

5 **Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation.**

a. $9x^2 - 12x + 4 = 0$

$a = 9, b = -12, c = 4$ Substitution

$b^2 - 4ac = (-12)^2 - 4(9)(4)$ Simplify.

$= 144 - 144$ Subtract.

$= 0$

The discriminant is 0, so there is one rational root.

b. $2x^2 - 16x + 33 = 0$

$a = 2, b = 16, c = 33$ Substitution

$b^2 - 4ac = (16)^2 - 4(2)(33)$ Simplify.

$= 256 - 264$ Subtract.

$= -8$

The discriminant is negative, so there are two complex roots.

CHECK Your Progress

5A. $-5x^2 + 8x - 1 = 0$ **5B.** $-7x + 15x^2 - 4 = 0$

You have studied a variety of methods for solving quadratic equations. The table below summarizes these methods.

Study Tip

Study Notebook

You may wish to copy this list of methods to your math notebook or Foldable to keep as a reference as you study.

CONCEPT SUMMARY *Solving Quadratic Equations*

Method	Can be Used	When to Use
Graphing	sometimes	Use only if an exact answer is not required. Best used to check the reasonableness of solutions found algebraically.
Factoring	sometimes	Use if the constant term is 0 or if the factors are easily determined. **Example** $x^2 - 3x = 0$
Square Root Property	sometimes	Use for equations in which a perfect square is equal to a constant. **Example** $(x + 13)^2 = 9$
Completing the Square	always	Useful for equations of the form $x^2 + bx + c = 0$, where b is even. **Example** $x^2 + 14x - 9 = 0$
Quadratic Formula	always	Useful when other methods fail or are too tedious. **Example** $3.4x^2 - 2.5x + 7.9 = 0$

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CHECK Your Understanding

Examples 1–4 (pp. 277–279)

Find the exact solutions by using the Quadratic Formula.

1. $8x^2 + 18x - 5 = 0$
2. $x^2 + 8x = 0$
3. $4x^2 + 4x + 1 = 0$
4. $x^2 + 6x + 9 = 0$
5. $2x^2 - 4x + 1 = 0$
6. $x^2 - 2x - 2 = 0$
7. $x^2 + 3x + 8 = 5$
8. $4x^2 + 20x + 25 = -2$

Examples 3 and 4 (pp. 278–279)

PHYSICS **For Exercises 9 and 10, use the following information.**
The height $h(t)$ in feet of an object t seconds after it is propelled straight up from the ground with an initial velocity of 85 feet per second is modeled by the equation $h(t) = -16t^2 + 85t$.

9. When will the object be at a height of 50 feet?
10. Will the object ever reach a height of 120 feet? Explain your reasoning.

Example 5 (p. 280)

Complete parts a and b for each quadratic equation.

a. Find the value of the discriminant.
b. Describe the number and type of roots. Do your answers for Exercises 1, 3, 5, and 7 fit these descriptions, respectively?

11. $8x^2 + 18x - 5 = 0$
12. $4x^2 + 4x + 1 = 0$
13. $2x^2 - 4x + 1 = 0$
14. $x^2 + 3x + 8 = 5$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
15, 16	1, 5
17, 18	2, 5
19–22	3, 5
23, 24	4, 5
25–33	1–4

Complete parts a–c for each quadratic equation.

a. Find the value of the discriminant.
b. Describe the number and type of roots.
c. Find the exact solutions by using the Quadratic Formula.

15. $-12x^2 + 5x + 2 = 0$
16. $-3x^2 - 5x + 2 = 0$
17. $9x^2 - 6x - 4 = -5$
18. $25 + 4x^2 = -20x$
19. $x^2 + 3x - 3 = 0$
20. $x^2 - 16x + 4 = 0$
21. $x^2 + 4x + 3 = 4$
22. $2x - 5 = -x^2$
23. $x^2 - 2x + 5 = 0$
24. $x^2 - x + 6 = 0$

Solve each equation by using the method of your choice. Find exact solutions.

25. $x^2 - 30x - 64 = 0$
26. $7x^2 + 3 = 0$
27. $x^2 - 4x + 7 = 0$
28. $2x^2 + 6x - 3 = 0$
29. $4x^2 - 8 = 0$
30. $4x^2 + 81 = 36x$

FOOTBALL **For Exercises 31 and 32, use the following information.**
The average NFL salary $A(t)$ (in thousands of dollars) can be estimated using $A(t) = 2.3t^2 - 12.4t + 73.7$, where t is the number of years since 1975.

31. Determine a domain and range for which this function makes sense.
32. According to this model, in what year did the average salary first exceed one million dollars?
33. **HIGHWAY SAFETY** Highway safety engineers can use the formula $d = 0.05s^2 + 1.1s$ to estimate the minimum stopping distance d in feet for a vehicle traveling s miles per hour. The speed limit on Texas highways is 70 mph. If a car is able to stop after 300 feet, was the car traveling faster than the Texas speed limit? Explain your reasoning.

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Complete parts a–c for each quadratic equation.
a. Find the value of the discriminant.
b. Describe the number and type of roots.
c. Find the exact solutions by using the Quadratic Formula.

34. $x^2 + 6x = 0$ **35.** $4x^2 + 7 = 9x$ **36.** $3x + 6 = -6x^2$

37. $\frac{3}{4}x^2 - \frac{1}{3}x - 1 = 0$ **38.** $0.4x^2 + x - 0.3 = 0$ **39.** $0.2x^2 + 0.1x + 0.7 = 0$

Solve each equation by using the method of your choice. Find exact solutions.

40. $-4(x + 3)^2 = 28$ **41.** $3x^2 - 10x = 7$ **42.** $x^2 + 9 = 8x$

43. $10x^2 + 3x = 0$ **44.** $2x^2 - 12x + 7 = 5$ **45.** $21 = (x - 2)^2 + 5$

Real-World Link
The Golden Gate, located in San Francisco, California, is the tallest bridge in the world, with its towers extending 746 feet above the water and the floor of the bridge extending 220 feet above the water.
Source: www.goldengatebridge.org

BRIDGES **For Exercises 46 and 47, use the following information.**
The supporting cables of the Golden Gate Bridge approximate the shape of a parabola. The parabola can be modeled by $y = 0.00012x^2 + 6$, where x represents the distance from the axis of symmetry and y represents the height of the cables. The related quadratic equation is $0.00012x^2 + 6 = 0$.

46. Calculate the value of the discriminant.

47. What does the discriminant tell you about the supporting cables of the Golden Gate Bridge?

48. ENGINEERING Civil engineers are designing a section of road that is going to dip below sea level. The road's curve can be modeled by the equation $y = 0.00005x^2 - 0.06x$, where x is the horizontal distance in feet between the points where the road is at sea level and y is the elevation (a positive value being above sea level and a negative being below). The engineers want to put stop signs at the locations where the elevation of the road is equal to sea level. At what horizontal distances will they place the stop signs?

H.O.T. Problems

EXTRA PRACTICE
See pages 901, 930.

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49. OPEN ENDED Graph a quadratic equation that has a
a. positive discriminant. **b.** negative discriminant. **c.** zero discriminant.

50. REASONING Explain why the roots of a quadratic equation are complex if the value of the discriminant is less than 0.

51. CHALLENGE Find the exact solutions of $2ix^2 - 3ix - 5i = 0$ by using the Quadratic Formula.

52. REASONING Given the equation $x^2 + 3x - 4 = 0$,
a. Find the exact solutions by using the Quadratic Formula.
b. Graph $f(x) = x^2 + 3x - 4$.
c. Explain how solving with the Quadratic Formula can help graph a quadratic function.

53. *Writing in Math* Use the information on page 276 to explain how a diver's height above the pool is related to time. Explain how you could determine how long it will take the diver to hit the water after jumping from the platform.

Bruce Hands/Getty Images
www.MathSchoolinternational.com

STANDARDIZED TEST PRACTICE

54. ACT/SAT If $2x^2 - 5x - 9 = 0$, then x could be approximately equal to which of the following?

A -1.12

B 1.54

C 2.63

D 3.71

55. REVIEW What are the x-intercepts of the graph of $y = -2x^2 - 5x + 12$?

F $-\frac{3}{2}, 4$

G $-4, \frac{3}{2}$

H $-2, \frac{1}{2}$

J $-\frac{1}{2}, 2$

Spiral Review

Solve each equation by using the Square Root Property. (Lesson 5-5)

56. $x^2 + 18x + 81 = 25$

57. $x^2 - 8x + 16 = 7$

58. $4x^2 - 4x + 1 = 8$

Simplify. (Lesson 5-4)

59. $\frac{2i}{3 + i}$

60. $\frac{4}{5 - i}$

61. $\frac{1 + i}{3 - 2i}$

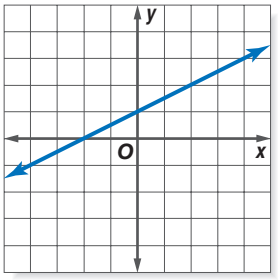
Solve each system of inequalities. (Lesson 3-3)

62. $x + y \leq 9$
$x - y \leq 3$
$y - x \geq 4$

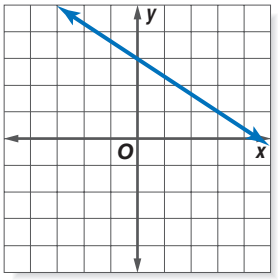
63. $x \geq 1$
$y \leq -1$
$y \leq x$

Write the slope-intercept form of the equation of the line with each graph shown. (Lesson 2-4)

64.



65.



66. PHOTOGRAPHY Desiree works in a photography studio and makes a commission of $8 per photo package she sells. On Tuesday, she sold 3 more packages than she sold on Monday. For the two days, Victoria earned $264. How many photo packages did she sell on these two days? (Lesson 1-3)

GET READY for the Next Lesson

PREREQUISITE SKILL **State whether each trinomial is a perfect square. If so, factor it.** (Lesson 5-3.)

67. $x^2 - 5x - 10$

68. $x^2 - 14x + 49$

69. $4x^2 + 12x + 9$

70. $25x^2 + 20x + 4$

71. $9x^2 - 12x + 16$

72. $36x^2 - 60x + 25$

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EXPLORE 5-7

Graphing Calculator Lab

The Family of Parabolas

The general form of a quadratic function is $y = a(x - h)^2 + k$. Changing the values of a, h, and k results in a different parabola in the family of quadratic functions. The parent graph of the family of parabolas is the graph of $y = x^2$.

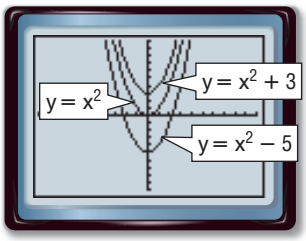
You can use a TI-83/84 Plus graphing calculator to analyze the effects that result from changing each of the parameters a, h, and k.

ACTIVITY 1

Graph the set of equations on the same screen in the standard viewing window.

$y = x^2$, $y = x^2 + 3$, $y = x^2 - 5$

Describe any similarities and differences among the graphs.



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Activity 1 shows how changing the value of k in the equation $y = a(x - h)^2 + k$ *translates* the parabola along the y-axis. If $k > 0$, the parabola is translated k units up, and if $k < 0$, it is translated k units down.

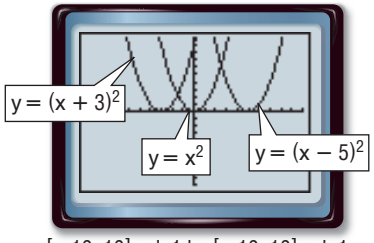
How do you think changing the value of h will affect the graph of $y = (x - h)^2$ as compared to the graph of $y = x^2$?

ACTIVITY 2

Graph the set of equations on the same screen in the standard viewing window.

$y = x^2$, $y = (x + 3)^2$, $y = (x - 5)^2$

Describe any similarities and differences among the graphs.



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Activity 2 shows how changing the value of h in the equation $y = a(x - h)^2 + k$ *translates* the graph horizontally. If $h > 0$, the graph translates to the right h units. If $h < 0$, the graph translates to the left $|h|$ units.

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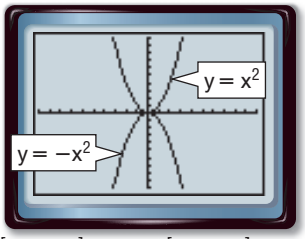
ACTIVITY 3

Graph each set of equations on the same screen. Describe any similarities and differences among the graphs.

a. $y = x^2, y = -x^2$

The graphs have the same vertex and the same shape. However, the graph of $y = x^2$ opens up and the graph of $y = -x^2$ opens down.



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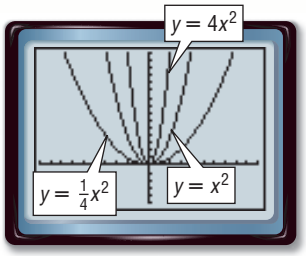
b. $y = x^2, y = 4x^2, y = \frac{1}{4}x^2$

The graphs have the same vertex, (0, 0), but each has a different shape. The graph of $y = 4x^2$ is narrower than the graph of $y = x^2$. The graph of $y = \frac{1}{4}x^2$ is wider than the graph of $y = x^2$.



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Changing the value of a in the equation $y = a(x - h)^2 + k$ can affect the direction of the opening and the shape of the graph. If $a > 0$, the graph opens up, and if $a < 0$, the graph opens down or is *reflected* over the x-axis. If $|a| > 1$, the graph is narrower than the graph of $y = x^2$. If $|a| < 1$, the graph is wider than the graph of $y = x^2$. Thus, a change in the absolute value of a results in a *dilation* of the graph of $y = x^2$.

Analyze the Results

Consider $y = a(x - h)^2 + k$, where $a \neq 0$.

1. How does changing the value of h affect the graph? Give an example.
2. How does changing the value of k affect the graph? Give an example.
3. How does using $-a$ instead of a affect the graph? Give an example.

Examine each pair of equations and predict the similarities and differences in their graphs. Use a graphing calculator to confirm your predictions. Write a sentence or two comparing the two graphs.

4. $y = x^2, y = x^2 + 2.5$
5. $y = -x^2, y = x^2 - 9$
6. $y = x^2, y = 3x^2$
7. $y = x^2, y = -6x^2$
8. $y = x^2, y = (x + 3)^2$
9. $y = -\frac{1}{3}x^2, y = -\frac{1}{3}x^2 + 2$
10. $y = x^2, y = (x - 7)^2$
11. $y = x^2, y = 3(x + 4)^2 - 7$
12. $y = x^2, y = -\frac{1}{4}x^2 + 1$
13. $y = (x + 3)^2 - 2, y = (x + 3)^2 + 5$
14. $y = 3(x + 2)^2 - 1,$ $y = 6(x + 2)^2 - 1$
15. $y = 4(x - 2)^2 - 3,$ $y = \frac{1}{4}(x - 2)^2 - 1$

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5-7 Analyzing Graphs of Quadratic Functions

Main Ideas

- Analyze quadratic functions of the form $y = a(x - h)^2 + k$.
- Write a quadratic function in the form $y = a(x - h)^2 + k$.

New Vocabulary

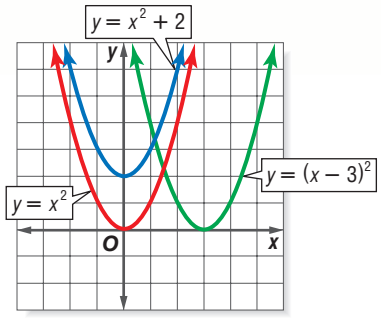
vertex form

GET READY for the Lesson

Concepts in Motion
Interactive Lab algebra2.com

A *family of graphs* is a group of graphs that displays one or more similar characteristics. The graph of $y = x^2$ is called the *parent graph* of the family of quadratic functions.

The graphs of other quadratic functions such as $y = x^2 + 2$ and $y = (x - 3)^2$ can be found by transforming the graph of $y = x^2$.



Analyze Quadratic Functions Each function above can be written in the form $y = (x - h)^2 + k$, where (h, k) is the vertex of the parabola and $x = h$ is its axis of symmetry. This is often referred to as the **vertex form** of a quadratic function.

Equation		Axis of
$y = x^2$ or $y = (x - 0)^2 + 0$	(0, 0)	$x = 0$
$y = x^2 + 2$ or $y = (x - 0)^2 + 2$	(0, 2)	$x = 0$
$y = (x - 3)^2$ or $y = (x - 3)^2 + 0$	(3, 0)	$x = 3$

Recall that a *translation* slides a figure without changing its shape or size. As the values of h and k change, the graph of $y = a(x - h)^2 + k$ is the graph of $y = x^2$ translated:

- $|h|$ units *left* if h is negative or $|h|$ units *right* if h is positive, and
- $|k|$ units *up* if k is positive or $|k|$ units *down* if k is negative.

EXAMPLE 1 Graph a Quadratic Equation in Vertex Form

Analyze $y = (x + 2)^2 + 1$. Then draw its graph.

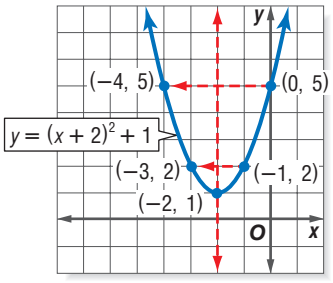
This function can be rewritten as $y = [x - (-2)]^2 + 1$. Then $h = -2$ and $k = 1$. The vertex is at (h, k) or $(-2, 1)$, and the axis of symmetry is $x = -2$. The graph is the graph of $y = x^2$ translated 2 units left and 1 unit up.

Now use this information to draw the graph.

Step 1 Plot the vertex, $(-2, 1)$.

Step 2 Draw the axis of symmetry, $x = -2$.

Step 3 Use symmetry to complete the graph.



CHECK Your Progress

1. Analyze $y = (x - 3)^2 - 2$. Then draw its graph.

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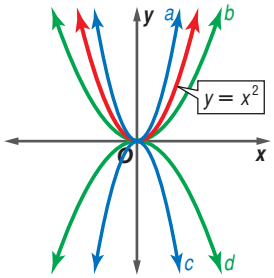
How does the value of a in the general form $y = a(x - h)^2 + k$ affect a parabola? Compare the graphs of the following functions to the parent function, $y = x^2$.



a. $y = 2x^2$ **b.** $y = \frac{1}{2}x^2$

c. $y = -2x^2$ **d.** $y = -\frac{1}{2}x^2$

All of the graphs have the vertex (0, 0) and axis of symmetry $x = 0$.

Notice that the graphs of $y = 2x^2$ and $y = \frac{1}{2}x^2$ are *dilations* of the graph of $y = x^2$. The graph of $y = 2x^2$ is narrower than the graph of $y = x^2$, while the graph of $y = \frac{1}{2}x^2$ is wider. The graphs of $y = -2x^2$ and $y = 2x^2$ are *reflections* of each other over the x-axis, as are the graphs of $y = -\frac{1}{2}x^2$ and $y = \frac{1}{2}x^2$.

Changing the value of a in the equation $y = a(x - h)^2 + k$ can affect the direction of the opening and the shape of the graph.

- If $a > 0$, the graph opens up.
- If $a < 0$, the graph opens down.
- If $|a| > 1$, the graph is narrower than the graph of $y = x^2$.
- If $0 < |a| < 1$, the graph is wider than the graph of $y = x^2$.

Study Tip

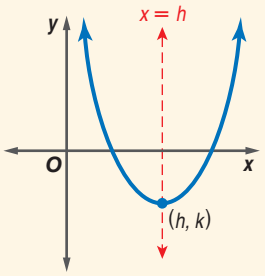
$0 < |a| < 1$ means that a is a real number between 0 and 1, such as $\frac{2}{5}$, or a real number between -1 and 0, such as $-\frac{\sqrt{2}}{2}$.

CONCEPT SUMMARY *Quadratic Functions in Vertex Form*

The vertex form of a quadratic function is $y = a(x - h)^2 + k$.

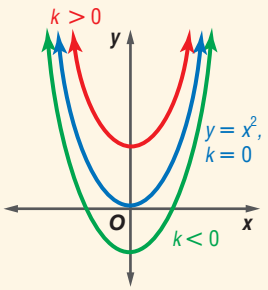
h* and *k

Vertex and Axis of Symmetry



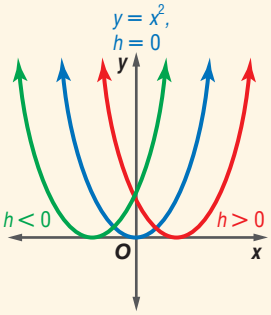
k

Vertical Translation



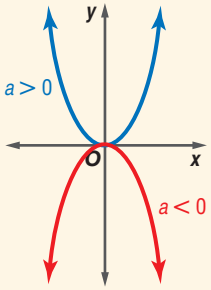
h

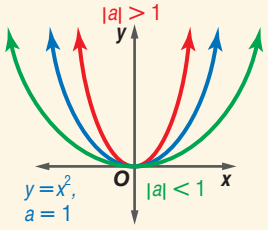
Horizontal Translation



a

Direction of Opening and Shape of Parabola





Concepts in Motion
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STANDARDIZED TEST EXAMPLE Vertex Form Parameters

2 Which function has the widest graph?

A $y = -2.5x^2$ **B** $y = -0.3x^2$ **C** $y = 2.5x^2$ **D** $y = 5x^2$

Read the Test Item

You are given four answer choices, each of which is in vertex form.

Solve the Test Item

The value of a determines the width of the graph. Since $|-2.5| = |2.5| > 1$ and $|5| > 1$, choices A, C, and D produce graphs that are narrower than $y = x^2$. Since $|-0.3| < 1$, choice B produces a graph that is wider than $y = x^2$. The answer is B.

Test-Taking Tip

The sign of a in the vertex form does not determine how wide the parabola will be. The sign determines whether the parabola opens up or down. The width is determined by the absolute value of a.

CHECK Your Progress

2. Which function has the narrowest graph?

F $y = -0.1x^2$ **G** $y = x^2$ **H** $y = 0.5x^2$ **J** $y = 2.3x^2$

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Write Quadratic Equations in Vertex Form Given a function of the form $y = ax^2 + bx + c$, you can complete the square to write the function in vertex form. If the coefficient of the quadratic term is not 1, the first step is to factor that coefficient from the quadratic and linear terms.

EXAMPLE Write Equations in Vertex Form

3 **Write each equation in vertex form. Then analyze the function.**

a. $y = x^2 + 8x - 5$

$y = x^2 + 8x - 5$	Notice that $x^2 + 8x - 5$ is not a perfect square.
$y = (x^2 + 8x + 16) - 5 - 16$	Complete the square by adding $\left(\frac{8}{2}\right)^2$ or 16. Balance this addition by subtracting 16.
$y = (x + 4)^2 - 21$	Write $x^2 + 8x + 16$ as a perfect square.

Since $h = -4$ and $k = -21$, the vertex is at $(-4, -21)$ and the axis of symmetry is $x = -4$. Since $a = 1$, the graph opens up and has the same shape as the graph of $y = x^2$, but it is translated 4 units left and 21 units down.

Study Tip

Check

As a check, graph the function in Example 3 to verify the location of its vertex and axis of symmetry.

b. $y = -3x^2 + 6x - 1$

$y = -3x^2 + 6x - 1$	Original equation
$y = -3(x^2 - 2x) - 1$	Group $ax^2 - bx$ and factor, dividing by a.
$y = -3(x^2 - 2x + 1) - 1 - (-3)(1)$	Complete the square by adding 1 inside the parentheses. Notice that this is an overall addition of $-3(1)$. Balance this addition by subtracting $-3(1)$.
$y = -3(x - 1)^2 + 2$	Write $x^2 - 2x + 1$ as a perfect square.

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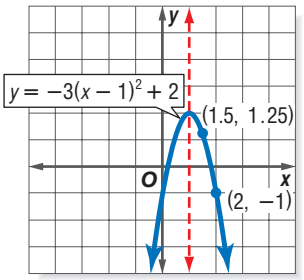
The vertex is at (1, 2), and the axis of symmetry is $x = 1$. Since $a = -3$, the graph opens downward and is narrower than the graph of $y = x^2$. It is also translated 1 unit right and 2 units up.

Now graph the function. Two points on the graph to the right of $x = 1$ are (1.5, 1.25) and (2, −1). Use symmetry to complete the graph.



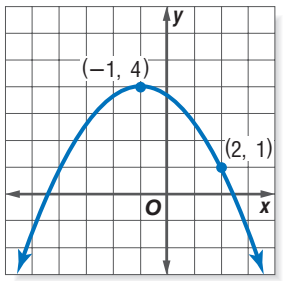
CHECK Your Progress

3A. $y = x^2 + 4x + 6$ **3B.** $y = 2x^2 + 12x + 17$

If the vertex and one other point on the graph of a parabola are known, you can write the equation of the parabola in vertex form.

EXAMPLE 4 Write an Equation Given a Graph

Write an equation for the parabola shown in the graph.



The vertex of the parabola is at (−1, 4), so $h = -1$ and $k = 4$. Since (2, 1) is a point on the graph of the parabola, let $x = 2$ and $y = 1$. Substitute these values into the vertex form of the equation and solve for a.

$y = a(x - h)^2 + k$	Vertex form
$1 = a[2 - (-1)]^2 + 4$	Substitute 1 for y, 2 for x, −1 for h, and 4 for k.
$1 = a(9) + 4$	Simplify.
$-3 = 9a$	Subtract 4 from each side.
$-\frac{1}{3} = a$	Divide each side by 9.

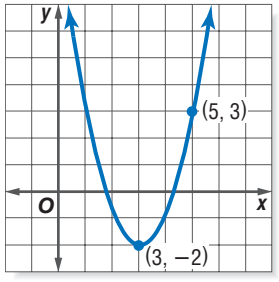
The equation of the parabola in vertex form is $y = -\frac{1}{3}(x + 1)^2 + 4$.

Cross-Curricular Project

You can use a quadratic function to model the world population. Visit algebra2.com to continue work on your project.

CHECK Your Progress

4. Write an equation for the parabola shown in the graph.



CHECK Your Understanding

Examples 1, 3 (pp. 286, 288)

Graph each function.

1. $y = 3(x + 3)^2$ **2.** $y = \frac{1}{3}(x - 1)^2 + 3$ **3.** $y = -2x^2 + 16x - 31$

Example 2 (p. 288)

4. STANDARDIZED TEST PRACTICE Which function has the widest graph?

A $y = -4x^2$ **B** $y = -1.2x^2$ **C** $y = 3.1x^2$ **D** $y = 11x^2$

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Example 3 (pp. 288–289)

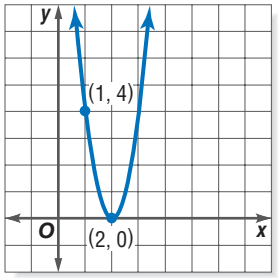
Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

5. $y = 5(x + 3)^2 - 1$ **6.** $y = x^2 + 8x - 3$ **7.** $y = -3x^2 - 18x + 11$

Example 4 (p. 289)

Write an equation in vertex form for the parabola shown in each graph.

8.



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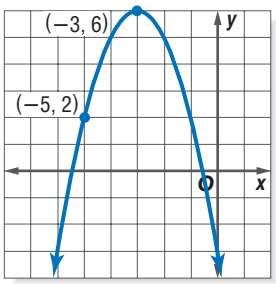

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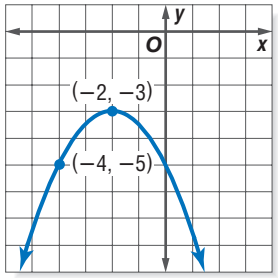
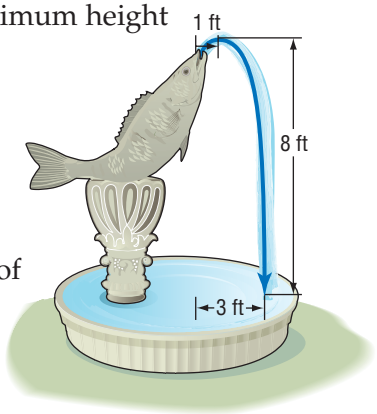


FOUNTAINS The height of a fountain's water stream can be modeled by a quadratic function. Suppose the water from a jet reaches a maximum height of 8 feet at a distance 1 foot away from the jet.



11. If the water lands 3 feet away from the jet, find a quadratic function that models the height $H(d)$ of the water at any given distance d feet from the jet. Then compare the graph of the function to the parent function.

12. Suppose a worker increases the water pressure so that the stream reaches a maximum height of 12.5 feet at a distance of 15 inches from the jet. The water now lands 3.75 feet from the jet. Write a new quadratic function for $H(d)$. How do the changes in h and k affect the shape of the graph?

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–16, 21, 22	1
17–18	1, 3
19, 20	2
23–26, 31, 32	3
27–30	4

Graph each function.

13. $y = 4(x + 3)^2 + 1$ **14.** $y = -(x - 5)^2 - 3$ **15.** $y = \frac{1}{4}(x - 2)^2 + 4$

16. $y = \frac{1}{2}(x - 3)^2 - 5$ **17.** $y = x^2 + 6x + 2$ **18.** $y = x^2 - 8x + 18$

19. What is the effect on the graph of the equation $y = x^2 + 2$ when the equation is changed to $y = x^2 - 5$?

20. What is the effect on the graph of the equation $y = x^2 + 2$ when the equation is changed to $y = 3x^2 - 5$?

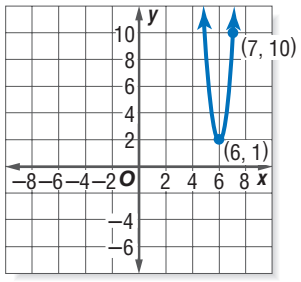
Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

21. $y = -2(x + 3)^2$ **22.** $y = \frac{1}{3}(x - 1)^2 + 2$ **23.** $y = -x^2 - 4x + 8$

24. $y = x^2 - 6x + 1$ **25.** $y = 5x^2 - 6$ **26.** $y = -8x^2 + 3$

Write an equation in vertex form for the parabola shown in each graph.

27.



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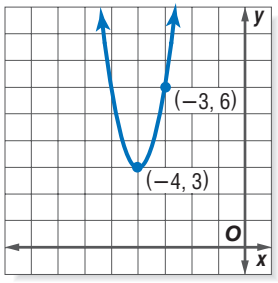

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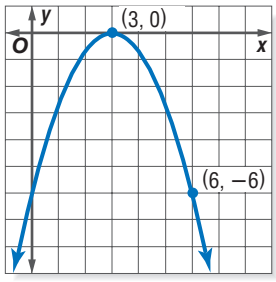

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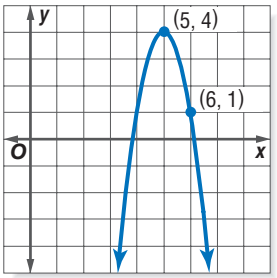
Write an equation in vertex form for the parabola shown in each graph.

30.



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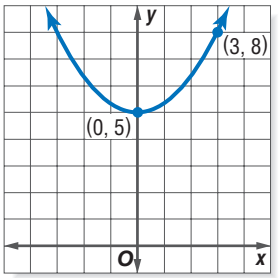

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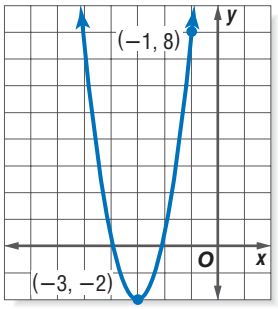

LAWN CARE **For Exercises 33 and 34, use the following information.**
The path of water from a sprinkler can be modeled by a quadratic function. The three functions below model paths for three different angles of the water.

Angle A: $y = -0.28(x - 3.09)^2 + 3.27$
Angle B: $y = -0.14(x - 3.57)^2 + 2.39$
Angle C: $y = -0.09(x - 3.22)^2 + 1.53$

33. Which sprinkler angle will send water the highest? Explain your reasoning.
34. Which sprinkler angle will send water the farthest? Explain your reasoning.
35. Which sprinkler angle will produce the widest path? The narrowest path?

Graph each function.

36. $y = -4x^2 + 16x - 11$
37. $y = -5x^2 - 40x - 80$
38. $y = -\frac{1}{2}x^2 + 5x - \frac{27}{2}$
39. $y = \frac{1}{3}x^2 - 4x + 15$

Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

40. $y = -3x^2 + 12x$
41. $y = 4x^2 + 24x$
42. $y = 4x^2 + 8x - 3$
43. $y = -2x^2 + 20x - 35$
44. $y = 3x^2 + 3x - 1$
45. $y = 4x^2 - 12x - 11$

46. Write an equation for a parabola with vertex at the origin and that passes through $(2, -8)$.
47. Write an equation for a parabola with vertex at $(-3, -4)$ and y-intercept 8.
48. Write one sentence that compares the graphs of $y = 0.2(x + 3)^2 + 1$ and $y = 0.4(x + 3)^2 + 1$.
49. Compare the graphs of $y = 2(x - 5)^2 + 4$ and $y = 2(x - 4)^2 - 1$.

Real-World Link
The KC135A has the nickname "Vomit Comet." It starts its ascent at 24,000 feet. As it approaches maximum height, the engines are stopped and the aircraft is allowed to free-fall at a determined angle. Zero gravity is achieved for 25 seconds as the plane reaches the top of its flight and begins its descent.

50. **AEROSPACE** NASA's KC135A aircraft flies in parabolic arcs to simulate the weightlessness experienced by astronauts in space. The height h of the aircraft (in feet) t seconds after it begins its parabolic flight can be modeled by the equation $h(t) = -9.09(t - 32.5)^2 + 34{,}000$. What is the maximum height of the aircraft during this maneuver and when does it occur?

DIVING **For Exercises 49–51, use the following information.**
The distance of a diver above the water $d(t)$ (in feet) t seconds after diving off a platform is modeled by the equation $d(t) = -16t^2 + 8t + 30$.

51. Find the time it will take for the diver to hit the water.
52. Write an equation that models the diver's distance above the water if the platform were 20 feet higher.
53. Find the time it would take for the diver to hit the water from this new height.

EXTRA PRACTICE
See pages 901, 930.
MathOnline
Self-Check Quiz at algebra2.com

NASA

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H.O.T. Problems

54. OPEN ENDED Write the equation of a parabola with a vertex of $(2, -1)$ and which opens downward.

55. CHALLENGE Given $y = ax^2 + bx + c$ with $a \neq 0$, derive the equation for the axis of symmetry by completing the square and rewriting the equation in the form $y = a(x - h)^2 + k$.

56. FIND THE ERROR Jenny and Ruben are writing $y = x^2 - 2x + 5$ in vertex form. Who is correct? Explain your reasoning.

Jenny

$y = x^2 - 2x + 5$

$y = (x^2 - 2x + 1) + 5 - 1$

$y = (x - 1)^2 + 4$

Ruben

$y = x^2 - 2x + 5$

$y = (x^2 - 2x + 1) + 5 + 1$

$y = (x - 1)^2 + 6$

57. CHALLENGE Explain how you can find an equation of a parabola using the coordinates of three points on its graph.

58. *Writing in Math* Use the information on page 286 to explain how the graph of $y = x^2$ can be used to graph any quadratic function. Include a description of the effects produced by changing a, h, and k in the equation $y = a(x - h)^2 + k$, and a comparison of the graph of $y = x^2$ and the graph of $y = a(x - h)^2 + k$ using values of your own choosing for a, h, and k.

STANDARDIZED TEST PRACTICE

59. ACT/SAT If $f(x) = x^2 - 5x$ and $f(n) = -4$, which of the following could be n?

A -5

B -4

C -1

D 1

60. REVIEW Which of the following most accurately describes the translation of the graph of $y = (x + 5)^2 - 1$ to the graph of $y = (x - 1)^2 + 3$?

F up 4 and 6 to the right

G up 4 and 1 to the left

H down 1 and 1 to the right

J down 1 and 5 to the left

Spiral Review

Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation. (Lesson 5-6)

61. $3x^2 - 6x + 2 = 0$

62. $4x^2 + 7x = 11$

63. $2x^2 - 5x + 6 = 0$

Solve each equation by completing the square. (Lesson 5-5)

64. $x^2 + 10x + 17 = 0$

65. $x^2 - 6x + 18 = 0$

66. $4x^2 + 8x = 9$

GET READY for the Next Lesson

PREREQUISITE SKILL Determine whether the given value satisfies the inequality. (Lesson 1-6)

67. $-2x^2 + 3 < 0$; $x = 5$

68. $4x^2 + 2x - 3 \geq 0$; $x = -1$

69. $4x^2 - 4x + 1 \leq 10$; $x = 2$

70. $6x^2 + 3x > 8$; $x = 0$

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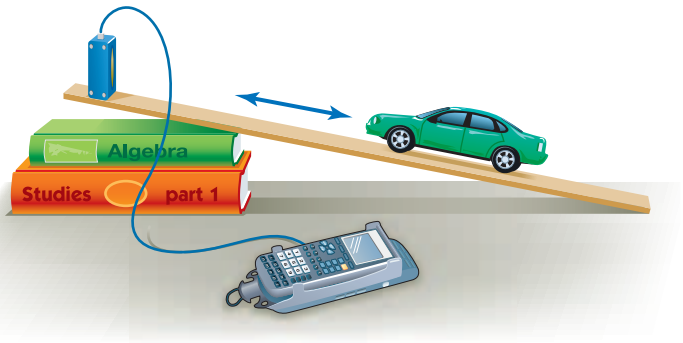
EXTEND 5-7

Graphing Calculator Lab

Modeling Motion

SET UP the Lab

- Place a board on a stack of books to create a ramp.
- Connect the data collection device to the graphing calculator. Place at the top of the ramp so that the data collection device can read the motion of the car on the ramp.
- Hold the car still about 6 inches up from the bottom of the ramp and zero the collection device.



ACTIVITY 1

Step 1 One group member should press the button to start collecting data.

Step 2 Another group member places the car at the bottom of the ramp. After data collection begins, gently but quickly push the car so it travels up the ramp toward the motion detector.

Step 3 Stop collecting data when the car returns to the bottom of the ramp. Save the data as Trial 1.

Step 4 Remove one book from the stack. Then repeat the experiment. Save the data as Trial 2. For Trial 3, create a steeper ramp and repeat the experiment.

ANALYZE THE RESULTS

1. What type of function could be used to represent the data? Justify your answer.
2. Use the CALC menu to find the vertex of the graph. Record the coordinates in a table like the one at the right.
3. Use the TRACE feature of the calculator to find the coordinates of another point on the graph. Then use the coordinates of the vertex and the point to find an equation of the graph.
4. Find an equation for each of the graphs of Trials 2 and 3.
5. How do the equations for Trials 1, 2, and 3 compare? Which graph is widest and which is most narrow? Explain what this represents in the context of the situation. How is this represented in the equations?
6. What do the x-intercepts and vertex of each graph represent?
7. Why were the values of h and k different in each trial?

Trial	Vertex (h, k)	Point (x, y)	Equation
1			
2			
3			

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5-8 Graphing and Solving Quadratic Inequalities

Main Ideas

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

New Vocabulary

quadratic inequality

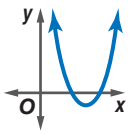
GET READY for the Lesson

Californian Jennifer Parilla is the only athlete from the United States to qualify for and compete in the Olympic trampoline event.

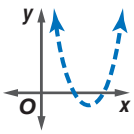
Suppose the height $h(t)$ in feet of a trampolinist above the ground during one bounce is modeled by the quadratic function $h(t) = -16t^2 + 42t + 3.75$. We can solve a quadratic inequality to determine how long this performer is more than a certain distance above the ground.

Graph Quadratic Inequalities You can graph **quadratic inequalities** in two variables using the same techniques you used to graph linear inequalities in two variables.

Step 1 Graph the related quadratic function, $y = ax^2 + bx + c$. Decide if the parabola should be solid or dashed.

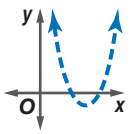


$\leq$ or $\geq$



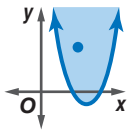
$<$ or $>$

Step 2 Test a point (x_1, y_1) inside the parabola. Check to see if this point is a solution of the inequality.

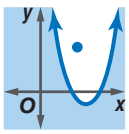


$y_1 \stackrel{?}{\geq} a(x_1)^2 + b(x_1) + c$

Step 3 If (x_1, y_1) is a solution, shade the region *inside* the parabola. If (x_1, y_1) is *not* a solution, shade the region *outside* the parabola.



(x_1, y_1) is a solution.



(x_1, y_1) is not a solution.

Clive Brunskill/Getty Images

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Study Tip

Look Back

For review of **graphing linear inequalities,** see Lesson 2-7.

EXAMPLE 1 Graph a Quadratic Inequality

Use a table to graph $y > -x^2 - 6x - 7$.

Step 1 Graph the related quadratic function, $y = -x^2 - 6x - 7$.

Since the inequality symbol is $>$, the parabola should be dashed.

x	-5	-4	-3	-2	-1
y	-2	1	2	1	-2

Step 2 Test a point inside the parabola, such as $(-3, 0)$.

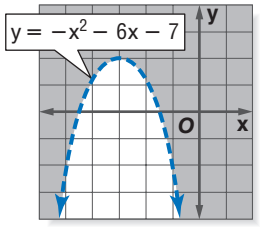
$y > -x^2 - 6x - 7$

$0 \stackrel{?}{>} -(-3)^2 - 6(-3) - 7$

$0 \stackrel{?}{>} -9 + 18 - 7$

$0 \stackrel{?}{>} 2$ ✗

So, $(-3, 0)$ is *not* a solution of the inequality.



Step 3 Shade the region outside the parabola.

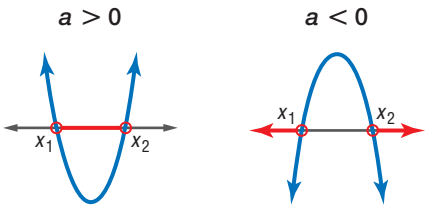
CHECK Your Progress **Graph each inequality.**

1A. $y \leq x^2 + 2x + 4$ **1B.** $y < -2x^2 + 3x + 5$

Solve Quadratic Inequalities To solve a quadratic inequality in one variable, you can use the graph of the related quadratic function.

To solve $ax^2 + bx + c < 0$, graph $y = ax^2 + bx + c$. Identify the x-values for which the graph lies *below* the x-axis.

For $\leq$, include the x-intercepts in the solution.

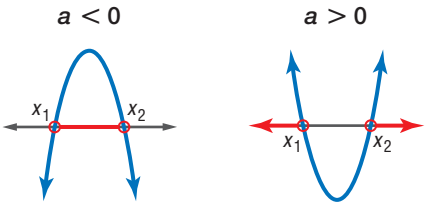


$\{x \mid x_1 < x < x_2\}$

$\{x \mid x < x_1 \text{ or } x > x_2\}$

To solve $ax^2 + bx + c > 0$, graph $y = ax^2 + bx + c$. Identify the x-values for which the graph lies *above* the x-axis.

For $\geq$, include the x-intercepts in the solution.



EXAMPLE 2 Solve $ax^2 + bx + c > 0$

Solve $x^2 + 2x - 3 > 0$ by graphing.

The solution consists of the x-values for which the graph of the related quadratic function lies *above* the x-axis. Begin by finding the roots.

$x^2 + 2x - 3 = 0$ Related equation

$(x + 3)(x - 1) = 0$ Factor.

$x + 3 = 0$ or $x - 1 = 0$ Zero Product Property

$x = -3$ $x = 1$ Solve each equation.

(continued on the next page)

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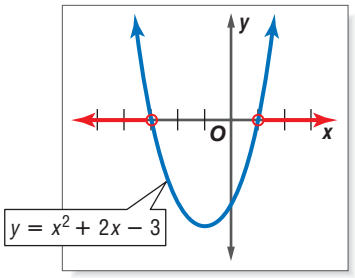
Study Tip

Solving Quadratic Inequalities by Graphing

A precise graph of the related quadratic function is not necessary since the zeros of the function were found algebraically.

Sketch the graph of a parabola that has x-intercepts at -3 and 1. The graph should open up since $a > 0$.

The graph lies above the x-axis to the left of $x = -3$ and to the right of $x = 1$. Therefore, the solution set is $\{x \mid x < -3 \text{ or } x > 1\}$.



CHECK Your Progress

Solve each inequality by graphing.

2A. $x^2 - 3x + 2 \geq 0$ **2B.** $0 \leq x^2 - 2x - 35$

EXAMPLE Solve $ax^2 + bx + c \leq 0$

3 **Solve $0 \geq 3x^2 - 7x - 1$ by graphing.**

This inequality can be rewritten as $3x^2 - 7x - 1 \leq 0$. The solution consists of the x-values for which the graph of the related quadratic function lies *on and below* the x-axis. Begin by finding the roots of the related equation.

$3x^2 - 7x - 1 = 0$ Related equation

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Use the Quadratic Formula.

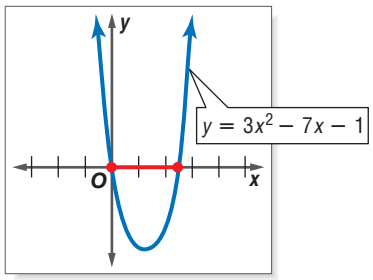
$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-1)}}{2(3)}$ Replace a with 3, b with -7, and c with -1.

$x = \dfrac{7 + \sqrt{61}}{6}$ or $x = \dfrac{7 - \sqrt{61}}{6}$ Simplify and write as two equations.

$x \approx 2.47$ $x \approx -0.14$ Simplify.

Sketch the graph of a parabola that has x-intercepts of 2.47 and -0.14. The graph should open up since $a > 0$.

The graph lies on and below the x-axis at $x = -0.14$ and $x = 2.47$ and between these two values. Therefore, the solution set of the inequality is approximately $\{x \mid -0.14 \leq x \leq 2.47\}$.



CHECK Test one value of x less than -0.14, one between -0.14 and 2.47, and one greater than 2.47 in the original inequality.

Test $x = -1$.	Test $x = 0$.	Test $x = 3$.
$0 \geq 3x^2 - 7x - 1$	$0 \geq 3x^2 - 7x - 1$	$0 \geq 3x^2 - 7x - 1$
$0 \stackrel{?}{\geq} 3(-1)^2 - 7(-1) - 1$	$0 \stackrel{?}{\geq} 3(0)^2 - 7(0) - 1$	$0 \stackrel{?}{\geq} 3(3)^2 - 7(3) - 1$
$0 \geq 9$ ✗	$0 \geq -1$ ✓	$0 \geq 5$ ✗

CHECK Your Progress

Solve each inequality by graphing.

3A. $0 > 2x^2 + 5x - 6$ **3B.** $5x^2 - 10x + 1 < 0$

Real-world problems that involve vertical motion can often be solved by using a quadratic inequality.

www.MathSchoolinternational.com

Real-World EXAMPLE

Real-World Link

A long hang time allows the kicking team time to provide good coverage on a punt return. The suggested hang time for high school and college punters is 4.5–4.6 seconds.

Source: www.takeaknee.com

4 **FOOTBALL** **The height of a punted football can be modeled by the function $H(x) = -4.9x^2 + 20x + 1$, where the height $H(x)$ is given in meters and the time x is in seconds. At what time in its flight is the ball within 5 meters of the ground?**

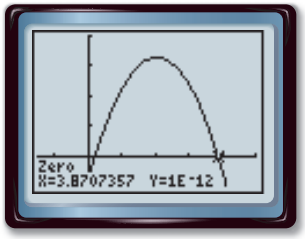
The function $H(x)$ describes the height of the football. Therefore, you want to find the values of x for which $H(x) \leq 5$.

$$H(x) \leq 5 \quad \text{Original inequality}$$

$$-4.9x^2 + 20x + 1 \leq 5 \quad H(x) = -4.9x^2 + 20x + 1$$

$$-4.9x^2 + 20x - 4 \leq 0 \quad \text{Subtract 5 from each side.}$$

Graph the related function $y = -4.9x^2 + 20x - 4$ using a graphing calculator. The zeros of the function are about 0.21 and 3.87, and the graph lies below the x-axis when $x < 0.21$ or $x > 3.87$.



[−1.5, 5] scl: 1 by [−5, 20] scl: 5

Thus, the ball is within 5 meters of the ground for the first 0.21 second of its flight and again after 3.87 seconds until the ball hits the ground at 4.13 seconds.

CHECK The ball starts 1 meter above the ground, so $x < 0.21$ makes sense. Based on the given information, a punt stays in the air about 4.5 seconds. So, it is reasonable that the ball is back within 5 meters of the ground after 3.87 seconds.

CHECK Your Progress

4. Use the function $H(x)$ above to find at what time in its flight the ball is at least 7 meters above the ground.

Online **Personal Tutor at** algebra2.com

EXAMPLE Solve a Quadratic Inequality

5 **Solve $x^2 + x > 6$ algebraically.**

First solve the related quadratic equation $x^2 + x = 6$.

$$x^2 + x = 6 \quad \text{Related quadratic equation}$$

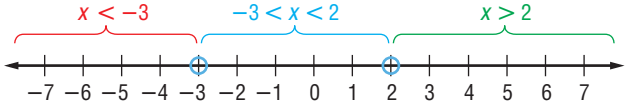
$$x^2 + x - 6 = 0 \quad \text{Subtract 6 from each side.}$$

$$(x + 3)(x - 2) = 0 \quad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Zero Product Property}$$

$$x = -3 \qquad x = 2 \quad \text{Solve each equation.}$$

Plot −3 and 2 on a number line. Use circles since these values are not solutions of the original inequality. Notice that the number line is now separated into three intervals.



(continued on the next page)

Study Tip

Solving Quadratic Inequalities Algebraically

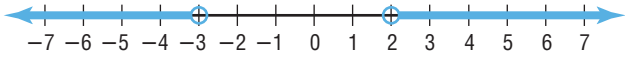
As with linear inequalities, the solution set of a quadratic inequality is sometimes all real numbers or the empty set, ∅. The solution is all real numbers when all three test points satisfy the inequality. It is the empty set when none of the test points satisfy the inequality.

Todd Rosenberg/Allsport/Getty Images
www.MathSchoolinternational.com

Test a value in each interval to see if it satisfies the original inequality.

$x < -3$	$-3 < x < 2$	$x > 2$
Test $x = -4$.	Test $x = 0$.	Test $x = 4$.
$x^2 + x > 6$	$x^2 + x > 6$	$x^2 + x > 6$
$(-4)^2 + (-4) \stackrel{?}{>} 6$	$0^2 + 0 \stackrel{?}{>} 6$	$4^2 + 4 \stackrel{?}{>} 6$
$12 > 6$ ✓	$0 > 6$ ✗	$20 > 6$ ✓

The solution set is $\{x \mid x < -3 \text{ or } x > 2\}$. This is shown on the number line below.



CHECK Your Progress

Solve each inequality algebraically.

5A. $x^2 + 5x < -6$ **5B.** $x^2 + 11x + 30 \leq 0$

CHECK Your Understanding

Example 1 (p. 295)

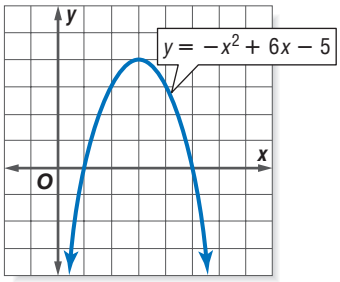
Graph each inequality.

1. $y \geq x^2 - 10x + 25$ **2.** $y < x^2 - 16$

3. $y > -2x^2 - 4x + 3$ **4.** $y \leq -x^2 + 5x + 6$

Examples 2, 3 (pp. 295–296)

5. Use the graph of the related function of $-x^2 + 6x - 5 < 0$, which is shown at the right, to write the solutions of the inequality.



Examples 2, 3, 5 (pp. 295–298)

Solve each inequality using a graph, a table, or algebraically.

6. $x^2 - 6x - 7 < 0$

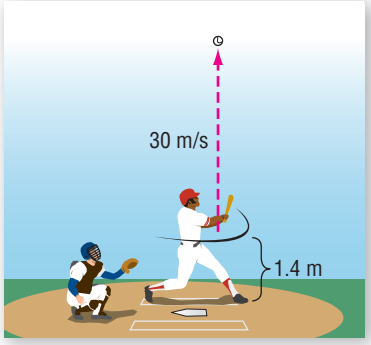
7. $x^2 - x - 12 > 0$

8. $x^2 < 10x - 25$

9. $x^2 \leq 3$

Example 4 (p. 297)

10. BASEBALL A baseball player hits a high pop-up with an initial upward velocity of 30 meters per second, 1.4 meters above the ground. The height $h(t)$ of the ball in meters t seconds after being hit is modeled by $h(t) = -4.9t^2 + 30t + 1.4$. How long does a player on the opposing team have to get under the ball if he catches it 1.7 meters above the ground? Does your answer seem reasonable? Explain.



Exercises

Graph each inequality.

11. $y \geq x^2 + 3x - 18$ **12.** $y < -x^2 + 7x + 8$ **13.** $y \leq x^2 + 4x + 4$

14. $y \leq x^2 + 4x$ **15.** $y > x^2 - 36$ **16.** $y > x^2 + 6x + 5$

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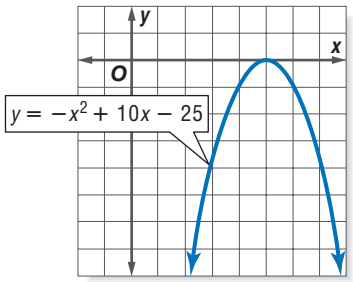
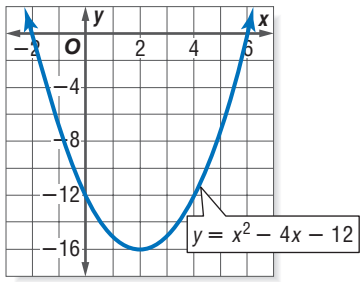
HOMEWORK HELP	
For Exercises	See Examples
11–16	1
17–20	2, 3
21–26	2, 3, 5
27, 28	4

Use the graph of the related function of each inequality to write its solutions.

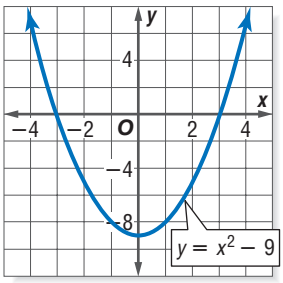
17. $-x^2 + 10x - 25 \geq 0$



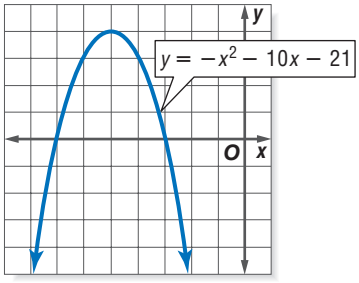
18. $x^2 - 4x - 12 \leq 0$



19. $x^2 - 9 > 0$



20. $-x^2 - 10x - 21 < 0$



Solve each inequality using a graph, a table, or algebraically.

21. $x^2 - 3x - 18 > 0$

22. $x^2 + 3x - 28 < 0$

23. $x^2 - 4x \leq 5$

24. $x^2 + 2x \geq 24$

25. $-x^2 - x + 12 \geq 0$

26. $-x^2 - 6x + 7 \leq 0$

Real-World Career

Landscape Architect

Landscape architects design outdoor spaces so that they are not only functional, but beautiful and compatible with the natural environment.

Math Online

For more information, go to algebra2.com.

27. LANDSCAPING Kinu wants to plant a garden and surround it with decorative stones. She has enough stones to enclose a rectangular garden with a perimeter of 68 feet, but she wants the garden to cover no more than 240 square feet. What could the width of her garden be?

28. GEOMETRY A rectangle is 6 centimeters longer than it is wide. Find the possible dimensions if the area of the rectangle is more than 216 square centimeters.

Graph each inequality.

29. $y \leq -x^2 - 3x + 10$

30. $y \geq -x^2 - 7x + 10$

31. $y > -x^2 + 10x - 23$

32. $y < -x^2 + 13x - 36$

33. $y < 2x^2 + 3x - 5$

34. $y \geq 2x^2 + x - 3$

Solve each inequality using a graph, a table, or algebraically.

35. $9x^2 - 6x + 1 \leq 0$

36. $4x^2 + 20x + 25 \geq 0$

37. $x^2 + 12x < -36$

38. $-x^2 + 14x - 49 \geq 0$

39. $18x - x^2 \leq 81$

40. $16x^2 + 9 < 24x$

41. $(x - 1)(x + 4)(x - 3) > 0$

42. BUSINESS A mall owner has determined that the relationship between monthly rent charged for store space r (in dollars per square foot) and monthly profit $P(r)$ (in thousands of dollars) can be approximated by the function $P(r) = -8.1r^2 + 46.9r - 38.2$. Solve each quadratic equation or inequality. Explain what each answer tells about the relationship between monthly rent and profit for this mall.

a. $-8.1r^2 + 46.9r - 38.2 = 0$

b. $-8.1r^2 + 46.9r - 38.2 > 0$

c. $-8.1r^2 + 46.9r - 38.2 > 10$

d. $-8.1r^2 + 46.9r - 38.2 < 10$

Aaron Haupt

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FUND-RAISING **For Exercises 43–45, use the following information.**
The girls' softball team is sponsoring a fund-raising trip to see a professional baseball game. They charter a 60-passenger bus for $525. In order to make a profit, they will charge $15 per person if all seats on the bus are sold, but for each empty seat, they will increase the price by $1.50 per person.

43. Write a quadratic function giving the softball team's profit $P(n)$ from this fund-raiser as a function of the number of passengers n.

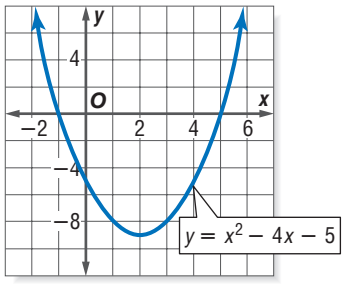
44. What is the minimum number of passengers needed in order for the softball team not to lose money?

45. What is the maximum profit the team can make with this fund-raiser, and how many passengers will it take to achieve this maximum?

EXTRA PRACTICE
See pages 902, 930.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

46. REASONING Examine the graph of $y = x^2 - 4x - 5$.



a. What are the solutions of $0 = x^2 - 4x - 5$?

b. What are the solutions of $x^2 - 4x - 5 \geq 0$?

c. What are the solutions of $x^2 - 4x - 5 \leq 0$?

47. OPEN ENDED List three points you might test to find the solution of $(x + 3)(x - 5) < 0$.

48. CHALLENGE Graph the intersection of the graphs of $y \leq -x^2 + 4$ and $y \geq x^2 - 4$.

49. *Writing in Math* Use the information on page 294 to explain how you can find the time a trampolinist spends above a certain height. Include a quadratic inequality that describes the time the performer spends more than 10 feet above the ground, and two approaches to solving this quadratic inequality.

STANDARDIZED TEST PRACTICE

50. ACT/SAT If $(x + 1)(x - 2)$ is positive, which statement must be true?

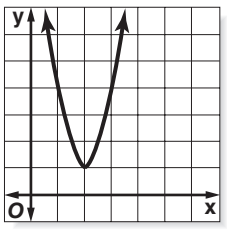
A $x < -1$ or $x > 2$

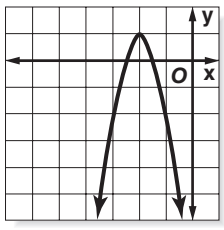
B $x > -1$ or $x < 2$

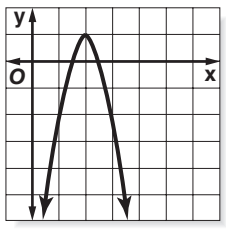
C $-1 < x < 2$

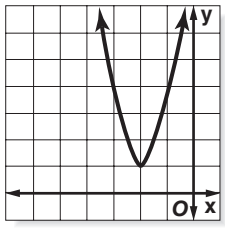
D $-2 < x < 1$

51. REVIEW Which is the graph of $y = -3(x - 2)^2 + 1$?

F 

G 

H 

J 

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Spiral Review

Write each equation in vertex form. Then identify the vertex, axis of symmetry, and direction of opening. (Lesson 5-7)

52. $y = x^2 - 2x + 9$

53. $y = -2x^2 + 16x - 32$

54. $y = \frac{1}{2}x^2 + 6x + 18$

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 5-6)

55. $x^2 + 12x + 32 = 0$

56. $x^2 + 7 = -5x$

57. $3x^2 + 6x - 2 = 3$

Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

58. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$

59. $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

60. $3j + 2k = 8$
$j - 7k = 18$

61. $5y + 2z = 11$
$10y - 4z = -2$

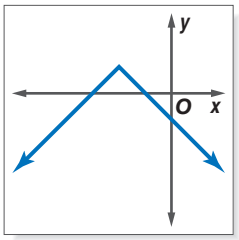
Find each product, if possible. (Lesson 4-3)

62. $\begin{bmatrix} -6 & 3 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 6 \end{bmatrix}$

63. $\begin{bmatrix} 2 & -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 \\ 9 & 0 \\ -2 & 4 \end{bmatrix}$

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

64.



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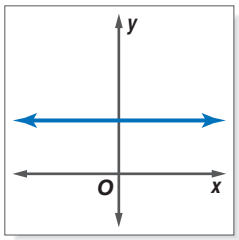
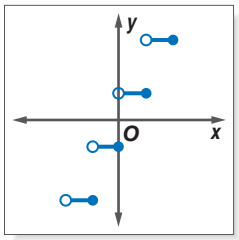


66.



67. EDUCATION The number of U.S. college students studying abroad in 2003 increased by about 8.57% over the previous year. The graph shows the number of U.S. students in study-abroad programs. (Lesson 2-5)

a. Write a prediction equation from the data given.

b. Use your equation to predict the number of students in these programs in 2010.

Source: Institute of International Education

68. LAW ENFORCEMENT A certain laser device measures vehicle speed to within 3 miles per hour. If a vehicle's actual speed is 65 miles per hour, write and solve an absolute value equation to describe the range of speeds that might register on this device. (Lesson 1-6)

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CHAPTER 5

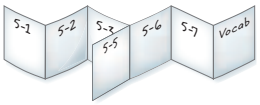
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Graphing Quadratic Functions (Lesson 5-1)

- The graph of $y = ax^2 + bx + c$, $a \neq 0$, opens up, and the function has a minimum value when $a > 0$. The graph opens down, and the function has a maximum value when $a < 0$.

Solving Quadratic Equations (Lessons 5-2 and 5-3)

- The solutions, or roots, of a quadratic equation are the zeros of the related quadratic function. You can find the zeros of a quadratic function by finding the x-intercepts of its graph.

Complex Numbers (Lesson 5-4)

- $\boldsymbol{i}$ is the imaginary unit. $\boldsymbol{i}^2 = -1$ and $\boldsymbol{i} = \sqrt{-1}$.

Solving Quadratic Equations (Lessons 5-5 and 5-6)

- Completing the square: **Step 1** Find one half of b, the coefficient of x. **Step 2** Square the result in Step 1. **Step 3** Add the result of Step 2 to $x^2 + bx$.
- Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Analyzing Graphs (Lesson 5-7)

- As the values of h and k change, the graph of $y = (x - h)^2 + k$ is the graph of $y = x^2$ translated $|h|$ units left if h is negative or $|h|$ units right if h is positive and $|k|$ units up if k is positive or $|k|$ units down if k is negative.
- Consider the equation $y = a(x - h)^2 + k$, $a \neq 0$. If $a > 0$, the graph opens up; if $a < 0$ the graph opens down. If $|a| > 1$, the graph is narrower than the graph of $y = x^2$. If $|a| < 1$, the graph is wider than the graph of $y = x^2$.

Key Vocabulary

axis of symmetry (p. 237)
completing the square (p. 269)
complex conjugates (p. 263)
complex number (p. 261)
constant term (p. 236)
discriminant (p. 279)
imaginary unit (p. 260)
linear term (p. 236)
maximum value (p. 238)
minimum value (p. 238)
parabola (p. 236)
pure imaginary number (p. 260)
quadratic equation (p. 246)
quadratic function (p. 236)
quadratic inequality (p. 294)
quadratic term (p. 236)
root (p. 246)
square root (p. 259)
vertex (p. 237)
vertex form (p. 286)
zero (p. 246)

Vocabulary Check

Choose the term from the list above that best matches each phrase.

1. the graph of any quadratic function
2. process used to create a perfect square trinomial
3. the line passing through the vertex of a parabola and dividing the parabola into two mirror images
4. a function described by an equation of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$
5. the solutions of an equation
6. $y = a(x - h)^2 + k$
7. in the Quadratic Formula, the expression under the radical sign, $b^2 - 4ac$
8. the square root of -1
9. a method used to solve a quadratic equation without using the quadratic formula
10. a number in the form $a + bi$

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Lesson-by-Lesson Review

5–1 Graphing Quadratic Functions (pp. 236–244)

Complete parts a–c for each quadratic function.

a. **Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.**

b. **Make a table of values that includes the vertex.**

c. **Use this information to graph the function.**

11. $f(x) = x^2 + 6x + 20$

12. $f(x) = x^2 - 8x + 7$

13. $f(x) = -2x^2 + 12x - 9$

14. FRAMES Josefina is making a rectangular picture frame. She has 72 inches of wood to make this frame. What dimensions will produce a picture frame that will frame the greatest area?

Example 1 **Find the maximum or minimum value of $f(x) = -x^2 + 4x - 12$.**

Since $a < 0$, the graph opens down and the function has a maximum value. The maximum value of the function is the y-coordinate of the vertex. The x-coordinate of the vertex is $x = \frac{-4}{2(-1)}$ or 2. Find the y-coordinate by evaluating the function for $x = 2$.

$f(x) = -x^2 + 4x - 12$ Original function

$f(2) = -(2)^2 + 4(2) - 12$ Replace x with 2.

$= -8$

Therefore, the maximum value of the function is -8.

5–2 Solving Quadratic Equations by Graphing (pp. 246–251)

Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

15. $x^2 - 36 = 0$ **16.** $-x^2 - 3x + 10 = 0$

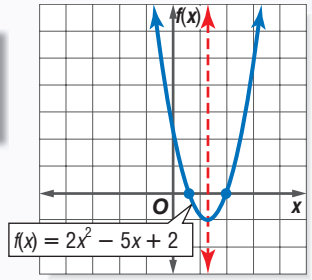
17. $-3x^2 - 6x - 2 = 0$ **18.** $\frac{1}{5}(x + 3)^2 - 5 = 0$

19. BASEBALL A baseball is hit upward at 100 feet per second. Use the formula $h(t) = v_0t - 16t^2$, where $h(t)$ is the height of an object in feet, v_0 is the object's initial velocity in feet per second, and t is the time in seconds. Ignoring the height of the ball when it was hit, how long does it take for the ball to hit the ground?

Example 2 **Solve $2x^2 - 5x + 2 = 0$ by graphing.**

The equation of the axis of symmetry is $x = -\frac{-5}{2(2)}$ or $x = \frac{5}{4}$.

x	0	$\frac{1}{2}$	$\frac{5}{4}$	2	$\frac{5}{2}$
$f(x)$	2	0	$-\frac{9}{8}$	0	2



The zeros of the related function are $\frac{1}{2}$ and 2. Therefore, the solutions of the equation are $\frac{1}{2}$ and 2.

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CHAPTER 5

Study Guide and Review

5–3 Solving Quadratic Equations by Factoring (pp. 253–258)

Write a quadratic equation in standard form with the given root(s).

20. $-4, -25$ **21.** $10, -7$ **22.** $\frac{1}{3}, 2$

Solve each equation by factoring.

23. $x^2 - 4x - 32 = 0$ **24.** $3x^2 + 6x + 3 = 0$

25. $5y^2 = 80$ **26.** $25x^2 - 30x = -9$

27. $6x^2 + 7x = 3$ **28.** $2c^2 + 18c - 44 = 0$

29. TRIANGLES Find the dimensions of a triangle if the base is $\frac{2}{3}$ the length of the height and the area is 12 square centimeters.

Example 3 **Write a quadratic equation in standard form with the roots 3 and −5.**

$(x - p)(x - q) = 0$ Write the pattern.

$(x - 3)(x + 5) = 0$ $p = 3$ and $q = -5$

$x^2 + 2x - 15 = 0$ Use FOIL.

Example 4 **Solve $x^2 + 9x + 20 = 0$ by factoring.**

$x^2 + 9x + 20 = 0$ Original equation

$(x + 4)(x + 5) = 0$ Factor the trinomial.

$x + 4 = 0$ or $x + 5 = 0$ Zero Product Property

$x = -4$ $x = -5$

The solution set is $\{-5, -4\}$.

5–4 Complex Numbers (pp. 259–266)

Simplify.

30. $\sqrt{45}$ **31.** $\sqrt{64n^3}$

32. $\sqrt{-64m^{12}}$

33. $(7 - 4i) - (-3 + 6i)$

34. $(3 + 4i)(5 - 2i)$ **35.** $(\sqrt{6} + i)(\sqrt{6} - i)$

36. $\frac{1 + i}{1 - i}$ **37.** $\frac{4 - 3i}{1 + 2i}$

38. ELECTRICITY The impedance in one part of a series circuit is $2 + 3j$ ohms, and the impedance in the other part of the circuit is $4 - 2j$. Add these complex numbers to find the total impedance in the circuit.

Example 5 **Simplify $(15 - 2i) + (-11 + 5i)$.**

$(15 - 2i) + (-11 + 5i)$

$= [15 + (-11)] + (-2 + 5)i$ Group the real and imaginary parts.

$= 4 + 3i$ Add.

Example 6 **Simplify $\frac{7i}{2 + 3i}$.**

$\frac{7i}{2 + 3i} = \frac{7i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i}$ $2 + 3i$ and $2 - 3i$ are conjugates.

$= \frac{14i - 21i^2}{4 - 9i^2}$ Multiply.

$= \frac{21 + 14i}{13}$ or $\frac{21}{13} + \frac{14}{13}i$ $i^2 = 1$

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Mixed Problem Solving
For mixed problem-solving practice, see page 930.

5–5 Completing the Square (pp. 268–275)

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

39. $x^2 + 34x + c$ **40.** $x^2 - 11x + c$

Solve each equation by completing the square.

41. $2x^2 - 7x - 15 = 0$

42. $2x^2 - 5x + 7 = 3$

43. GARDENING Antoinette has a rectangular rose garden with the length 8 feet longer than the width. If the area of her rose garden is 128 square feet, find the dimensions of the garden.

Example 7 **Solve $x^2 + 10x - 39 = 0$ by completing the square.**

$$x^2 + 10x - 39 = 0$$
$$x^2 + 10x = 39$$
$$x^2 + 10x + 25 = 39 + 25$$
$$(x + 5)^2 = 64$$
$$x + 5 = \pm 8$$
$$x + 5 = 8 \text{ or } x + 5 = -8$$
$$x = 3 \qquad x = -13$$

The solution set is $\{-13, 3\}$.

5–6 The Quadratic Formula and the Discriminant (pp. 276–283)

Complete parts a–c for each quadratic equation.

a. Find the value of the discriminant.
b. Describe the number and type of roots.
c. Find the exact solutions by using the Quadratic Formula.

44. $x^2 + 2x + 7 = 0$

45. $-2x^2 + 12x - 5 = 0$

46. $3x^2 + 7x - 2 = 0$

47. FOOTBALL The path of a football thrown across a field is given by the equation $y = -0.005x^2 + x + 5$, where x represents the distance, in feet, the ball has traveled horizontally and y represents the height, in feet, of the ball above ground level. About how far has the ball traveled horizontally when it returns to the ground?

Example 8 **Solve $x^2 - 5x - 66 = 0$ by using the Quadratic Formula.**

In $x^2 - 5x - 66 = 0$, $a = 1$, $b = -5$, and $c = -66$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-66)}}{2(1)}$$
$$= \frac{5 \pm 17}{2}$$

Write as two equations.

$$x = \frac{5 + 17}{2} \text{ or } x = \frac{5 - 17}{2}$$
$$= 11 \qquad = -6$$

The solution set is $\{-6, 11\}$.

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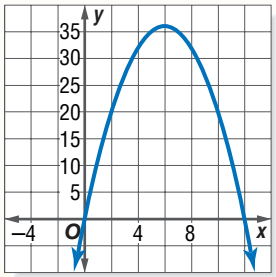
5–7 Analyzing Graphs of Quadratic Functions (pp. 286–292)

Write each equation in vertex form, if not already in that form. Identify the vertex, axis of symmetry, and direction of opening. Then graph the function.

48. $y = -6(x + 2)^2 + 3$ **49.** $y = -\frac{1}{3}x^2 + 8x$

50. $y = (x - 2)^2 - 2$ **51.** $y = 2x^2 + 8x + 10$

52. NUMBER THEORY The graph shows the product of two numbers with a sum of 12. Find an equation that models this product and use it to determine the two numbers that would give a maximum product.



Example 9 **Write the quadratic function $y = 3x^2 + 42x + 142$ in vertex form. Then identify the vertex, axis of symmetry, and the direction of opening.**

$y = 3x^2 + 42x + 142$ Original equation

$y = 3(x^2 + 14x) + 142$ Group and factor.

$y = 3(x^2 + 14x + 49) + 142 - 3(49)$ Complete the square.

$y = 3(x + 7)^2 - 5$ Write $x^2 + 14x + 49$ as a perfect square.

So, $a = 3$, $h = -7$, and $k = -5$. The vertex is at $(-7, -5)$, and the axis of symmetry is $x = -7$. Since a is positive, the graph opens up.

5–8 Graphing and Solving Quadratic Inequalities (pp. 294–301)

Graph each inequality.

53. $y > x^2 - 5x + 15$ **54.** $y \geq -x^2 + 7x - 11$

Solve each inequality using a graph, a table, or algebraically.

55. $6x^2 + 5x > 4$ **56.** $8x + x^2 \geq -16$

57. $4x^2 - 9 \leq -4x$ **58.** $3x^2 - 5 > 6x$

59. GAS MILEAGE The gas mileage y in miles per gallon for a particular vehicle is given by the equation $y = 10 + 0.9x - 0.01x^2$, where x is the speed of the vehicle between 10 and 75 miles per hour. Find the range of speeds that would give a gas mileage of at least 25 miles per gallon.

Example 10 **Solve $x^2 + 3x - 10 < 0$.**

Find the roots of the related equation.

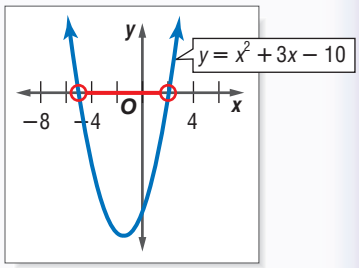
$0 = x^2 + 3x - 10$ Related equation

$0 = (x + 5)(x - 2)$ Factor.

$x + 5 = 0$ or $x - 2 = 0$ Zero Product Property

$x = -5$ $x = 2$ Solve each equation.

The graph opens up since $a > 0$. The graph lies below the x-axis between $x = -5$ and $x = 2$. The solution set is $\{x \mid -5 < x < 2\}$.



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CHAPTER 5

Practice Test

Complete parts a–c for each quadratic function.

a. Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.

b. Make a table of values that includes the vertex.

c. Use this information to graph the function.

1. $f(x) = x^2 - 2x + 5$

2. $f(x) = -3x^2 + 8x$

3. $f(x) = -2x^2 - 7x - 1$

Determine whether each function has a maximum or a minimum value. State the maximum or minimum value of each function.

4. $f(x) = x^2 + 6x + 9$

5. $f(x) = 3x^2 - 12x - 24$

6. $f(x) = -x^2 + 4x$

7. Write a quadratic equation with roots -4 and 5 in standard form.

Solve each equation using the method of your choice. Find exact solutions.

8. $x^2 + x - 42 = 0$

9. $-1.6x^2 - 3.2x + 18 = 0$

10. $15x^2 + 16x - 7 = 0$

11. $x^2 + 8x - 48 = 0$

12. $x^2 + 12x + 11 = 0$

13. $x^2 - 9x - \frac{19}{4} = 0$

14. $3x^2 + 7x - 31 = 0$

15. $10x^2 + 3x = 1$

16. $-11x^2 - 174x + 221 = 0$

17. BALLOONING At a hot-air balloon festival, you throw a weighted marker straight down from an altitude of 250 feet toward a bull's-eye below. The initial velocity of the marker when it leaves your hand is 28 feet per second. Find out how long it will take the marker to hit the target by solving the equation $-16t^2 - 28t + 250 = 0$.

Simplify.

18. $(5 - 2i) - (8 - 11i)$

19. $(14 - 5i)^2$

Write each equation in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

20. $y = (x + 2)^2 - 3$

21. $y = x^2 + 10x + 27$

22. $y = -9x^2 + 54x - 8$

Graph each inequality.

23. $y \leq x^2 + 6x - 7$

24. $y > -2x^2 + 9$

25. $y \geq -\frac{1}{2}x^2 - 3x + 1$

Solve each inequality using a graph, a table, or algebraically.

26. $(x - 5)(x + 7) < 0$

27. $3x^2 \geq 16$

28. $-5x^2 + x + 2 < 0$

29. PETS A rectangular turtle pen is 6 feet long by 4 feet wide. The pen is enlarged by increasing the length and width by an equal amount in order to double its area. What are the dimensions of the new pen?

30. MULTIPLE CHOICE Which of the following is the sum of both solutions of the equation $x^2 + 8x - 48 = 0$?

A -16

B -8

C -4

D 12

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CHAPTER 5

Standardized Test Practice

Cumulative, Chapters 1–5

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. What is the effect on the graph of the equation $y = x^2 + 4$ when it is changed to $y = x^2 - 3$?

A The slope of the graph changes.

B The graph widens.

C The graph is the same shape, and the vertex of the graph is moved down.

D The graph is the same shape, and the vertex of the graph is shifted to the left.

2. What is the solution set for the equation $3(2x + 1)^2 = 27$?

F $\{-5, 4\}$

G $\{-2, 1\}$

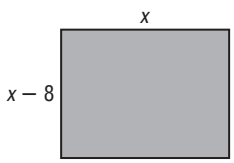
H $\{2, -1\}$

J $\{-3, 3\}$

TEST-TAKING TIP

Question 2 To solve equations or inequalities, you can replace the variables in the question with the values given in each answer choice. The answer choice that results in true statements is the correct answer choice.

3. For what value of x would the rectangle below have an area of 48 square units?



A 4

B 6

C 8

D 12

4. Which shows the functions correctly listed in order from widest to narrowest graph?

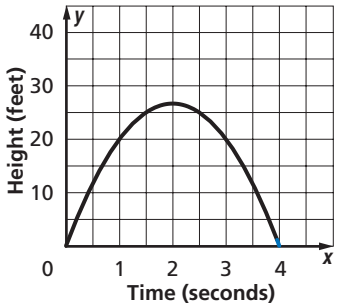
F $y = 8x^2, y = 2x^2, y = \frac{1}{2}x^2, y = -\frac{4}{5}x^2$

G $y = -\frac{4}{5}x^2, y = \frac{1}{2}x^2, y = 2x^2, y = 8x^2$

H $y = \frac{1}{2}x^2, y = -\frac{4}{5}x^2, y = 2x^2, y = 8x^2$

J $y = 8x^2, y = 2x^2, y = -\frac{4}{5}x^2, y = \frac{1}{2}x^2$

5. The graph below shows the height of an object from the time it is propelled from Earth.



For how long is the object above a height of 20 feet?

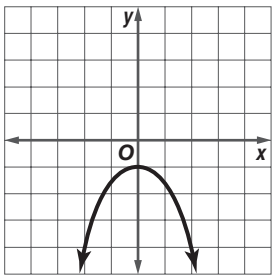
A 0.5 second

B 1 second

C 2 seconds

D 4 seconds

6. Which equation is the parent function of the graph represented below?



F $y = x^2$

G $y = |x|$

H $y = x$

J $y = \sqrt{x}$

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7. An object is shot straight upward into the air with an initial speed of 800 feet per second. The height h that the object will be after t seconds is given by the equation $h = -16t^2 + 800t$. When will the object reach a height of 10,000 feet?

A 10 seconds

B 25 seconds

C 100 seconds

D 625 seconds

8. What are the roots of the quadratic equation $3x^2 + x = 4$?

F $-1, \frac{4}{3}$

G $-\frac{4}{3}, 1$

H $-2, \frac{2}{3}$

J $-\frac{2}{3}, 2$

9. Which equation will produce the narrowest parabola when graphed?

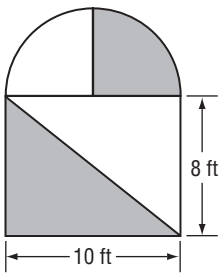
A $y = 3x^2$

B $y = \frac{3}{4}x^2$

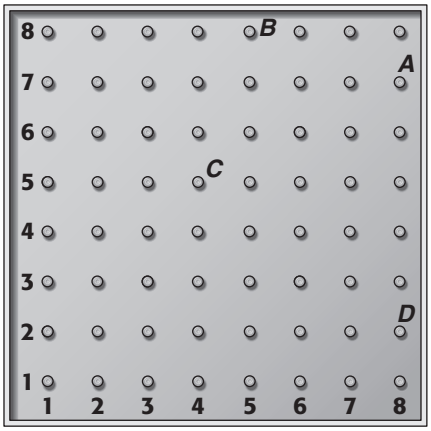
C $y = -\frac{3}{4}x^2$

D $y = -6x^2$

10. GRIDDABLE To the nearest tenth, what is the area in square feet of the shaded region below?



11. Mary was given this geoboard to model the slope $-\frac{3}{4}$.



If the peg in the upper right-hand corner represents the origin on a coordinate plane, where could Mary place a rubber band to represent the given slope?

F from peg A to peg B

G from peg A to peg C

H from peg B to peg D

J from peg C to peg D

Pre-AP

Record your answers on a sheet of paper. Show your work.

12. Scott launches a model rocket from ground level. The rocket's height h in meters is given by the equation $h = -4.9t^2 + 56t$, where t is the time in seconds after the launch.

a. What is the maximum height the rocket will reach? Round to the nearest tenth of a meter. Show each step and explain your method.

b. How long after it is launched will the rocket reach its maximum height? Round to the nearest tenth of a second.

NEED EXTRA HELP?												
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12
Go to Lesson...	5-7	5-5	5-3	5-7	5-7	5-1	5-3	5-3	2-3	1-4	2-3	5-7

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CHAPTER 6

Polynomial Functions

BIG Ideas

- Add, subtract, multiply, divide, and factor polynomials.
- Analyze and graph polynomial functions.
- Evaluate polynomial functions and solve polynomial equations.
- Find factors and zeroes of polynomial functions.

Key Vocabulary

polynomial function (p. 332)
scientific notation (p. 315)
synthetic division (p. 327)
synthetic substitution (p. 356)

Real-World Link

Power Generation Many real-world situations can be modeled using linear equations. But there are also many situations for which a linear equation would not be an accurate model. The power generated by a windmill can be best modeled using a polynomial function.

FOLDABLES Study Organizer

Polynomial Functions Make this Foldable to help you organize your notes. Begin with five sheets of grid paper.

1 **Stack** sheets of paper with edges $\frac{3}{4}$-inch apart. Fold up the bottom edges to create equal tabs.

2 **Staple** along the fold. Label the tabs with lesson numbers.

Guy Grenier/Masterfile

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GET READY for Chapter 6

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

QUICKCheck

Rewrite each difference as a sum (Prerequisite Skill)

1. $2 - 7$
2. $-6 - 11$
3. $x - y$
4. $8 - 2x$
5. $2xy - 6yz$
6. $6a^2b - 12b^2c$
7. **CANDY** Janet has \$4. She buys x candy bars for \$0.50 each. Rewrite the amount of money she has left as a sum. (Prerequisite Skill)

Use the Distributive Property to rewrite each expression without parentheses. (Lesson 1-2)

8. $-2(4x^3 + x - 3)$
9. $-1(x + 2)$
10. $-1(x - 3)$
11. $-3(2x^4 - 5x^2 - 2)$
12. $-\frac{1}{2}(3a + 2)$
13. $-\frac{2}{3}(2 + 6z)$

SCHOOL SHOPPING For Exercises 14 and 15, use the following information.

Students, ages 12 to 17, plan on spending an average of \$113 on clothing for school. The students plan on spending 36% of their money at specialty stores and 19% at department stores. (Lesson 1-2)

14. Write an expression to represent the amount that the average student spends shopping for clothes at specialty and department stores.
15. Evaluate the expression from Exercise 14 by using the Distributive Property.

Solve each equation. (Lesson 5-6)

16. $x^2 - 17x + 60 = 0$
17. $14x^2 + 23x + 3 = 0$
18. $2x^2 + 5x + 1 = 0$
19. $3x^2 - 5x + 2 = 0$

QUICKReview

EXAMPLE 1

Rewrite $a - b - c$ as a sum.

$a - b - c$ — Write the expression.

$= a + (-b) + (-c)$ — Rewrite by adding $(-b)$ and $(-c)$.

EXAMPLE 2

Use the Distributive Property to rewrite $-x(y - z + y)$ without parentheses.

$-x(y - z + y)$ — Original expression

$= -x(y) + (-x)(-z) + (-x)(y)$ — Distributive Property

$= -xy + xz - xy$ — Simplify.

EXAMPLE 3

Solve $4x^2 - 6x - 5 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Quadratic Formula

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(4)(-5)}}{2(4)}$ — Substitute.

$x = \frac{6 \pm \sqrt{116}}{8}$ — Simplify.

$x = \frac{6 \pm 2\sqrt{29}}{8}$ or $x = \frac{3 \pm \sqrt{29}}{4}$ — $\sqrt{116} = \sqrt{4 \cdot 29}$ or $2\sqrt{29}$

The exact solutions are $\frac{3 + \sqrt{29}}{4}$ and $\frac{3 - \sqrt{29}}{4}$.
The approximate solutions are 2.1 and -0.6.

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6-1 Properties of Exponents

Main Ideas

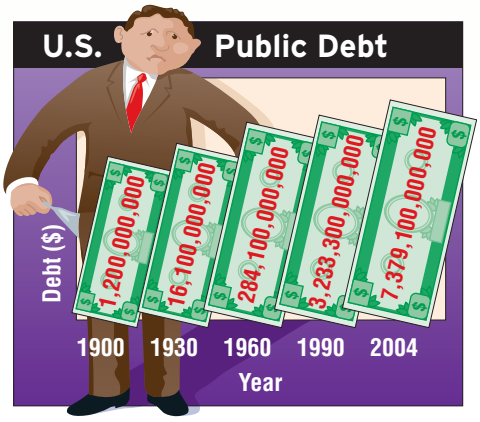
- Use properties of exponents to multiply and divide monomials.
- Use expressions written in scientific notation.

New Vocabulary

simplify
standard notation
scientific notation
dimensional analysis

GET READY for the Lesson

Economists often deal with very large numbers. For example, the table shows the U.S. public debt for several years. Such numbers, written in standard notation, are difficult to work with because they contain so many digits. Scientific notation uses powers of ten to make very large or very small numbers more manageable.



Source: Bureau of the Public Debt

Multiply and Divide Monomials To **simplify** an expression containing powers means to rewrite the expression without parentheses or negative exponents. Negative exponents are a way of expressing the multiplicative inverse of a number. For example, $\frac{1}{x^2}$ can be written as x^{-2}. Note that an expression such as x^{-2} is not a monomial. *Why?*

KEY CONCEPT — ***Negative Exponents***

Words For any real number $a \neq 0$ and any integer n, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.

Examples $2^{-3} = \frac{1}{2^3}$ and $\frac{1}{b^{-8}} = b^8$

Study Tip

Look Back
You can review **monomials** in Lesson 1-1.

EXAMPLE **Simplify Expressions with Multiplication**

1 **Simplify each expression. Assume that no variable equals 0.**

a. $(3x^3y^2)(-4x^2y^4)$

$$(3x^3y^2)(-4x^2y^4)$$
$$= (3 \cdot x \cdot x \cdot x \cdot y \cdot y) \cdot (-4 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y) \quad \text{Definition of exponents}$$
$$= 3(-4) \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \quad \text{Commutative Property}$$
$$= -12x^5y^6 \quad \text{Definition of exponents}$$

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b. $(a^{-3})(a^2b^4)(c^{-1})$

$$(a^{-3})(a^2b^4)(c^{-1}) = \left(\frac{1}{a^3}\right)(a^2b^4)\left(\frac{1}{c}\right)$$ Definition of negative exponents

$$= \left(\frac{1}{a \cdot a \cdot a}\right)(a \cdot a \cdot b \cdot b \cdot b \cdot b)\left(\frac{1}{c}\right)$$ Definition of exponents

$$= \left(\frac{1}{\cancel{a} \cdot \cancel{a} \cdot a}\right)(\cancel{a} \cdot \cancel{a} \cdot b \cdot b \cdot b \cdot b)\left(\frac{1}{c}\right)$$ Cancel out common factors.

$$= \frac{b^4}{ac}$$ Definition of exponents and fractions

CHECK Your Progress

1A. $(-5x^4y^3)(-3xy^5)$ **1B.** $(2x^{-3}y^3)(-7x^5y^{-6})$

Example 1 suggests the following property of exponents.

KEY CONCEPT — ***Product of Powers***

Words For any real number a and integers m and n, $a^m \cdot a^n = a^{m+n}$.

Examples $4^2 \cdot 4^9 = 4^{11}$ and $b^3 \cdot b^5 = b^8$

To multiply powers of the same variable, add the exponents. Knowing this, it seems reasonable to expect that when dividing powers, you would subtract exponents. Consider $\frac{x^9}{x^5}$.

$$\frac{x^9}{x^5} = \frac{\overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot x \cdot x \cdot x \cdot x}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}}}$$ Remember that $x \neq 0$.

$$= x \cdot x \cdot x \cdot x$$ Simplify.

$$= x^4$$ Definition of exponents

It appears that our conjecture is true. To divide powers of the same base, you subtract exponents.

KEY CONCEPT — ***Quotient of Powers***

Words For any real number $a \neq 0$, and any integers m and n, $\frac{a^m}{a^n} = a^{m-n}$.

Examples $\frac{5^3}{5} = 5^{3-1}$ or 5^2 and $\frac{x^7}{x^3} = x^{7-3}$ or x^4

Study Tip

Check

You can check your answer using the definition of exponents.

$\frac{p^3}{p^8} = \frac{\overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}} \cdot \overset{1}{\cancel{p}}}{\underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot \underset{1}{\cancel{p}} \cdot p \cdot p \cdot p \cdot p \cdot p}$

or $\frac{1}{p^5}$

EXAMPLE **Simplify Expressions with Division**

2 **Simplify $\frac{p^3}{p^8}$. Assume that $p \neq 0$.**

$$\frac{p^3}{p^8} = p^{3-8}$$ Subtract exponents.

$$= p^{-5} \text{ or } \frac{1}{p^5}$$ Remember that a simplified expression cannot contain negative exponents.

CHECK Your Progress

Simplify each expression. Assume that no variable equals 0.

2A. $\frac{y^{12}}{y^4}$ **2B.** $\frac{15c^5d^3}{-3c^2d^7}$

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You can use the Quotient of Powers property and the definition of exponents to simplify $\frac{y^4}{y^4}$, if $y \neq 0$.

Method 1

$\frac{y^4}{y^4} = y^{4-4}$ Quotient of Powers

$= y^0$ Subtract.

Method 2

$\frac{y^4}{y^4} = \frac{\overset{1}{y} \cdot \overset{1}{y} \cdot \overset{1}{y} \cdot \overset{1}{y}}{\underset{1}{y} \cdot \underset{1}{y} \cdot \underset{1}{y} \cdot \underset{1}{y}}$ Definition of exponents

$= 1$ Divide.

In order to make the results of these two methods consistent, we define $y^0 = 1$, where $y \neq 0$. In other words, any nonzero number raised to the zero power is equal to 1. *Notice that 0^0 is undefined.*

The properties we have presented can be used to verify the properties of powers that are listed below.

KEY CONCEPT — ***Properties of Powers***

Words	**Examples**
Suppose a and b are real numbers and m and n are integers. Then the following properties hold.	
Power of a Power: $(a^m)^n = a^{mn}$	$(a^2)^3 = a^6$
Power of a Product: $(ab)^m = a^m b^m$	$(xy)^2 = x^2y^2$
Power of a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$ and $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ or $\frac{b^n}{a^n}$, $a \neq 0, b \neq 0$	$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$, $\left(\frac{x}{y}\right)^{-4} = \frac{y^4}{x^4}$
Zero Power: $a^0 = 1$, $a \neq 0$	$2^0 = 1$

Study Tip

Simplified Expressions

A monomial expression is in simplified form when:

- there are no powers of powers,
- each base appears exactly once,
- all fractions are in simplest form, and
- there are no negative exponents.

EXAMPLE **Simplify Expressions with Powers**

3 **Simplify each expression.**

a. $(a^3)^6$

$(a^3)^6 = a^{3(6)}$ Power of a power

$= a^{18}$ Simplify.

b. $\left(\frac{-3x}{y}\right)^4$

$\left(\frac{-3x}{y}\right)^4 = \frac{(-3x)^4}{y^4}$ Power of a quotient

$= \frac{(-3)^4 x^4}{y^4}$ Power of a product

$= \frac{81x^4}{y^4}$ $(-3)^4 = 81$

CHECK Your Progress

3A. $(-2p^3s^2)^5$ **3B.** $\left(\frac{a}{4}\right)^{-3}$

With complicated expressions, you often have a choice of which way to start simplifying.

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EXAMPLE Simplify Expressions Using Several Properties

4 Simplify $\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4$.

Method 1

Raise the numerator and denominator to the fourth power before simplifying.

$$\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4 = \frac{(-2x^{3n})^4}{(x^{2n}y^3)^4}$$

$$= \frac{(-2)^4(x^{3n})^4}{(x^{2n})^4\,(y^3)^4}$$

$$= \frac{16x^{12n}}{x^{8n}\,y^{12}}$$

$$= \frac{16x^{12n-8n}}{y^{12}} \text{ or } \frac{16x^{4n}}{y^{12}}$$

Method 2

Simplify the fraction before raising to the fourth power.

$$\left(\frac{-2x^{3n}}{x^{2n}y^3}\right)^4 = \left(\frac{-2x^{3n-2n}}{y^3}\right)^4$$

$$= \left(\frac{-2x^{n}}{y^3}\right)^4$$

$$= \frac{16x^{4n}}{y^{12}}$$

CHECK Your Progress

4A. $\left(\frac{3x^2y}{2xy^4}\right)^3$ **4B.** $\left(\frac{-3x^{-5}y^{-2n}}{5x^{-6}}\right)^4$

Online Personal Tutor at algebra2.com

Scientific Notation The form that you usually write numbers in is **standard notation.** A number is in **scientific notation** when it is in the form $a \times 10^n$, where $1 \le a < 10$ and n is an integer. Real-world problems using numbers in scientific notation often involve units of measure. Performing operations with units is known as **dimensional analysis.**

Real-World EXAMPLE

5 **ASTRONOMY** **After the Sun, the next-closest star to Earth is Alpha Centauri C, which is about 4×10^{16} meters away. How long does it take light from Alpha Centauri C to reach Earth? Use the information at the left.**

Begin with the formula $d = rt$, where d is distance, r is rate, and t is time.

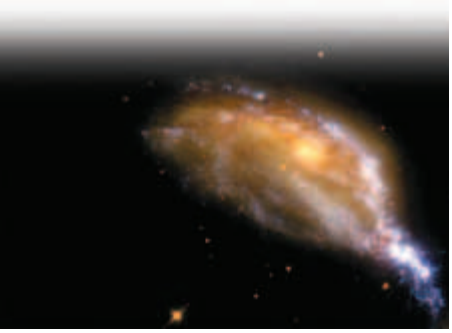
$t = \frac{d}{r}$ Solve the formula for time.

$= \frac{4 \times 10^{16} \text{ m}}{3.00 \times 10^8 \text{ m/s}}$ ← Distance from Alpha Centauri C to Earth

$= \frac{4}{3.00} \cdot \frac{10^{16}}{10^8} \cdot \frac{\text{m}}{\text{m/s}}$ **Estimate:** The result should be slightly greater than $\frac{10^{16}}{10^8}$ or 10^8.

$\approx 1.33 \times 10^8$ s $\frac{4}{3.00} \approx 1.33$, $\frac{10^{16}}{10^8} = 10^{16-8}$ or 10^8, $\frac{\text{m}}{\text{m/s}} = \text{m} \cdot \frac{\text{s}}{\text{m}} = \text{s}$

It takes about 1.33×10^8 seconds or 4.2 years for light from Alpha Centauri C to reach Earth.



AFP/CORBIS

Real-World Link

Light travels at a speed of about 3.00×10^8 m/s. The distance that light travels in a year is called a *light-year.*

Source: www.britannica.com

CHECK Your Progress

5. The density D of an object in grams per milliliter is found by dividing the mass m of the substance by the volume V of the object. A sample of platinum has a mass of 8.4×10^{-2} kilogram and a volume of 4×10^{-6} cubic meter. Use this information to calculate the density of platinum.

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CHECK Your Understanding

Simplify. Assume that no variable equals 0.

Examples 1, 2 (pp. 312–313)

1. $(-3x^2y^3)(5x^5y^6)$
2. $\dfrac{30y^4}{-5y^2}$
3. $\dfrac{-2a^3b^6}{18a^2b^2}$

Example 3 (p. 314)

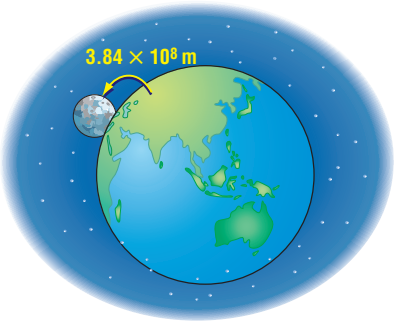
4. $(2b)^4$
5. $\left(\dfrac{1}{w^4z^2}\right)^3$
6. $\left(\dfrac{cd}{3}\right)^{-2}$

Example 4 (p. 315)

7. $(n^3)^3(n^{-3})^3$
8. $\dfrac{81p^6q^5}{(3p^2q)^2}$
9. $\left(\dfrac{-6x^6}{3x^3}\right)^{-2}$

Example 5 (p. 315)

10. **ASTRONOMY** Refer to Example 5 on page 315. The average distance from Earth to the Moon is about 3.84×10^8 meters. How long would it take a radio signal traveling at the speed of light to cover that distance?



Exercises

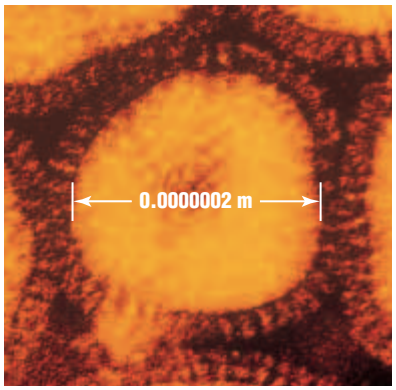
HOMEWORK HELP

For Exercises	See Examples
11–14	1
15–18	2
16–19	3
23–26	4
27, 28	5

Simplify. Assume that no variable equals 0.

11. $\left(\dfrac{1}{3}a^8b^2\right)(2a^2b^2)$
12. $(5cd^2)(-c^4d)$
13. $(7x^3y^{-5})(4xy^3)$
14. $(-3b^3c)(7b^2c^2)$
15. $\dfrac{a^2n^6}{an^5}$
16. $\dfrac{-y^5z^7}{y^2z^5}$
17. $\dfrac{-5x^3y^3z^4}{20x^3y^7z^4}$
18. $\dfrac{3a^5b^3c^3}{9a^3b^7c}$
19. $(n^4)^4$
20. $(z^2)^5$
21. $(2x)^4$
22. $(-2c)^3$
23. $(a^3b^3)(ab)^{-2}$
24. $(-2r^2s)^3(3rs^2)$
25. $\dfrac{2c^3d(3c^2d^5)}{30c^4d^2}$
26. $\dfrac{-12m^4n^8\,(m^3n^2)}{36m^3n}$

27. **BIOLOGY** Use the diagram at the right to write the diameter of a typical flu virus in scientific notation. Then estimate the area of a typical flu virus. (*Hint:* Treat the virus as a circle.)



28. **POPULATION** The population of Earth is about 6.445×10^9. The land surface area of Earth is 1.483×10^8 km^2. What is the population density for the land surface area of Earth?

Simplify. Assume that no variable equals 0.

29. $2x^2(6y^3)(2x^2y)$
30. $3a(5a^2b)(6ab^3)$
31. $\dfrac{30a^{-2}b^{-6}}{60a^{-6}b^{-8}}$
32. $\dfrac{12x^{-3}y^{-2}z^{-8}}{30x^{-6}y^{-4}z^{-1}}$
33. $\left(\dfrac{x}{y^{-1}}\right)^{-2}$
34. $\left(\dfrac{v}{w^{-2}}\right)^{-3}$
35. $\left(\dfrac{8a^3b^2}{16a^2b^3}\right)^4$
36. $\left(\dfrac{6x^2y^4}{3x^4y^3}\right)^3$
37. $\left(\dfrac{4x^{-3}y^2}{xy^{-5}}\right)^{-2}$

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38. If $2^{r+5} = 2^{2r-1}$, what is the value of r?

39. What value of r makes $y^{28} = y^{3r} \cdot y^7$ true?

40. INCOME In 2003, the population of Texas was about 2.21×10^7. The personal income for the state that year was about 6.43×10^{11} dollars. What was the average personal income?

EXTRA PRACTICE
See pages 902, 931.
Math Online
Self-Check Quiz at
algebra2.com

41. RESEARCH Use the Internet or other source to find the masses of Earth and the Sun. About how many times as large as Earth is the Sun?

H.O.T. Problems

42. OPEN ENDED Write an example that illustrates a property of powers. Then use multiplication or division to explain why it is true.

43. FIND THE ERROR Alejandra and Kyle both simplified $\frac{2a^2b}{(-2ab^3)^{-2}}$. Who is correct? Explain your reasoning.

Alejandra

$$\frac{2a^2b}{(-2ab^3)^{-2}} = (2a^2b)(-2ab^3)^2$$
$$= (2a^2b)(-2)^2a^2(b^3)^2$$
$$= (2a^2b)2^2a^2b^6$$
$$= 8a^4b^7$$

Kyle

$$\frac{2a^2b}{(-2ab^3)^{-2}} = \frac{2a^2b}{(-2)^2a(b^3)^{-2}}$$
$$= \frac{2a^2b}{4ab^{-6}}$$
$$= \frac{2a^2bb^6}{4a}$$
$$= \frac{ab^7}{2}$$

44. REASONING Determine whether $x^y \cdot x^z = x^{yz}$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

45. CHALLENGE Determine which is greater, 100^{10} or 10^{100}. Explain.

46. *Writing in Math* Use the information on page 312 to explain why scientific notation is useful in economics. Include the 2004 national debt of \$7,379,100,000,000 and the U.S. population of 293,700,000, both written in words and in scientific notation, and an explanation of how to find the amount of debt per person with the result written in scientific notation and in standard notation.

STANDARDIZED TEST PRACTICE

47. ACT/SAT Which expression is equal to $\frac{(2x^2)^3}{12x^4}$?

A $\frac{x}{2}$

B $\frac{2x}{3}$

C $\frac{1}{2x^2}$

D $\frac{2x^2}{3}$

48. REVIEW Four students worked the same math problem. Each student's work is shown below.

Student F
$$x^2 x^{-5} = \frac{x^2}{x^5} = \frac{1}{x^3}, x \neq 0$$

Student G
$$x^2 x^{-5} = \frac{x^2}{x^{-5}} = x^7, x \neq 0$$

Student H
$$x^2 x^{-5} = \frac{x^2}{x^{-5}} = x^{-7}, x \neq 0$$

Student J
$$x^2 x^{-5} = \frac{x^2}{x^5} = x^3, x \neq 0$$

Which is a completely correct solution?

F Student F
G Student G
H Student H
J Student J

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Spiral Review

Solve each inequality algebraically. (Lesson 5-8)

49. $x^2 - 8x + 12 < 0$

50. $x^2 + 2x - 86 \geq -23$

51. $15x^2 + 4x + 12 \leq 0$

Graph each function. (Lesson 5-7)

52. $y = -2(x - 2)^2 + 3$

53. $y = \frac{1}{3}(x + 5)^2 - 1$

54. $y = \frac{1}{2}x^2 + x + \frac{3}{2}$

Evaluate each determinant. (Lesson 4-3)

55. $\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix}$

56. $\begin{vmatrix} 1 & 0 & -3 \\ 2 & -1 & 4 \\ -3 & 0 & 2 \end{vmatrix}$

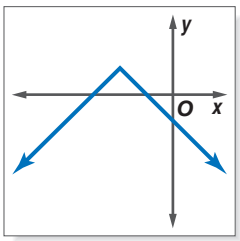
Solve each system of equations. (Lesson 3-5)

57.
$x + y = 5$
$x + y + z = 4$
$2x - y + 2z = -1$

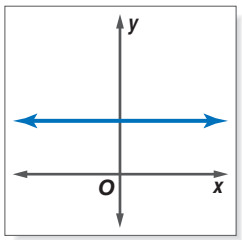
58.
$a + b + c = 6$
$2a - b + 3c = 16$
$a + 3b - 2c = -6$

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

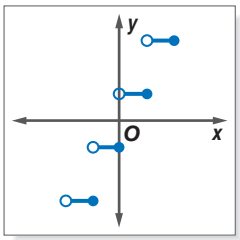
59.



60.



61.

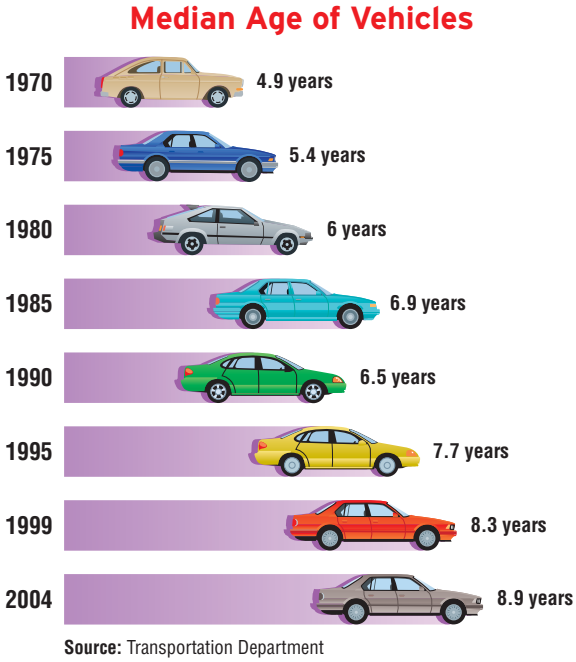


TRANSPORTATION **For Exercises 62–64, refer to the graph at the right.** (Lesson 2-5)

62. Make a scatter plot of the data, where the horizontal axis is the number of years since 1975.

63. Write a prediction equation.

64. Predict the median age of vehicles on the road in 2015.



Solve each equation. (Lesson 1-3)

65. $2x + 11 = 25$

66. $-12 - 5x = 3$

GET READY for the Next Lesson

PREREQUISITE SKILL **Use the Distributive Property to find each product.** (Lesson 1-2)

67. $2(x + y)$

68. $3(x - z)$

69. $4(x + 2)$

70. $-2(3x - 5)$

71. $-5(x - 2y)$

72. $-3(-y + 5)$

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READING MATH

Dimensional Analysis

Real-world problems often involve units of measure. Performing operations with units is called **dimensional analysis.** You can use dimensional analysis to convert units or to perform calculations.

Example **A car's gas tank holds 14 gallons of gasoline and the car gets 16 miles per gallon. How many miles can be driven on a full tank of gasoline?**

You want to find the number of miles that can be driven on 1 tank of gasoline, or the number of *miles per tank*. You know that there are 14 gallons per tank and 16 miles per gallon. Translate these into fractions that you can multiply.

$$\frac{14 \text{ gal}}{1 \text{ tank}} \cdot \frac{16 \text{ mi}}{1 \text{ gal}} = \frac{14 \cancel{\text{gal}}}{1 \text{ tank}} \cdot \frac{16 \text{ mi}}{1 \cancel{\text{gal}}}$$ The units *gallons* cancel out.

$$= (14)(16) \text{ mi/tank}$$ Simplify.

$$= 224 \text{ mi/tank}$$ Multiply.

So, 224 miles can be driven on a full tank of gasoline. This answer is reasonable because the final units are mi/tank, not mi/gal, gal/mi, or mi.

Reading to Learn

Solve each problem using dimensional analysis. Include the appropriate units with your answer.

1. How many miles will a person run during a 5-kilometer race? (*Hint:* 1 km $\approx$ 0.62 mi)

2. A zebra can run 40 miles per hour. How far can a zebra run in 3 minutes?

3. A cyclist traveled 43.2 miles at an average speed of 12 miles per hour. How long did the cyclist ride?

4. The average student is in class 315 minutes/day. How many hours per day is this?

5. If you are going 50 miles per hour, how many feet per second are you traveling?

6. The equation $d = \frac{1}{2}(9.8 \text{ m/s}^2)(3.5 \text{ s})^2$ represents the distance d that a ball falls 3.5 seconds after it is dropped from a tower. Find the distance.

7. Explain what the following statement means.
Dimensional analysis tells you what to multiply or divide.

8. Explain how dimensional analysis can be useful in checking the reasonableness of your answer.

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6-2 Operations with Polynomials

Main Ideas

- Add and subtract polynomials.
- Multiply polynomials.

New Vocabulary

degree of a polynomial

GET READY for the Lesson

Shenequa has narrowed her choice for which college to attend. She is most interested in Coastal Carolina University, where the current year's tuition is $3430. Shenequa assumes that tuition will increase at a rate of 6% per year. You can use polynomials to represent the increasing tuition costs.

College Choices

College	Tuition
Allegheny College	$26,650
University of Maryland	$7821
Coastal Carolina University	$3430

Add and Subtract Polynomials If r represents the rate of increase of tuition, then the tuition for the second year will be $3430(1 + r)$. For the third year, it will be $3430(1 + r)^2$, or $3430r^2 + 6860r + 3430$ in expanded form. The **degree of a polynomial** is the degree of the monomial with the greatest degree. For example, the degree of this polynomial is 2.

Study Tip

Look Back

You can review **polynomials** in Lesson 1-1.

EXAMPLE Degree of a Polynomial

1 **Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.**

a. $\frac{1}{6}x^3y^5 - 9x^4$

This expression is a polynomial because each term is a monomial. The degree of the first term is $3 + 5$ or 8, and the degree of the second term is 4. The degree of the polynomial is 8.

b. $x + \sqrt{x} + 5$

This expression is not a polynomial because $\sqrt{x}$ is not a monomial.

c. $x^{-2} + 3x^{-1} - 4$

This expression is not a polynomial because x^{-2} and x^{-1} are not monomials. $x^{-2} = \frac{1}{x^2}$ and $x^{-1} = \frac{1}{x}$. Monomials cannot contain variables in the denominator.

CHECK Your Progress

1A. $\frac{x}{y} + 3x^2$ **1B.** $x^5y + 9x^4y^3 - 2xy$

To *simplify* a polynomial means to perform the operations indicated and combine like terms.

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EXAMPLE Simplify Polynomials

Study Tip

Alternate Methods

Notice that Example 2a uses a horizontal method and Example 2b uses a vertical method to simplify. Either method will yield a correct solution.

2 **Simplify each expression.**

a. $(3x^2 - 2x + 3) - (x^2 + 4x - 2)$

Remove parentheses and group like terms together.

$(3x^2 - 2x + 3) - (x^2 + 4x - 2)$

$= 3x^2 - 2x + 3 - x^2 - 4x + 2$ Distribute the -1.

$= (3x^2 - x^2) + (-2x - 4x) + (3 + 2)$ Group like terms.

$= 2x^2 - 6x + 5$ Combine like terms.

b. $(5x^2 - 4x + 1) + (-3x^2 + x - 3)$

Align like terms vertically and add.

$$\begin{array}{r} 5x^2 - 4x + 1 \\ (+)\ -3x^2 + x - 3 \\ \hline 2x^2 - 3x - 2 \end{array}$$

CHECK Your Progress

2A. $(-x^2 - 3x + 4) - (x^2 + 2x + 5)$ **2B.** $(3x^2 - 6) + (-x + 1)$

Multiply Polynomials You can use the Distributive Property to multiply polynomials.

EXAMPLE Simplify Using the Distributive Property

3 **Find $2x(7x^2 - 3x + 5)$.**

$2x(7x^2 - 3x + 5) = 2x(7x^2) + 2x(-3x) + 2x(5)$ Distributive Property

$= 14x^3 - 6x^2 + 10x$ Multiply the monomials.

CHECK Your Progress **Find each product.**

3A. $\frac{4}{3}x^2(6x^2 + 9x - 12)$ **3B.** $-2a(-3a^2 - 11a + 20)$

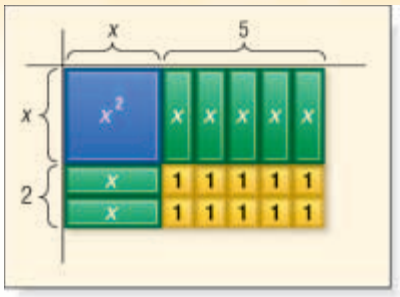
You can use algebra tiles to model the product of two binomials.

ALGEBRA LAB

Multiplying Binomials

Use algebra tiles to find the product of $x + 5$ and $x + 2$.

- Draw a 90° angle on your paper.
- Use an x tile and a 1 tile to mark off a length equal to $x + 5$ along the top.
- Use the tiles to mark off a length equal to $x + 2$ along the side.
- Draw lines to show the grid formed.
- Fill in the lines with the appropriate tiles to show the area product. The model shows the polynomial $x^2 + 7x + 10$.



The area of the rectangle is the product of its length and width.
So, $(x + 5)(x + 2) = x^2 + 7x + 10$.

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EXAMPLE 4 Multiply Polynomials

Find $(n^2 + 6n - 2)(n + 4)$.

Method 1 Horizontally

$(n^2 + 6n - 2)(n + 4)$

$= n^2(n + 4) + 6n(n + 4) + (-2)(n + 4)$ Distributive Property

$= n^2 \cdot n + n^2 \cdot 4 + 6n \cdot n + 6n \cdot 4 + (-2) \cdot n + (-2) \cdot 4$ Distributive Property

$= n^3 + 4n^2 + 6n^2 + 24n - 2n - 8$ Multiply monomials.

$= n^3 + 10n^2 + 22n - 8$ Combine like terms.

Concepts in Motion
Animation
algebra2.com

Method 2 Vertically

$$\begin{array}{rr} & n^2 + 6n - 2 \\ (\times) & n + 4 \\ \hline & 4n^2 + 24n - 8 \\ n^3 + 6n^2 - 2n & \\ \hline n^3 + 10n^2 + 22n - 8 & \end{array}$$

CHECK Your Progress **Find each product.**

4A. $(x^2 + 4x + 16)(x - 4)$ **4B.** $(2x^2 - 4x + 5)(3x - 1)$

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CHECK Your Understanding

Example 1 (p. 320)

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

1. $2a + 5b$ **2.** $\frac{1}{3}x^3 - 9y$ **3.** $\frac{mw^2 - 3}{nz^3 + 1}$

Simplify.

Examples 2–4 (pp. 321–322)

4. $(2a + 3b) + (8a - 5b)$ **5.** $(x^2 - 4x + 3) - (4x^2 + 3x - 5)$

6. $2x(3y + 9)$ **7.** $2p^2q(5pq - 3p^3q^2 + 4pq^4)$

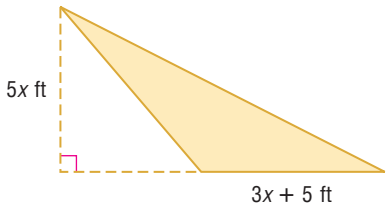
8. $(y - 10)(y + 7)$ **9.** $(x + 6)(x + 3)$

10. $(2z - 1)(2z + 1)$ **11.** $(2m - 3n)^2$

12. $(x + 1)(x^2 - 2x + 3)$ **13.** $(2x - 1)(x^2 - 4x + 4)$

Example 4 (p. 322)

14. GEOMETRY Find the area of the triangle.



Exercises

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

15. $3z^2 - 5z + 11$ **16.** $x^3 - 9$ **17.** $\frac{6xy}{z} - \frac{3c}{d}$

18. $\sqrt{m - 5}$ **19.** $5x^2y^4 + x\sqrt{3}$ **20.** $\frac{4}{3}y^2 + \frac{5}{6}y^7$

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HOMEWORK HELP

For Exercises	See Examples
15–20	1
21–24	2
25–28	3
29–36	4

Simplify.

21. $(3x^2 - x + 2) + (x^2 + 4x - 9)$

22. $(5y + 3y^2) + (-8y - 6y^2)$

23. $(9r^2 + 6r + 16) - (8r^2 + 7r + 10)$

24. $(7m^2 + 5m - 9) + (3m^2 - 6)$

25. $4b(cb - zd)$

26. $4a(3a^2 + b)$

27. $-5ab^2(-3a^2b + 6a^3b - 3a^4b^4)$

28. $2xy(3xy^3 - 4xy + 2y^4)$

29. $(p + 6)(p - 4)$

30. $(a + 6)(a + 3)$

31. $(b + 5)(b - 5)$

32. $(6 - z)(6 + z)$

33. $(3x + 8)(2x + 6)$

34. $(4y - 6)(2y + 7)$

35. $(3b - c)^3$

36. $(x^2 + xy + y^2)(x - y)$

37. PERSONAL FINANCE Toshiro has $850 to invest. He can invest in a savings account that has an annual interest rate of 1.7%, and he can invest in a money market account that pays about 3.5% per year. Write a polynomial to represent the amount of interest he will earn in 1 year if he invests x dollars in the savings account and the rest in the money market account.

E-SALES For Exercises 38 and 39, use the following information.
A small online retailer estimates that the cost, in dollars, associated with selling x units of a particular product is given by the expression $0.001x^2 + 5x + 500$. The revenue from selling x units is given by $10x$.

38. Write a polynomial to represent the profit generated by the product.

39. Find the profit from sales of 1850 units.

40. Simplify $(c^2 - 6cd - 2d^2) + (7c^2 - cd + 8d^2) - (-c^2 + 5cd - d^2)$.

41. Find the product of $x^2 + 6x - 5$ and $-3x + 2$.

Simplify.

42. $(4x^2 - 3y^2 + 5xy) - (8xy + 3y^2)$

43. $(10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)$

44. $\frac{3}{4}x^2(8x + 12y - 16xy^2)$

45. $\frac{1}{2}a^3(4a - 6b + 8ab^4)$

46. $d^{-3}(d^5 - 2d^3 + d^{-1})$

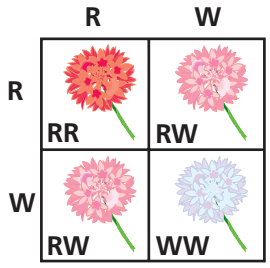
47. $x^{-3}y^2(yx^4 + y^{-1}x^3 + y^{-2}x^2)$

48. $(a^3 - b)(a^3 + b)$

49. $(m^2 - 5)(2m^2 + 3)$

50. $(x - 3y)^2$

51. $(1 + 4c)^2$



Real-World Link

Genetics

The possible genes of parents and offspring can be summarized in a *Punnett square*, such as the one above.

Source: *Biology: The Dynamics of Life*

52. GENETICS Suppose R and W represent two genes that a plant can inherit from its parents. The terms of the expansion of $(R + W)^2$ represent the possible pairings of the genes in the offspring. Write $(R + W)^2$ as a polynomial.

H.O.T. Problems

53. OPEN ENDED Write a polynomial of degree 5 that has three terms.

54. Which One Doesn't Belong? Identify the expression that does not belong with the other three. Explain your reasoning.

$3xy + 6x^2$	$\frac{5}{x^2}$	$x + 5$	$5b + 11c - 9ad^2$

55. CHALLENGE What is the degree of the product of a polynomial of degree 8 and a polynomial of degree 6? Include an example to support your answer.

56. *Writing in Math* Use the information about tuition increases to explain how polynomials can be applied to financial situations. Include an explanation of how a polynomial can be applied to a situation with a fixed percent rate of increase and an explanation of how to use an expression and the 6% rate of increase to estimate Shenequa's tuition in the fourth year.

EXTRA PRACTICE
See pages 902, 931.

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STANDARDIZED TEST PRACTICE

57. ACT/SAT Which polynomial has degree 3?

A $x^3 + x^2 - 2x^4$

B $-2x^2 - 3x + 4$

C $x^2 + x + 12^3$

D $1 + x + x^3$

58. REVIEW $(-4x^2 + 2x + 3) - 3(2x^2 - 5x + 1) =$

F $2x^2$

G $-10x^2$

H $-10x^2 + 17x$

J $2x^2 + 17x$

Spiral Review

Simplify. Assume that no variable equals 0. (Lesson 6-1)

59. $(-4d^2)^3$ **60.** $5rt^2(2rt)^2$ **61.** $\frac{x^2yz^4}{xy^3z^2}$ **62.** $\left(\frac{3ab^2}{6a^2b}\right)^2$

Graph each inequality. (Lesson 5-8)

63. $y > x^2 - 4x + 6$ **64.** $y \leq -x^2 + 6x - 3$ **65.** $y < x^2 - 2x$

Determine whether each function has a maximum or a minimum value. Then find the maximum or minimum value of each function. (Lesson 5-1)

66. $f(x) = x^2 - 8x + 3$ **67.** $f(x) = -3x^2 - 18x + 5$ **68.** $f(x) = -7 + 4x^2$

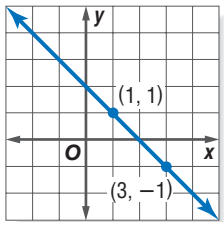
Use matrices A, B, C, and D to find the following. (Lesson 4-2)

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -3 \\ 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 6 & -2 \end{bmatrix} \quad C = \begin{bmatrix} -4 & -5 \\ -3 & 1 \\ 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -3 & 4 \end{bmatrix}$$

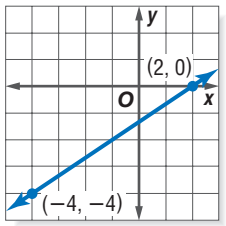
69. $A + D$ **70.** $B - C$ **71.** $3B - 2A$

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

72.



73.



74. In 1990, 2,573,225 people attended St. Louis Cardinals home games. In 2004, the attendance was 3,048,427. What was the average annual rate of increase in attendance?

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify. Assume that no variable equals 0.** (Lesson 6-1)

75. $\frac{x^3}{x}$ **76.** $\frac{4y^5}{2y^2}$ **77.** $\frac{x^2y^3}{xy}$ **78.** $\frac{9a^3b}{3ab}$

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6-3 Dividing Polynomials

Main Ideas

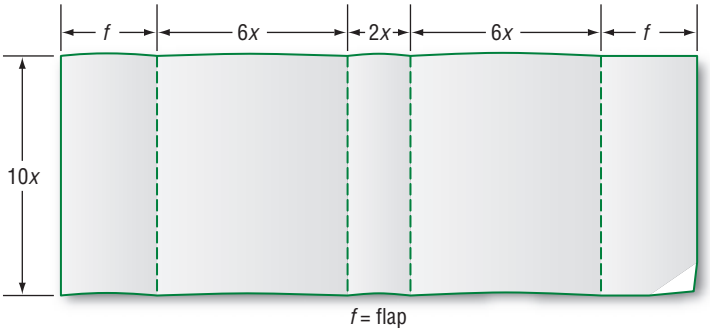
- Divide polynomials using long division.
- Divide polynomials using synthetic division.

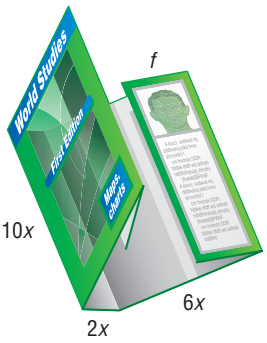
New Vocabulary

synthetic division

GET READY for the Lesson

Arianna needed $140x^2 + 60x$ square inches of paper to make a book jacket $10x$ inches tall. In figuring the area she needed, she allowed for a front and back flap. If the spine of the book jacket is $2x$ inches, and the front and back of the book jacket are $6x$ inches, how wide are the front and back flaps? You can use a quotient of polynomials to help you find the answer.





Use Long Division In Lesson 6-1, you learned to divide monomials. You can divide a polynomial by a monomial by using those same skills.

EXAMPLE Divide a Polynomial by a Monomial

1 Simplify $\dfrac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy}$.

$$\frac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy} = \frac{4x^3y^2}{4xy} + \frac{8xy^2}{4xy} - \frac{12x^2y^3}{4xy}$$ Sum of quotients

$$= \frac{4}{4} \cdot x^{3-1}y^{2-1} + \frac{8}{4} \cdot x^{1-1}y^{2-1} - \frac{12}{4} \cdot x^{2-1}y^{3-1}$$ Divide.

$$= x^2y + 2y - 3xy^2$$ $x^{1-1} = x^0$ or 1

CHECK Your Progress

Simplify.

1A. $\dfrac{9x^2y^3 - 15xy^2 + 12xy^3}{3xy^2}$

1B. $\dfrac{16a^5b^3 + 12a^3b^4 - 20ab^5}{4ab^3}$

1C. $(20c^4d^2f - 16cf + 4cdf)(4cdf)^{-1}$

1D. $(18x^2y + 27x^3y^2z)(3xy)^{-2}$

You can use a process similar to long division to divide a polynomial by a polynomial with more than one term. The process is known as the *division algorithm*. When doing the division, remember that you can only add or subtract like terms.

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EXAMPLE Division Algorithm

2 **Use long division to find $(z^2 + 2z - 24) \div (z - 4)$.**

$$\begin{array}{r} z \\ z-4\overline{)z^2+2z-24} \\ \underline{(-)z^2-4z} \\ 6z-24 \end{array} \qquad \begin{array}{l} \\ z(z-4) = z^2 - 4z \\ 2z - (-4z) = 6z \end{array}$$

$$\begin{array}{r} z+6 \\ z-4\overline{)z^2+2z-24} \\ \underline{(-)z^2-4z} \\ 6z-24 \\ \underline{(-)6z-24} \\ 0 \end{array}$$

The quotient is $z + 6$. The remainder is 0.

CHECK Your Progress

Use long division to find each quotient.

2A. $(x^2 + 7x - 30) \div (x - 3)$ **2B.** $(x^2 - 13x + 12) \div (x - 1)$

Just as with the division of whole numbers, the division of two polynomials may result in a quotient with a remainder. Remember that $9 \div 4 = 2 + \text{R1}$ and is often written as $2\frac{1}{4}$. The result of a division of polynomials with a remainder can be written in a similar manner.

STANDARDIZED TEST EXAMPLE Quotient with Remainder

3 Which expression is equal to $(t^2 + 3t - 9)(5 - t)^{-1}$?

A $t + 8 - \frac{31}{5 - t}$

B $-t - 8$

C $-t - 8 + \frac{31}{5 - t}$

D $-t - 8 - \frac{31}{5 - t}$

Read the Test Item

Since the second factor has an exponent of -1, this is a division problem.

$(t^2 + 3t - 9)(5 - t)^{-1} = \frac{t^2 + 3t - 9}{5 - t}$

Test-Taking Tip

You may be able to eliminate some of the answer choices by substituting the same value for t in the original expression and the answer choices and evaluating.

Solve the Test Item

$$\begin{array}{r} -t-8 \\ -t+5\overline{)t^2+3t-9} \\ \underline{(-)t^2-5t} \\ 8t-9 \\ \underline{(-)8t-40} \\ 31 \end{array} \qquad \begin{array}{l} \text{For ease in dividing, rewrite } 5 - t \text{ as } -t + 5. \\ -t(-t + 5) = t^2 - 5t \\ 3t - (-5t) = 8t \\ -8(-t + 5) = 8t - 40 \\ \text{Subtract. } -9 - (-40) = 31 \end{array}$$

The quotient is $-t - 8$, and the remainder is 31. Therefore, $(t^2 + 3t - 9)(5 - t)^{-1} = -t - 8 + \frac{31}{5 - t}$. The answer is C.

CHECK Your Progress

3. Which expression is equal to $(r^2 + 5r + 7)(1 - r)^{-1}$?

F $-r - 6 + \frac{13}{1 - r}$ **G** $r + 6$ **H** $r - 6 + \frac{13}{1 - r}$ **J** $r + 6 - \frac{13}{1 - r}$

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Use Synthetic Division

Synthetic division is a simpler process for dividing a polynomial by a binomial. Suppose you want to divide $5x^3 - 13x^2 + 10x - 8$ by $x - 2$ using long division. Compare the coefficients in this division with those in Example 4.

$$
\begin{array}{r}
5x^2 - 3x + 4 \\
x - 2 \overline{)\,5x^3 - 13x^2 + 10x - 8} \\
\underline{(-)5x^3 - 10x^2} \\
-3x^2 + 10x \\
\underline{(-)-3x^2 + 6x} \\
4x - 8 \\
\underline{(-)4x - 8} \\
0
\end{array}
$$

EXAMPLE Synthetic Division

4 **Use synthetic division to find $(5x^3 - 13x^2 + 10x - 8) \div (x - 2)$.**

Step 1 Write the terms of the dividend so that the degrees of the terms are in descending order. Then write just the coefficients as shown at the right.

$$
\begin{array}{cccc}
5x^3 & -13x^2 & +10x & -8 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
5 & -13 & 10 & -8
\end{array}
$$

Step 2 Write the constant r of the divisor $x - r$ to the left. In this case, $r = 2$. Bring the first coefficient, 5, down.

$$
\begin{array}{c|cccc}
2 & 5 & -13 & 10 & -8 \\
\hline
 & 5 & & &
\end{array}
$$

Step 3 Multiply the first coefficient by r: $2 \cdot 5 = 10$. Write the product under the second coefficient. Then add the product and the second coefficient: $-13 + 10 = -3$.

$$
\begin{array}{c|cccc}
2 & 5 & -13 & 10 & -8 \\
 & & 10 & & \\
\hline
 & 5 & -3 & &
\end{array}
$$

Step 4 Multiply the sum, -3, by r: $2(-3) = -6$. Write the product under the next coefficient and add: $10 + (-6) = 4$.

$$
\begin{array}{c|cccc}
2 & 5 & -13 & 10 & -8 \\
 & & & 10 & -6 \\
\hline
 & 5 & -3 & 4 &
\end{array}
$$

Step 5 Multiply the sum, 4, by r: $2 \cdot 4 = 8$. Write the product under the next coefficient and add: $-8 + 8 = 0$. The remainder is 0.

$$
\begin{array}{c|cccc}
2 & 5 & -13 & 10 & -8 \\
 & & & 10 & -6 & 8 \\
\hline
 & 5 & -3 & 4 &
\end{array}
$$

The numbers along the bottom row are the coefficients of the quotient. Start with the power of x that is one less than the degree of the dividend. Thus, the quotient is $5x^2 - 3x + 4$.

CHECK Your Progress **Use synthetic division to find each quotient.**

4A. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$ **4B.** $(3x^3 - 8x^2 + 11x - 14) \div (x - 2)$

To use synthetic division, the divisor must be of the form $x - r$. If the coefficient of x in a divisor is not 1, you can rewrite the division expression so that you can use synthetic division.

EXAMPLE Divisor with First Coefficient Other than 1

5 **Use synthetic division to find $(8x^4 - 4x^2 + x + 4) \div (2x + 1)$.**

Use division to rewrite the divisor so it has a first coefficient of 1.

$$\frac{8x^4 - 4x^2 + x + 4}{2x + 1} = \frac{(8x^4 - 4x^2 + x + 4) \div 2}{(2x + 1) \div 2}$$ Divide numerator and denominator by 2.

$$= \frac{4x^4 - 2x^2 + \frac{1}{2}x + 2}{x + \frac{1}{2}}$$ Simplify the numerator and denominator.

(continued on the next page)

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Since the numerator does not have an x^3-term, use a coefficient of 0 for x^3.

$x - r = x + \frac{1}{2}$, so $r = -\frac{1}{2}$.

$$\begin{array}{r|cccc|c} -\frac{1}{2} & 4 & 0 & -2 & \frac{1}{2} & 2 \\ & & -2 & 1 & \frac{1}{2} & -\frac{1}{2} \\ \hline & 4 & -2 & -1 & 1 & \frac{3}{2} \end{array}$$

The result is $4x^3 - 2x^2 - x + 1 + \dfrac{\frac{3}{2}}{x + \frac{1}{2}}$. Now simplify the fraction.

$$\frac{\frac{3}{2}}{x + \frac{1}{2}} = \frac{3}{2} \div \left(x + \frac{1}{2}\right) \quad \text{Rewrite as a division expression.}$$

$$= \frac{3}{2} \div \frac{2x + 1}{2} \quad x + \tfrac{1}{2} = \tfrac{2x}{2} + \tfrac{1}{2} = \tfrac{2x+1}{2}$$

$$= \frac{3}{2} \cdot \frac{2}{2x + 1} \quad \text{Multiply by the reciprocal.}$$

$$= \frac{3}{2x + 1} \quad \text{The solution is } 4x^3 - 2x^2 - x + 1 + \frac{3}{2x + 1}.$$

CHECK Divide using long division.

$$\begin{array}{r} 4x^3 - 2x^2 - x + 1 \\ 2x + 1 \overline{)\,8x^4 + 0x^3 - 4x^2 + x + 4} \\ \underline{(-)8x^4 + 4x^3} \\ -4x^3 - 4x^2 \\ \underline{(-)-4x^3 - 2x^2} \\ -2x^2 + x \\ \underline{(-)-2x^2 - x} \\ 2x + 4 \\ \underline{(-)2x + 1} \\ 3 \end{array}$$

The result is $4x^3 - 2x^2 - x + 1 + \dfrac{3}{2x + 1}$. ✓

CHECK Your Progress **Use synthetic division to find each quotient.**

5A. $(3x^4 - 5x^3 + x^2 + 7x) \div (3x + 1)$ **5B.** $(8y^5 - 2y^4 - 16y^2 + 4) \div (4y - 1)$

CHECK Your Understanding

Example 1 (p. 325) **Simplify.**

1. $\dfrac{6xy^2 - 3xy + 2x^2y}{xy}$

2. $(5ab^2 - 4ab + 7a^2b)(ab)^{-1}$

3. BAKING The number of cookies produced in a factory each day can be estimated by $C(w) = -w^2 + 16w + 1000$, where w is the number of workers and C is the number of cookies produced. Divide to find the average number of cookies produced per worker.

Examples 2, 4 (pp. 326–327) **Simplify.**

4. $(x^2 - 10x - 24) \div (x + 2)$

5. $(3a^4 - 6a^3 - 2a^2 + a - 6) \div (a + 1)$

6. $(z^5 - 3z^2 - 20) \div (z - 2)$

7. $(x^3 + y^3) \div (x + y)$

8. $\dfrac{x^3 + 13x^2 - 12x - 8}{x + 2}$

9. $(b^4 - 2b^3 + b^2 - 3b + 4)(b - 2)^{-1}$

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Example 3 (p. 326)

10. STANDARDIZED TEST PRACTICE Which expression is equal to $(x^2 - 4x + 6)(x - 3)^{-1}$?

A $x - 1$ **B** $x - 1 + \frac{3}{x-3}$ **C** $x - 1 - \frac{3}{x-3}$ **D** $-x + 1 - \frac{3}{x-3}$

Example 5 (pp. 327–328)

Simplify.

11. $(12y^2 + 36y + 15) \div (6y + 3)$ **12.** $\frac{9b^2 + 9b - 10}{3b - 2}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
13–16	1
17–22	2, 4
23–28	3, 4
29–34	2, 3, 5

Simplify.

13. $\frac{9a^3b^2 - 18a^2b^3}{3a^2b}$ **14.** $\frac{5xy^2 - 6y^3 + 3x^2y^3}{xy}$

15. $(28c^3d - 42cd^2 + 56cd^3) \div (14cd)$ **16.** $(a^3b^2 - a^2b + 2a)(-ab)^{-1}$

17. $(x^3 - 4x^2) \div (x - 4)$ **18.** $(x^3 - 27) \div (x - 3)$

19. $(b^3 + 8b^2 - 20b) \div (b - 2)$ **20.** $(g^2 + 8g + 15)(g + 3)^{-1}$

21. $\frac{y^3 + 3y^2 - 5y - 4}{y + 4}$ **22.** $\frac{m^3 + 3m^2 - 7m - 21}{m + 3}$

23. $(t^5 - 3t^2 - 20)(t - 2)^{-1}$ **24.** $(y^5 + 32)(y + 2)^{-1}$

25. $(2c^3 - 3c^2 + 3c - 4) \div (c - 2)$ **26.** $(2b^3 + b^2 - 2b + 3)(b + 1)^{-1}$

27. $\frac{x^5 - 7x^3 + x + 1}{x + 3}$ **28.** $\frac{3c^5 + 5c^4 + c + 5}{c + 2}$

29. $\frac{4x^3 + 5x^2 - 3x + 1}{4x + 1}$ **30.** $\frac{x^3 - 3x^2 + x - 3}{x^2 + 1}$

31. $(6t^3 + 5t^2 + 9) \div (2t + 3)$ **32.** $\frac{x^4 + x^2 - 3x + 5}{x^2 + 2}$

33. $\frac{2x^4 + 3x^3 - 2x^2 - 3x - 6}{2x + 3}$ **34.** $\frac{6x^4 + 5x^3 + x^2 - 3x + 1}{3x + 1}$

35. ENTERTAINMENT A magician gives these instructions to a volunteer.

- Choose a number and multiply it by 4.
- Then add the sum of your number and 15 to the product you found.
- Now divide by the sum of your number and 3.

What number will the volunteer always have at the end? Explain.

Larry Dale Gordon/Getty Images

Real-World Career

Cost Analyst

Cost analysts study and write reports about the factors involved in the cost of production.

Math Online

For more information, go to algebra2.com.

BUSINESS For Exercises 36 and 37, use the following information.

The number of sports magazines sold can be estimated by $n = \frac{3500a^2}{a^2 + 100}$, where a is the amount of money spent on advertising in hundreds of dollars and n is the number of subscriptions sold.

36. Perform the division indicated by $\frac{3500a^2}{a^2 + 100}$.

37. About how many subscriptions will be sold if \$1500 is spent on advertising?

PHYSICS For Exercises 38–40, suppose an object moves in a straight line so that, after t seconds, it is $t^3 + t^2 + 6t$ feet from its starting point.

38. Find the distance the object travels between the times $t = 2$ and $t = x$, where $x > 2$.

39. How much time elapses between $t = 2$ and $t = x$?

40. Find a simplified expression for the average speed of the object between times $t = 2$ and $t = x$.

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H.O.T. Problems

41. OPEN ENDED Write a quotient of two polynomials such that the remainder is 5.

42. REASONING Review any of the division problems in this lesson. What is the relationship between the degrees of the dividend, the divisor, and the quotient?

43. FIND THE ERROR Shelly and Jorge are dividing $x^3 - 2x^2 + x - 3$ by $x - 4$. Who is correct? Explain your reasoning.

Shelly

4	1	−2	1	−3
		4	−24	100
	1	−6	25	−103

Jorge

4	1	−2	1	−3
		4	8	36
	1	2	9	33

44. CHALLENGE Suppose the result of dividing one polynomial by another is $r^2 - 6r + 9 - \frac{1}{r-3}$. What two polynomials might have been divided?

45. *Writing in Math* Use the information on page 325 to explain how you can use division of polynomials in manufacturing. Include the dimensions of the piece of paper that the publisher needs, the formula from geometry that applies to this situation, and an explanation of how to use division of polynomials to find the width of the flap.

EXTRA PRACTICE
See pages 903, 931.
Math Online
Self-Check Quiz at algebra2.com

STANDARDIZED TEST PRACTICE

46. ACT/SAT What is the remainder when $x^3 - 7x + 5$ is divided by $x + 3$?

A −11 C 1

B −1 D 11

47. REVIEW If $i = \sqrt{-1}$, then $5i(7i) =$

F 70 H −35

G 35 J −70

Spiral Review

Simplify. (Lesson 6-2)

48. $(2x^2 - 3x + 5) - (3x^2 + x - 9)$

49. $y^2z(y^2z^3 - yz^2 + 3)$

50. $(y + 5)(y - 3)$

51. $(a - b)^2$

52. ASTRONOMY Earth is an average of 1.5×10^{11} meters from the Sun. Light travels at 3×10^8 meters per second. About how long does it take sunlight to reach Earth? (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Given $f(x) = x^2 - 5x + 6$, find each value. (Lesson 2-1)

53. $f(-2)$ **54.** $f(2)$ **55.** $f(2a)$ **56.** $f(a + 1)$

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6-4 Polynomial Functions

Main Ideas

- Evaluate polynomial functions.
- Identify general shapes of graphs of polynomial functions.

New Vocabulary

polynomial in one variable
leading coefficient
polynomial function
end behavior

GET READY for the Lesson

A cross section of a honeycomb has a pattern with one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function $f(r) = 3r^2 - 3r + 1$, where r is the number of rings and $f(r)$ is the number of hexagons.

Polynomial Functions The expression $3r^2 - 3r + 1$ is a **polynomial in one variable** since it only contains one variable, r.

KEY CONCEPT *Polynomial in One Variable*

Words A polynomial of degree n in one variable x is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $3x^5 + 2x^4 - 5x^3 + x^2 + 1$
$n = 5, a_5 = 3, a_4 = 2, a_3 = -5, a_2 = 1, a_1 = 0$, and $a_0 = 1$

The degree of a polynomial in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

Polynomial	Expression	Degree	Leading Coefficient
Constant	9	0	9
Linear	$x - 2$	1	1
Quadratic	$3x^2 + 4x - 5$	2	3
Cubic	$4x^3 - 6$	3	4
General	$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$	n	a_n

EXAMPLE 1 **Find Degrees and Leading Coefficients**

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

a. $7x^4 + 5x^2 + x - 9$

This is a polynomial in one variable.

The degree is 4, and the leading coefficient is 7.

(continued on the next page)

Brownie Harris/CORBIS

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b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, x and y.

CHECK Your Progress

1A. $7x^6 - 4x^3 + \frac{1}{x}$ **1B.** $\frac{1}{2}x^2 + 2x^3 - x^5$

A polynomial equation used to represent a function is called a **polynomial function**. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

KEY CONCEPT *Definition of a Polynomial Function*

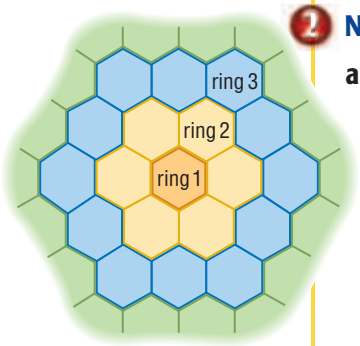
Words A polynomial function of degree n is a continuous function that can be described by an equation of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, a_n is not zero, and n represents a nonnegative integer.

Example $f(x) = 4x^2 - 3x + 2$
$n = 2, a_2 = 4, a_1 = -3, a_0 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

Real-World EXAMPLE

2 **NATURE** **Refer to the application at the beginning of the lesson.**



Rings of a Honeycomb

a. Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1$, 2, and 3.

Find the values of $f(1)$, $f(2)$, and $f(3)$.

$f(r) = 3r^2 - 3r + 1$ $f(r) = 3r^2 - 3r + 1$ $f(r) = 3r^2 - 3r + 1$

$f(1) = 3(1)^2 - 3(1) + 1$ $f(2) = 3(2)^2 - 3(2) + 1$ $f(3) = 3(3)^2 - 3(3) + 1$

$= 3 - 3 + 1$ or 1 $= 12 - 6 + 1$ or 7 $= 27 - 9 + 1$ or 19

You know the numbers of hexagons in the first three rings are 1, 6, and 12. So, the total number of hexagons with one ring is 1, two rings is $6 + 1$ or 7, and three rings is $12 + 6 + 1$ or 19. These match the functional values for $r = 1$, 2, and 3, respectively. That is 1, 7, and 19 are the range values corresponding to the domain values of 1, 2, and 3.

b. Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$ Original function

$f(12) = 3(12)^2 - 3(12) + 1$ Replace r with 12.

$= 432 - 36 + 1$ or 397 Simplify.

CHECK Your Progress

2A. Show that $f(r)$ gives the total number of hexagons when $r = 4$.

2B. Find the total number of hexagons in a honeycomb with 20 rings.

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You can also evaluate functions for variables and algebraic expressions.

EXAMPLE Function Values of Variables

3 **Find $q(a + 1) - 2q(a)$ if $q(x) = x^2 + 3x + 4$.**

To evaluate $q(a + 1)$, replace x in $q(x)$ with $a + 1$.

$$\begin{aligned} q(x) &= x^2 + 3x + 4 && \text{Original function} \\ q(a+1) &= (a+1)^2 + 3(a+1) + 4 && \text{Replace } x \text{ with } a+1. \\ &= a^2 + 2a + 1 + 3a + 3 + 4 && \text{Simplify } (a+1)^2 \text{ and } 3(a+1). \\ &= a^2 + 5a + 8 && \text{Simplify.} \end{aligned}$$

Study Tip

Function Values

When finding function values of expressions, be sure to take note of where the coefficients occur. In Example 3, $2q(a)$ is 2 times the function value of a, not $q(2a)$, the function value of $2a$.

To evaluate $2q(a)$, replace x with a in $q(x)$, then multiply the expression by 2.

$$\begin{aligned} q(x) &= x^2 + 3x + 4 && \text{Original function} \\ 2q(a) &= 2(a^2 + 3a + 4) && \text{Replace } x \text{ with } a. \\ &= 2a^2 + 6a + 8 && \text{Distributive Property} \end{aligned}$$

Now evaluate $q(a + 1) - 2q(a)$.

$$\begin{aligned} q(a+1) - 2q(a) &= a^2 + 5a + 8 - (2a^2 + 6a + 8) && \text{Replace } q(a+1) \text{ and } 2q(a). \\ &= a^2 + 5a + 8 - 2a^2 - 6a - 8 \\ &= -a^2 - a && \text{Simplify.} \end{aligned}$$

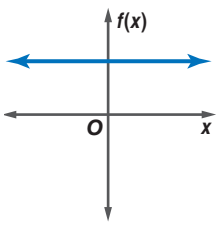
CHECK Your Progress

3A. Find $f(b^2)$ if $f(x) = 2x^2 + 3x - 1$.

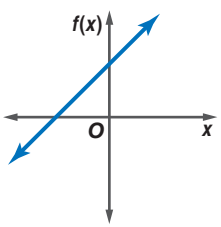
3B. Find $2g(c + 2) + 3g(2c)$ if $g(x) = x^2 - 4$.

Graphs of Polynomial Functions The general shapes of the graphs of several polynomial functions are shown below. These graphs show the *maximum* number of times the graph of each type of polynomial may intersect the x-axis. Recall that the x-coordinate of the point at which the graph intersects the x-axis is called a *zero* of a function. How does the degree compare to the maximum number of real zeros?

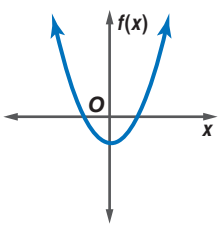
Constant function
Degree 0



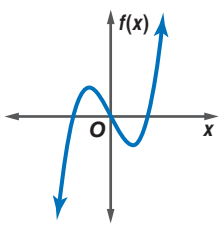
Linear function
Degree 1



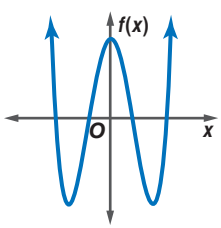
Quadratic function
Degree 2



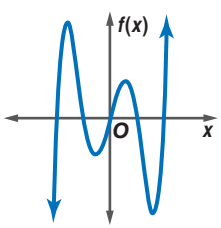
Cubic function
Degree 3



Quartic function
Degree 4



Quintic function
Degree 5



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Concepts in Motion
Animation
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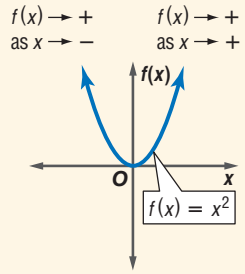
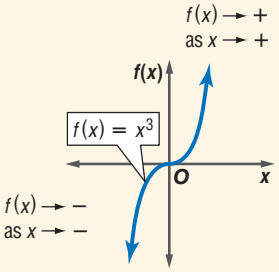
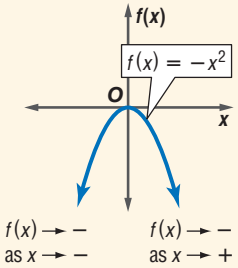
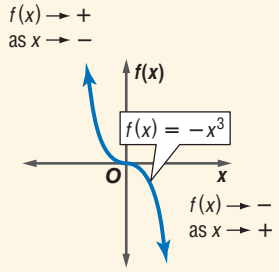
The **end behavior** is the behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). This is represented as $x \to +\infty$ and $x \to -\infty$, respectively. $x \to +\infty$ is read *x approaches positive infinity.* Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph's end behavior.

CONCEPT SUMMARY — End Behavior of a Polynomial Function

Degree: even **Leading Coefficient:** positive **End Behavior:**	**Degree:** odd **Leading Coefficient:** positive **End Behavior:**	**Degree:** even **Leading Coefficient:** negative **End Behavior:**	**Degree:** odd **Leading Coefficient:** negative **End Behavior:**
			
Domain: all reals **Range:** all reals ≥ minimum	**Domain:** all reals **Range:** all reals	**Domain:** all reals **Range:** all reals ≤ maximum	**Domain:** all reals **Range:** all reals

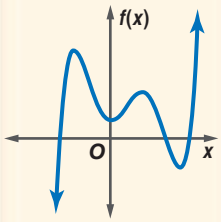
For any polynomial function, the domain is all real numbers. For any polynomial function of odd degree, the range is all real numbers. For polynomial functions of even degree, the range is all real numbers greater than or equal to some number or all real numbers less than or equal to some number; it is never all real numbers.

The graph of an even-degree function may or may not intersect the x-axis. If it intersects the x-axis in two places, the function has two real zeros. If it does not intersect the x-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the x-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the x-axis at least once, and thus the function always has at least one real zero.

Study Tip

Number of Zeros

The number of zeros of an odd-degree function may be less than the maximum by a multiple of 2. For example, the graph of a quintic function may only cross the x-axis 1, 3, or 5 times.



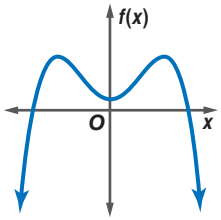
The same is true for an even-degree function. One exception is when the graph of $f(x)$ touches the x-axis.

EXAMPLE 4 Graphs of Polynomial Functions

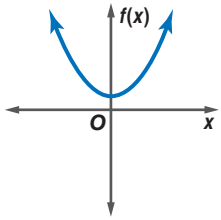
For each graph,

- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function, and**
- **state the number of real zeros.**

a.



b.



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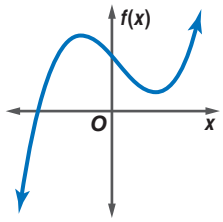
a. • $f(x) \to -\infty$ as $x \to +\infty$. $f(x) \to -\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- The graph intersects the x-axis at two points, so the function has two real zeros.

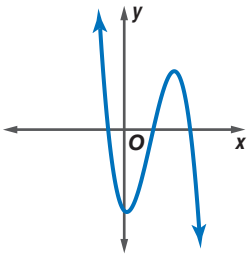
b. • $f(x) \to +\infty$ as $x \to +\infty$. $f(x) \to +\infty$ as $x \to -\infty$.
- It is an even-degree polynomial function.
- This graph does not intersect the x-axis, so the function has no real zeros.

CHECK Your Progress

4A.



4B.



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CHECK Your Understanding

Example 1 (pp. 331–332)

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. $5x^6 - 8x^2$

2. $2b + 4b^3 - 3b^5 - 7$

Example 2 (p. 332)

Find $p(3)$ and $p(-1)$ for each function.

3. $p(x) = -x^3 + x^2 - x$

4. $p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$

5. BIOLOGY The intensity of light emitted by a firefly can be determined by $L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3$, where t is temperature in degrees Celsius and $L(t)$ is light intensity in lumens. If the temperature is 30°C, find the light intensity.

Example 3 (p. 333)

If $p(x) = 2x^3 + 6x - 12$ and $q(x) = 5x^2 + 4$, find each value.

6. $p(a^3)$

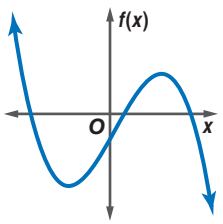
7. $5[q(2a)]$

8. $3p(a) - q(a + 1)$

Example 4 (pp. 334–335)

For each graph,
a. describe the end behavior,
b. determine whether it represents an odd-degree or an even-degree polynomial function, and
c. state the number of real zeros.

9.



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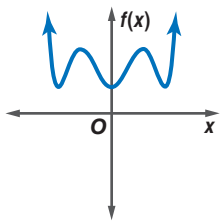

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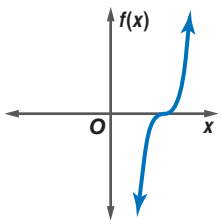

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–17	1
18–21, 34, 35	2
22–27	3
28–33	4

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

12. $7 - x$

13. $(a + 1)(a^2 - 4)$

14. $a^2 + 2ab + b^2$

15. $c^2 + c - \frac{1}{c}$

16. $6x^4 + 3x^2 + 4x - 8$

17. $7 + 3x^2 - 5x^3 + 6x^2 - 2x$

Find $p(4)$ and $p(-2)$ for each function.

18. $p(x) = 2 - x$

19. $p(x) = x^2 - 3x + 8$

20. $p(x) = 2x^3 - x^2 + 5x - 7$

21. $p(x) = x^5 - x^2$

If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

22. $r(3a)$

23. $4p(a)$

24. $p(a^2)$

25. $p(2a^3)$

26. $r(x + 1)$

27. $p(x^2 + 3)$

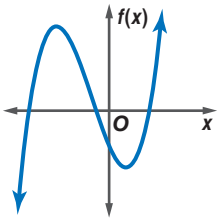
For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

28.



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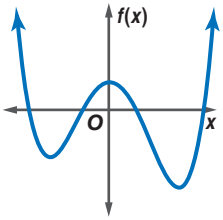

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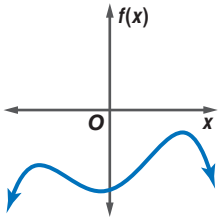

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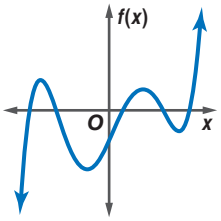

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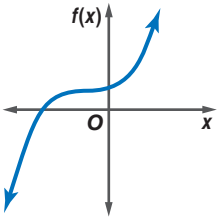

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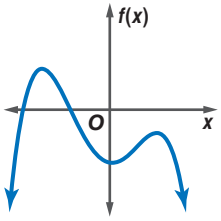

34. ENERGY The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function $P(s) = \frac{s^3}{1000}$, where s represents the speed of the wind in kilometers per hour. Find the units of power $P(s)$ generated by a windmill when the wind speed is 18 kilometers per hour.

35. PHYSICS For a moving object with mass m in kilograms, the kinetic energy KE in joules is given by the function $KE(v) = \frac{1}{2}mv^2$, where v represents the speed of the object in meters per second. Find the kinetic energy of an all-terrain vehicle with a mass of 171 kilograms moving at a speed of 11 meters/second.

Find $p(4)$ and $p(-2)$ for each function.

36. $p(x) = x^4 - 7x^3 + 8x - 6$

37. $p(x) = 7x^2 - 9x + 10$

38. $p(x) = \frac{1}{2}x^4 - 2x^2 + 4$

39. $p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5$

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If $p(x) = 3x^2 - 2x + 5$ and $r(x) = x^3 + x + 1$, find each value.

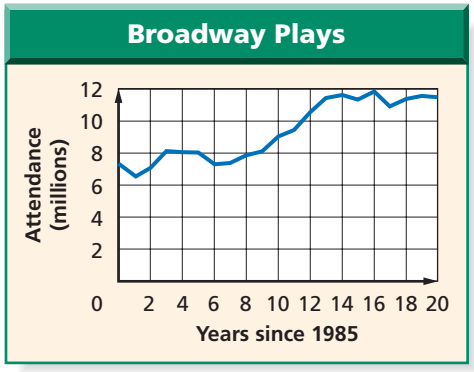
40. $2[p(x + 4)]$ **41.** $r(x + 1) - r(x^2)$ **42.** $3[p(x^2 - 1)] + 4p(x)$

Real-World Link

The Phantom of the Opera is the longest-running Broadway show in history.

Source: playbill.com

THEATER **For Exercises 43–45, use the graph that models the attendance at Broadway plays (in millions) from 1985–2005.**

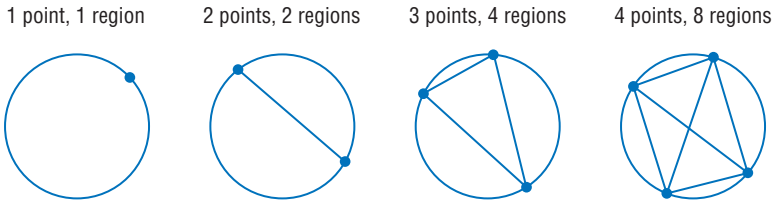


43. Is the graph an odd-degree or even-degree function?

44. Discuss the end behavior.

45. Do you think attendance at Broadway plays will increase or decrease after 2005? Explain your reasoning.

PATTERNS **For Exercises 46–48, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.**



46. The number of regions formed by connecting n points of a circle can be described by the function $f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$. What is the degree of this polynomial function?

47. Find the number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.

48. How many points would you have to connect to form 99 regions?

EXTRA PRACTICE
See pages 903, 931.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

49. **REASONING** Explain why a constant polynomial such as $f(x) = 4$ has degree 0 and a linear polynomial such as $f(x) = x + 5$ has degree 1.

50. **OPEN ENDED** Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots.

51. **REASONING** Determine whether the following statement is *always*, *sometimes* or *never* true. Explain.

A polynomial function that has four real roots is a fourth-degree polynomial.

CHALLENGE **For Exercises 52–55, use the following information.**
The graph of the polynomial function $f(x) = ax(x - 4)(x + 1)$ goes through the point at (5, 15).

52. Find the value of a.

53. For what value(s) of x will $f(x) = 0$?

54. Simplify and rewrite the function as a cubic function.

55. Sketch the graph of the function.

56. ***Writing in Math*** Use the information on page 331 to explain where polynomial functions are found in nature. Include an explanation of how you could use the equation to find the number of hexagons in the tenth ring and any other examples of patterns found in nature that might be modeled by a polynomial equation.

Joan Marcus

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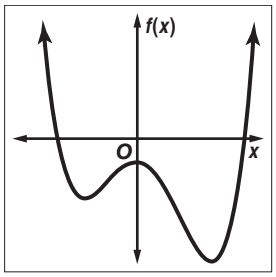
STANDARDIZED TEST PRACTICE

57. ACT/SAT The figure at the right shows the graph of a polynomial function $f(x)$. Which of the following could be the degree of $f(x)$?



A 2 **C** 4

B 3 **D** 5

58. REVIEW Which polynomial represents $(4x^2 + 5x - 3)(2x - 7)$?

F $8x^3 - 18x^2 - 41x - 21$

G $8x^3 + 18x^2 + 29x - 21$

H $8x^3 - 18x^2 - 41x + 21$

J $8x^3 + 18x^2 - 29x + 21$

Spiral Review

Simplify. (Lesson 6-3)

59. $(t^3 - 3t + 2) \div (t + 2)$

60. $(y^2 + 4y + 3)(y + 1)^{-1}$

61. $\dfrac{x^3 - 3x^2 + 2x - 6}{x - 3}$

62. $\dfrac{3x^4 + x^3 - 8x^2 + 10x - 3}{3x - 2}$

63. BUSINESS Ms. Schifflet is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by p. Marty's salary is \$23,450 now. Write a polynomial to represent Marty's salary in one year and another to represent Marty's salary after three years. Assume that the rate of increase will be the same for each of the three years. (Lesson 6-2)

Solve each equation by completing the square. (Lesson 5-5)

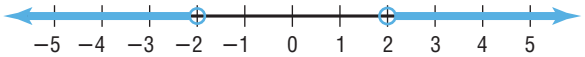
64. $x^2 - 8x - 2 = 0$

65. $x^2 + \frac{1}{3}x - \frac{35}{36} = 0$

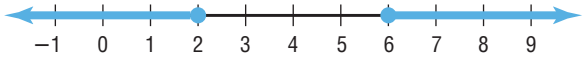
Write an absolute value inequality for each graph. (Lesson 1-6)

66.

67.



68.



69.

Name the property illustrated by each statement. (Lesson 1-3)

70. If $3x = 4y$ and $4y = 15z$, then $3x = 15z$.

71. $5y(4a - 6b) = 20ay - 30by$

72. $2 + (3 + x) = (2 + 3) + x$

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each equation by making a table of values.** (Lesson 5-1)

73. $y = x^2 + 4$

74. $y = -x^2 + 6x - 5$

75. $y = \frac{1}{2}x^2 + 2x - 6$

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6-5 Analyzing Graphs of Polynomial Functions

Main Ideas

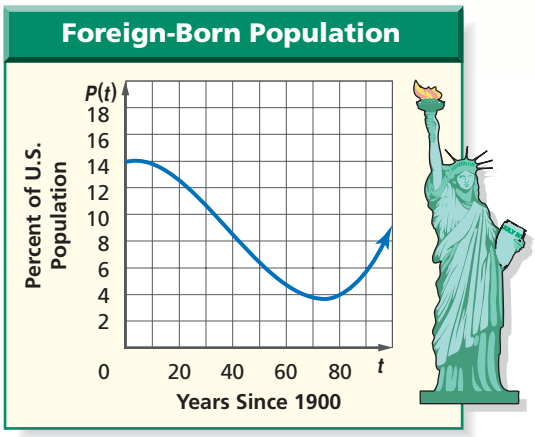
- Graph polynomial functions and locate their real zeros.
- Find the relative maxima and minima of polynomial functions.

New Vocabulary

Location Principle
relative maximum
relative minimum

GET READY for the Lesson

The percent of the United States population that was foreign-born since 1900 can be modeled by $P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14$, where $t = 0$ in 1900. Notice that the graph is decreasing from $t = 5$ to $t = 75$ and then it begins to increase. The points at $t = 5$ and $t = 75$ are turning points in the graph.



Graph Polynomial Functions To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth continuous curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

Study Tip

Graphing Polynomial Functions

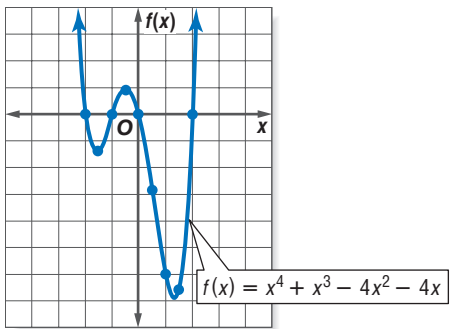
To graph polynomial functions it will often be necessary to include x-values that are not integers.

EXAMPLE **Graph a Polynomial Function**

1 Graph $f(x) = x^4 + x^3 - 4x^2 - 4x$ by making a table of values.

x	$f(x)$
−2.5	≈ 8.4
−2.0	0.0
−1.5	≈ −1.3
−1.0	0.0
−0.5	≈ 0.9

x	$f(x)$
0.0	0.0
0.5	≈ −2.8
1.0	−6.0
1.5	≈ −6.6
2.0	0.0



This is an even-degree polynomial with a positive leading coefficient, so $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$. Notice that the graph intersects the x-axis at four points, indicating there are four real zeros of this function.

CHECK Your Progress

1. Graph $f(x) = x^4 - x^3 - x^2 + x$ by making a table of values.

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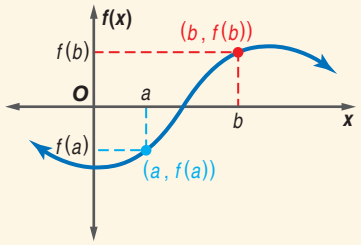
In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, because it is a continuous function, the graph of a polynomial function will cross the x-axis somewhere between pairs of x-values at which the corresponding $f(x)$-values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x-values. This property for locating zeros is called the **Location Principle**.

KEY CONCEPT — **Location Principle**

Words Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

Model

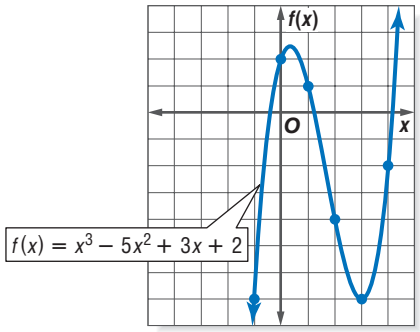


EXAMPLE 2 Locate Zeros of a Function

Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 - 5x^2 + 3x + 2$ is located. Then draw the graph.

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 1, 2, or 3 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

x	$f(x)$	
−2	−32	
−1	−7	change in sign
0	2	
1	1	change in sign
2	−4	
3	−7	
4	−2	change in sign
5	17	



The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$.

Concepts in Motion
Animation
algebra2.com

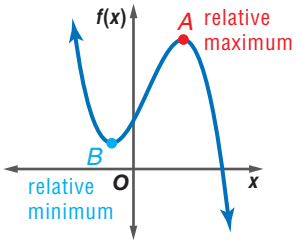
CHECK Your Progress

2. Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 + 4x^2 - 6x - 7$ is located. Then draw the graph.

Reading Math

Maximum and Minimum The plurals of *maximum* and *minimum* are *maxima* and *minima*.

Maximum and Minimum Points The graph at the right shows the shape of a general third-degree polynomial function.



Point A on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. These points are often referred to as *turning points*. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

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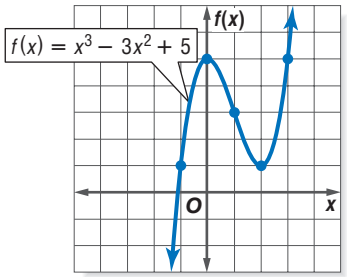
EXAMPLE Maximum and Minimum Points

3 **Graph $f(x) = x^3 - 3x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.**

Make a table of values and graph the equation.

x	$f(x)$	
−2	−15	
−1	1	zero between $x = -2$ and $x = -1$
0	5	← indicates a relative maximum
1	3	
2	1	← indicates a relative minimum
3	5	



Look at the table of values and the graph.

- The values of $f(x)$ change signs between $x = -2$ and $x = -1$, indicating a zero of the function.
- The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.
- The value of $f(x)$ at $x = 2$ is less than the surrounding points, so it is a relative minimum.

CHECK Your Progress

3. Graph $f(x) = x^3 + 4x^2 - 3$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

The graph of a polynomial function can reveal trends in real-world data.

Real-World Link

Gasoline and diesel fuels are the most familiar transportation fuels in this country, but other energy sources are available, including ethanol, a grain alcohol that can be produced from corn or other crops.

Source: U.S. Environmental Protection Agency

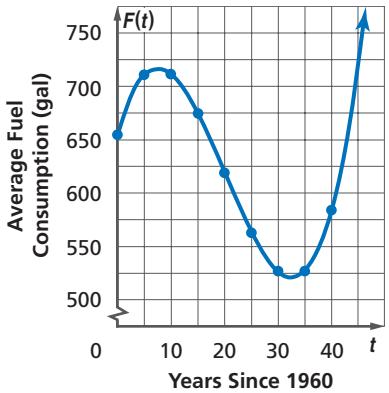
Real-World EXAMPLE Graph a Polynomial Model

4 **ENERGY** **The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation $F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654$, where t is the number of years since 1960.**

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

t	$F(t)$
0	654
5	710.88
10	711.5
15	674.63
20	619
25	563.38
30	526.5
35	527.13
40	584



(continued on the next page)

VCG/Getty Images
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b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as t increases, $F(t)$ increases.

c. What trends in fuel consumption does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise. The trend may continue for some years, but it is unlikely that consumption will rise this quickly indefinitely. Fuel supplies will limit consumption.

CHECK Your Progress

4. The price of one share of stock of a company is given by the function $f(x) = 0.001x^4 - 0.03x^3 + 0.15x^2 + 1.01x + 18.96$, where x is the number of months since January 2006. Graph the equation. Describe the turning points of the graph and its end behavior. What trends in the stock price does the graph suggest? Is it reasonable to assume the trend will continue indefinitely?

Online **Personal Tutor at algebra2.com**

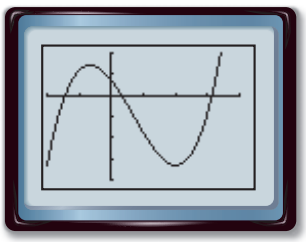
A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

GRAPHING CALCULATOR LAB

Maximum and Minimum Points

You can use a TI-83/84 Plus to find the coordinates of relative maxima and relative minima. Enter the polynomial function in the Y= list and graph the function. Make sure that all the turning points are visible in the viewing window. Find the coordinates of the minimum and maximum points, respectively.

The graphing calculator screen at the right shows one relative maximum and one relative minimum of the function that is graphed.



KEYSTROKES: *Refer to page 243 to review finding maxima and minima.*

THINK AND DISCUSS

1. Graph $f(x) = x^3 - 3x^2 + 4$. Estimate the x-coordinates of the relative maximum and relative minimum points from the graph.

2. Use the maximum and minimum options from the CALC menu to find the exact coordinates of these points. You will need to use the arrow keys to select points to the left and to the right of the point.

3. Graph $f(x) = \frac{1}{2}x^4 - 4x^3 + 7x^2 - 8$. How many relative maximum and relative minimum points does the graph contain? What are the coordinates?

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CHECK Your Understanding

Example 1 (p. 339)

Graph each polynomial function by making a table of values.

1. $f(x) = x^3 - x^2 - 4x + 4$

2. $f(x) = x^4 - 7x^2 + x + 5$

Example 2 (p. 340)

Determine the consecutive integer values of x between which each real zero of each function is located. Then draw the graph.

3. $f(x) = x^3 - x^2 + 1$

4. $f(x) = x^4 - 4x^2 + 2$

Example 3 (p. 341)

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.

5. $f(x) = x^3 + 2x^2 - 3x - 5$

6. $f(x) = x^4 - 8x^2 + 10$

Example 4 (pp. 341–342)

CABLE TV For Exercises 7–10, use the following information.

The number of cable TV systems after 1985 can be modeled by the function $C(t) = -43.2t^2 + 1343t + 790$, where t represents the number of years since 1985.

7. Graph this equation for the years 1985 to 2005.

8. Describe the turning points of the graph and its end behavior.

9. What is the domain of the function? Use the graph to estimate the range.

10. What trends in cable TV subscriptions does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–18	1–3
19–25	4

For Exercises 11–18, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

11. $f(x) = -x^3 - 4x^2$

12. $f(x) = x^3 - 2x^2 + 6$

13. $f(x) = x^3 - 3x^2 + 2$

14. $f(x) = x^3 + 5x^2 - 9$

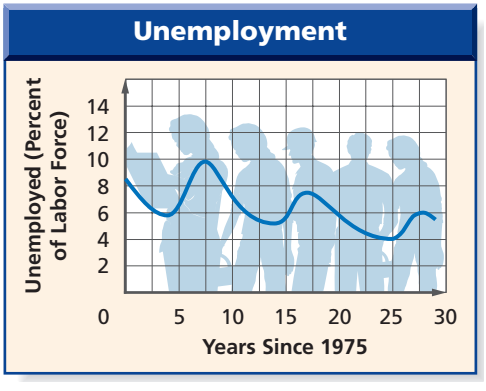
15. $f(x) = -3x^3 + 20x^2 - 36x + 16$

16. $f(x) = x^3 - 4x^2 + 2x - 1$

17. $f(x) = x^4 - 8$

18. $f(x) = x^4 - 10x^2 + 9$

EMPLOYMENT For Exercises 19–22, use the graph that models the unemployment rates from 1975–2004.



19. In what year was the unemployment rate the highest? the lowest?

20. Describe the turning points and the end behavior of the graph.

21. If this graph was modeled by a polynomial equation, what is the least degree the equation could have?

22. Do you expect the unemployment rate to increase or decrease from 2005 to 2010? Explain your reasoning.

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Real-World Link

As children develop, their sleeping needs change. Infants sleep about 16–18 hours a day. Toddlers usually sleep 10–12 hours at night and take one or two daytime naps. School-age children need 9–11 hours of sleep, and teens need at least 9 hours of sleep.

Source: www.kidshealth.org

HEALTH **For Exercises 23–25, use the following information.** During a regular respiratory cycle, the volume of air in liters in human lungs can be described by $V(t) = 0.173t + 0.152t^2 - 0.035t^3$, where t is the time in seconds.

23. Estimate the real zeros of the function by graphing.

24. About how long does a regular respiratory cycle last?

25. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air.

For Exercises 26–31, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

26. $f(x) = -x^4 + 5x^2 - 2x - 1$

27. $f(x) = -x^4 + x^3 + 8x^2 - 3$

28. $f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$

29. $f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5$

30. $f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$

31. $f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6$

CHILD DEVELOPMENT **For Exercises 32 and 33, use the following information.** The average height (in inches) for boys ages 1 to 20 can be modeled by the equation $B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25$, where x is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation $G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26$.

32. Graph both equations by making a table of values. Use $x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ as the domain. Round values to the nearest inch.

33. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls?

Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

34. $f(x) = x^3 + x^2 - 7x - 3$

35. $f(x) = -x^3 + 6x^2 - 6x - 5$

36. $f(x) = -x^4 + 3x^2 - 8$

37. $f(x) = 3x^4 - 7x^3 + 4x - 5$

EXTRA PRACTICE
See pages 903, 931.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

38. **OPEN ENDED** Sketch a graph of a function that has one relative maximum point and two relative minimum points.

CHALLENGE **For Exercises 39–41, sketch a graph of each polynomial.**

39. even-degree polynomial function with one relative maximum and two relative minima

40. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative

41. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

42. **REASONING** Explain the Location Principle and how to use it.

43. ***Writing in Math*** Use the information about foreign-born population on page 339 to explain how graphs of polynomial functions can be used to show trends in data. Include a description of the types of data that are best modeled by polynomial functions and an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.

Michael Newman/PhotoEdit

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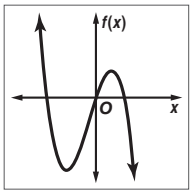
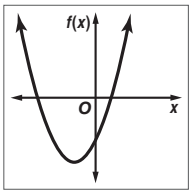
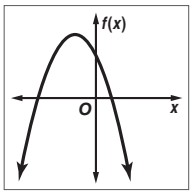
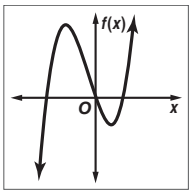
STANDARDIZED TEST PRACTICE

44. ACT/SAT Which of the following could be the graph of $f(x) = x^3 + x^2 - 3x$?

A 

C 

B 

D 

45. REVIEW Mandy went shopping. She spent two-fifths of her money in the first store. She spent three-fifths of what she had left in the next store. In the last store she visited, she spent three-fourths of the money she had left. When she finished shopping, Mandy had \$6. How much money in dollars did Mandy have when she started shopping?

F \$16 H \$100

G \$56 J \$106

Spiral Review

If $p(x) = 2x^2 - 5x + 4$ and $r(x) = 3x^3 - x^2 - 2$, find each value. (Lesson 6-4)

46. $r(2a)$ **47.** $5p(c)$ **48.** $p(2a^2)$

49. $r(x - 1)$ **50.** $p(x^2 + 4)$ **51.** $2[p(x^2 + 1)] - 3r(x - 1)$

Simplify. (Lesson 6-3)

52. $(4x^3 - 7x^2 + 3x - 2) \div (x - 2)$ **53.** $\dfrac{x^4 + 4x^3 - 4x^2 + 5x}{x - 5}$

Simplify. (Lesson 6-2)

54. $(3x^2 - 2xy + y^2) + (x^2 + 5xy - 4y^2)$ **55.** $(2x + 4)(7x - 1)$

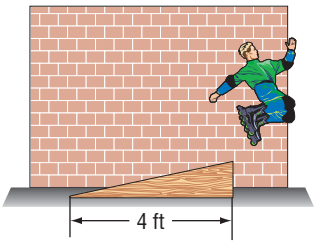
Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

56. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$ **57.** $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

58. $3j + 2k = 8$
$j - 7k = 18$

59. $5y + 2z = 11$
$10y - 4z = -2$

60. SPORTS Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of $\frac{1}{4}$, how tall should they make the ramp? (Lesson 2-3)



GET READY for the Next Lesson

PREREQUISITE SKILL Find the greatest common factor of each set of numbers.

61. 18, 27 **62.** 24, 84 **63.** 16, 28

64. 12, 27, 48 **65.** 12, 30, 54 **66.** 15, 30, 65

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EXTEND 6-5

Graphing Calculator Lab

Modeling Data Using Polynomial Functions

You can use a TI-83/84 Plus graphing calculator to model data for which the curve of best fit is a polynomial function.

EXAMPLE

Concepts in Motion
Interactive Lab algebra2.com

The table shows the distance a seismic wave can travel based on its distance from an earthquake's epicenter. Draw a scatter plot and a curve of best fit that relates distance to travel time. Then determine approximately how far from the epicenter the wave will be felt 8.5 minutes after the earthquake occurs.

Travel Time (min)	1	2	5	7	10	12	13
Distance (km)	400	800	2500	3900	6250	8400	10,000

Source: University of Arizona

Step 1 Enter the travel times in L1 and the distances in L2.

KEYSTROKES: *Refer to page 92 to review how to enter lists.*

Step 2 Graph the scatter plot.

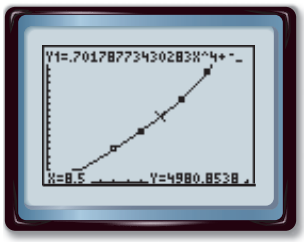
KEYSTROKES: *Refer to page 92 to review how to graph a scatter plot.*

Step 3 Compute and graph the equation for the curve of best fit. A quartic curve is the best fit for these data. You can verify this by comparing the R^2 values for each type of graph.

KEYSTROKES: STAT ▶ 7 2nd [L1] , 2nd [L2] ENTER Y= VARS 5 ▶ ▶ 1 GRAPH

Step 4 Use the [CALC] feature to find the value of the function for $x = 8.5$.

KEYSTROKES: 2nd [CALC] 1 8.5 ENTER



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After 8.5 minutes, you would expect the wave to be felt approximately 5000 kilometers away.

Exercises

For Exercises 1–3, use the table that shows how many minutes out of each eight-hour workday are used to pay one day's worth of taxes.

Year	Minutes
1940	83
1950	117
1960	130
1970	141
1980	145
1990	145
2000	160

Source: Tax Foundation

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the number of years since 1930. Try LinReg, QuadReg, and CubicReg.
2. Write the equation for the curve that best fits the data.
3. Based on this equation, how many minutes should you expect to work each day in the year 2010 to pay one day's taxes?

Other Calculator Keystrokes at algebra2.com

www.MathSchoolinternational.com

For Exercises 4–7, use the table that shows the estimated number of alternative-fueled vehicles in use in the United States per year.

Year	Estimated Alternative-Fueled Vehicles in Use in the United States
1995	333,049
1996	352,421
1997	367,526
1998	383,847
1999	411,525
2000	455,906
2001	490,019
2002	518,919

Source: eia.doe.gov

4. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of vehicles to the year. Try LinReg, QuadReg, and CubicReg. (*Hint:* Enter the x-values as years since 1994.)
5. Write the equation for the curve that best fits the data. Round to the nearest tenth.
6. Based on this equation before rounding, how many Alternative-Fueled Vehicles would you expect to be in use in the year 2008?
7. Find a curve of best fit that is quartic. Is it a better fit than the equation you wrote in Exercise 5? Explain.

For Exercises 8–11, use the table that shows the distance from the Sun to the Earth for each month of the year.

Month	Distance
January	0.9840
February	0.9888
March	0.9962
April	1.0050
May	1.0122
June	1.0163
July	1.0161
August	1.0116
September	1.0039
October	0.9954
November	0.9878
December	0.9837

Source: astronomycafe.net

8. Draw a scatter plot of the data. Then graph several curves of best fit that relate the distance to the month. Try LinReg, QuadReg, and CubicReg.
9. Write the equation for the curve that best fits the data.
10. Based on this equation, what is the distance from the Sun to the Earth halfway through September?
11. Would you use this model to find the distance from the Sun to Earth in subsequent years? Explain your reasoning.

Extension

For Exercises 12–15, design and complete your own data analysis.

12. Write a question that could be answered by examining data. For example, you might estimate the number of people living in your town 5 years from now or predict the future cost of a car.
13. Collect and organize the data you need to answer the question you wrote. You may need to research your topic on the internet or conduct a survey to collect the data you need.
14. Make a scatter plot and find a regression equation for your data. Then use the regression equation to answer the question.

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CHAPTER 6

Mid-Chapter Quiz

Lessons 6-1 through 6-5

Simplify. Assume that no variable equals 0. (Lesson 6-1)

1. $(-3x^2y)^3(2x)^2$ **2.** $\dfrac{a^6b^{-2}c}{a^3b^2c^4}$ **3.** $\left(\dfrac{x^2z}{xz^4}\right)^2$

4. CHEMISTRY One gram of water contains about 3.34×10^{22} molecules. About how many molecules are in 5×10^2 grams of water? (Lesson 6-1)

Simplify. (Lesson 6-2)

5. $(9x + 2y) - (7x - 3y)$ **6.** $(t + 2)(3t - 4)$

7. $(n + 2)(n^2 - 3n + 1)$ **8.** $4a(ab + 5a^2)$

9. MULTIPLE CHOICE The area of the base of a rectangular suitcase measures $3x^2 + 5x - 4$ square units. The height of the suitcase measures $2x$ units. Which polynomial expression represents the volume of the suitcase? (Lesson 6-2)

A $3x^3 + 5x^2 - 4x$

B $6x^2 + 10x - 8$

C $6x^3 + 10x^2 - 8x$

D $3x^3 + 10x^2 - 4$

Simplify. (Lesson 6-3)

10. $(m^3 - 4m^2 - 3m - 7) \div (m - 4)$

11. $\dfrac{2d^3 - d^2 - 9d + 9}{2d - 3}$

12. $(x^3 + x^2 - 13x - 28) \div (x - 4)$

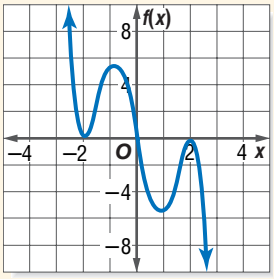
13. $\dfrac{3y^3 + 7y^2 - y - 5}{y + 2}$

14. WOODWORKING Arthur is building a rectangular table with an area of $3x^2 - 17x - 28$ square feet. If the length of the table is $3x + 4$ feet, what should the width of the rectangular table be? (Lesson 6-3)

15. PETS A pet food company estimates that it costs $0.02x^2 + 3x + 250$ dollars to produce x bags of dog food. Find an expression for the average cost per unit. (Lesson 6-3)

16. If $p(x) = 2x^3 - x$, find $p(a - 1)$. (Lesson 6-4)

17. Describe the end behavior of the graph. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeroes. (Lesson 6-4)



18. WIND CHILL The function $C(s) = 0.013s^2 - s - 7$ estimates the wind chill temperature $C(s)$ at 0°F for wind speeds s from 5 to 30 miles per hour. Estimate the wind chill temperature at 0°F if the wind speed is 27 miles per hour. (Lesson 6-4)

19. The formula $L(t) = \dfrac{8t^2}{\pi^2}$ represents the swing of a pendulum. L is the length of the pendulum in feet, and t is the time in seconds to swing back and forth. Find the length of a pendulum $L(t)$ that makes one swing in 2 seconds. (Lesson 6-4)

20. MULTIPLE CHOICE The function $f(x) = x^2 - 4x + 3$ has a relative minimum located at which of the following x-values? (Lesson 6-5)

F -2 **H** 3

G 2 **J** 4

21. Graph $y = x^3 + 2x^2 - 4x - 6$. Estimate the x-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

22. MARKET PRICE Prices of oranges from January to August can be modeled by (1, 2.7), (2, 4.4), (3, 4.9), (4, 5.5), (5, 4.3), (6, 5.3), (7, 3.5), (8, 3.9). How many turning points would the graph of a polynomial function through these points have? Describe them. (Lesson 6-5)

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6-6 Solving Polynomial Equations

Main Ideas

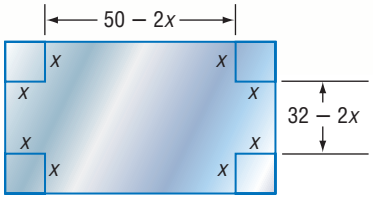
- Factor polynomials.
- Solve polynomial equations by factoring.

New Vocabulary

quadratic form

GET READY for the Lesson

The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner.



The volume of the box depends on the side length x of the cut squares. It is given by $V(x) = 4x^3 - 164x^2 + 1600x$. You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

Factor Polynomials Whole numbers are factored using prime numbers. For example, $100 = 2 \cdot 2 \cdot 5 \cdot 5$. Many polynomials can also be factored. Their factors, however, are other polynomials. Polynomials that cannot be factored are called *prime*. One method for finding the dimensions of the square to cut to make a box involves factoring the polynomial that represents the volume.

The table below summarizes the most common factoring techniques used with polynomials. Some of these techniques were introduced in Lesson 5-3. The others will be presented in this lesson.

CONCEPT SUMMARY *Factoring Techniques*

Number of Terms	Factoring Technique	General Case
any number	Greatest Common Factor (GCF)	$a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b)$
two	Difference of Two Squares	$a^2 - b^2 = (a + b)(a - b)$
	Sum of Two Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
	Difference of Two Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
three	Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$
	General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
four or more	Grouping	$ax + bx + ay + by = x(a + b) + y(a + b)$ $= (a + b)(x + y)$

Whenever you factor a polynomial, always look for a common factor first. Then determine whether the resulting polynomial factor can be factored again using one or more of the methods listed above.

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EXAMPLE GCF

Study Tip

Checking You can check the result when factoring by finding the product.

1 **Factor $6x^2y^2 - 2xy^2 + 6x^3y$.**

$$6x^2y^2 - 2xy^2 + 6x^3y = (2 \cdot 3 \cdot x \cdot x \cdot y \cdot y) - (2 \cdot x \cdot y \cdot y) + (2 \cdot 3 \cdot x \cdot x \cdot x \cdot y)$$
$$= (2xy \cdot 3xy) - (2xy \cdot y) + (2xy \cdot 3x^2)$$
The GCF is $2xy$. The remaining polynomial cannot be factored using the methods above.
$$= 2xy(3xy - y + 3x^2)$$

CHECK Your Progress **Factor completely.**

1A. $18x^3y^4 + 12x^2y^3 - 6xy^2$ **1B.** $a^4b^4 + 3a^3b^4 + a^2b^3$

EXAMPLE Grouping

2 **Factor $a^3 - 4a^2 + 3a - 12$.**

$a^3 - 4a^2 + 3a - 12 = (a^3 - 4a^2) + (3a - 12)$ Group to find a GCF.

$= a^2(a - 4) + 3(a - 4)$ Factor the GCF of each binomial.

$= (a - 4)(a^2 + 3)$ Distributive Property

CHECK Your Progress **Factor completely.**

2A. $x^2 + 3xy + 2xy^2 + 6y^3$ **2B.** $6a^3 - 9a^2b + 4ab - 6b^2$

Factoring by grouping is the only method that can be used to factor polynomials with four or more terms. For polynomials with two or three terms, it may be possible to factor the polynomial according to one of the patterns shown on page 349.

EXAMPLE Two or Three Terms

3 **Factor each polynomial.**

a. **$8x^3 - 24x^2 + 18x$**

This trinomial does not fit any of the factoring patterns. First, factor out the GCF. Then the remaining trinomial is a perfect square trinomial.

$8x^3 - 24x^2 + 18x = 2x(4x^2 - 12x + 9)$ Factor out the GCF.

$= 2x(2x - 3)^2$ Perfect square trinomial

b. **$m^6 - n^6$**

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$ Difference of two squares

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$ Sum and difference of two cubes

CHECK Your Progress

3A. $3xy^2 - 48x$ **3B.** $c^3d^3 + 27$

Extra Examples at algebra2.com

www.MathSchoolinternational.com

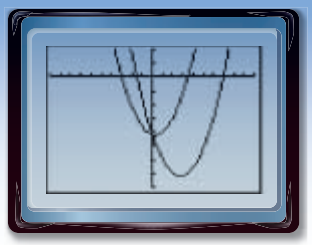
You can use a graphing calculator to check that the factored form of a polynomial is correct.

GRAPHING CALCULATOR LAB

Factoring Polynomials

Is the factored form of $2x^2 - 11x - 21$ equal to $(2x - 7)(x + 3)$? You can find out by graphing $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$. If the two graphs coincide, the factored form is probably correct.

- Enter $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$ on the **Y=** screen.
- Graph the functions. Since two different graphs appear, $2x^2 - 11x - 21 \neq (2x - 7)(x + 3)$.



[−10, 10] scl: 1 by [−40, 10] scl: 5

THINK AND DISCUSS

1. Determine if $x^2 + 5x - 6 = (x - 3)(x - 2)$ is a true statement. If not, write the correct factorization.
2. Does this method guarantee a way to check the factored form of a polynomial? Why or why not?

In some cases, you can rewrite a polynomial in x in the form $au^2 + bu + c$. For example, by letting $u = x^2$ the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form.**

KEY CONCEPT — *Quadratic Form*

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers a, b, and c, $a \neq 0$, where u is some expression in x. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

EXAMPLE 4 Write Expressions in Quadratic Form

Write each expression in quadratic form, if possible.

a. $x^4 + 13x^2 + 36$

$x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36$ $\quad (x^2)^2 = x^4$

b. $12x^8 - x^2 + 10$

This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

CHECK Your Progress

4A. $16x^6 - 625$ **4B.** $9x^{10} - 15x^4 + 9$

Solve Equations Using Quadratic Form In Chapter 5, you learned to solve quadratic equations by factoring and using the Zero Product Property. You can extend these techniques to solve higher-degree polynomial equations.

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EXAMPLE Solve Polynomial Equations

Study Tip

Substitution

To avoid confusion, you can substitute another variable for the expression in parentheses. For example, $x^4 - 13x^2 + 36 = 0$ could be written as $u^2 - 13u + 36 = 0$. Then once you have solved the equation for u, substitute x^2 for u and solve for x.

5 Solve each equation.

a. $x^4 - 13x^2 + 36 = 0$

$x^4 - 13x^2 + 36 = 0$ Original equation

$(x^2)^2 - 13(x^2) + 36 = 0$ Write the expression on the left in quadratic form.

$(x^2 - 9)(x^2 - 4) = 0$ Factor the trinomial.

$(x - 3)(x + 3)(x - 2)(x + 2) = 0$ Factor each difference of squares.

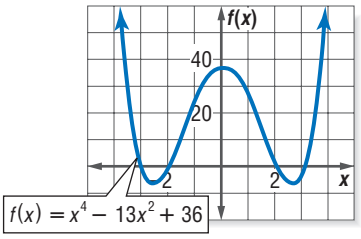
Use the Zero Product Property.

$x - 3 = 0$ or $x + 3 = 0$ or $x - 2 = 0$ or $x + 2 = 0$

$x = 3$ $x = -3$ $x = 2$ $x = -2$

The solutions are -3, -2, 2, and 3.

CHECK The graph of $f(x) = x^4 - 13x^2 + 36$ shows that the graph intersects the x-axis at -3, -2, 2, and 3.



b. $x^3 + 343 = 0$

$x^3 + 343 = 0$ Original equation

$(x)^3 + 7^3 = 0$ This is the sum of two cubes.

$(x + 7)[x^2 - x(7) + 7^2] = 0$ Sum of two cubes formula with $a = x$ and $b = 7$

$(x + 7)(x^2 - 7x + 49) = 0$ Simplify.

$(x + 7) = 0$ or $x^2 - 7x + 49 = 0$ Zero Product Property

The solution of the first equation is -7. The second equation can be solved by using the Quadratic Formula.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

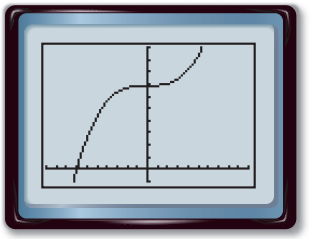
$= \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)}$ Replace a with 1, b with -7, and c with 49.

$= \dfrac{7 \pm \sqrt{-147}}{2}$ Simplify.

$= \dfrac{7 \pm i\sqrt{147}}{2}$ or $\dfrac{7 \pm 7i\sqrt{3}}{2}$ $\sqrt{147} \times \sqrt{-1} = 7i\sqrt{3}$

Thus, the solutions of the original equation are -7, $\dfrac{7 + 7i\sqrt{3}}{2}$, and $\dfrac{7 - 7i\sqrt{3}}{2}$.

CHECK The graph of $f(x) = x^3 + 343$ confirms the solution.



[−10, 10] sci: 1 by [−50, 500] sci: 50

CHECK Your Progress

5A. $x^4 - 29x^2 + 100 = 0$ **5B.** $x^3 + 8 = 0$

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CHECK Your Understanding

Examples 1–3 (p. 350)

Factor completely. If the polynomial is not factorable, write *prime*.

1. $-12x^2 - 6x$
2. $a^2 + 5a + ab$
3. $21 - 7y + 3x - xy$
4. $y^2 + 4y + 2y + 8$
5. $z^2 - 4z - 12$
6. $3b^2 - 48$
7. $16w^2 - 169$
8. $h^3 + 8000$

Example 4 (p. 351)

Write each expression in quadratic form, if possible.

9. $5y^4 + 7y^3 - 8$
10. $84n^4 - 62n^2$

Example 5 (p. 352)

Solve each equation.

11. $x^4 - 50x^2 + 49 = 0$
12. $x^3 - 125 = 0$

13. **POOL** The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are x feet deep by $7x - 6$ feet wide by $9x - 2$ feet long. How deep is the pool?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18, 19	2
20–23	3
24–29	4
30–39	5

Factor completely. If the polynomial is not factorable, write *prime*.

14. $2xy^3 - 10x$
15. $6a^2b^2 + 18ab^3$
16. $12cd^3 - 8c^2d^2 + 10c^5d^3$
17. $3a^2bx + 15cx^2y + 25ad^3y$
18. $8yz - 6z - 12y + 9$
19. $3ax - 15a + x - 5$
20. $y^2 - 5y + 4$
21. $2b^2 + 13b - 7$
22. $z^3 + 125$
23. $t^3 - 8$

Write each expression in quadratic form, if possible.

24. $2x^4 + 6x^2 - 10$
25. $a^8 + 10a^2 - 16$
26. $11n^6 + 44n^3$
27. $7b^5 - 4b^3 + 2b$
28. $7x^{\frac{1}{9}} - 3x^{\frac{1}{3}} + 4$
29. $6x^{\frac{2}{5}} - 4x^{\frac{1}{5}} - 16$

Solve each equation.

30. $x^4 - 34x^2 + 225 = 0$
31. $x^4 - 15x^2 - 16 = 0$
32. $x^4 + 6x^2 - 27 = 0$
33. $x^3 + 64 = 0$
34. $27x^3 + 1 = 0$
35. $8x^3 - 27 = 0$

Gregg Mancuso/Stock Boston

Real-World Career

Designer

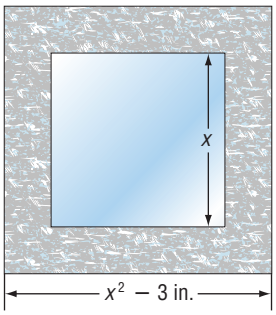
Designers combine practical knowledge with artistic ability to turn abstract ideas into formal designs.

MathOnline

For more information, go to algebra2.com.

DESIGN **For Exercises 36–38, use the following information.**

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.



36. Write a polynomial equation that models the area of the frame.
37. What are the dimensions of the glass piece?
38. What are the dimensions of the frame?

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39. GEOMETRY The width of a rectangular prism is w centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

40. Find the factorization of $3x^2 + x - 2$.

41. What are the factors of $2y^2 + 9y + 4$?

Factor completely. If the polynomial is not factorable, write prime.

42. $3n^2 + 21n - 24$

43. $y^4 - z^2$

44. $16a^2 + 25b^2$

45. $3x^2 - 27y^2$

46. $x^4 - 81$

47. $3a^3 + 2a^2 - 5a + 9a^2b + 6ab - 15b$

PACKAGING **For Exercises 48 and 49, use the following information.**

A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation $V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24$, where h is the height of the original piece.

48. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

49. How much did each dimension of the packaging increase for the new foam piece?

EXTRA PRACTICE
See pages 904, 931.
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Self-Check Quiz at
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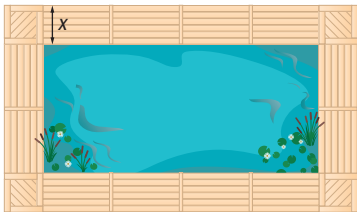
50. LANDSCAPING A boardwalk that is x feet wide is built around a rectangular pond. The pond is 30 feet wide and 40 feet long. The combined area of the pond and the boardwalk is 2000 square feet. What is the width of the boardwalk?



Graphing Calculator

CHECK FACTORING **Use a graphing calculator to determine if each polynomial is factored correctly. Write *yes* or *no*. If the polynomial is not factored correctly, find the correct factorization.**

51. $3x^2 + 5x + 2 \stackrel{?}{=} (3x + 2)(x + 1)$

52. $x^3 + 8 \stackrel{?}{=} (x + 2)(x^2 - x + 4)$

53. $2x^2 - 5x - 3 \stackrel{?}{=} (x - 1)(2x + 3)$

54. $3x^2 - 48 \stackrel{?}{=} 3(x + 4)(x - 4)$

H.O.T. Problems

55. OPEN ENDED Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form.

56. CHALLENGE Factor $64p^{2n} + 16p^n + 1$.

57. REASONING Find a counterexample to the statement $a^2 + b^2 = (a + b)^2$.

58. CHALLENGE Explain how you would solve $|a - 3|^2 - 9|a - 3| = -8$. Then solve the equation.

59. *Writing in Math* Use the information on page 349 to explain how solving a polynomial equation can help you find dimensions. Explain how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches. Explain why there can be more than one square that can be cut to produce the same volume.

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STANDARDIZED TEST PRACTICE

60. ACT/SAT Which is *not* a factor of $x^3 - x^2 - 2x$?

A x

B $x + 1$

C $x - 1$

D $x - 2$

61. ACT/SAT The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?

F 46 **G** 48 H 50 **J** 82

62. REVIEW $27x^3 + y^3 =$

A $(3x + y)(3x + y)(3x + y)$

B $(3x + y)(9x^2 - 3xy + y^2)$

C $(3x - y)(9x^2 + 3xy + y^2)$

D $(3x - y)(9x^2 + 9xy + y^2)$

Spiral Review

Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

63. $f(x) = x^3 - 6x^2 + 4x + 3$

64. $f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4$

Find $p(7)$ and $p(-3)$ for each function. (Lesson 6-4)

65. $p(x) = x^2 - 5x + 3$

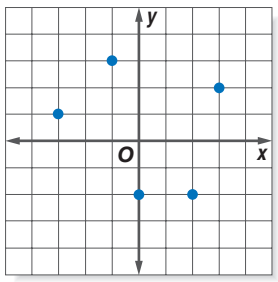
66. $p(x) = x^3 - 11x - 4$

67. $p(x) = \frac{2}{3}x^4 - 3x^3$

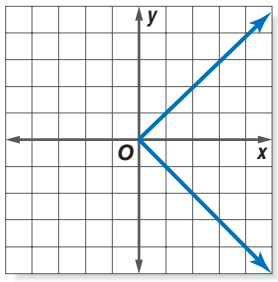
68. PHOTOGRAPHY The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3-2)

Determine whether each relation is a function. Write *yes* or *no*. (Lesson 2-1)

69.



70.



GET READY for the Next Lesson

PREREQUISITE SKILL Find each quotient. (Lesson 6-3)

71. $(x^3 + 4x^2 - 9x + 4) \div (x - 1)$

72. $(4x^3 - 8x^2 - 5x - 10) \div (x + 2)$

73. $(x^4 - 9x^2 - 2x + 6) \div (x - 3)$

74. $(x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1)$

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6-7 The Remainder and Factor Theorems

Main Ideas

- Evaluate functions using synthetic substitution.
- Determine whether a binomial is a factor of a polynomial by using synthetic substitution.

New Vocabulary

synthetic substitution
depressed polynomial

GET READY for the Lesson

The number of international travelers to the United States since 1986 can be modeled by the equation $T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9$, where x is the number of years since 1986 and $T(x)$ is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function by substituting 20 for x, or you can use synthetic substitution.

Synthetic Substitution Synthetic division can be used to find the value of a function. Consider the polynomial function $f(a) = 4a^2 - 3a + 6$. Divide the polynomial by $a - 2$.

Method 1 Long Division

$$\begin{array}{r} 4a + 5 \\ a - 2 \overline{)\,4a^2 - 3a + 6} \\ \underline{4a^2 - 8a} \\ 5a + 6 \\ \underline{5a - 10} \\ 16 \end{array}$$

Method 2 Synthetic Division

$$\begin{array}{r|rrr} 2 & 4 & -3 & 6 \\ & & 8 & 10 \\ \hline & 4 & 5 & 16 \end{array}$$

Compare the remainder of 16 to $f(2)$.

$$\begin{aligned} f(2) &= 4(2)^2 - 3(2) + 6 && \text{Replace } a \text{ with 2.} \\ &= 16 - 6 + 6 && \text{Multiply.} \\ &= 16 && \text{Simplify.} \end{aligned}$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This illustrates the **Remainder Theorem**.

KEY CONCEPT *Remainder Theorem*

If a polynomial $f(x)$ is divided by $x - a$, the remainder is the constant $f(a)$, and

Dividend	equals	quotient	times	divisor	plus	remainder.
$f(x)$	$=$	$q(x)$	$\cdot$	$(x - a)$	$+$	$f(a)$,

where $q(x)$ is a polynomial with degree one less than the degree of $f(x)$.

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.

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EXAMPLE 1 Synthetic Substitution

If $f(x) = 2x^4 - 5x^2 + 8x - 7$, find $f(6)$.

Method 1 Synthetic Substitution

By the Remainder Theorem, $f(6)$ should be the remainder when you divide the polynomial by $x - 6$.

$$\begin{array}{r|rrrrr} 6 & 2 & 0 & -5 & 8 & -7 \\ & & 12 & 72 & 402 & 2460 \\ \hline & 2 & 12 & 67 & 410 & 2453 \end{array}$$

Notice that there is no x^3 term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, $f(6) = 2453$.

Method 2 Direct Substitution

Replace x with 6.

$f(x) = 2x^4 - 5x^2 + 8x - 7$ Original function

$f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7$ Replace x with 6.

$= 2592 - 180 + 48 - 7$ or 2453 Simplify.

By using direct substitution, $f(6) = 2453$. Both methods give the same result.

CHECK Your Progress

1A. If $f(x) = 3x^3 - 6x^2 + x - 11$, find $f(3)$.

1B. If $g(x) = 4x^5 + 2x^3 + x^2 - 1$, find $f(-1)$.

Factors of Polynomials The synthetic division below shows that the quotient of $x^4 + x^3 - 17x^2 - 20x + 32$ and $x - 4$ is $x^3 + 5x^2 + 3x - 8$.

$$\begin{array}{r|rrrrr} 4 & 1 & 1 & -17 & -20 & 32 \\ & & 4 & 20 & 12 & -32 \\ \hline & 1 & 5 & 3 & -8 & 0 \end{array}$$

Study Tip

Depressed Polynomial

A *depressed polynomial* has a degree that is one less than the original polynomial.

When you divide a polynomial by one of its binomial factors, the quotient is called a **depressed polynomial**. From the results of the division and by using the Remainder Theorem, we can make the following statement.

Dividend	equals	quotient	times	divisor	plus	remainder.
$x^4 + x^3 - 17x^2 - 20x + 32$	$=$	$(x^3 + 5x^2 + 3x - 8)$	$\cdot$	$(x - 4)$	$+$	0

Since the remainder is 0, $f(4) = 0$. This means that $x - 4$ is a factor of $x^4 + x^3 - 17x^2 - 20x + 32$. This illustrates the **Factor Theorem**, which is a special case of the Remainder Theorem.

KEY CONCEPT *Factor Theorem*

The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

If $x - a$ is a factor of $f(x)$, then $f(a)$ has a factor of $(a - a)$, or 0. Since a factor of $f(a)$ is 0, $f(a) = 0$. Now assume that $f(a) = 0$. If $f(a) = 0$, then the Remainder Theorem states that the remainder is 0 when $f(x)$ is divided by $x - a$. This means that $x - a$ is a factor of $f(x)$. This proves the Factor Theorem.

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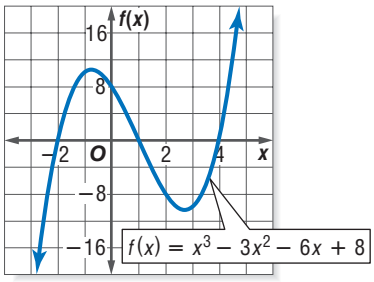
Suppose you wanted to find the factors of $x^3 - 3x^2 - 6x + 8$. One approach is to graph the related function, $f(x) = x^3 - 3x^2 - 6x + 8$. From the graph, you can see that the graph of $f(x)$ crosses the x-axis at -2, 1, and 4. These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.



$$f(x) = [x - (-2)](x - 1)(x - 4)$$
$$= (x + 2)(x - 1)(x - 4)$$

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed, and once graphed, the exact zeros are often difficult to determine.

EXAMPLE Use the Factor Theorem

2 **Show that $x + 3$ is a factor of $x^3 + 6x^2 - x - 30$. Then find the remaining factors of the polynomial.**

The binomial $x + 3$ is a factor of the polynomial if -3 is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

-3	1	6	-1	-30
		-3	-9	30
	1	3	-10	0

Since the remainder is 0, $x + 3$ is a factor of the polynomial. The polynomial $x^3 + 6x^2 - x - 30$ can be factored as $(x + 3)(x^2 + 3x - 10)$. The polynomial $x^2 + 3x - 10$ is the depressed polynomial. Check to see if this polynomial can be factored.

$x^2 + 3x - 10 = (x - 2)(x + 5)$ Factor the trinomial.

So, $x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5)$.

Study Tip

Factoring

The factors of a polynomial do not have to be binomials. For example, the factors of $x^3 + x^2 - x + 15$ are $x + 3$ and $x^2 - 2x + 5$.

CHECK Your Progress

2. Show that $x - 2$ is a factor of $x^3 - 7x^2 + 4x + 12$. Then find the remaining factors of the polynomial.

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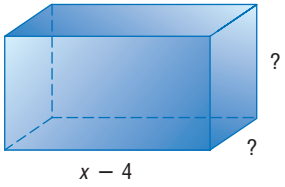
EXAMPLE Find All Factors

3 **GEOMETRY The volume of the rectangular prism is given by $V(x) = x^3 + 3x^2 - 36x + 32$. Find the missing measures.**



The volume of a rectangular prism is $\ell \times w \times h$.

You know that one measure is $x - 4$, so $x - 4$ is a factor of $V(x)$.

4	1	3	-36	32
		4	28	-32
	1	7	-8	0

The quotient is $x^2 + 7x - 8$. Use this to factor $V(x)$.

Extra Examples at algebra2.com

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$V(x) = x^3 + 3x^2 - 36x + 32$ Volume function

$= (x - 4)(x^2 + 7x - 8)$ Factor.

$= (x - 4)(x + 8)(x - 1)$ Factor the trinomial $x^2 + 7x - 8$.

So, the missing measures of the prism are $x + 8$ and $x - 1$.

CHECK Your Progress

3. The volume of a rectangular prism is given by $V(x) = x^3 + 7x^2 - 36$. Find the expressions for the dimensions of the prism.

CHECK Your Understanding

Example 1 (p. 357)

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

1. $f(x) = x^3 - 2x^2 - x + 1$

2. $f(x) = 5x^4 - 6x^2 + 2$

For Exercises 3–5, use the following information.

The projected sales of e-books in millions of dollars can be modeled by the function $S(x) = -17x^3 + 200x^2 - 113x + 44$, where x is the number of years since 2000.

3. Use synthetic substitution to estimate the sales for 2008.

4. Use direct substitution to evaluate $S(8)$.

5. Which method—synthetic substitution or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer.

Examples 2, 3 (pp. 358–359)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. $x^3 - x^2 - 5x - 3; x + 1$

7. $x^3 - 3x + 2; x - 1$

8. $6x^3 - 25x^2 + 2x + 8; 3x - 2$

9. $x^4 + 2x^3 - 8x - 16; x + 2$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–17	1
18–29	2, 3
30–33	3

Use synthetic substitution to find $g(3)$ and $g(-4)$ for each function.

10. $g(x) = x^2 - 8x + 6$

11. $g(x) = x^3 + 2x^2 - 3x + 1$

12. $g(x) = x^3 - 5x + 2$

13. $g(x) = x^4 - 6x - 8$

14. $g(x) = 2x^3 - 8x^2 - 2x + 5$

15. $g(x) = 3x^4 + x^3 - 2x^2 + x + 12$

16. $g(x) = x^5 + 8x^3 + 2x - 15$

17. $g(x) = x^6 - 4x^4 + 3x^2 - 10$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

18. $x^3 + 2x^2 - x - 2; x - 1$

19. $x^3 - x^2 - 10x - 8; x + 1$

20. $x^3 + x^2 - 16x - 16; x + 4$

21. $x^3 - 6x^2 + 11x - 6; x - 2$

22. $2x^3 - 5x^2 - 28x + 15; x - 5$

23. $3x^3 + 10x^2 - x - 12; x + 3$

24. $2x^3 + 7x^2 - 53x - 28; 2x + 1$

25. $2x^3 + 17x^2 + 23x - 42; 2x + 7$

26. $x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1$

27. $16x^5 - 32x^4 - 81x + 162; x - 2$

28. Use synthetic substitution to show that $x - 8$ is a factor of $x^3 - 4x^2 - 29x - 24$. Then find any remaining factors.

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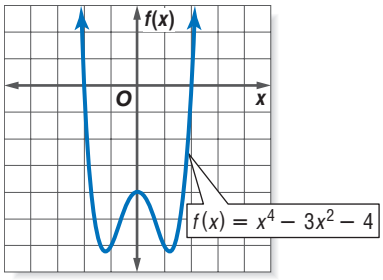
29. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial.



Cross-Curricular Project

Changes in world population can be modeled by a polynomial equation. Visit algebra2.com to continue work on your project.

BOATING **For Exercises 30 and 31, use the following information.**

A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in feet per second is given by the function $f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t$, where t is the time in seconds.

30. Find the speed of the boat at 1, 2, and 3 seconds.

31. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find $f(6)$ and explain what this means.

ENGINEERING **For Exercises 32 and 33, use the following information.**

When a certain type of plastic is cut into sections, the length of each section determines its strength. The function $f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100$ can describe the relative strength of a section of length x feet. Sections of plastic x feet long, where $f(x) = 0$, are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

32. Show that $x - 5$ is a factor of the polynomial function.

33. Are there other lengths of plastic that are extremely weak? Explain your reasoning.

Find values of k so that each remainder is 3.

34. $(x^2 - x + k) \div (x - 1)$

35. $(x^2 + kx - 17) \div (x - 2)$

36. $(x^2 + 5x + 7) \div (x + k)$

37. $(x^3 + 4x^2 + x + k) \div (x + 2)$

PERSONAL FINANCE **For Exercises 38–41, use the following information.**

Zach has purchased some home theater equipment for $2000, which he is financing through the store. He plans to pay $340 per month and wants to have the balance paid off after six months. The formula $B(x) = 2000x^6 - 340(x^5 + x^4 + x^3 + x^2 + x + 1)$ represents his balance after six months if x represents 1 plus the monthly interest rate (expressed as a decimal).

38. Find his balance after 6 months if the annual interest rate is 12%. (*Hint*: The monthly interest rate is the annual rate divided by 12, so $x = 1.01$.)

39. Find his balance after 6 months if the annual interest rate is 9.6%.

40. How would the formula change if Zach wanted to pay the balance in five months?

41. Suppose he finances his purchase at 10.8% and plans to pay $410 every month. Will his balance be paid in full after five months?

EXTRA PRACTICE
See pages 904, 931.
Math nline
Self-Check Quiz at algebra2.com

H.O.T. Problems

42. **OPEN ENDED** Give an example of a polynomial function that has a remainder of 5 when divided by $x - 4$.

43. **REASONING** Determine the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.

−2	1	0	6	32
		−2	4	−20
	1	−2	10	12

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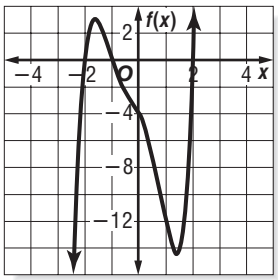
44. CHALLENGE Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

45. *Writing in Math* Use the information on page 356 to explain how you can use the Remainder Theorem to evaluate polynomials. Include an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution. Evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

STANDARDIZED TEST PRACTICE

46. ACT/SAT Use the graph of the polynomial function at the right. Which is *not* a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?



A $(x - 2)$

B $(x + 2)$

C $(x - 1)$

D $(x + 1)$

47. REVIEW The total area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

F $(5a^2 + 4b)(5a^2 + 4b)$

G $(5a^2 + 4b)(5a^2 - 4b)$

H $(5a - 4b)(5a - 4b)$

J $(5a + 4b)(5a - 4b)$

Spiral Review

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

48. $7xy^3 - 14x^2y^5 + 28x^3y^2$

49. $ab - 5a + 3b - 15$

50. $2x^2 + 15x + 25$

51. $c^3 - 216$

Graph each function by making a table of values. (Lesson 6-5)

52. $f(x) = x^3 - 4x^2 + x + 5$

53. $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

54. CITY PLANNING City planners have laid out streets on a coordinate grid before beginning construction. One street lies on the line with equation $y = 2x + 1$. Another street that intersects the first street passes through the point $(2, -3)$ and is perpendicular to the first street. What is the equation of the line on which the second street lies? (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL Find the exact solutions of each equation by using the Quadratic Formula. (Lesson 5-6)

55. $x^2 + 7x + 8 = 0$

56. $3x^2 - 9x + 2 = 0$

57. $2x^2 + 3x + 2 = 0$

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6-8 Roots and Zeros

Main Ideas

- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

GET READY for the Lesson

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time.

Suppose the equation $M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t$ models the number of milligrams of a certain medication in the bloodstream t hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

Types of Roots You have already learned that a zero of a function $f(x)$ is any value c such that $f(c) = 0$. When the function is graphed, the real zeros of the function are the x-intercepts of the graph.

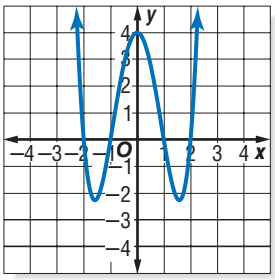
KEY CONCEPT *Zeros, Factors, and Roots*

Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial function. Then the following statements are equivalent.

- c is a zero of the polynomial function $f(x)$.
- $x - c$ is a factor of the polynomial $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.

In addition, if c is a real number, then $(c, 0)$ is an intercept of the graph of $f(x)$.

The graph of $f(x) = x^4 - 5x^2 + 4$ is shown at the right. The zeros of the function are -2, -1, 1, and 2. The factors of the polynomial are $x + 2$, $x + 1$, $x - 1$, and $x - 2$. The solutions of the equation $f(x) = 0$ are -2, -1, 1, and 2. The x-intercepts of the graph of $f(x)$ are $(-2, 0)$, $(-1, 0)$, $(1, 0)$, and $(2, 0)$.



Study Tip

Look Back

For review of **complex numbers**, see Lesson 5-4.

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the **Fundamental Theorem of Algebra.**

KEY CONCEPT *Fundamental Theorem of Algebra*

Every polynomial equation with complex coordinates and degree greater than zero has at least one root in the set of complex numbers.

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EXAMPLE Determine Number and Type of Roots

1 **Solve each equation. State the number and type of roots.**

a. $x^2 - 8x + 16 = 0$

$x^2 - 8x + 16 = 0$ Original equation

$(x - 4)^2 = 0$ Factor the left side as a perfect square trinomial.

$x = 4$ Solve for x using the Square Root Property.

Since $x - 4$ is twice a factor of $x^2 - 8x + 16$, 4 is a double root. So this equation has one real repeated root, 4.

b. $x^4 - 1 = 0$

$$x^4 - 1 = 0$$
$$(x^2 + 1)(x^2 - 1) = 0$$
$$(x^2 + 1)(x + 1)(x - 1) = 0$$
$$x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x^2 = -1 \qquad x = -1 \qquad x = 1$$
$$x = \pm\sqrt{-1} \text{ or } \pm i$$

This equation has two real roots, 1 and -1, and two imaginary roots, i and $-i$.

CHECK Your Progress

1A. $x^3 + 2x = 0$ **1B.** $x^4 - 16 = 0$

Reading Math

Roots In addition to double roots, equations can have triple or quadruple roots. In general, these roots are referred to as *repeated roots*.

Real-World Link

René Descartes (1596–1650) was a French mathematician and philosopher. One of his best-known quotations comes from his *Discourse on Method*: "I think, therefore I am."

Source: *A History of Mathematics*

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

KEY CONCEPT *Corollary*

A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers.

Similarly, a polynomial function of nth degree has exactly n zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

KEY CONCEPT *Descartes' Rule of Signs*

If $P(x)$ is a polynomial with real coefficients, the terms of which are arranged in descending powers of the variable,

- the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

National Library of Medicine/Mark Marte
www.MathSchoolinternational.com

EXAMPLE Find Numbers of Positive and Negative Zeros

2 **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$.**

Since $p(x)$ has degree 5, it has five zeros. However, some of them may be imaginary. Use Descartes' Rule of Signs to determine the number and type of real zeros. Count the number of changes in sign for the coefficients of $p(x)$.

Concepts in Motion
Animation
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$$p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$$

yes: + to −; no: − to −; yes: − to +; yes: + to −; yes: − to +

Since there are 4 sign changes, there are 4, 2, or 0 positive real zeros.

Find $p(-x)$ and count the number of changes in signs for its coefficients.

$$p(x) = (-x)^5 - 6(-x)^4 - 3(-x)^3 + 7(-x)^2 - 8(-x) + 1$$
$$= -x^5 - 6x^4 + 3x^3 + 7x^2 + 8x + 1$$

no: − to −; yes: − to +; no: + to +; no: + to +; no: + to +

Since there is 1 sign change, there is exactly 1 negative real zero.

Thus, the function $p(x)$ has either 4, 2, or 0 positive real zeros and exactly 1 negative real zero. Make a chart of the possible combinations of real and imaginary zeros.

Number of Positive Real Zeros	Number of Negative Real Zeros	Number of Imaginary Zeros	Total Number of Zeros
4	1	0	4 + 1 + 0 = 5
2	1	2	2 + 1 + 2 = 5
0	1	4	0 + 1 + 4 = 5

Study Tip

Zero at the Origin

Recall that the number 0 has no sign. Therefore, if 0 is a zero of a function, the sum of the number of positive real zeros, negative real zeros, and imaginary zeros is reduced by how many times 0 is a zero of the function.

CHECK Your Progress

2. State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $h(x) = 2x^5 + x^4 + 3x^3 - 4x^2 - x + 9$.

Find Zeros We can find all of the zeros of a function using some of the strategies you have already learned.

EXAMPLE Use Synthetic Substitution to Find Zeros

3 **Find all of the zeros of $f(x) = x^3 - 4x^2 + 6x - 4$.**

Since $f(x)$ has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for $f(x)$ and $f(-x)$.

$$f(x) = x^3 - 4x^2 + 6x - 4 \qquad f(-x) = -x^3 - 4x^2 - 6x - 4$$

yes yes yes; no no no

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of $f(-x)$, $f(x)$ has no negative real zeros. Thus, $f(x)$ has either 3 real zeros, or 1 real zero and 2 imaginary zeros.

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Study Tip

Finding Zeros

While direct substitution could be used to find each real zero of a polynomial, using synthetic substitution provides you with a depressed polynomial that can be used to find any imaginary zeros.

To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and $f(0)$ is -4, begin by evaluating $f(x)$ for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find $f(a)$ for several values of a.

x	1	-4	6	-4
1	1	-3	3	-1
2	1	-2	2	0
3	1	-1	3	5
4	1	0	6	20

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at $x = 2$. Since the depressed polynomial of this zero, $x^2 - 2x + 2$, is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, $x^2 - 2x + 2 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

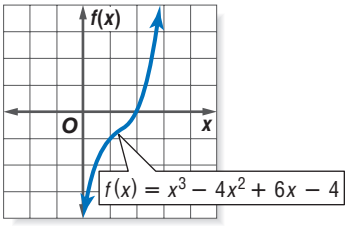
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$ Replace a with 1, b with -2, and c with 2.

$$= \frac{2 \pm \sqrt{-4}}{2}$$ Simplify.

$$= \frac{2 \pm 2i}{2}$$ $\sqrt{4} \times \sqrt{-1} = 2i$

$$= 1 \pm i$$ Simplify.

Thus, the function has one real zero at $x = 2$ and two imaginary zeros at $x = 1 + i$ and $x = 1 - i$. The graph of the function verifies that there is only one real zero.



CHECK Your Progress

3. Find all of the zeros of $h(x) = x^3 + 2x^2 + 9x + 18$.

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In Chapter 5, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

KEY CONCEPT — ***Complex Conjugates Theorem***

Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

EXAMPLE 4 Use Zeros to Write a Polynomial Function

Write a polynomial function of least degree with integral coefficients the zeros of which include 3 and $2 - i$.

Explore If $2 - i$ is a zero, then $2 + i$ is also a zero according to the Complex Conjugates Theorem. So, $x - 3$, $x - (2 - i)$, and $x - (2 + i)$ are factors of the polynomial function.

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Plan Write the polynomial function as a product of its factors.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$

Solve Multiply the factors to find the polynomial function.

$f(x) = (x - 3)[x - (2 - i)][x - (2 + i)]$ Write an equation.

$= (x - 3)[(x - 2) + i][(x - 2) - i]$ Regroup terms.

$= (x - 3)[(x - 2)^2 - i^2]$ Rewrite as the difference of two squares.

$= (x - 3)[x^2 - 4x + 4 - (-1)]$ Square x − 2 and replace i^2 with −1.

$= (x - 3)(x^2 - 4x + 5)$ Simplify.

$= x^3 - 4x^2 + 5x - 3x^2 + 12x - 15$ Multiply using the Distributive Property.

$= x^3 - 7x^2 + 17x - 15$ Combine like terms.

Check Since there are three zeros, the degree of the polynomial function must be three, so $f(x) = x^3 - 7x^2 + 17x - 15$ is a polynomial function of least degree with integral coefficients and zeros of 3, $2 - i$, and $2 + i$.

CHECK Your Progress

4. Write a polynomial function of least degree with integral coefficients the zeros of which include -1 and $1 + 2i$.

CHECK Your Understanding

Example 1 (p. 363)

Solve each equation. State the number and type of roots.

1. $x^2 + 4 = 0$

2. $x^3 + 4x^2 - 21x = 0$

Example 2 (p. 364)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

3. $f(x) = 5x^3 + 8x^2 - 4x + 3$

4. $r(x) = x^5 - x^3 - x + 1$

Example 3 (pp. 364–365)

Find all of the zeros of each function.

5. $p(x) = x^3 + 2x^2 - 3x + 20$

6. $f(x) = x^3 - 4x^2 + 6x - 4$

7. $v(x) = x^3 - 3x^2 + 4x - 12$

8. $f(x) = x^3 - 3x^2 + 9x + 13$

Example 4 (pp. 365–366)

9. Write a polynomial function of least degree with integral coefficients the zeros of which include 2 and $4i$.

10. Write a polynomial function of least degree with integral coefficients the zeros of which include $\frac{1}{2}$, 3, and -3.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–16	1
17–22	3
23–32	2
33–38	4

Solve each equation. State the number and type of roots.

11. $3x + 8 = 0$

12. $2x^2 - 5x + 12 = 0$

13. $x^3 + 9x = 0$

14. $x^4 - 81 = 0$

15. $x^4 - 16 = 0$

16. $x^5 - 8x^3 + 16x = 0$

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State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

17. $f(x) = x^3 - 6x^2 + 1$

18. $g(x) = 5x^3 + 8x^2 - 4x + 3$

19. $h(x) = 4x^3 - 6x^2 + 8x - 5$

20. $q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9$

21. $p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1$

22. $f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1$

Find all of the zeros of each function.

23. $g(x) = x^3 + 6x^2 + 21x + 26$

24. $h(x) = x^3 - 6x^2 + 10x - 8$

25. $f(x) = x^3 - 5x^2 - 7x + 51$

26. $f(x) = x^3 - 7x^2 + 25x - 175$

27. $g(x) = 2x^3 - x^2 + 28x + 51$

28. $q(x) = 2x^3 - 17x^2 + 90x - 41$

29. $h(x) = 4x^4 + 17x^2 + 4$

30. $p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40$

31. $r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13$

32. $h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156$

Write a polynomial function of least degree with integral coefficients that has the given zeros.

33. $-4, 1, 5$

34. $-2, 2, 4, 6$

35. $4i, 3, -3$

36. $2i, 3i, 1$

37. $9, 1 + 2i$

38. $6, 2 + 2i$

PROFIT **For Exercises 39–41, use the following information.**

A computer manufacturer determines that for each employee the profit for producing x computers per day is $P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x$.

39. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (*Hint:* Notice that 0, which is neither positive nor negative, is a zero of this function since $d(0) = 0$.)

40. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator.

41. What is the meaning of the roots in this problem?

VCG/Getty Images

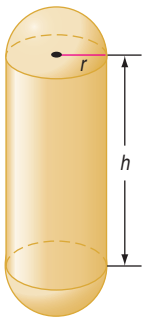
Real-World Link

A space shuttle is a reusable vehicle, launched like a rocket, which can put people and equipment in orbit around Earth. The first space shuttle was launched in 1981.

Source: kidsastronomy.about.com

SPACE EXPLORATION **For Exercises 42 and 43, use the following information.**

The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of 336π cubic meters and a height of 17 meters more than the radius of the tank. (*Hint:* $V(r) = \pi r^2 h$)

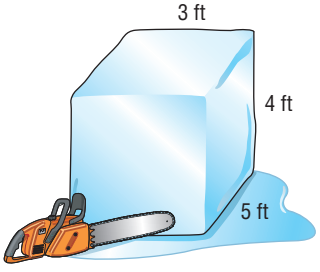


42. Write an equation that represents the volume of the cylinder.

43. What are the dimensions of the cylindrical part of the tank?

SCULPTING **For Exercises 44 and 45, use the following information.**

Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet.



44. Write a polynomial equation to model this situation.

45. How much should he take from each dimension?

EXTRA PRACTICE
See pages 904, 931.
MathOnline
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H.O.T. Problems

46. OPEN ENDED Sketch the graph of a polynomial function that has the indicated number and type of zeros.

a. 3 real, 2 imaginary **b.** 4 real **c.** 2 imaginary

47. CHALLENGE If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation.

48. REASONING State the least degree a polynomial equation with real coefficients can have if it has roots at $x = 5 + i$, $x = 3 - 2i$, and a double root at $x = 0$. Explain.

49. CHALLENGE Find a counterexample to disprove the following statement. *The polynomial function of least degree with integral coefficients with zeros at $x = 4$, $x = -1$, and $x = 3$, is unique.*

50. *Writing in Math* Use the information about medication on page 362 to explain how the roots of an equation can be used in pharmacology. Include an explanation of what the roots of this equation represent and an explanation of what the roots of this equation reveal about how often a patient should take this medication.

STANDARDIZED TEST PRACTICE

51. ACT/SAT How many negative real zeros does $f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ have?

A 3

B 2

C 1

D 0

52. REVIEW Tiles numbered from 1 to 6 are placed in a bag and are drawn out to determine which of six tasks will be assigned to six people. What is the probability that the tiles numbered 5 and 6 are drawn consecutively?

F $\frac{2}{3}$ **G** $\frac{2}{5}$ **H** $\frac{1}{2}$ **J** $\frac{1}{3}$

Spiral Review

Use synthetic substitution to find $f(-3)$ and $f(4)$ for each function. (Lesson 6-7)

53. $f(x) = x^3 - 5x^2 + 16x - 7$

54. $f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5$

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

55. $15a^2b^2 - 5ab^2c^2$

56. $12p^2 - 64p + 45$

57. $4y^3 + 24y^2 + 36y$

58. BASKETBALL In a recent season, Monique Currie of the Duke Blue Devils scored 635 points. She made a total of 356 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 76 more 2-point field goals than free throws and 77 more free throws than 3-point field goals. Find the number of each type of shot she made. (Lesson 3-5)

GET READY for the Next Lesson

PREREQUISITE SKILL Find all values of $\pm\frac{a}{b}$ given each replacement set.

59. $a = \{1, 5\}$; $b = \{1, 2\}$

60. $a = \{1, 2\}$; $b = \{1, 2, 7, 14\}$

61. $a = \{1, 3\}$; $b = \{1, 3, 9\}$

62. $a = \{1, 2, 4\}$; $b = \{1, 2, 4, 8, 16\}$

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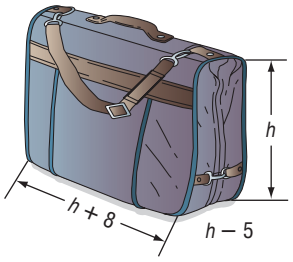
6-9 Rational Zero Theorem

Main Ideas

- Identify the possible rational zeros of a polynomial function.
- Find all the rational zeros of a polynomial function.

GET READY for the Lesson

On an airplane, carry-on baggage must fit into the overhead compartment above the passenger's seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation $h(h + 8)(h - 5) = 2772$, where h is the height, $h + 8$ is the length, and $h - 5$ is the width, to find the dimensions of the overhead compartment.



Identify Rational Zeros Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The **Rational Zero Theorem** can help you choose some possible zeros to test.

Study Tip

The Rational Zero Theorem only applies to rational zeros. Not *all* of the roots of a polynomial are found using the divisibility of the coefficients.

KEY CONCEPT *Rational Zero Theorem*

Words Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ represent a polynomial function with integral coefficients. If $\frac{p}{q}$ is a rational number in simplest form and is a zero of $y = f(x)$, then p is a factor of a_0 and q is a factor of a_n.

Example Let $f(x) = 2x^3 + 3x^2 - 17x + 12$. If $\frac{3}{2}$ is a zero of $f(x)$, then 3 is a factor of 12 and 2 is a factor of 2.

In addition, if the coefficient of the x term with the highest degree is 1, we have the following corollary.

KEY CONCEPT *Corollary (Integral Zero Theorem)*

If the coefficients of a polynomial function are integers such that $a_n = 1$ and $a_0 \neq 0$, any rational zeros of the function must be factors of a_n.

EXAMPLE **Identify Possible Zeros**

1 **List all of the possible rational zeros of each function.**

a. $f(x) = 2x^3 - 11x^2 + 12x + 9$

If $\frac{p}{q}$ is a rational zero, then p is a factor of 9 and q is a factor of 2. The possible values of p are ±1, ±3, and ±9. The possible values for q are ±1 and ±2. So, $\frac{p}{q} = \pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}$, and $\pm\frac{9}{2}$.

(continued on the next page)

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b. $f(x) = x^3 - 9x^2 - x + 105$

Since the coefficient of x^3 is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers ± 1, ± 3, ± 5, ± 7, ± 15, ± 21, ± 35, and ± 105.

CHECK Your Progress

1A. $g(x) = 3x^3 - 4x + 10$ **1B.** $h(x) = x^3 + 11x^2 + 24$

Find Rational Zeros Once you have found the possible rational zeros of a function, you can test each number using synthetic substitution to determine the zeros of the function.

EXAMPLE **Find Rational Zeros**

2 **GEOMETRY** **The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.**

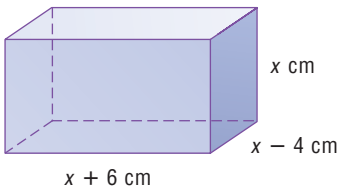
Let x = the height, $x - 4$ = the width, and $x + 6$ = the length.

Study Tip

Descartes' Rule of Signs

Examine the signs of the coefficients in the equation, $+ + - -$. There is one change of sign, so there is only one positive real zero.

Write an equation for the volume.



$$\ell wh = V \quad \text{Formula for volume}$$
$$(x - 4)(x + 6)x = 675 \quad \text{Substitute.}$$
$$x^3 + 2x^2 - 24x = 675 \quad \text{Multiply.}$$
$$x^3 + 2x^2 - 24x - 675 = 0 \quad \text{Subtract 675.}$$

The leading coefficient is 1, so the possible integer zeros are factors of 675, ± 1, ± 3, ± 5, ± 9, ± 15, ± 25, ± 27, ± 45, ± 75, ± 135, ± 225, and ± 675. Since length can only be positive, we only need to check positive zeros. From Descartes' Rule of Signs, we also know there is only one positive real zero. Make a table for the synthetic division and test possible real zeros.

p	1	2	−24	−675
1	1	3	−21	−696
3	1	5	−9	−702
5	1	7	11	−620
9	1	11	75	0

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are $9 - 4$ or 5 centimeters and $9 + 6$ or 15 centimeters.

CHECK Verify that the dimensions are correct. $5 \times 9 \times 15 = 675$ ✓

CHECK Your Progress

2. The volume of a rectangular solid is 1056 cubic inches. The length is 1 inch more than the width, and the height is 3 inches less than the width. Find the dimensions of the solid.

You usually do not need to test all of the possible zeros. Once you find a zero, you can try to factor the depressed polynomial to find any other zeros.

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EXAMPLE 3 Find All Zeros

Find all of the zeros of $f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60$.

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes' Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are ±1, ±2, ±3, ±4, ±5, ±6, ±10, ±12, ±15, ±20, ±30, ±60, $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, and $\pm\frac{15}{2}$. Make a table and test some possible rational zeros.

$\frac{p}{q}$	2	−13	23	−52	60
1	2	−11	12	−40	20
2	2	−9	5	−42	−24
3	2	−7	2	−46	−78
5	2	−3	8	−12	0

Since $f(5) = 0$, you know that $x = 5$ is a zero. The depressed polynomial is $2x^3 - 3x^2 + 8x - 12$.

Factor $2x^3 - 3x^2 + 8x - 12$.

$2x^3 - 3x^2 + 8x - 12 = 0$ Write the depressed polynomial.

$2x^3 + 8x - 3x^2 - 12 = 0$ Regroup terms.

$2x(x^2 + 4) - 3(x^2 + 4) = 0$ Factor by grouping.

$(x^2 + 4)(2x - 3) = 0$ Distributive Property

$x^2 + 4 = 0$ or $2x - 3 = 0$ Zero Product Property

$x^2 = -4$ $\qquad$ $2x = 3$

$x = \pm 2i$ $\qquad$ $x = \frac{3}{2}$

There is another real zero at $x = \frac{3}{2}$ and two imaginary zeros at $x = 2i$ and $x = -2i$.

The zeros of this function are 5, $\frac{3}{2}$, $2i$ and $-2i$.

CHECK Your Progress

Find all of the zeros of each function.

3A. $h(x) = 9x^4 + 5x^2 - 4$ $\qquad$ **3B.** $k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18$

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CHECK Your Understanding

Example 1 (pp. 369–370)

List all of the possible rational zeros of each function.

1. $p(x) = x^4 - 10$ $\qquad$ **2.** $d(x) = 6x^3 + 6x^2 - 15x - 2$

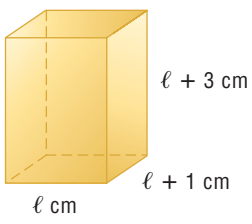
Example 2 (p. 370)

Find all of the rational zeros of each function.

3. $p(x) = x^3 - 5x^2 - 22x + 56$ $\qquad$ **4.** $f(x) = x^3 - x^2 - 34x - 56$

5. $t(x) = x^4 - 13x^2 + 36$ $\qquad$ **6.** $f(x) = 2x^3 - 7x^2 - 8x + 28$

7. GEOMETRY The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid.



Example 3 (p. 371)

Find all of the zeros of each function.

8. $f(x) = 6x^3 + 5x^2 - 9x + 2$

9. $f(x) = x^4 - x^3 - x^2 - x - 2$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16–21	2
22–29	3

List all of the possible rational zeros of each function.

10. $f(x) = x^3 + 6x + 2$

11. $h(x) = x^3 + 8x + 6$

12. $f(x) = 3x^4 + 15$

13. $n(x) = x^5 + 6x^3 - 12x + 18$

14. $p(x) = 3x^3 - 5x^2 - 11x + 3$

15. $h(x) = 9x^6 - 5x^3 + 27$

Find all of the rational zeros of each function.

16. $f(x) = x^3 + x^2 - 80x - 300$

17. $p(x) = x^3 - 3x - 2$

18. $f(x) = 2x^5 - x^4 - 2x + 1$

19. $f(x) = x^5 - 6x^3 + 8x$

20. $g(x) = x^4 - 3x^3 + x^2 - 3x$

21. $p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$

Find all of the zeros of each function.

22. $p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40$

23. $g(x) = 5x^4 - 29x^3 + 55x^2 - 28x$

24. $h(x) = 6x^3 + 11x^2 - 3x - 2$

25. $p(x) = x^3 + 3x^2 - 25x + 21$

26. $h(x) = 10x^3 - 17x^2 - 7x + 2$

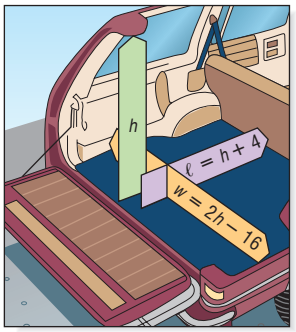
27. $g(x) = 48x^4 - 52x^3 + 13x - 3$

28. $p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10$

29. $h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12$

AUTOMOBILES **For Exercises 30 and 31, use the following information.**

The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.



30. Use a rectangular prism to model the cargo space. Write a polynomial function that represents the volume of the cargo space.

31. Will a package 34 inches long, 44 inches wide, and 34 inches tall fit in the cargo space? Explain.

FOOD **For Exercises 32–34, use the following information.**

A restaurant orders spaghetti sauce in cylindrical metal cans. The volume of each can is about 160π cubic inches, and the height of the can is 6 inches more than the radius.

32. Write a polynomial equation that represents the volume of a can. Use the formula for the volume of a cylinder, $V = \pi r^2 h$.

33. What are the possible values of r? Which values are reasonable here?

34. Find the dimensions of the can.

AMUSEMENT PARKS **For Exercises 35–37, use the following information.**

An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a square pyramid. Before building the new attraction, engineers must build and test a scale model.

35. If the height of the scale model is 9 inches less than its length, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$.

36. If the volume is 6300 cubic inches, write an equation for the situation.

37. What are the dimensions of the scale model?

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EXTRA PRACTICE
See pages 905, 931.
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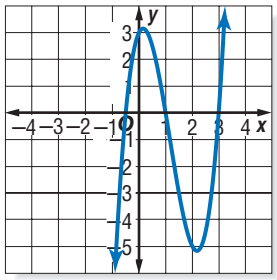
For Exercises 38 and 39, use the following information.

38. Find all of the zeros of $f(x) = x^3 - 2x^2 + 3$ and $g(x) = 2x^3 - 7x^2 + 2x + 3$.

39. Determine which function, f or g, is shown in the graph at the right.



H.O.T. Problems

40. FIND THE ERROR Lauren and Luis are listing the possible rational zeros of $f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6$. Who is correct? Explain your reasoning.

Lauren	Luis
$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$	$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$

41. OPEN ENDED Write a polynomial function that has possible rational zeros of $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

42. CHALLENGE If k and $2k$ are zeros of $f(x) = x^3 + 4x^2 + 9kx - 90$, find k and all three zeros of $f(x)$.

43. *Writing in Math* Use the information on page 369 to explain how the Rational Zero Theorem can be used to solve problems involving large numbers. Include the polynomial equation that represents the volume of the overhead baggage compartment and a list of all measures of the width of the compartment, assuming that the width is a whole number.

STANDARDIZED TEST PRACTICE

44. Which of the following is a zero of the function $f(x) = 12x^5 - 5x^3 + 2x - 9$?

A -6

B $\frac{3}{8}$

C $-\frac{2}{3}$

D 1

45. REVIEW A window is in the shape of an equilateral triangle. Each side of the triangle is 8 feet long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support?

F 5.7 ft **H** 11.3 ft

G 6.9 ft **J** 13.9 ft

Spiral Review

Given a function and one of its zeros, find all of the zeros of the function. (Lesson 6-8)

46. $g(x) = x^3 + 4x^2 - 27x - 90; -3$

47. $h(x) = x^3 - 11x + 20; 2 + i$

48. $f(x) = x^3 + 5x^2 + 9x + 45; -5$

49. $g(x) = x^3 - 3x^2 - 41x + 203; -7$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 6-7)

50. $20x^3 - 29x^2 - 25x + 6; x - 2$

51. $3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3$

52. GEOMETRY The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? (Lesson 3-5)

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CHAPTER 6

Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.

Key Concepts

Properties of Exponents (Lesson 6-1)

- The properties of powers for real numbers a and b and integers m and n are as follows.

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
$a^m \cdot a^n = a^{m+n}$	$(a^m)^n = a^{mn}$
$(ab)^m = a^m b^m$	$a^{-n} = \frac{1}{a^n}, a \neq 0$

Operations with Polynomials (Lesson 6-2)

- To add or subtract: Combine like terms.
- To multiply: Use the Distributive Property.
- To divide: Use long division or synthetic division.

Polynomial Functions and Graphs (Lessons 6-4 and 6-5)

- Turning points of a function are called *relative maxima* and *relative minima.*

Solving Polynomial Equations (Lesson 6-6)

- You can factor polynomials using the GCF, grouping, or quadratic techniques.

The Remainder and Factor Theorems (Lesson 6-7)

- Factor Theorem: The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Roots, Zeros, and the Rational Zero Theorem (Lessons 6-8 and 6-9)

- Complex Conjugates Theorem: If $a + bi$ is a zero of a function, then $a - bi$ is also a zero.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n = 0$, any rational zeros of the function must be factors of a_n.

Key Vocabulary

- degree of a polynomial (p. 320)
- depressed polynomial (p. 357)
- end behavior (p. 334)
- leading coefficient (p. 331)
- polynomial function (p. 332)
- polynomial in one variable (p. 331)
- quadratic form (p. 351)
- relative maximum (p. 340)
- relative minimum (p. 340)
- scientific notation (p. 315)
- simplify (p. 312)
- standard notation (p. 315)
- synthetic division (p. 327)
- synthetic substitution (p. 356)

Vocabulary Check

Choose a term from the list above that best completes each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser y-coordinates is a ____.
2. The _____ is the coefficient of the term in a polynomial function with the highest degree.
3. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in ____.
4. A shortcut method known as ____ is used to divide polynomials by binomials.
5. A number is expressed in _____ when it is in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.
6. The _____ is the sum of the exponents of the variables of a monomial.
7. When a polynomial is divided by one of its binomial factors, the quotient is called a(n) _____.
8. When we _____ an expression, we rewrite it without parentheses or negative exponents.
9. What a graph does as x approaches positive infinity or negative infinity is called the _____ of the graph.
10. The use of synthetic division to evaluate a function is called_____.

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Lesson-by-Lesson Review

6–1 Properties of Exponents (pp. 312–318)

Simplify. Assume that no variable equals 0.

11. $f^{-7} \cdot f^4$ **12.** $(3x^2)^3$

13. $(2y)(4xy^3)$ **14.** $\left(\frac{3}{5}c^2f\right)\left(\frac{4}{3}cd\right)^2$

15. MARATHON Assume that there are 10,000 runners in a marathon and each runner runs a distance of 26.2 miles. If you add together the total number of miles for all runners, how many times around the world would the marathon runners have gone? Consider the circumference of Earth to be 2.5×10^4 miles.

Example 1 **Simplify $(3x^4y^6)(-8x^3y)$.**

$(3x^4y^6)(-8x^3y)$

$= (3)(-8)x^{4+3}y^{6+1}$ Commutative Property and Product of Powers

$= -24x^7y^7$ Simplify.

Example 2 **Light travels at approximately 3.0×10^8 meters per second. How far does light travel in one week?**

Determine the number of seconds in one week.

$60 \cdot 60 \cdot 24 \cdot 7 = 604{,}800$ or 6.048×10^5 seconds

Multiply by the speed of light.

$(3.0 \times 10^8) \cdot (6.048 \times 10^5) = 1.8144 \times 10^{14}$ m

6–2 Operations with Polynomials (pp. 320–324)

Simplify.

16. $(4c - 5) - (c + 11) + (-6c + 17)$

17. $(11x^2 + 13x - 15) - (7x^2 - 9x + 19)$

18. $(d - 5)(d + 3)$ **19.** $(2a^2 + 6)^2$

20. CAR RENTAL The cost of renting a car is \$40 per day plus \$0.10 per mile. If a car is rented for d days and driven m miles a day, represent the cost C.

Example 3 **Find $(9k + 4)(7k - 6)$.**

$(9k + 4)(7k - 6)$

$= (9k)(7k) + (9k)(-6) + (4)(7k) + (4)(-6)$

$= 63k^2 - 54k + 28k - 24$

$= 63k^2 - 26k - 24$

6–3 Dividing Polynomials (pp. 325–330)

Simplify.

21. $(2x^4 - 6x^3 + x^2 - 3x - 3) \div (x - 3)$

22. $x^4 + 18x^3 + 10x^2 + 3x) \div (x^2 + 3x)$

23. SAILING The area of a triangular sail is $16x^4 - 60x^3 - 28x^2 + 56x - 32$ square meters. The base of the triangle is $x - 4$ meters. What is the height of the sail?

Example 4 **Use synthetic division to find $(4x^4 - x^3 - 19x^2 + 11x - 2) \div (x - 2)$.**

2	4	−1	−19	11	−2
		8	14	−10	2
	4	7	−5	1	0

The quotient is $4x^3 + 7x^2 - 5x + 1$.

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6–4 Polynomial Functions (pp. 331–338)

Find $p(-4)$ and $p(x + h)$ for each function.

24. $p(x) = x - 2$ **25.** $p(x) = -x + 4$

26. $p(x) = 6x + 3$ **27.** $p(x) = x^2 + 5$

28. $p(x) = x^2 - x$ **29.** $p(x) = 2x^3 - 1$

30. STORMS The average depth of a tsunami can be modeled by $d(s) = \left(\frac{s}{356}\right)^2$, where s is the speed in kilometers per hour and d is the average depth of the water in kilometers. Find the average depth of a tsunami when the speed is 250 kilometers per hour.

Example 5 **Find $p(a + 1)$ if $p(x) = 5x - x^2 + 3x^3$.**

$$\begin{aligned} p(a + 1) &= 5(a + 1) - (a + 1)^2 + 3(a + 1)^3 \\ &= 5a + 5 - (a^2 + 2a + 1) + \\ &\quad 3(a^3 + 3a^2 + 3a + 1) \\ &= 5a + 5 - a^2 - 2a - 1 + 3a^3 + \\ &\quad 9a^2 + 9a + 3 \\ &= 3a^3 + 8a^2 + 12a + 7 \end{aligned}$$

6–5 Analyzing Graphs of Polynomial Functions (pp. 339–347)

For Exercises 31–36, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which the real zeros are located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

31. $h(x) = x^3 - 6x - 9$

32. $f(x) = x^4 + 7x + 1$

33. $p(x) = x^5 + x^4 - 2x^3 + 1$

34. $g(x) = x^3 - x^2 + 1$

35. $r(x) = 4x^3 + x^2 - 11x + 3$

36. $f(x) = x^3 + 4x^2 + x - 2$

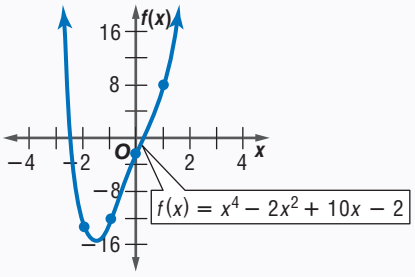
37. PROFIT A small business' monthly profits for the first half of 2006 can be modeled by (1, 550), (2, 725), (3, 680), (4, 830), (5, 920), (6, 810). How many turning points would the graph of a polynomial function through these points have? Describe them.

Example 6 **Graph $f(x) = x^4 - 2x^2 + 10x - 2$ by making a table of values.**

Make a table of values for several values of x.

x	−3	−2	−1	0	1	2
$f(x)$	31	−14	−13	−2	7	26

Plot the points and connect the points with a smooth curve.



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6–6 Solving Polynomial Equations (pp. 349–355)

Factor completely. If the polynomial is not factorable, write *prime*.

38. $10a^3 - 20a^2 - 2a + 4$

39. $5w^3 - 20w^2 + 3w - 12$

40. $x^4 - 7x^3 + 12x^2$ **41.** $x^2 - 7x + 5$

Solve each equation.

42. $3x^3 + 4x^2 - 15x = 0$

43. $m^4 + 3m^3 = 40m^2$

44. $x^4 - 8x^2 + 16 = 0$ **45.** $a^3 - 64 = 0$

46. HOME DECORATING The area of a dining room is 160 square feet. A rectangular rug placed in the center of the room is twice as long as it is wide. If the rug is bordered by 2 feet of hardwood floor on all sides, find the dimensions of the rug.

Example 7 **Factor $3m^2 + m - 4$.**

Find two numbers with a product of $3(-4)$ or -12 and a sum of 1. The two numbers must be 4 and -3 because $4(-3) = -12$ and $4 + (-3) = 1$.

$$\begin{aligned} 3m^2 + m - 4 &= 3m^2 + 4m - 3m - 4 \\ &= (3m^2 + 4m) - (3m + 4) \\ &= m(3m + 4) + (-1)(3m + 4) \\ &= (3m + 4)(m - 1) \end{aligned}$$

Example 8 **Solve $x^3 - 3x^2 - 54x = 0$.**

$$x^3 - 3x^2 - 54x = 0$$
$$x(x - 9)(x + 6) = 0$$
$$x(x^2 - 3x - 54) = 0$$
$$x = 0 \quad \text{or} \quad x - 9 = 0 \quad \text{or} \quad x + 6 = 0$$
$$x = 0 \qquad x = 9 \qquad x = -6$$

6–7 The Remainder and Factor Theorems (pp. 356–361)

Use synthetic substitution to find $f(3)$ and $f(-2)$ for each function.

47. $f(x) = x^2 - 5$ **48.** $f(x) = x^2 - 4x + 4$

49. $f(x) = x^3 - 3x^2 + 4x + 8$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

50. $x^3 + 5x^2 + 8x + 4;\ x + 1$

51. $x^3 + 4x^2 + 7x + 6;\ x + 2$

52. PETS The volume of water in a rectangular fish tank can be modeled by the polynomial $3x^3 - x^2 - 34x - 40$. If the depth of the tank is given by the polynomial $3x + 5$, what polynomials express the length and width of the fish tank?

Example 9 **Show that $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Then find any remaining factors of the polynomial.**

$$\begin{array}{r|cccc} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

The remainder is 0, so $x + 2$ is a factor of $x^3 - 2x^2 - 5x + 6$. Since $x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)$, the remaining factors of $x^3 - 2x^2 - 5x + 6$ are $x - 3$ and $x - 1$.

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6–8 Roots and Zeroes (pp. 362–368)

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

53. $f(x) = 2x^4 - x^3 + 5x^2 + 3x - 9$

54. $f(x) = -4x^4 - x^2 - x + 1$

55. $f(x) = 3x^4 - x^3 + 8x^2 + x - 7$

56. $f(x) = 2x^4 - 3x^3 - 2x^2 + 3$

DESIGN For Exercises 57 and 58, use the following information.

An artist has a piece he wants displayed in a gallery. The gallery told him the biggest piece they would display is 72 cubic feet. The artwork is currently 5 feet long, 8 feet wide, and 6 feet high. Joe decides to cut off the same amount from the length, width, and height.

57. Assume that a rectangular prism is a good model for the artwork. Write a polynomial equation to model this situation.

58. How much should he take from each dimension?

Example 10 **State the possible number of positive real zeros, negative real zeros, and imaginary zeros of $f(x) = 5x^4 + 6x^3 - 8x + 12$.**

Since $f(x)$ has two sign changes, there are 2 or 0 real positive zeros.

$f(-x) = 5x^4 - 6x^3 + 8x + 12$

Since $f(-x)$ has two sign changes, there are 0 or 2 negative real zeros.

There are 0, 2, or 4 imaginary zeros.

6–9 Rational Zero Theorem (pp. 369–373)

Find all of the rational zeros of each function.

59. $f(x) = 2x^3 - 13x^2 + 17x + 12$

60. $f(x) = x^3 - 3x^2 - 10x + 24$

61. $f(x) = x^4 - 4x^3 - 7x^2 + 34x - 24$

62. $f(x) = 2x^3 - 5x^2 - 28x + 15$

63. $f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10$

64. SHIPPING The height of a shipping cylinder is 4 feet more than the radius. If the volume of the cylinder is 5π cubic feet, how tall is it? Use the formula $V = \pi \cdot r^2 \cdot h$.

Example 11 **Find all of the zeros of $f(x) = x^3 + 7x^2 - 36$.**

There are exactly three complex zeros. There are one positive real zero and two negative real zeros. The possible rational zeros are ±1, ±2, ±3, ±4, ±6, ±9, ±12, ±18, ±36.

$$\begin{array}{r|rrrr} 2 & 1 & 7 & 0 & -36 \\ & & 2 & 18 & 36 \\ \hline & 1 & 9 & 18 & 0 \end{array}$$

$x^3 + 7x^2 - 36 = (x - 2)(x^2 + 9x + 18)$

$= (x - 2)(x + 3)(x + 6)$

Therefore, the zeros are 2, -3, and -6.

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CHAPTER 6

Practice Test

Simplify.

1. $(5b)^4(6c)^2$
2. $(13x - 1)(x + 3)$
3. $(3x^2 - 5x + 2) - (x^2 + 12x - 7)$
4. $(8x^3 + 9x^2 + 2x - 10) + (10x - 9)$
5. $(x^4 - x^3 - 10x^2 + 4x + 24) \div (x - 2)$
6. $(2x^3 + 9x^2 - 2x + 7) \div (x + 2)$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

7. $x^3 - x^2 - 5x - 3$; $x + 1$
8. $x^3 + 8x + 24$; $x + 2$

Factor completely. If the polynomial is not factorable, write *prime*.

9. $3x^3y + x^2y^2 + x^2y$
10. $3x^2 - 2x - 2$
11. $ax^2 + 6ax + 9a$
12. $8r^3 - 64s^6$
13. $x^2 - 14x + 45$
14. $2r^2 + 3pr - 2p^2$

For Exercises 15–18, complete each of the following.

a. Graph each function by making a table of values.

b. Determine consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

15. $g(x) = x^3 + 6x^2 + 6x - 4$
16. $h(x) = x^4 + 6x^3 + 8x^2 - x$
17. $f(x) = x^3 + 3x^2 - 2x + 1$
18. $g(x) = x^4 - 2x^3 - 6x^2 + 8x + 5$

Solve each equation.

19. $a^4 = 6a^2 + 27$
20. $p^3 + 8p^2 = 18p$
21. $16x^4 - x^2 = 0$
22. $r^4 - 9r^2 + 18 = 0$
23. $p^{\frac{3}{2}} - 8 = 0$
24. $n^3 + n - 27 = n$
25. **TRAVEL** While driving in a straight line from Milwaukee to Madison, your velocity is given by $v(t) = 5t^2 - 50t + 120$, where t is driving time in hours. Estimate your speed after 1 hour of driving.

Use synthetic substitution to find $f(-2)$ and $f(3)$ for each function.

26. $f(x) = 7x^5 - 25x^4 + 17x^3 - 32x^2 + 10x - 22$
27. $f(x) = 3x^4 - 12x^3 - 21x^2 + 30x$
28. Write $36x^{\frac{2}{3}} + 18x^{\frac{1}{3}} + 5$ in quadratic form.
29. Write the polynomial equation of degree 4 with leading coefficient 1 that has roots at -2, -1, 3, and 4.

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

30. $f(x) = x^3 - x^2 - 14x + 24$
31. $f(x) = 2x^3 - x^2 + 16x - 5$

Find all rational zeros of each function.

32. $g(x) = x^3 - 3x^2 - 53x - 9$
33. $h(x) = x^4 + 2x^3 - 23x^2 + 2x - 24$
34. $f(x) = 5x^3 - 29x^2 + 55x - 28$
35. $g(x) = 4x^3 + 16x^2 - x - 24$

FINANCIAL PLANNING For Exercises 36 and 37, use the following information.

Toshi will start college in six years. According to their plan, Toshi's parents will save $1000 each year for the next three years. During the fourth and fifth years, they will save $1200 each year. During the last year before he starts college, they will save $2000.

36. In the formula $A = P(1 + r)^t$, A = the balance, P = the amount invested, r = the interest rate, and t = the number of years the money has been invested. Use this formula to write a polynomial equation to describe the balance of the account when Toshi starts college.
37. Find the balance of the account if their investment yields 6% annually.

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CHAPTER 6

Standardized Test Practice

Cumulative, Chapters 1–6

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Which expression is equivalent to $3a(2a + 1) - (2a - 2)(a + 3)$?

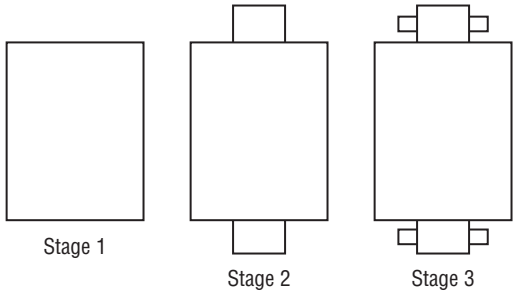
A $2a^2 + 6a + 7$

B $4a^2 - a + 6$

C $4a^2 + 6a - 6$

D $4a^2 - 3a + 7$

2. The figure below shows the first 3 stages of a fractal.



How many rectangles will the nth stage of this fractal contain?

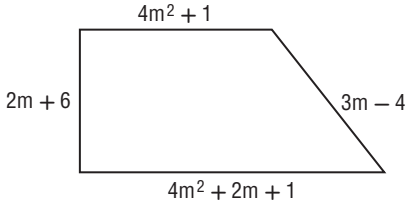
F $2n$

G 2^n

H $2n - 1$

J $2^n - 1$

3. GRIDDABLE Miguel is finding the perimeter of the quadrilateral below. What is the value of the constant term of the perimeter?



4. Which expression best represents the simplification of $(-2a^{-2}b^{-6})(-3a^{-1}b^8)$?

A $-\frac{1}{6a^3b^2}$

B $\frac{6b^2}{a^3}$

C $\frac{a^2}{6b^{14}}$

D $\frac{6a^2}{b^{48}}$

5. Which expression is equivalent to $(6a - 2b) - \frac{1}{4}(4a + 12b)$?

F $5a + 10b$

G $10a + 10b$

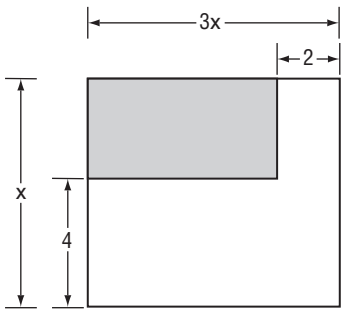
H $5a + b$

J $5a - 5b$

TEST-TAKING TIP

Question 5 If you simplify an expression and do not find your answer among the given answer choices, follow these steps. First, check your answer. Then, compare your answer with each of the given answer choices to determine whether it is equivalent to any of them.

6. What is the area of the shaded region of the rectangle expressed as a polynomial in simplest form?



A $3x^2 - 14x + 8$

B $3x^2 + 14x + 8$

C $3x^2 - 8$

D $4x + 6$

Standardized Test Practice at algebra2.com

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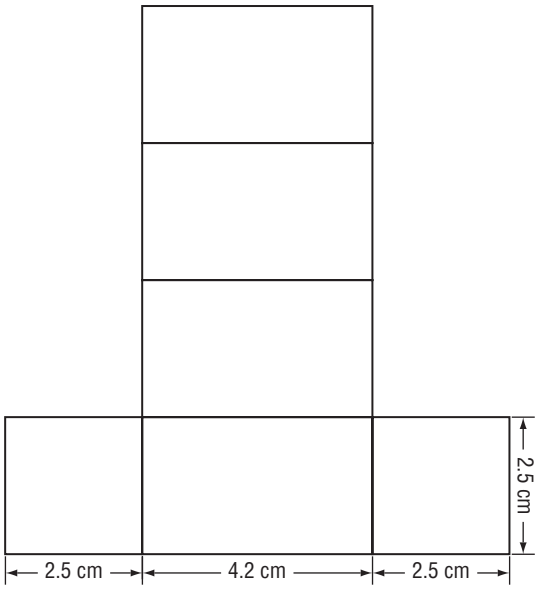
7. The figure below is the net of a rectangular prism. Use a ruler to measure the dimensions of the net to the nearest tenth of a centimeter.



Which measurement best approximates the volume of the rectangular prism represented by the net?

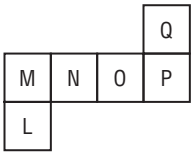
F 6.3 cm^3

G 10.5 cm^3

H 26.3 cm^3

J 44.1 cm^3

8. Which of the following is a true statement about the cube whose net is shown below?



A Faces L and M are parallel.

B Faces N and O are parallel.

C Faces M and P are perpendicular.

D Faces Q and L are perpendicular.

9. Kelly is designing a 12-inch by 12-inch scrapbook page. She cuts one picture that is 4 inches by 6 inches. She decides that she wants the next picture to be 75% as big as the first picture and the third picture to be 150% larger than the second picture. What are the approximate dimensions of the third picture?

F 0.45 in. by 0.68 in.

G 3.0 in. by 4.5 in.

H 4.5 in. by 6.75 in.

J 6.0 in. by 9.0 in.

10. GRIDDABLE Jalisa is a waitress. She recorded the following data about the amount that she made in tips for a certain number of hours.

Amount of Tips	Hours Worked
$12	1
$36	3
$60	5

If Jalisa continues to make the same amount of tips as shown in the table above, how much, in dollars, will she make in tips for working 9 hours?

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Consider the polynomial function $f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2$.

a. What is the degree of the function?

b. What is the leading coefficient of the function?

c. Evaluate $f(1)$, $f(-2)$, and $f(2a)$. Show your work.

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson or Page...	6-2	6-3	6-2	6-1	6-2	6-2	6-7	754	8-4	2-4	7-1

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CHAPTER 7

Radical Equations and Inequalities

BIG Ideas

- Find the composition of functions.
- Determine the inverses of functions or relations.
- Graph and analyze square root functions and inequalities.
- Simplify and solve equations involving roots, radicals, and rational exponents.

Key Vocabulary

extraneous solution (p. 422)
inverse function (p. 392)
principal root (p. 402)
rationalizing the denominator (p. 409)

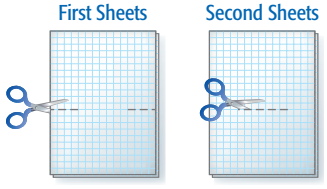
Real-World Link

Thrill Rides Many formulas involve square roots. For example, equations involving speeds of objects are often expressed with square roots. You can use such an equation to find the speed of a thrill ride such as the *Power Tower* free-fall ride at Cedar Point in Sandusky, Ohio.

FOLDABLES Study Organizer

Radical Equations and Inequalities Make this Foldable to help you organize your notes. Begin with four sheets of grid paper.

1. **Fold** in half along the width. On the first two sheets, cut 5 centimeters along the fold at the ends. On the second two sheets, cut in the center, stopping 5 centimeters from the ends.



2. **Insert** the first sheets through the second sheets and align the folds. Label the pages with lesson numbers.

Joyrides
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GET READY for Chapter 7

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

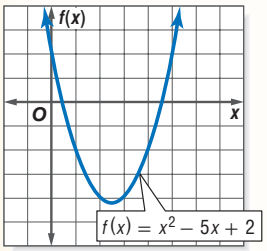
Option 1

Take the Quick Check below. Refer to the Quick Review for help.

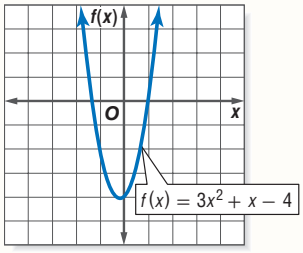
QUICKCheck

Use the related graph of each equation to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located. (Lesson 5-2)

1. $x^2 - 5x + 2 = 0$



2. $3x^2 + x - 4 = 0$



Simplify each expression using synthetic division. (Lesson 6-3)

3. $(3x^2 - 14x - 24) \div (x - 6)$

4. $(a^2 - 2a - 30) \div (a + 7)$

MEDICINE **For Exercises 5 and 6, use the following information.**

The number of students at a large high school who will catch the flu during an outbreak can be estimated by $n = \frac{170t^2}{t^2 + 1}$, where t is the number of weeks from the beginning of the epidemic and n is the number of ill people. (Lesson 6-3)

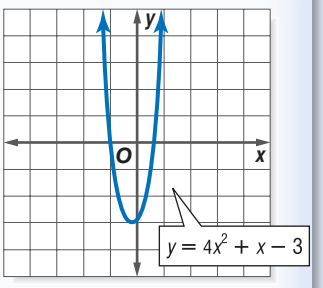
5. Perform the division indicated by $\frac{170t^2}{t^2 + 1}$.

6. Use the formula to estimate how many people will become ill during the first week.

QUICKReview

EXAMPLE 1

Use the related graph of $0 = 4x^2 + x - 3$ to determine its roots. If exact roots cannot be found, state the consecutive integers between which the roots are located.



The roots are the x-values where the graph crosses the x-axis.

The graph crosses the x-axis at -1 and between 0 and 1.

EXAMPLE 2

Simplify $(16x^4 - 8x^2 + 2x + 8) \div (2x + 2)$ using synthetic division.

Use division to rewrite the divisor so it has a first coefficient of 1.

$$\frac{16x^4 - 8x^2 + 2x + 8}{2x + 2} = \frac{(16x^4 - 8x^2 + 2x + 8) \div 2}{(2x + 2) \div 2}$$

Divide numerator and denominator by 2.

$\frac{8x^4 - 4x^2 + x + 4}{x + 1}$ Simplify.

Since the numerator does not have an x^3-term, use a coefficient of 0 for x^3.

$x - r = x + 1$, so $r = -1$.

$$\begin{array}{r|rrrrr} -1 & 8 & 0 & -4 & 1 & 4 \\ & \downarrow & -8 & 8 & -4 & 3 \\ \hline & 8 & -8 & 4 & -3 & 7 \end{array}$$

The result is $8x^3 - 8x^2 + 4x - 3 + \frac{7}{x + 1}$.

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7-1 Operations on Functions

Main Ideas

- Find the sum, difference, product, and quotient of functions.
- Find the composition of functions.

New Vocabulary

composition of functions

GET READY for the Lesson

Carol Coffmon owns a store where she sells birdhouses. The revenue from birdhouse sales is given by $r(x) = 125x$. The cost of making the birdhouses is given by $c(x) = 65x + 5400$. Her profit p is the revenue minus the cost or $p = r - c$. So the profit function $p(x)$ can be defined as $p(x) = (r - c)(x)$.

Arithmetic Operations Let $f(x)$ and $g(x)$ be any two functions. You can add, subtract, multiply, and divide functions according to these rules.

KEY CONCEPT *Operations with Functions*

Operation	Definition	Examples if $f(x) = x + 2$, $g(x) = 3x$
Sum	$(f + g)(x) = f(x) + g(x)$	$(x + 2) + 3x = 4x + 2$
Difference	$(f - g)(x) = f(x) - g(x)$	$(x + 2) - 3x = -2x + 2$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$(x + 2)3x = 3x^2 + 6x$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$	$\frac{x + 2}{3x}, x \neq 0$

EXAMPLE **Add and Subtract Functions**

1 Given $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$, find each function.

a. $(f + g)(x)$

$(f + g)(x) = f(x) + g(x)$ Addition of functions

$= (x^2 - 3x + 1) + (4x + 5)$ $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 + x + 6$ Simplify.

b. $(f - g)(x)$

$(f - g)(x) = f(x) - g(x)$ Subtraction of functions

$= (x^2 - 3x + 1) - (4x + 5)$ $f(x) = x^2 - 3x + 1$ and $g(x) = 4x + 5$

$= x^2 - 7x - 4$ Simplify.

CHECK Your Progress

Given $f(x) = x^2 + 5x - 2$ and $g(x) = 3x - 2$, find each function.

1A. $(f + g)(x)$ **1B.** $(f - g)(x)$

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Notice that the functions f and g have the same domain of all real numbers. The functions $f + g$ and $f - g$ also have domains that include all real numbers. For each new function, the domain consists of the intersection of the domains of $f(x)$ and $g(x)$. The domain of the quotient function is further restricted by excluded values that make the denominator equal to zero.

EXAMPLE 2 Multiply and Divide Functions

Given $f(x) = x^2 + 5x - 1$ and $g(x) = 3x - 2$, find each function.

a. $(f \cdot g)(x)$

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) && \text{Product of functions}\\ &= (x^2 + 5x - 1)(3x - 2) && \text{Substitute.}\\ &= x^2(3x - 2) + 5x(3x - 2) - 1(3x - 2) && \text{Distributive Property}\\ &= 3x^3 - 2x^2 + 15x^2 - 10x - 3x + 2 && \text{Distributive Property}\\ &= 3x^3 + 13x^2 - 13x + 2 && \text{Simplify.}\end{aligned}$$

b. $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Division of functions}$$

$$= \frac{x^2 + 5x - 1}{3x - 2}, x \neq \frac{2}{3} \quad f(x) = x^2 + 5x - 1 \text{ and } g(x) = 3x - 2$$

Because $x = \frac{2}{3}$ makes $3x - 2 = 0$, $\frac{2}{3}$ is excluded from the domain of $\left(\frac{f}{g}\right)(x)$.

CHECK Your Progress

Given $f(x) = x^2 - 7x + 2$ and $g(x) = x + 4$, find each function.

2A. $(f \cdot g)(x)$ **2B.** $\left(\frac{f}{g}\right)(x)$

Cross-Curricular Project

Math Online

You can use operations on functions to find a function to compare the populations of different cities, states, or countries over time. Visit algebra2.com.

Composition of Functions Functions can also be combined using **composition of functions**. In a composition, a function is performed, and then a second function is performed on the result of the first function.

KEY CONCEPT *Composition of Functions*

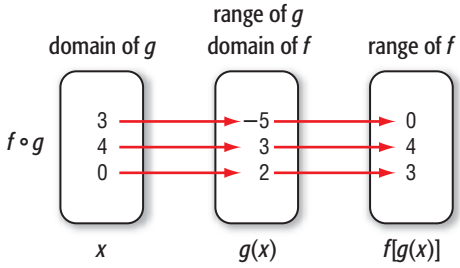
Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composite function $f \circ g$ can be described by

$$[f \circ g](x) = f[g(x)].$$

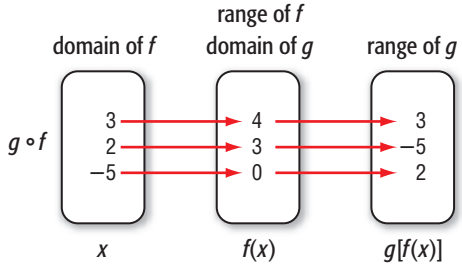
Reading Math

Composite Functions The composition of f and g is denoted by $f \circ g$. This is read *f of g*.

Suppose $f = \{(3, 4), (2, 3), (-5, 0)\}$ and $g = \{(3, -5), (4, 3), (0, 2)\}$.



$f \circ g = \{(3, 0), (4, 4), (0, 3)\}$



$g \circ f = \{(3, 3), (2, -5), (-5, 2)\}$

Concepts in Motion

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The composition of two functions may not exist. Given two functions f and g, $[f \circ g](x)$ is defined only if the range of $g(x)$ is a subset of the domain of $f(x)$.

EXAMPLE 3 Evaluate Composition of Functions

If $f = \{(7, 8), (5, 3), (9, 8), (11, 4)\}$ and $g = \{(5, 7), (3, 5), (7, 9), (9, 11)\}$, find $f \circ g$ and $g \circ f$.

To find $f \circ g$, evaluate $g(x)$ first. Then use the range of g as the domain of f and evaluate $f(x)$.

$f[g(5)] = f(7)$ or 8 $\quad g(5) = 7$ $\qquad$ $f[g(7)] = f(9)$ or 8 $\quad g(7) = 9$

$f[g(3)] = f(5)$ or 3 $\quad g(3) = 5$ $\qquad$ $f[g(9)] = f(11)$ or 4 $\quad g(9) = 11$

$f \circ g = \{(5, 8), (3, 3), (7, 8), (9, 4)\}$

To find $g \circ f$, evaluate $f(x)$ first. Then use the range of f as the domain of g and evaluate $g(x)$.

$g[f(7)] = g(8)$ $\quad g(8)$ is undefined. $\qquad$ $g[f(9)] = g(8)$ $\quad g(8)$ is undefined.

$g[f(5)] = g(3)$ or 5 $\quad f(5) = 3$ $\qquad$ $g[f(11)] = g(4)$ $\quad g(4)$ is undefined.

Since 8 and 4 are not in the domain of g, $g \circ f$ is undefined for $x = 7$, $x = 9$, and $x = 11$. However, $g[f(5)] = 5$ so $g \circ f = \{(5, 5)\}$.

CHECK Your Progress

3. If $f = \{(3, -2), (-1, -5), (4, 7), (10, 8)\}$ and $g = \{(4, 3), (2, -1), (9, 4), (3, 10)\}$, find $f \circ g$ and $g \circ f$.

Notice that in most instances $f \circ g \neq g \circ f$. Therefore, the order in which you compose two functions is very important.

EXAMPLE 4 Simplify Composition of Functions

Study Tip

Composing Functions

To remember the correct order for composing functions, think of starting with x and working outward from the grouping symbols.

a. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x + 3$ and $g(x) = x^2 + x - 1$.

$$\begin{aligned}
[f \circ g](x) &= f[g(x)] && \text{Composition of functions} \\
&= f(x^2 + x - 1) && \text{Replace } g(x) \text{ with } x^2 + x - 1. \\
&= (x^2 + x - 1) + 3 && \text{Substitute } x^2 + x - 1 \text{ for } x \text{ in } f(x). \\
&= x^2 + x + 2 && \text{Simplify.}
\end{aligned}$$

$$\begin{aligned}
[g \circ f](x) &= g[f(x)] && \text{Composition of functions} \\
&= g(x + 3) && \text{Replace } f(x) \text{ with } x + 3. \\
&= (x + 3)^2 + (x + 3) - 1 && \text{Substitute } x + 3 \text{ for } x \text{ in } g(x). \\
&= x^2 + 6x + 9 + x + 3 - 1 && \text{Evaluate } (x + 3)^2. \\
&= x^2 + 7x + 11 && \text{Simplify.}
\end{aligned}$$

So, $[f \circ g](x) = x^2 + x + 2$ and $[g \circ f](x) = x^2 + 7x + 11$.

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b. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = 2$.

$[f \circ g](x) = x^2 + x + 2$ Function from part a

$[f \circ g](2) = (2)^2 + 2 + 2$ or 8 Replace x with 2 and simplify.

$[g \circ f](x) = x^2 + 7x + 11$ Function from part a

$[g \circ f](2) = (2)^2 + 7(2) + 11$ or 29 Replace x with 2 and simplify.

So, $[f \circ g](2) = 8$ and $[g \circ f](2) = 29$.

CHECK Your Progress

4A. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = x - 5$ and $g(x) = x^2 + 2x + 3$.

4B. Evaluate $[f \circ g](x)$ and $[g \circ f](x)$ for $x = -3$.

Real-World EXAMPLE **Use Composition of Functions**

Study Tip

Combining Functions

By combining functions, you can make the evaluation of the functions more efficient.

5 TAXES **Tyrone Davis has $180 deducted from every paycheck for retirement. He can have these deductions taken before taxes are applied, which reduces his taxable income. His federal income tax rate is 18%. If Tyrone earns $2200 every pay period, find the difference in his net income if he has the retirement deduction taken before taxes or after taxes.**

Explore Let x = Tyrone's income per paycheck, $r(x)$ = his income after the deduction for retirement, and $t(x)$ = his income after the deduction for federal income tax.

Plan Write equations for $r(x)$ and $t(x)$.
$180 is deducted from every paycheck for retirement:
$r(x) = x - 180$.

Tyrone's tax rate is 18%: $t(x) = x - 0.18x$.

Solve If Tyrone has his retirement deducted *before* taxes, then his net income is represented by $[t \circ r](2200)$.

$[t \circ r](2200) = t(2200 - 180)$ Replace x with 2200 in $r(x) = x - 180$.

$= t(2020)$

$= 2020 - 0.18(2020)$ Replace x with 2020 in $t(x) = x - 0.18x$.

$= 1656.40$

If Tyrone has his retirement deducted *after* taxes, then his net income is represented by $[r \circ t](2200)$.

$[r \circ t](2200) = r[2200 - 0.18(2200)]$ Replace x with 2200 in $t(x) \circ x - 0.18x$.

$= r(1804)$

$= 1804 - 180$ Replace x with 1804 in $r(x) = x - 180$.

$= 1624$

$[t \circ r](2200) = 1656.40$ and $[r \circ t](2200) = 1624$. The difference is $1656.40 - $1624, or $32.40. So, his net pay is $32.40 more by having his retirement deducted before taxes.

Check The answer makes sense. Since the taxes are being applied to a smaller amount, less tax will be deducted from his paycheck.

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CHECK Your Progress

5. All-Mart is offering both an in-store $35 rebate and a 15% discount on an MP3 player that normally costs $300. Which provides the better price: taking the discount before the rebate, or taking the rebate before the discount?

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CHECK Your Understanding

Examples 1, 2 (pp. 384–385)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 3x + 4$, $g(x) = 5 + x$

2. $f(x) = x^2 + 3$, $g(x) = x - 4$

Example 3 (p. 386)

For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

3. $f = \{(-1, 9), (4, 7)\}$, $g = \{(-5, 4), (7, 12), (4, -1)\}$

4. $f = \{(0, -7), (1, 2), (2, -1)\}$, $g = \{(-1, 10), (2, 0)\}$

Example 4 (pp. 386–387)

Find $[g \circ h](x)$ and $[h \circ g](x)$.

5. $g(x) = 2x$, $h(x) = 3x - 4$

6. $g(x) = x + 5$, $h(x) = x^2 + 6$

If $f(x) = 3x$, $g(x) = x + 7$, and $h(x) = x^2$, find each value.

7. $f[g(3)]$ **8.** $g[h(-2)]$ **9.** $h[h(1)]$

Example 5 (p. 387)

SHOPPING For Exercises 10–13, use the following information.
Mai-Lin is shopping for computer software. She finds a CD-ROM that costs $49.99, but is on sale at a 25% discount. She also has a $5 coupon she can use.

10. Express the price of the CD after the discount and the price of the CD after the coupon. Let x represent the price of the CD, $p(x)$ represent the price after the 25% discount, and $c(x)$ represent the price after the coupon.

11. Find $c[p(x)]$ and explain what this value represents.

12. Find $p[c(x)]$ and explain what this value represents.

13. Which method results in the lower sale price? Explain your reasoning.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–21	1, 2
22–27	3
28–45	4
46, 47	5

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

14. $f(x) = x + 9$, $g(x) = x - 9$

15. $f(x) = 2x - 3$, $g(x) = 4x + 9$

16. $f(x) = 2x^2$, $g(x) = 8 - x$

17. $f(x) = x^2 + 6x + 9$, $g(x) = 2x + 6$

18. $f(x) = x^2 - 1$, $g(x) = \frac{x}{x + 1}$

19. $f(x) = x^2 - x - 6$, $g(x) = \frac{x - 3}{x + 2}$

WALKING For Exercises 20 and 21, use the following information.
Carlos is walking on a moving walkway. His speed is given by the function $C(x) = 3x^2 + 3x - 4$, and the speed of the walkway is $W(x) = x^2 - 4x + 7$.

20. What is his total speed as he walks along the moving walkway?

21. Carlos turned around because he left his cell phone at a restaurant. What was his speed as he walked against the moving walkway?

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For each pair of functions, find $f \circ g$ and $g \circ f$, if they exist.

22. $f = \{(1, 1), (0, -3)\}$
$g = \{(1, 0), (-3, 1), (2, 1)\}$

23. $f = \{(1, 2), (3, 4), (5, 4)\}$
$g = \{(2, 5), (4, 3)\}$

24. $f = \{(3, 8), (4, 0), (6, 3), (7, -1)\}$
$g = \{(0, 4), (8, 6), (3, 6), (-1, 8)\}$

25. $f = \{(4, 5), (6, 5), (8, 12), (10, 12)\}$
$g = \{4, 6), (2, 4), (6, 8), (8, 10)\}$

26. $f = \{(2, 5), (3, 9), (-4, 1)\}$
$g = \{(5, -4), (8, 3), (2, -2)\}$

27. $f = \{(7, 0), (-5, 3), (8, 3), (-9, 2)\}$
$g = \{(2, -5), (1, 0), (2, -9), (3, 6)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

28. $g(x) = 4x$
$h(x) = 2x - 1$

29. $g(x) = -5x$
$h(x) = -3x + 1$

30. $g(x) = x + 2$
$h(x) = x^2$

31. $g(x) = x - 4$
$h(x) = 3x^2$

32. $g(x) = 2x$
$h(x) = x^3 + x^2 + x + 1$

33. $g(x) = x + 1$
$h(x) = 2x^2 - 5x + 8$

If $f(x) = 4x$, $g(x) = 2x - 1$, and $h(x) = x^2 + 1$, find each value.

34. $f[g(-1)]$

35. $h[g(4)]$

36. $g[f(5)]$

37. $f[h(-4)]$

38. $g[g(7)]$

39. $f[f(-3)]$

40. $h\left[f\left(\frac{1}{4}\right)\right]$

41. $g\left[h\left(-\frac{1}{2}\right)\right]$

42. $[g \circ (f \circ h)](3)$

43. $[f \circ (h \circ g)](3)$

44. $[h \circ (g \circ f)](2)$

45. $[f \circ (g \circ h)](2)$

Real-World Link
In 2003, there were an estimated 19.2 million people who participated in inline skating.

Source: Inline Skating Resource Center

POPULATION GROWTH **For Exercises 46 and 47, use the following information.**
From 1990 to 2002, the number of births $b(x)$ in the United States can be modeled by the function $b(x) = -8x + 4045$, and the number of deaths $d(x)$ can be modeled by the function $d(x) = 24x + 2160$, where x is the number of years since 1990 and $b(x)$ and $d(x)$ are in thousands.

46. The net increase in population P is the number of births per year minus the number of deaths per year, or $P = b - d$. Write an expression that can be used to model the population increase in the U.S. from 1990 to 2002 in function notation.

47. Assume that births and deaths continue at the same rates. Estimate the net increase in population in 2015.

SHOPPING **For Exercises 48–50, use the following information.**
Liluye wants to buy a pair of inline skates that are on sale for 30% off the original price of $149. The sales tax is 5.75%.

48. Express the price of the inline skates after the discount and the price of the inline skates after the sales tax using function notation. Let x represent the price of the inline skates, $p(x)$ represent the price after the 30% discount, and $s(x)$ represent the price after the sales tax.

49. Which composition of functions represents the price of the inline skates, $p[s(x)]$ or $s[p(x)]$? Explain your reasoning.

50. How much will Liluye pay for the inline skates?

51. **FINANCE** Regina pays $50 each month on a credit card that charges 1.6% interest monthly. She has a balance of $700. The balance at the beginning of the nth month is given by $f(n) = f(n - 1) + 0.016 f(n - 1) - 50$. Find the balance at the beginning of the first five months. No additional charges are made on the card. (*Hint*: $f(1) = 700$)

EXTRA PRACTICE
See pages 905 and 932.
MathOnline
Self-Check Quiz at algebra2.com

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H.O.T. Problems

52. OPEN ENDED Write a set of ordered pairs for functions f and g, given that $f \circ g = \{(4, 3), (-1, 9), (-2, 7)\}$.

53. FIND THE ERROR Danette and Marquan are trying to find $[g \circ f](3)$ for $f(x) = x^2 + 4x + 5$ and $g(x) = x - 7$. Who is correct? Explain your reasoning.

Danette

$[g \cdot f](3) = g((3)\,2 + 4(3) + 5)$
$= g(26)$
$= 26 - 7$
$= 19$

Marquan

$[g \cdot f](3) = f(3 - 7)$
$= f(-4)$
$= (-4)^2 + 4(-4) + 5$
$= 5$

54. CHALLENGE If $f(0) = 4$ and $f(x + 1) = 3f(x) - 2$, find $f(4)$.

55. *Writing in Math* Refer to the information on page 384 to explain how combining functions can be important to business. Describe how to write a new function that represents the profit, using the revenue and cost functions. What are the benefits of combining two functions into one function?

STANDARDIZED TEST PRACTICE

56. ACT/SAT What is the value of $f(g(6))$ if $f(x) = 2x + 4$ and $g(x) = x^2 + 5$?

A 38

B 43

C 86

D 261

57. REVIEW If $g(x) = x^2 + 9x + 21$ and $h(x) = 2(x + 5)^2$, which is an equivalent form of $h(x) - g(x)$?

F $-x^2 - 11x - 29$

G $x^2 + 11x + 29$

H $x + 4$

J $x^2 + 7x + 11$

Spiral Review

List all of the possible rational zeros of each function. (Lesson 6-9)

58. $r(x) = x^2 - 6x + 8$ **59.** $f(x) = 4x^3 - 2x^2 + 6$ **60.** $g(x) = 9x^2 - 1$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function. (Lesson 6-8)

61. $f(x) = 7x^4 + 3x^3 - 2x^2 - x + 1$ **62.** $g(x) = 2x^4 - x^3 - 3x + 7$

63. CHEMISTRY The mass of a proton is about 1.67×10^{-27} kilogram. The mass of an electron is about 9.11×10^{-31} kilogram. About how many times as massive as an electron is a proton? (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation or formula for the specified variable.** (Lesson 1-3)

64. $2x - 3y = 6$, for x **65.** $4x^2 - 5xy + 2 = 3$, for y **66.** $3x + 7xy = -2$, for x

67. $I = prt$, for t **68.** $C = \frac{5}{9}(F - 32)$, for F **69.** $F = G\frac{Mm}{r^2}$, for m

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7-2 Inverse Functions and Relations

Main Ideas

- Find the inverse of a function or relation.
- Determine whether two functions or relations are inverses.

New Vocabulary

inverse relation
inverse function
identity function
one-to-one

GET READY for the Lesson

Most scientific formulas involve measurements given in SI (International System) units. The SI units for speed are meters per second. However, the United States uses customary measurements such as miles per hour.

To convert x miles per hour to an approximate equivalent in meters per second, you can evaluate the following.

$$f(x) = \frac{x \text{ miles}}{1 \text{ hour}} \cdot \frac{1600 \text{ meters}}{1 \text{ mile}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \text{ or } f(x) = \frac{4}{9}x$$

To convert x meters per second to an approximate equivalent in miles per hour, you can evaluate the following.

$$g(x) = \frac{x \text{ meters}}{1 \text{ second}} \cdot \frac{3600 \text{ seconds}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{1600 \text{ meters}} \text{ or } g(x) = \frac{9}{4}x$$

Notice that $f(x)$ multiplies a number by 4 and divides it by 9. The function $g(x)$ does the inverse operation of $f(x)$. It divides a number by 4 and multiplies it by 9. These functions are inverses.

Find Inverses Recall that a relation is a set of ordered pairs. The **inverse relation** is the set of ordered pairs obtained by reversing the coordinates of each ordered pair. The domain of a relation becomes the range of the inverse, and the range of a relation becomes the domain of the inverse.

KEY CONCEPT *Inverse Relations*

Words Two relations are inverse relations if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

Examples $Q = \{(1, 2), (3, 4), (5, 6)\}$ $S = \{(2, 1), (4, 3), (6, 5)\}$
Q and S are inverse relations.

EXAMPLE **Find an Inverse Relation**

1 **GEOMETRY** **The ordered pairs of the relation $\{(2, 1), (5, 1), (2, -4)\}$ are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.**

To find the inverse of this relation, reverse the coordinates of the ordered pairs.

(continued on the next page)

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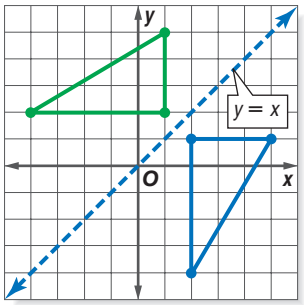
The inverse of the relation is {(1, 2), (1, 5), (−4, 2)}.

Plotting the points shows that the ordered pairs also describe the vertices of a right triangle. Notice that the graphs of the relation and the inverse relation are reflections over the graph of $y = x$.



CHECK Your Progress

1. The ordered pairs of the relation {(−8, −3), (−8, −6), (−3, −6)} are the coordinates of the vertices of a right triangle. Find the inverse of this relation and determine whether the resulting ordered pairs are also the vertices of a right triangle.

Reading Math

f^{-1} is read *f inverse* or the *inverse of f*. Note that −1 is not an exponent.

The ordered pairs of **inverse functions** are also related. We can write the inverse of function $f(x)$ as $f^{-1}(x)$.

KEY CONCEPT *Property of Inverse Functions*

Suppose f and f^{-1} are inverse functions. Then, $f(a) = b$ if and only if $f^{-1}(b) = a$.

Let's look at the inverse functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$.

Evaluate $f(5)$.	Now, evaluate $f^{-1}(7)$.
$f(x) = x + 2$	$f^{-1}(x) = x - 2$
$f(5) = 5 + 2$ or 7	$f^{-1}(7) = 7 - 2$ or 5

Since $f(x)$ and $f^{-1}(x)$ are inverses, $f(5) = 7$ and $f^{-1}(7) = 5$. The inverse function can be found by exchanging the domain and range of the function.

EXAMPLE 2 **Find and Graph an Inverse Function**

a. Find the inverse of $f(x) = \frac{x + 6}{2}$.

Step 1 Replace $f(x)$ with y in the original equation.

$$f(x) = \frac{x + 6}{2} \qquad y = \frac{x + 6}{2}$$

Step 2 Interchange x and y.

$$x = \frac{y + 6}{2}$$

Step 3 Solve for y.

$$x = \frac{y + 6}{2}$$ Inverse

$$2x = y + 6$$ Multiply each side by 2.

$$2x - 6 = y$$ Subtract 6 from each side.

Step 4 Replace y with $f^{-1}(x)$.

$$y = 2x - 6 \qquad f^{-1}(x) = 2x - 6$$

The inverse of $f(x) = \frac{x + 6}{2}$ is $f^{-1}(x) = 2x - 6$.

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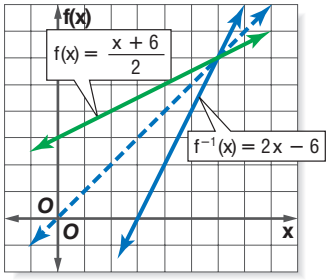
b. Graph the function and its inverse.

Graph both functions on the coordinate plane. The graph of $f^{-1}(x) = 2x - 6$ is the reflection of the graph of $f(x) = \frac{x + 6}{2}$ over the line $y = x$.



CHECK Your Progress

2A. Find the inverse of $f(x) = \frac{x - 3}{5}$.

2B. Graph the function and its inverse.

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Inverses of Relations and Functions You can determine whether two functions are inverses by finding both of their compositions. If both equal the **identity function** $I(x) = x$, then the functions are inverse functions.

KEY CONCEPT — ***Inverse Functions***

Words Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

Symbols $[f \circ g](x) = x$ and $[g \circ f](x) = x$

EXAMPLE **Verify that Two Functions are Inverses**

3 **Determine whether $f(x) = 5x + 10$ and $g(x) = \frac{1}{5}x - 2$ are inverse functions.**

Check to see if the compositions of $f(x)$ and $g(x)$ are identity functions.

$$[f \circ g](x) = f[g(x)] \qquad [g \circ f](x) = g[f(x)]$$
$$= f\left(\frac{1}{5}x - 2\right) \qquad = g(5x + 10)$$
$$= 5\left(\frac{1}{5}x - 2\right) + 10 \qquad = \frac{1}{5}(5x + 10) - 2$$
$$= x - 10 + 10 \qquad = x + 2 - 2$$
$$= x \qquad = x$$

The functions are inverses since both $[f \circ g](x)$ and $[g \circ f](x)$ equal x.

Study Tip

Inverse Functions

Both compositions of $f(x)$ and $g(x)$ must be the identity function for $f(x)$ and $g(x)$ to be inverses. It is necessary to check them both.

CHECK Your Progress

3. Determine whether $f(x) = 3x - 3$ and $g(x) = \frac{1}{3}x + 4$ are inverse functions.

You can also determine whether two functions are inverse functions by graphing. The graphs of a function and its inverse are mirror images with respect to the graph of the identity function $I(x) = x$.

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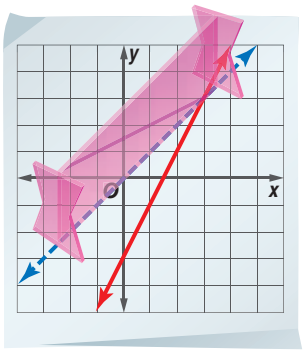
ALGEBRA LAB

Inverses of Functions

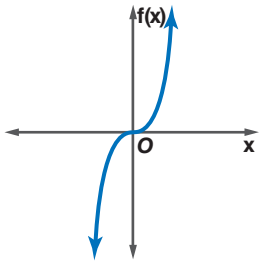


- Use a full sheet of grid paper. Draw and label the x- and y-axes.
- Graph $y = 2x - 3$. Then graph $y = x$ as a dashed line.
- Place a geomirror so that the drawing edge is on the line $y = x$. Carefully plot the points that are part of the reflection of the original line. Draw a line through the points.

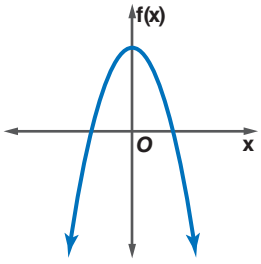
ANALYZE

1. What is the equation of the drawn line?
2. What is the relationship between the line $y = 2x - 3$ and the line that you drew? Justify your answer.
3. Try this activity with the function $y = |x|$. Is the inverse also a function? Explain.

When the inverse of a function is a function, then the original function is said to be **one-to-one.** Recall that the *vertical line test* can be used to determine if a graph represents a function. Similarly, the *horizontal line test* can be used to determine if the inverse of a function is a function.



No horizontal line can be drawn so that it passes through more than point. The inverse of this function is a function.



A horizontal line can be drawn so that it passes through more than one point. The inverse of this function is not a function.

CHECK Your Understanding

Example 1 (pp. 391–392)

Find the inverse of each relation.

1. $\{(2, 4), (-3, 1), (2, 8)\}$
2. $\{(1, 3), (1, -1), (1, -3), (1, 1)\}$

Example 2 (pp. 392–393)

Find the inverse of each function. Then graph the function and its inverse.

3. $f(x) = -x$
4. $g(x) = 3x + 1$
5. $y = \frac{1}{2}x + 5$

PHYSICS **For Exercises 6 and 7, use the following information.**
The acceleration due to gravity is 9.8 meters per second squared (m/s^2). To convert to feet per second squared, you can use the following operations.

$$\frac{9.8 \text{ m}}{\text{s}^2} \times \frac{100 \text{ cm}}{1 \text{ m}} \times \frac{1 \text{ in.}}{2.54 \text{ cm}} \times \frac{1 \text{ ft}}{12 \text{ in.}}$$

6. Find the value of the acceleration due to gravity in feet per second squared.
7. An object is accelerating at 50 feet per second squared. How fast is it accelerating in meters per second squared?

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Example 3 (p. 393)

Determine whether each pair of functions are inverse functions.

8. $f(x) = x + 7$, $g(x) = x - 7$

9. $g(x) = 3x - 2$, $f(x) = \frac{x - 2}{3}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
10–15	1
16–29	2
30–35	3

Find the inverse of each relation.

10. $\{(2, 6), (4, 5), (-3, -1)\}$

11. $\{(3, 8), (4, -2), (5, -3)\}$

12. $\{(7, -4), (3, 5), (-1, 4), (7, 5)\}$

13. $\{(-1, -2), (-3, -2), (-1, -4), (0, 6)\}$

14. $\{(6, 11), (-2, 7), (0, 3), (-5, 3)\}$

15. $\{(2, 8), (-6, 5), (8, 2), (5, -6)\}$

Find the inverse of each function. Then graph the function and its inverse.

16. $y = -3$

17. $g(x) = -2x$

18. $f(x) = x - 5$

19. $g(x) = x + 4$

20. $f(x) = 3x + 3$

21. $y = -2x - 1$

22. $y = \frac{1}{3}x$

23. $f(x) = \frac{5}{8}x$

24. $f(x) = \frac{1}{3}x + 4$

25. $f(x) = \frac{4}{5}x - 7$

26. $g(x) = \frac{2x + 3}{6}$

27. $f(x) = \frac{7x - 4}{8}$

GEOMETRY **The formula for the area of a circle is $A = \pi r^2$.**

28. Find the inverse of the function.

29. Use the inverse to find the radius of the circle whose area is 36 square centimeters.

Determine whether each pair of functions are inverse functions.

30. $f(x) = x - 5$, $g(x) = x + 5$

31. $f(x) = 3x + 4$, $g(x) = 3x - 4$

32. $f(x) = 6x + 2$, $g(x) = x - \frac{1}{3}$

33. $g(x) = 2x + 8$, $f(x) = \frac{1}{2}x - 4$

34. $h(x) = 5x - 7$, $g(x) = \frac{1}{5}(x + 7)$

35. $g(x) = 2x + 1$, $f(x) = \frac{x - 1}{2}$

NUMBER GAMES **For Exercises 36–38, use the following information.**

Damaso asked Emilia to choose a number between 1 and 35. He told her to subtract 12 from that number, multiply by 2, add 10, and divide by 4.

36. Write an equation that models this problem.

37. Find the inverse.

38. Emilia's final number was 9. What was her original number?

TEMPERATURE **For Exercises 39 and 40, use the following information.**

A formula for converting degrees Celsius to Fahrenheit is $F(x) = \frac{9}{5}x + 32$.

39. Find the inverse $F^{-1}(x)$. Show that $F(x)$ and $F^{-1}(x)$ are inverses.

40. Explain what purpose $F^{-1}(x)$ serves.

41. **REASONING** Determine the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

42. **OPEN ENDED** Sketch a graph of a function f that satisfies the following conditions: f does not have an inverse function, $f(x) > x$ for all x, and $f(1) > 0$.

43. **CHALLENGE** Give an example of a function that is its own inverse.

Real-World Career

Meteorologist

Meteorologists use observations from ground and space, along with formulas and rules based on past weather patterns to make their forecast.

Math Online

For more information, go to algebra2.com.

H.O.T. Problems

EXTRA PRACTICE

See pages 905 and 932.

Math Online

Self-Check Quiz at algebra2.com

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44. ***Writing in Math*** Refer to the information on page 391 to explain how inverse functions can be used in measurement conversions. Point out why it might be helpful to know the customary units if you are given metric units. Demonstrate how to convert the speed of light $c = 3.0 \times 10^8$ meters per second to miles per hour.

STANDARDIZED TEST PRACTICE

45. ACT/SAT Which of the following is the inverse of the function $f(x) = \frac{3x - 5}{2}$?

A $g(x) = \frac{2x + 5}{3}$ **C** $g(x) = 2x + 5$

B $g(x) = \frac{3x + 5}{2}$ **D** $g(x) = \frac{2x - 5}{3}$

46. REVIEW Which expression represents $f(g(x))$ if $f(x) = x^2 + 3$ and $g(x) = -x + 1$?

F $x^2 - x + 2$ **H** $-x^3 + x^2 - 3x + 3$

G $-x^2 - 2$ **J** $x^2 - 2x + 4$

Spiral Review

If $f(x) = 2x + 4$, $g(x) = x - 1$, and $h(x) = x^2$, find each value. (Lesson 7-1)

47. $f[g(2)]$ **48.** $g[h(-1)]$ **49.** $h[f(-3)]$

List all of the possible rational zeros of each function. (Lesson 6-9)

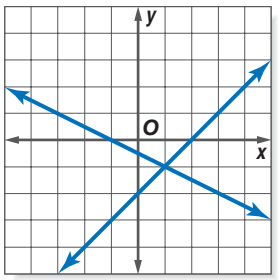
50. $f(x) = x^3 + 6x^2 - 13x - 42$ **51.** $h(x) = -4x^3 - 86x^2 + 57x + 20$

Perform the indicated operations. (Lesson 4-2)

52. $\begin{bmatrix} 3 & -4 \\ 2 & 8 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 7 & 7 \\ 3 & -6 \end{bmatrix}$ **53.** $\begin{bmatrix} 3 & 3 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 5 & 2 \end{bmatrix}$

54. Find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ for the polygonal region with vertices at (2, 4), (−1, 3), (−3, −3), and (2, −5). (Lesson 3-4)

55. State whether the system of equations shown at the right is *consistent and independent*, *consistent and dependent*, or *inconsistent*. (Lesson 3-1)



56. BUSINESS The amount that a mail-order company charges for shipping and handling is given by the function $c(x) = 3 + 0.15x$, where x is the weight in pounds. Find the charge for an 8-pound order. (Lesson 2-2)

Solve each equation or inequality. Check your solutions. (Lessons 1-3, 1-4, and 1-5)

57. $2x + 7 = -3$ **58.** $-5x + 6 = -4$ **59.** $|x - 1| = 3$

60. $|3x + 2| = 5$ **61.** $2x - 4 > 8$ **62.** $-x - 3 \leq 4$

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each inequality.** (Lesson 2-7)

63. $y > \frac{2}{3}x - 3$ **64.** $y \leq -4x + 5$ **65.** $y < -x - 1$

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7-3 Square Root Functions and Inequalities

Main Ideas

- Graph and analyze square root functions.
- Graph square root inequalities.

New Vocabulary

square root function

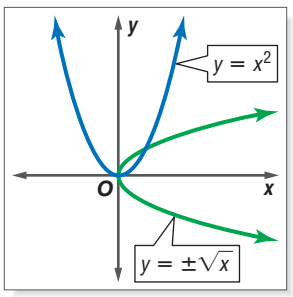
square root inequality

GET READY for the Lesson

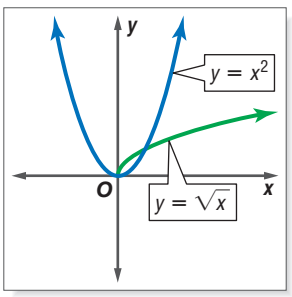
The Sunshine Skyway Bridge across Tampa Bay, Florida, is supported by 21 steel cables, each 9 inches in diameter. The amount of weight that a steel cable can support is given by $w = 8d^2$, where d is the diameter of the cable in inches and w is the weight in tons.

If you need to know what diameter a steel cable should have to support a given weight, you can use the equation $d = \sqrt{\frac{w}{8}}$.

Square Root Functions If a function contains a square root of a variable, it is called a **square root function**. The parent function of the family of square root functions is $y = \sqrt{x}$. The inverse of a quadratic function is a square root function only if the range is restricted to nonnegative numbers.



$y = \pm\sqrt{x}$ is not a function.



$y = \sqrt{x}$ is a function.

In order for a square root to be a real number, the radicand cannot be negative. When graphing a square root function, determine when the radicand would be negative and exclude those values from the domain.

Raymond Gehman/CORBIS

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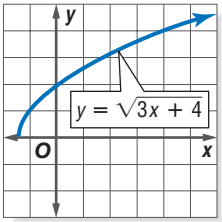
EXAMPLE 1 Graph a Square Root Function

Graph $y = \sqrt{3x + 4}$. State the domain, range, and x- and y-intercepts.

x	y
$-\frac{4}{3}$	0
-1	1
0	2
2	3.2
4	4



Since the radicand cannot be negative, identify the domain.

$3x + 4 \geq 0$ Write the expression inside the radicand as ≥ 0.

$x \geq -\frac{4}{3}$ Solve for x.

The x-intercept is $-\frac{4}{3}$.

Make a table of values and graph the function. From the graph, you can see that the domain is $x \geq -\frac{4}{3}$, and the range is $y \geq 0$. The y-intercept is 2.

CHECK Your Progress

1. Graph $y = \sqrt{-2x + 3}$. State the domain, range, and x- and y-intercepts.

Real-World EXAMPLE 2

Real-World Link

Submarines were first used by The United States in 1776 during the Revolutionary War.

Source: www.infoplease.com

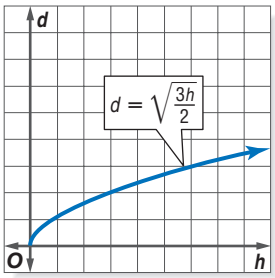
SUBMARINES **A lookout on a submarine is h feet above the surface of the water. The greatest distance d in miles that the lookout can see on a clear day is given by the square root of the quantity h multiplied by $\frac{3}{2}$.**

a. Graph the function. State the domain and range.

The function is $d = \sqrt{\frac{3h}{2}}$. Make a table of values and graph the function.

The domain is $h \geq 0$, and the range is $d \geq 0$.

h	d
0	0
2	$\sqrt{3}$ or 1.73
4	$\sqrt{6}$ or 2.45
6	$\sqrt{9}$ or 3.00
8	$\sqrt{12}$ or 3.46
10	$\sqrt{15}$ or 3.87



b. A ship is 3 miles from a submarine. How high would the submarine have to raise its periscope in order to see the ship?

$d = \sqrt{\frac{3h}{2}}$ Original equation

$3 = \sqrt{\frac{3h}{2}}$ Replace d with 3.

$9 = \frac{3h}{2}$ Square each side.

$18 = 3h$ Multiply each side by 2.

$6 = h$ Divide each side by 3.

The periscope would have to be 6 feet above the water. Check the reasonableness of this result by comparing it to the graph.

Frank Rossotto/Stocktreck/CORBIS

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CHECK Your Progress

The speed v of a ball can be determined by the equation $v = \sqrt{\frac{2k}{m}}$, where k is the kinetic energy and m is the mass of the ball. Assume that the mass of the ball is 5 kg.

2A. Graph the function. State the domain and range.

2B. The ball is traveling 6 meters per second. What is the ball's kinetic energy in Joules?

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Like quadratic functions, graphs of square root functions can be transformed.

GRAPHING CALCULATOR LAB

Square Root Functions

You can use a TI-83/84 Plus graphing calculator to graph square root functions. Use [2nd] [$\sqrt{\ }$] to enter the functions in the Y=list.

THINK AND DISCUSS

1. Graph $y = \sqrt{x}$, $y = \sqrt{x} + 1$, and $y = \sqrt{x} - 2$ in the viewing window $[-2, 8]$ by $[-4, 6]$. State the domain and range of each function and describe the similarities and differences among the graphs.
2. Graph $y = \sqrt{x}$, $y = \sqrt{2x}$, and $y = \sqrt{8x}$ in the viewing window $[0, 10]$ by $[0, 10]$. State the domain and range of each function and describe the similarities and differences among the graphs.
3. Make a conjecture about an equation that translates the graph of $y = \sqrt{x}$ to the left three units. Test your conjecture with the graphing calculator.

Square Root Inequalities A **square root inequality** is an inequality involving square roots.

EXAMPLE **Graph a Square Root Inequality**

3 **Graph $y < \sqrt{2x - 6}$.**

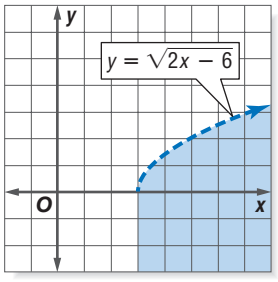
Graph $y = \sqrt{2x - 6}$. Since the boundary should not be included, the graph should be dashed.

The domain includes values for $x \geq 3$, so the graph includes $x = 3$ and values for which $x > 3$. Select a point to see if it is in the shaded region.

Test (4, 1): $1 \overset{?}{<} \sqrt{2(4) - 6}$

$1 < \sqrt{2}$ true

Shade the region that includes the point (4, 1).



Study Tip

Domain of a Square Root Inequality

The domain of a square root inequality includes only those values for which the expression under the radical sign is greater than or equal to 0.

CHECK Your Progress

3. Graph $y \geq \sqrt{x + 1}$.

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CHECK Your Understanding

Example 1 (p. 398)

Graph each function. State the domain and range of the function.

1. $y = \sqrt{x} + 2$ **2.** $y = \sqrt{4x}$ **3.** $y = \sqrt{x - 1} + 3$

Example 2 (p. 398)

FIREFIGHTING For Exercises 4 and 5, use the following information.
When fighting a fire, the velocity v of water being pumped into the air is the square root of twice the product of the maximum height h and g, the acceleration due to gravity (32 ft/s^2).

4. Determine an equation that will give the maximum height of the water as a function of its velocity.

5. The Coolville Fire Department must purchase a pump that will propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 ft/s meet the fire department's need? Explain.

Example 3 (p. 399)

Graph each inequality.

6. $y \leq \sqrt{x - 4} + 1$ **7.** $y > \sqrt{2x + 4}$ **8.** $y \geq \sqrt{x + 2} - 1$

Exercises

HOMEWORK HELP

For Exercises	See Examples
9–20	1
21–23	2
24–29	3

Graph each function. State the domain and range of each function.

9. $y = \sqrt{3x}$ **10.** $y = -\sqrt{5x}$ **11.** $y = -4\sqrt{x}$

12. $y = \frac{1}{2}\sqrt{x}$ **13.** $y = \sqrt{x + 2}$ **14.** $y = \sqrt{x - 7}$

15. $y = -\sqrt{2x + 1}$ **16.** $y = \sqrt{5x - 3}$ **17.** $y = \sqrt{x + 6} - 3$

18. $y = 5 - \sqrt{x + 4}$ **19.** $y = \sqrt{3x - 6} + 4$ **20.** $y = 2\sqrt{3 - 4x} + 3$

21. ROLLER COASTERS The velocity of a roller coaster as it moves down a hill is $v = \sqrt{{v_0}^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. An engineer wants a new coaster to have a velocity greater than 90 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 10 feet per second, how high should the engineer make the hill? Is your answer reasonable?

AEROSPACE For Exercises 22 and 23, use the following information.
The force due to gravity decreases with the square of the distance from the center of Earth. As an object moves farther from Earth, its weight decreases. The radius of Earth is approximately 3960 miles. The formula relating weight and distance is $r = \sqrt{\frac{3960^2 W_E}{W_S}} - 3960$, where W_E represents the weight of a body on Earth, W_S represents its weight a certain distance from the center of Earth, and r represents the distance above Earth's surface.

22. An astronaut weighs 140 pounds on Earth and 120 pounds in space. How far is he above Earth's surface?

23. An astronaut weighs 125 pounds on Earth. What is her weight in space if she is 99 miles above the surface of Earth?

EXTRA PRACTICE
See pages 906 and 932.
MathOnline
Self-Check Quiz at algebra2.com

Graph each inequality.

24. $y \leq -6\sqrt{x}$ **25.** $y < \sqrt{x + 5}$ **26.** $y > \sqrt{2x + 8}$

27. $y \geq \sqrt{5x - 8}$ **28.** $y \geq \sqrt{x - 3} + 4$ **29.** $y < \sqrt{6x - 2} + 1$

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H.O.T. Problems

30. OPEN ENDED Write a square root function with a domain of $\{x \mid x \geq 2\}$.

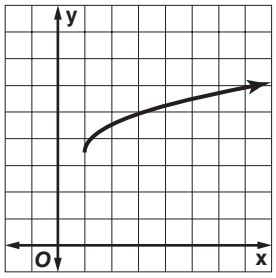
31. CHALLENGE Recall how values of a, h, and k can affect the graph of a quadratic function of the form $y = a(x - h)^2 + k$. Describe how values of a, h, and k can affect the graph of a square root function of the form $y = a\sqrt{x - h} + k$.

32. REASONING Describe the difference between the graphs of $y = \sqrt{x} - 4$ and $y = \sqrt{x - 4}$.

33. *Writing in Math* Refer to the information on page 397 to explain how square root functions can be used in bridge design. Assess the weights for which a diameter less than 1 is reasonable. Evaluate the amount of weight that the Sunshine Skyway Bridge can support.

STANDARDIZED TEST PRACTICE

34. ACT/SAT Given the graph of the square root function at the right, which must be true?



I. The domain is all real numbers.

II. The function is $y = \sqrt{x} + 3.5$.

III. The range is about $\{y \mid y \geq 3.5\}$.

A I only **C** II and III

B I, II, and III **D** III only

35. REVIEW For a game, Patricia must roll a die and draw a card from a deck of 26 cards, with each one having a letter of the alphabet on it. What is the probability that Patricia will roll an odd number and draw a letter in her name?

F $\frac{2}{3}$ **H** $\frac{1}{13}$

G $\frac{3}{26}$ **J** $\frac{1}{26}$

Spiral Review

Determine whether each pair of functions are inverse functions. (Lesson 7-2)

36. $f(x) = 3x$, $g(x) = \frac{1}{3}x$

37. $f(x) = 4x - 5$, $g(x) = \frac{1}{4}x - \frac{5}{16}$

38. $f(x) = \frac{3x + 2}{7}$, $g(x) = \frac{7x - 2}{3}$

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 7-1)

39. $f(x) = x + 5$, $g(x) = x - 3$

40. $f(x) = 10x - 20$, $g(x) = x - 2$

41. $f(x) = 4x^2 - 9$, $g(x) = \frac{1}{2x + 3}$

42. BIOLOGY Humans blink their eyes about once every 5 seconds. How many times do humans blink their eyes in two hours? (Lesson 1-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Determine whether each number is *rational* or *irrational*.** (Lesson 1-2)

43. 4.63 **44.** π **45.** $\frac{16}{3}$ **46.** 8.333… **47.** 7.323223222…

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7-4 *n*th Roots

Main Ideas

- Simplify radicals.
- Use a calculator to approximate radicals.

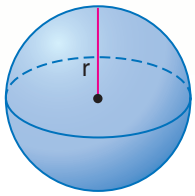
New Vocabulary

*n*th root

principal root

GET READY for the Lesson

The radius r of a sphere with volume V can be found using the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$. This is an example of an equation that contains an nth root. In this case, $n = 3$.



Simplify Radicals Finding the square root of a number and squaring a number are inverse operations. To find the square root of a number n, you must find a number whose square is n.

Study Tip

Look Back

Review **square roots** in Lesson 5-4.

Similarly, the inverse of raising a number to the nth power is finding the **nth root** of a number. The table below shows the relationship between raising a number to a power and taking that root of a number.

Powers	Factors	Roots
$a^3 = 125$	$5 \cdot 5 \cdot 5 = 125$	5 is a cube root of 125.
$a^4 = 81$	$3 \cdot 3 \cdot 3 \cdot 3 = 81$	3 is a fourth root of 81.
$a^5 = 32$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$	2 is a fifth root of 32.
$a^n = b$	$\underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a} = b$	a is an nth root of b.

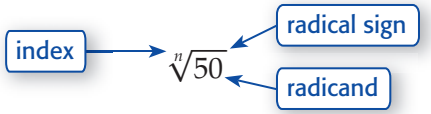
This pattern suggests the following formal definition of an nth root.

KEY CONCEPT — ***Definition of nth Root***

Word For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b.

Example Since $2^5 = 32$, 2 is a fifth root of 32.

The symbol $\sqrt[n]{\ }$ indicates an nth root.



Some numbers have more than one real nth root. For example, 36 has two square roots, 6 and -6. When there is more than one real root, the nonnegative root is called the **principal root**. When no index is given, as in $\sqrt{36}$, the radical sign indicates the principal square root. The symbol $\sqrt[n]{b}$ stands for the principal nth root of b. If n is odd and b is negative, there will be no nonnegative root. In this case, the principal root is negative.

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$\sqrt{16} = 4$ $\sqrt{16}$ indicates the principal square root of 16.

$-\sqrt{16} = -4$ $-\sqrt{16}$ indicates the opposite of the principal square root of 16.

$\pm\sqrt{16} = \pm 4$ $\pm\sqrt{16}$ indicates both square roots of 16. $\pm$ means positive or negative.

$\sqrt[3]{-125} = -5$ $\sqrt[3]{-125}$ indicates the principal cube root of -125.

$-\sqrt[4]{81} = -3$ $-\sqrt[4]{81}$ indicates the opposite of the principal fourth root of 81.

CONCEPT SUMMARY *Real nth roots of b, $\sqrt[n]{b}$, or $-\sqrt[n]{b}$*

n	$\sqrt[n]{b}$ if $b > 0$	$\sqrt[n]{b}$ if $b < 0$	$b = 0$
even	one positive root, one negative root $\pm\sqrt[4]{625} = \pm 5$	no real roots $\sqrt{-4}$ not a real number	one real root, 0 $\sqrt[n]{0} = 0$
odd	one positive root, no negative roots $\sqrt[3]{8} = 2$	no positive roots, one negative root $\sqrt[5]{-32} = -2$	

EXAMPLE **Find Roots**

1 Simplify.

a. $\pm\sqrt{25x^4}$

$\pm 25x^4 = \pm\sqrt{(5x^2)^2}$

$= \pm 5x^2$

The square roots of $25x^4$ are $\pm 5x^2$.

b. $-\sqrt{(y^2+2)^8}$

$-\sqrt{(y^2+2)^8} = -\sqrt{[(y^2+2)^4]^2}$

$= -(y^2+2)^4$

The opposite of the principal square root of $(y^2+2)^8$ is $-(y^2+2)^4$.

c. $\sqrt[5]{32x^{15}y^{20}}$

$\sqrt[5]{32x^{15}y^{20}} = \sqrt[5]{(2x^3y^4)^5}$

$= 2x^3y^4$

The principal fifth root of $32x^{15}y^{20}$ is $2x^3y^4$.

d. $\sqrt{-9}$

$\sqrt{-9} = \sqrt[2]{-9}$ (n is even. b is negative.)

Thus, $\sqrt{-9}$ is not a real number.

Study Tip

Fractional Exponents

For any real number b and any positive integer n, $\sqrt[n]{b} = b^{\frac{1}{n}}$.

CHECK Your Progress

1A. $\pm\sqrt{81y^6}$ **1B.** $-\sqrt{(x-3)^{12}}$

1C. $\sqrt[6]{729x^{30}y^{18}}$ **1D.** $\sqrt{-25}$

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When you find the nth root of an even power and the result is an odd power, you must take the absolute value of the result to ensure that the answer is nonnegative.

$$\sqrt{(-5)^2} = |-5| \text{ or } 5 \qquad \sqrt{(-2)^6} = |(-2)^3| \text{ or } 8$$

If the result is an even power or you find the nth root of an odd power, there is no need to take the absolute value. *Why?*

EXAMPLE Simplify Using Absolute Value

2 **Simplify.**

a. $\sqrt[8]{x^8}$

Note that x is an eighth root of x^8. The index is even, so the principal root is nonnegative. Since x could be negative, you must take the absolute value of x to identify the principal root.

$$\sqrt[8]{x^8} = |x|$$

b. $\sqrt[4]{81(a+1)^{12}}$

$$\sqrt[4]{81(a+1)^{12}} = \sqrt[4]{[3(a+1)^3]^4}$$

Since the index 4 is even and the exponent 3 is odd, you must use an absolute value.

$$\sqrt[4]{81(a+1)^{12}} = 3|(a+1)^3|$$

CHECK Your Progress

2A. $\sqrt{100x^{10}}$ **2B.** $\sqrt{64(y+1)^{14}}$

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Approximate Radicals with a Calculator Recall that real numbers that cannot be expressed as terminating or repeating decimals are *irrational numbers*. Approximations for irrational numbers are often used in real-world problems.

EXAMPLE

3 **PHYSICS** **The distance a planet is from the Sun is a function of the length of its year. The formula is $d = \sqrt[3]{6t^2}$, where d is the distance of the planet from the Sun in millions of miles and t is the number of Earth-days in the planet's year. If the length of a year on Mars is 687 Earth-days, how far from the Sun is Mars?**

$d = \sqrt[3]{6t^2}$ Original formula

$= \sqrt[3]{6(687)^2}$ or about 141.48 $t = 687$

Mars is approximately 141.48 million miles from the Sun.

CHECK According to NASA, Mars is approximately 142 million miles from the Sun. So, 141.48 million miles is reasonable. ✓

Study Tip

Graphing Calculators

To find a root of index greater than 2, first type the index. Then select $\sqrt[x]{\ }$ from the MATH menu. Finally, enter the radicand.

CHECK Your Progress

3. Approximately how far away from the Sun is Earth?

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CHECK Your Understanding

Examples 1, 2 (pp. 403–404)

Simplify.

1. $\sqrt[3]{64}$
2. $\sqrt{(-2)^2}$
3. $\sqrt[5]{-243}$
4. $\sqrt[4]{-4096}$
5. $\sqrt[3]{x^3}$
6. $\sqrt[4]{y^4}$
7. $\sqrt{36a^2b^4}$
8. $\sqrt{(4x + 3y)^2}$

Example 3 (p. 404)

Use a calculator to approximate each value to three decimal places.

9. $\sqrt{77}$
10. $-\sqrt[3]{19}$
11. $\sqrt[4]{48}$

12. **SHIPPING** Golden State Manufacturing wants to increase the size of the boxes it uses to ship its products. The new volume N is equal to the old volume V times the scale factor F cubed, or $N = V \cdot F^3$. What is the scale factor if the old volume was 8 cubic feet and the new volume is 216 cubic feet?

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–22	1
23–36	2
37–50	3

Simplify.

13. $\sqrt{225}$
14. $\pm\sqrt{169}$
15. $\sqrt{-(-7)^2}$
16. $\sqrt{(-18)^2}$
17. $\sqrt[3]{-27}$
18. $\sqrt[7]{-128}$
19. $\sqrt{\frac{1}{16}}$
20. $\sqrt[3]{\frac{1}{125}}$
21. $\sqrt{0.25}$
22. $\sqrt[3]{-0.064}$
23. $\sqrt[4]{z^8}$
24. $-\sqrt[6]{x^6}$
25. $\sqrt{49m^6}$
26. $\sqrt{64a^8}$
27. $\sqrt[3]{27r^3}$
28. $\sqrt[3]{-c^6}$
29. $\sqrt{(5g)^4}$
30. $\sqrt[3]{(2z)^6}$
31. $\sqrt{25x^4y^6}$
32. $\sqrt{36x^4z^4}$
33. $\sqrt{169x^8y^4}$
34. $\sqrt{9p^{12}q^6}$
35. $\sqrt[3]{8a^3b^3}$
36. $\sqrt[3]{-27c^9d^{12}}$

Use a calculator to approximate each value to three decimal places.

37. $\sqrt{129}$
38. $-\sqrt{147}$
39. $\sqrt{0.87}$
40. $\sqrt{4.27}$
41. $\sqrt[3]{59}$
42. $\sqrt[3]{-480}$
43. $\sqrt[4]{602}$
44. $\sqrt[5]{891}$
45. $\sqrt[6]{4123}$
46. $\sqrt[7]{46{,}815}$
47. $\sqrt[6]{(723)^3}$
48. $\sqrt[4]{(3500)^2}$

49. **AEROSPACE** The radius r of the orbit of a satellite is given by $r = \sqrt[3]{\frac{GMt^2}{4\pi^2}}$, where G is the universal gravitational constant, M is the mass of the central object, and t is the time it takes the satellite to complete one orbit. Find the radius of the orbit if G is 6.67×10^{-11} N $\cdot$ m^2/kg^2, M is 5.98×10^{24} kg, and t is 2.6×10^6 seconds.

EXTRA PRACTICE
See pages 906 and 932.
MathOnline
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50. **SHOPPING** A certain store found that the number of customers that will attend a limited time sale can be modeled by $N = 125\sqrt[3]{100Pt}$, where N is the number of customers expected, P is the percent of the sale discount, and t is the number of hours the sale will last. Find the number of customers the store should expect for a sale that is 50% off and will last four hours.

H.O.T. Problems

51. **OPEN ENDED** Write a number whose principal square root and cube root are both integers.

52. **REASONING** Determine whether the statement $\sqrt[4]{(-x)^4} = x$ is *sometimes*, *always*, or *never* true.

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53. CHALLENGE Under what conditions is $\sqrt{x^2 + y^2} = x + y$ true?

54. REASONING Explain why it is not always necessary to take the absolute value of a result to indicate the principal root.

55. *Writing in Math* Refer to the information on page 402 to explain how nth roots apply to geometry. Analyze what happens to the value of r as the value of V increases.

STANDARDIZED TEST PRACTICE

56. ACT/SAT Which of the following is closest to $\sqrt[3]{7.32}$?

A 1.8

B 1.9

C 2.0

D 2.1

57. REVIEW What is the product of the complex numbers $(5 + i)$ and $(5 - i)$?

F 24

G 26

H $25 - i$

J $26 - 10i$

Spiral Review

Graph each function. State the domain and range. (Lesson 7-3)

58. $y = \sqrt{x - 2}$ **59.** $y = \sqrt{x} - 1$ **60.** $y = 2\sqrt{x} + 1$

61. Determine whether the functions $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. (Lesson 7-2)

Simplify. (Lesson 5-4)

62. $(3 + 2i) - (1 - 7i)$ **63.** $(8 - i)(4 - 3i)$ **64.** $\dfrac{2 + 3i}{1 + 2i}$

Solve each system of equations. (Lesson 3-2)

65. $2x - y = 7$
$x + 3y = 0$

66. $4x + y = 7$
$3x + \frac{4}{5}y = 5.5$

67. $\frac{1}{4}x + \frac{2}{3}y = 3$
$2x + y = -2$

68. BUSINESS A dry cleaner ordered 7 drums of two different types of cleaning fluid. One type costs \$30 per drum, and the other type costs \$20 per drum. The total cost was \$160. How much of each type of fluid did the company order? Write a system of equations and solve by graphing. (Lesson 3-1)

Graph each function. (Lesson 2-6)

69. $f(x) = 5$ **70.** $f(x) = |x - 3|$ **71.** $f(x) = |2x| + 3$

GET READY for the Next Lesson

PREREQUISITE SKILL Find each product. (Lesson 6-2)

72. $(x + 3)(x + 8)$ **73.** $(y - 2)(y + 5)$ **74.** $(a + 2)(a - 9)$

75. $(a + b)(a + 2b)$ **76.** $(x - 3y)(x + 3y)$ **77.** $(2w + z)(3w - 5z)$

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CHAPTER 7

Mid-Chapter Quiz

Lessons 7-1 through 7-4

Given $f(x) = 2x^2 - 5x + 3$ and $g(x) = 6x + 4$, find each function. (Lesson 7-1)

1. $(f + g)(x)$
2. $(f - g)(x)$
3. $(f \cdot g)(x)$
4. $\left(\frac{f}{g}\right)(x)$
5. $[f \circ g](x)$
6. $[g \circ f](x)$

DINING For Exercises 7 and 8, use the following information. (Lesson 7-1)

The Rockwell family goes out to dinner at Jack's Fancy Steak House. They have a coupon for 10% off their meal, but this restaurant adds an 18% gratuity.

7. Express the price of the meal after the discount and the price of the meal after the gratuity gets added using function notation. Let x represent the price of the meal, $p(x)$ represent the price after the 10% discount, and $g(x)$ represent the price after the gratuity is added to the bill.
8. Which composition of functions represents the price of the meal, $p[g(x)]$ or $g[p(x)]$? Explain your reasoning.

Determine whether each pair of functions are inverse functions. (Lesson 7-2)

9. $f(x) = x + 73$, $g(x) = x - 73$
10. $g(x) = 7x - 11$, $h(x) = \frac{1}{7}x + 11$

REMODELING For Exercises 11 and 12, use the following information. (Lesson 7-2)

Kimi is replacing the carpet in her 12-foot by 15-foot living room. The new carpet costs $13.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet.

11. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$ for Kimi?
12. What will the new carpet cost Kimi?

Graph each inequality. (Lesson 7-3)

13. $y < \sqrt{x + 3}$
14. $y \geq -5\sqrt{x}$

Graph each function. State the domain and range of each function. (Lesson 7-3)

15. $y = 3 - \sqrt{x}$
16. $y = \sqrt{5x}$
17. $y = \sqrt{2x - 7} + 4$
18. $y = -2\sqrt{6x - 1}$

19. **MULTIPLE CHOICE** What is the domain of $f(x) = \sqrt{5x - 3}$?

A $\left\{x \middle| x > \frac{3}{5}\right\}$

B $\left\{x \middle| x > -\frac{3}{5}\right\}$

C $\left\{x \middle| x \geq \frac{3}{5}\right\}$

D $\left\{x \middle| x \geq -\frac{3}{5}\right\}$

Simplify. (Lesson 7-4)

20. $\sqrt{36x^2y^6}$
21. $\sqrt[3]{-64a^6b^9}$
22. $\sqrt{4n^2 + 12n + 9}$
23. $\sqrt{\frac{x^4}{y^3}}$

24. **MULTIPLE CHOICE** The relationship between the length and mass of Pacific Halibut can be approximated by the equation $L = 0.46\sqrt[3]{M}$, where L is the length in meters and M is the mass in kilograms. Use this equation to predict the length of a 25-kilogram Pacific Halibut. (Lesson 7-4)

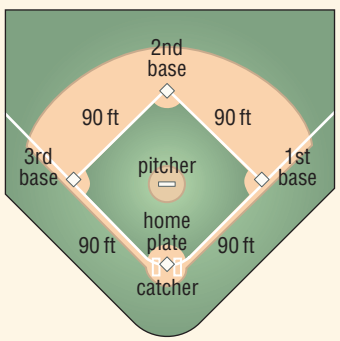
F 1.03 m

G 1.35 m

H 1.97 m

J 2.30 m

25. **BASEBALL** Refer to the drawing below. How far does the catcher have to throw a ball from home plate to second base? (Lesson 7-4)



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7-5 Operations with Radical Expressions

Main Ideas

- Simplify radical expressions.
- Add, subtract, multiply, and divide radical expressions.

New Vocabulary

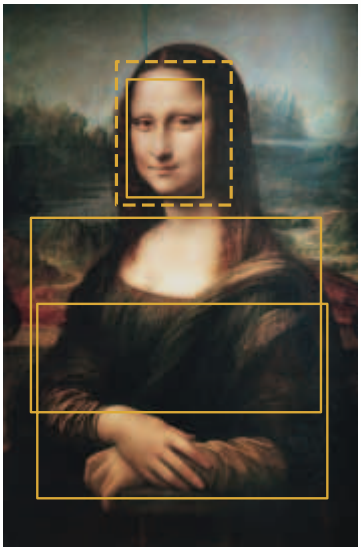
rationalizing the denominator

like radical expressions

conjugates

GET READY for the Lesson

Golden rectangles have been used by artists and architects to create beautiful designs. For example, if you draw a rectangle around the Mona Lisa's face, the resulting quadrilateral is the golden rectangle. The ratio of the lengths of the sides of a golden rectangle is $\frac{2}{\sqrt{5}-1}$. In this lesson, you will learn how to simplify radical expressions like $\frac{2}{\sqrt{5}-1}$.



Simplify Radicals The properties you have used to simplify radical expressions involving square roots also hold true for expressions involving nth roots.

KEY CONCEPT *Properties of Radicals*

For any real numbers a and b and any integer $n > 1$, the following properties hold true.

Property	Words	Examples
Product Property	**1.** If n is even and a and b are both nonnegative, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, and **2.** If n is odd, then $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.	$\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$, or 4, and $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{27}$, or 3
Quotient Property	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, if all roots are defined and $b \neq 0$.	$\frac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\frac{54}{2}} = \sqrt[3]{27}$, or 3

Follow these steps to simplify a square root.

Step 1 Factor the radicand into as many squares as possible.

Step 2 Use the Product Property to isolate the perfect squares.

Step 3 Simplify each radical.

Gianni Dagli Orti/CORBIS

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You can use the properties of radicals to write expressions in simplified form.

CONCEPT SUMMARY *Simplifying Radical Expressions*

A radical expression is in simplified form when the following conditions are met.

- The index n is as small as possible.
- The radicand contains no factors (other than 1) that are nth powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in a denominator.

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize a denominator, multiply the numerator and denominator by a quantity so that the radicand has an exact root.

EXAMPLE Simplify Expressions

1 **Simplify.**

a. $\sqrt{16p^8q^7}$

$$\sqrt{16p^8q^7} = \sqrt{4^2 \cdot (p^4)^2 \cdot (q^3)^2 \cdot q}$$ Factor into squares where possible.

$$= \sqrt{4^2} \cdot \sqrt{(p^4)^2} \cdot \sqrt{(q^3)^2} \cdot \sqrt{q}$$ Product Property of Radicals

$$= 4p^4|q^3|\sqrt{q}$$ Simplify.

However, for $\sqrt{16p^8q^7}$ to be defined, $16p^8q^7$ must be nonnegative. If that is true, q must be nonnegative, since it is raised to an odd power. Thus, the absolute value is unnecessary, and $\sqrt{16p^8q^7} = 4p^4q^3\sqrt{q}$.

Study Tip

Rationalizing the Denominator

You may want to think of rationalizing the denominator as making the denominator a rational number.

b. $\sqrt{\dfrac{x^4}{y^5}}$

$$\sqrt{\frac{x^4}{y^5}} = \frac{\sqrt{x^4}}{\sqrt{y^5}}$$ Quotient Property

$$= \frac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2 \cdot y}}$$ Factor into squares.

$$= \frac{\sqrt{(x^2)^2}}{\sqrt{(y^2)^2} \cdot \sqrt{y}}$$ Product Property

$$= \frac{x^2}{y^2\sqrt{y}}$$ $\sqrt{(x^2)^2} = x^2$

$$= \frac{x^2}{y^2\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}}$$ Rationalize the denominator.

$$= \frac{x^2\sqrt{y}}{y^3}$$ $\sqrt{y} \cdot \sqrt{y} = y$

c. $\sqrt[5]{\dfrac{5}{4a}}$

$$\sqrt[5]{\frac{5}{4a}} = \frac{\sqrt[5]{5}}{\sqrt[5]{4a}}$$ Quotient Property

$$= \frac{\sqrt[5]{5}}{\sqrt[5]{4a}} \cdot \frac{\sqrt[5]{8a^4}}{\sqrt[5]{8a^4}}$$ Rationalize the denominator.

$$= \frac{\sqrt[5]{5 \cdot 8a^4}}{\sqrt[5]{4a \cdot 8a^4}}$$ Product Property

$$= \frac{\sqrt[5]{40a^4}}{\sqrt[5]{32a^5}}$$ Multiply.

$$= \frac{\sqrt[5]{40a^4}}{2a}$$ $\sqrt[5]{32a^5} = 2a$

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CHECK Your Progress Simplify.

1A. $\sqrt{36r^5s^{10}}$ **1B.** $\sqrt{\frac{m^9}{n^7}}$ **1C.** $\sqrt[4]{\frac{3x}{2}}$

Operations with Radicals You can use the Product and Quotient Properties to multiply and divide some radicals, respectively.

EXAMPLE Multiply Radicals

2 Simplify $6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n}$.

$$6\sqrt[3]{9n^2} \cdot 3\sqrt[3]{24n} = 6 \cdot 3 \cdot \sqrt[3]{9n^2 \cdot 24n}$$ Product Property of Radicals

$$= 18 \cdot \sqrt[3]{2^3 \cdot 3^3 \cdot n^3}$$ Factor into cubes where possible.

$$= 18 \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{3^3} \cdot \sqrt[3]{n^3}$$ Product Property of Radicals

$$= 18 \cdot 2 \cdot 3 \cdot n \text{ or } 108n$$ Multiply.

CHECK Your Progress

Simplify.

2A. $5\sqrt[4]{24x^3} \cdot 4\sqrt[4]{54x}$ **2B.** $7\sqrt[3]{75a^4} \cdot 3\sqrt[3]{45a^2}$

Can you add radicals in the same way that you multiply them? In other words, if $\sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a}$, does $\sqrt{a} + \sqrt{a} = \sqrt{a + a}$?

ALGEBRA LAB

Adding Radicals

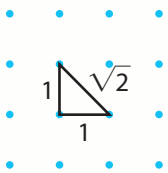
You can use dot paper to show the sum of two like radicals, such as $\sqrt{2} + \sqrt{2}$.

Step 1 First, find a segment of length $\sqrt{2}$ units by using the Pythagorean Theorem with the dot paper.

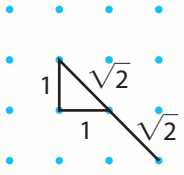
$a^2 + b^2 = c^2$

$1^2 + 1^2 = c^2$

$2 = c^2$



Step 2 Extend the segment to twice its length to represent $\sqrt{2} + \sqrt{2}$.



ANALYZE THE RESULTS

1. Is $\sqrt{2} + \sqrt{2} = \sqrt{2 + 2}$ or 2? Justify your answer using the geometric models above.
2. Use this method to model other irrational numbers. Do these models support your conjecture?

Study Tip

Simplifying Radicals

In general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$. In fact, $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$ only when $a = 0$, $b = 0$, or both $a = 0$ and $b = 0$.

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You add radicals in the same manner as adding monomials. That is, you can add only the like terms or like radicals. Two radical expressions are called **like radical expressions** if both the indices and the radicands are alike.

Like: $2\sqrt[4]{3a}$ and $5\sqrt[4]{3a}$ Radicands are $3a$; indices are 4.

Unlike: $\sqrt{3}$ and $\sqrt[3]{3}$ Different indices

$\sqrt[4]{5x}$ and $\sqrt[4]{5}$ Different radicands

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EXAMPLE Add and Subtract Radicals

3 **Simplify $2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$.**

$2\sqrt{12} - 3\sqrt{27} + 2\sqrt{48}$

$= 2\sqrt{2^2 \cdot 3} - 3\sqrt{3^2 \cdot 3} + 2\sqrt{2^2 \cdot 2^2 \cdot 3}$ Factor using squares.

$= 2\sqrt{2^2} \cdot \sqrt{3} - 3\sqrt{3^2} \cdot \sqrt{3} + 2\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{3}$ Product Property

$= 2 \cdot 2 \cdot \sqrt{3} - 3 \cdot 3 \cdot \sqrt{3} + 2 \cdot 2 \cdot 2 \cdot \sqrt{3}$ $\sqrt{2^2} = 2$, $\sqrt{3^2} = 3$

$= 4\sqrt{3} - 9\sqrt{3} + 8\sqrt{3}$ Multiply.

$= 3\sqrt{3}$ Combine like radicals.

CHECK Your Progress **Simplify.**

3A. $3\sqrt{8} + 5\sqrt{32} - 4\sqrt{18}$ **3B.** $5\sqrt{12} - 2\sqrt{27} + 6\sqrt{108}$

Just as you can add and subtract radicals like monomials, you can multiply radicals using the FOIL method as you do when multiplying binomials.

EXAMPLE Multiply Radicals

4 **Simplify.**

a. $(3\sqrt{5} - 2\sqrt{3})(2 + \sqrt{3})$

F O I L

$(3\sqrt{5} - 2\sqrt{3})(2 + \sqrt{3}) = 3\sqrt{5} \cdot 2 + 3\sqrt{5} \cdot \sqrt{3} - 2\sqrt{3} \cdot 2 - 2\sqrt{3} \cdot \sqrt{3}$

$= 6\sqrt{5} + 3\sqrt{5 \cdot 3} - 4\sqrt{3} - 2\sqrt{3^2}$ Product Property

$= 6\sqrt{5} + 3\sqrt{15} - 4\sqrt{3} - 6$ $2\sqrt{3^2} = 2 \cdot 3$ or 6

b. $(5\sqrt{3} - 6)(5\sqrt{3} + 6)$

$(5\sqrt{3} - 6)(5\sqrt{3} + 6) = 5\sqrt{3} \cdot 5\sqrt{3} + 5\sqrt{3} \cdot 6 - 6 \cdot 5\sqrt{3} - 6 \cdot 6$ FOIL

$= 25\sqrt{3^2} + 30\sqrt{3} - 30\sqrt{3} - 36$ Multiply.

$= 75 - 36$ $25\sqrt{3^2} = 25 \cdot 3$ or 75

$= 39$ Subtract.

Study Tip
Conjugates
The product of conjugates of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ is always a rational number.

CHECK Your Progress

4A. $(4\sqrt{2} + 2\sqrt{6})(\sqrt{5} - 3)$ **4B.** $(3\sqrt{5} + 4)(3\sqrt{5} - 4)$

Binomials like those in Example 4b, of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates** of each other. You can use conjugates to rationalize denominators.

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EXAMPLE Use a Conjugate to Rationalize a Denominator

5 Simplify $\dfrac{1-\sqrt{3}}{5+\sqrt{3}}$.

$$\frac{1-\sqrt{3}}{5+\sqrt{3}} = \frac{(1-\sqrt{3})(5-\sqrt{3})}{(5+\sqrt{3})(5-\sqrt{3})}$$ Multiply by $\frac{5-\sqrt{3}}{5-\sqrt{3}}$ because $5-\sqrt{3}$ is the conjugate of $5+\sqrt{3}$.

$$= \frac{1 \cdot 5 - 1 \cdot \sqrt{3} - \sqrt{3} \cdot 5 + (\sqrt{3})^2}{5^2 - (\sqrt{3})^2}$$ FOIL

Difference of squares

$$= \frac{5 - \sqrt{3} - 5\sqrt{3} + 3}{25 - 3}$$ Multiply.

$$= \frac{8 - 6\sqrt{3}}{22}$$ Combine like terms.

$$= \frac{4 - 3\sqrt{3}}{11}$$ Divide numerator and denominator by 2.

CHECK Your Progress **Simplify.**

5A. $\dfrac{4+\sqrt{2}}{5-\sqrt{2}}$ **5B.** $\dfrac{3-2\sqrt{5}}{6+\sqrt{5}}$

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CHECK Your Understanding

Example 1 (pp. 409–410)

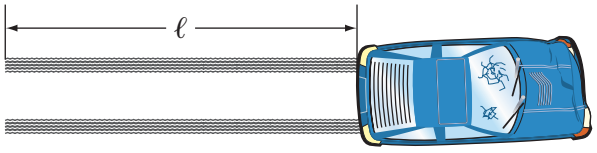
Simplify.

1. $5\sqrt{63}$ **2.** $\sqrt[4]{16x^5y^4}$ **3.** $\sqrt{75x^3y^6}$

4. $\sqrt{\dfrac{7}{8y}}$ **5.** $\sqrt{\dfrac{a^7}{b^9}}$ **6.** $\sqrt[3]{\dfrac{2}{3x}}$

LAW ENFORCEMENT **For Exercises 7 and 8, use the following information.**

Under certain conditions, a police accident investigator can use the formula $s = \dfrac{10\sqrt{\ell}}{\sqrt{5}}$ to estimate the speed s of a car in miles per hour based on the length ℓ in feet of the skid marks it left.



7. Write the formula without a radical in the denominator.

8. How fast was a car traveling that left skid marks 120 feet long?

Examples 2–5 (pp. 410–412)

Simplify.

9. $(-2\sqrt{15})(4\sqrt{21})$ **10.** $\sqrt{2ab^2} \cdot \sqrt{6a^3b^2}$ **11.** $\dfrac{\sqrt[3]{625}}{\sqrt[3]{25}}$

12. $\sqrt{3} - 2\sqrt[4]{3} + 4\sqrt{3} + 5\sqrt[4]{3}$ **13.** $3\sqrt[3]{128} + 5\sqrt[3]{16}$

14. $(3-\sqrt{5})(1+\sqrt{3})$ **15.** $(2+\sqrt{2})(2-\sqrt{2})$

16. $\dfrac{1+\sqrt{5}}{3-\sqrt{5}}$ **17.** $\dfrac{4-\sqrt{7}}{3+\sqrt{7}}$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
18–23	1
34–35	2
36–41	3
42–45	4
46–51	5

Simplify.

18. $\sqrt{243}$ **19.** $\sqrt{72}$ **20.** $\sqrt[3]{54}$ **21.** $\sqrt[4]{96}$

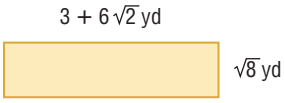
22. $\sqrt{50x^4}$ **23.** $\sqrt[3]{16y^3}$ **24.** $\sqrt{18x^2y^3}$ **25.** $\sqrt{40a^3b^4}$

26. $3\sqrt[3]{56y^6z^3}$ **27.** $2\sqrt[3]{24m^4n^5}$ **28.** $\sqrt[4]{\frac{1}{81}c^5d^4}$ **29.** $\sqrt[5]{\frac{1}{32}w^6z^7}$

30. $\sqrt[3]{\frac{3}{4}}$ **31.** $\sqrt[4]{\frac{2}{3}}$ **32.** $\sqrt{\frac{a^4}{b^3}}$ **33.** $\sqrt{\frac{4r^8}{t^9}}$

34. $(3\sqrt{12})(2\sqrt{21})$ **35.** $(-3\sqrt{24})(5\sqrt{20})$

36. GEOMETRY Find the perimeter and area of the rectangle.



37. GEOMETRY Find the perimeter of a regular pentagon whose sides measure $(2\sqrt{3} + 3\sqrt{12})$ feet.

Simplify.

38. $\sqrt{12} + \sqrt{48} - \sqrt{27}$ **39.** $\sqrt{98} - \sqrt{72} + \sqrt{32}$

40. $\sqrt{3} + \sqrt{72} - \sqrt{128} + \sqrt{108}$ **41.** $5\sqrt{20} + \sqrt{24} - \sqrt{180} + 7\sqrt{54}$

42. $(5 + \sqrt{6})(5 - \sqrt{2})$ **43.** $(3 + \sqrt{7})(2 + \sqrt{6})$

44. $(\sqrt{11} - \sqrt{2})^2$ **45.** $(\sqrt{3} - \sqrt{5})^2$

46. $\frac{7}{4 - \sqrt{3}}$ **47.** $\frac{\sqrt{6}}{5 + \sqrt{3}}$ **48.** $\frac{-2 - \sqrt{3}}{1 + \sqrt{3}}$

49. $\frac{2 + \sqrt{2}}{5 - \sqrt{2}}$ **50.** $\frac{x + 1}{\sqrt{x^2 - 1}}$ **51.** $\frac{x - 1}{\sqrt{x} - 1}$

52. What is $\sqrt{39}$ divided by $\sqrt{26}$?

53. Divide $\sqrt{14}$ by $\sqrt{35}$.

AMUSEMENT PARKS **For Exercises 54 and 55, use the following information.**
The velocity v in feet per second of a roller coaster at the bottom of a hill is related to the vertical drop h in feet and the velocity v_0 in feet per second of the coaster at the top of the hill by the formula $v_0 = \sqrt{v^2 - 64h}$.

54. Explain why $v_0 = v - 8\sqrt{h}$ is not equivalent to the given formula.

55. What velocity must a coaster have at the top of a 225-foot hill to achieve a velocity of 120 feet per second at the bottom?

SPORTS **For Exercises 56 and 57, use the following information.**
A ball that is hit or thrown horizontally with a velocity of v meters per second will travel a distance of d meters before hitting the ground, where $d = v\sqrt{\frac{h}{4.9}}$ and h is the height in meters from which the ball is hit or thrown.

56. Use the properties of radicals to rewrite the formula.

57. How far will a ball that is hit with a velocity of 45 meters per second at a height of 0.8 meter above the ground travel before hitting the ground?

EXTRA PRACTICE
See pages 906 and 932.
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Self-Check Quiz at algebra2.com

H.O.T. Problems

58. REASONING Determine whether the statement $\frac{1}{\sqrt[n]{a}} = \sqrt[n]{a}$ is *sometimes*, *always*, or *never* true. Explain.

59. OPEN ENDED Write a sum of three radicals that contains two like terms. Explain how you would combine the terms. Defend your answer.

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60. FIND THE ERROR Ethan and Alexis are simplifying $\frac{4+\sqrt{5}}{2-\sqrt{5}}$. Who is correct? Explain your reasoning.

Ethan

$$\frac{4+\sqrt{5}}{2-\sqrt{5}} = \frac{4+\sqrt{5}}{2-\sqrt{5}} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}$$
$$= \frac{13+6\sqrt{5}}{-1}$$

Alexis

$$\frac{4+\sqrt{5}}{2-\sqrt{5}} = \frac{4+\sqrt{5}}{2-\sqrt{5}} \cdot \frac{4-\sqrt{5}}{4-\sqrt{5}}$$
$$= \frac{11}{13-6\sqrt{5}}$$

61. *Writing in Math* Refer to the information given on page 408 to explain how radical expressions relate to the Mona Lisa. Use the properties in this lesson to explain how you could rewrite the radical expression.

STANDARDIZED TEST PRACTICE

62. ACT/SAT The expression $\sqrt{180a^2b^8}$ is equivalent to which of the following?

A $5\sqrt{6}|a|b^4$

B $6\sqrt{5}|a|b^4$

C $3\sqrt{10}|a|b^4$

D $36\sqrt{5}|a|b^4$

63. REVIEW When the number of a year is divisible by 4, then a leap year occurs. However, when the year is divisible by 100, then a leap year does not occur unless the year is divisible by 400. Which is *not* an example of a leap year?

F 1884 **H** 1904

G 1900 **J** 1940

Spiral Review

Simplify. (Lesson 7-4)

64. $\sqrt{144z^8}$ **65.** $\sqrt[3]{216a^3b^9}$ **66.** $\sqrt{(y+2)^2}$

67. Graph $y \leq \sqrt{x+1}$. (Lesson 7-3)

68. ELECTRONICS There are three basic things to be considered in an electrical circuit: the flow of the electrical current I, the resistance to the flow Z, called impedance, and electromotive force E, called voltage. These quantities are related in the formula $E = I \cdot Z$. The current of a circuit is to be $35 - 40j$ amperes. Electrical engineers use the letter j to represent the imaginary unit. Find the impedance of the circuit if the voltage is to be $430 - 330j$ volts. (Lesson 5-4)

Find the inverse of each matrix, if it exists. (Lesson 4-7)

69. $\begin{bmatrix} 8 & 6 \\ 7 & 5 \end{bmatrix}$ **70.** $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ **71.** $\begin{bmatrix} 8 & 4 \\ 6 & 3 \end{bmatrix}$

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.**

72. $2\left(\frac{1}{8}\right)$ **73.** $3\left(\frac{1}{6}\right)$ **74.** $\frac{1}{2}+\frac{1}{3}$ **75.** $\frac{1}{3}+\frac{3}{4}$

76. $\frac{1}{8}+\frac{5}{12}$ **77.** $\frac{5}{6}-\frac{1}{5}$ **78.** $\frac{5}{8}-\frac{1}{4}$ **79.** $\frac{1}{4}-\frac{2}{3}$

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7-6 Rational Exponents

Main Ideas

- Write expressions with rational exponents in radical form and vice versa.
- Simplify expressions in exponential or radical form.

GET READY for the Lesson

Astronomers refer to the space around a planet where the planet's gravity is stronger than the Sun's as the *sphere of influence* of the planet. The radius r of the sphere of influence is given by the formula $r = D\left(\frac{M_p}{M_S}\right)^{\frac{2}{5}}$, where M_p is the mass of the planet, M_S is the mass of the Sun, and D is the distance between the planet and the Sun.

Rational Exponents and Radicals You know that squaring a number and taking the square root of a number are inverse operations. But how would you evaluate an expression that contains a fractional exponent such as the one above? You can investigate such an expression by assuming that fractional exponents behave as integral exponents.

$$\left(b^{\frac{1}{2}}\right)^2 = b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} \quad \text{Write the square as multiplication.}$$

$$= b^{\frac{1}{2}+\frac{1}{2}} \quad \text{Add the exponents.}$$

$$= b^1 \text{ or } b \quad \text{Simplify.}$$

Thus, $b^{\frac{1}{2}}$ is a number whose square equals b. So $b^{\frac{1}{2}} = \sqrt{b}$.

KEY CONCEPT $b^{\frac{1}{n}}$

Words For any real number b and for any positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except when $b < 0$ and n is even.

Example $8^{\frac{1}{3}} = \sqrt[3]{8}$ or 2

EXAMPLE **Radical and Exponential Forms**

1 **a. Write $a^{\frac{1}{4}}$ in radical form.**

$a^{\frac{1}{4}} = \sqrt[4]{a}$ Definition of $b^{\frac{1}{n}}$

b. Write $\sqrt[3]{y}$ in exponential form.

$\sqrt[3]{y} = y^{\frac{1}{3}}$ Definition of $b^{\frac{1}{n}}$

CHECK Your Progress

1A. $x^{\frac{1}{5}}$

1B. $\sqrt[8]{c}$

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Study Tip

Negative Base

Suppose the base of a monomial is negative, such as $(-9)^2$ or $(-9)^3$. The expression is undefined if the exponent is even because there is no number that, when multiplied an even number of times, results in a negative number. However, the expression is defined for an odd exponent.

EXAMPLE 2 Evaluate Expressions with Rational Exponents

Evaluate each expression.

a. $16^{-\frac{1}{4}}$

Method 1

$$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} \qquad b^{-n} = \frac{1}{b^n}$$

$$= \frac{1}{\sqrt[4]{16}} \qquad 16^{\frac{1}{4}} = \sqrt[4]{16}$$

$$= \frac{1}{\sqrt[4]{2^4}} \qquad 16 = 2^4$$

$$= \frac{1}{2} \qquad \text{Simplify.}$$

Method 2

$$16^{-\frac{1}{4}} = (2^4)^{-\frac{1}{4}} \qquad 16 = 2^4$$

$$= 2^{4\left(-\frac{1}{4}\right)} \qquad \text{Power of a Power}$$

$$= 2^{-1} \qquad \text{Multiply exponents.}$$

$$= \frac{1}{2} \qquad 2^{-1} = \frac{1}{2^1}$$

b. $243^{\frac{3}{5}}$

Method 1

$$243^{\frac{3}{5}} = 243^{3\left(\frac{1}{5}\right)} \qquad \text{Factor.}$$

$$= (243^3)^{\frac{1}{5}} \qquad \text{Power of a Power}$$

$$= \sqrt[5]{243^3} \qquad b^{\frac{1}{5}} = \sqrt[5]{b}$$

$$= \sqrt[5]{(3^5)^3} \qquad 243 = 3^5$$

$$= \sqrt[5]{3^5 \cdot 3^5 \cdot 3^5} \qquad \text{Expand the cube.}$$

$$= 3 \cdot 3 \cdot 3 \text{ or } 27 \qquad \text{Find the fifth root.}$$

Method 2

$$243^{\frac{3}{5}} = (3^5)^{\frac{3}{5}} \qquad 243 = 3^5$$

$$= 3^{5\left(\frac{3}{5}\right)} \qquad \text{Power of a Power}$$

$$= 3^3 \qquad \text{Multiply exponents.}$$

$$= 27 \qquad 3^3 = 3 \cdot 3 \cdot 3$$

CHECK Your Progress

2A. $27^{-\frac{1}{3}}$ **2B.** $64^{\frac{2}{3}}$

In Example 2b, Method 1 uses a combination of the definition of $b^{\frac{1}{n}}$ and the properties of powers. This example suggests the following general definition of rational exponents.

KEY CONCEPT *Rational Exponents*

Words For any nonzero real number b, and any integers m and n, with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$, except when $b < 0$ and n is even.

Example $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2$, or 4

In general, we define $b^{\frac{m}{n}}$ as $\left(b^{\frac{1}{n}}\right)^m$ or $(b^m)^{\frac{1}{n}}$. Now apply the definition of $b^{\frac{1}{n}}$ to $\left(b^{\frac{1}{n}}\right)^m$ and $(b^m)^{\frac{1}{n}}$.

$$\left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m \qquad (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$$

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Real-World Link

With origins in both the ancient Egyptian and Greek societies, weightlifting was among the sports on the program of the first Modern Olympic Games, in 1896, in Athens, Greece.

Source: International Weightlifting Association

Real-World EXAMPLE

3 **WEIGHTLIFTING** **The formula $M = 512 - 146{,}230B^{-\frac{8}{5}}$ can be used to estimate the maximum total mass that a weightlifter of mass B kilograms can lift using the snatch and the clean and jerk.**

a. According to the formula, what is the maximum amount that 2004 Olympic champion Hossein Reza Zadeh of Iran can lift if he weighs 163 kilograms?

$M = 512 - 146{,}230B^{-\frac{8}{5}}$ Original formula

$= 512 - 146{,}230(163)^{-\frac{8}{5}}$ or about 470 $B = 163$

The formula predicts that he can lift at most 470 kilograms.

b. Hossein Reza Zadeh's winning total in the 2004 Olympics was 472.50 kg. Compare this to the value predicted by the formula.

The formula prediction is close to the actual weight, but slightly lower.

CHECK Your Progress

3. The radius r of a sphere with volume V is given by $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$. Find the radius of a ball whose volume is 77 cm^3.

Simplify Expressions All of the properties of powers you learned in Lesson 6-1 apply to rational exponents. When simplifying expressions containing rational exponents, leave the exponent in rational form rather than writing the expression as a radical. Write the expression with all positive exponents. Also, any exponents in the denominator of a fraction must be positive *integers*. So, it may be necessary to rationalize a denominator.

EXAMPLE Simplify Expressions with Rational Exponents

4 **Simplify each expression.**

a. $x^{\frac{1}{5}} \cdot x^{\frac{7}{5}}$

$x^{\frac{1}{5}} \cdot x^{\frac{7}{5}} = x^{\frac{1}{5}+\frac{7}{5}}$ Multiply powers.

$= x^{\frac{8}{5}}$ Add exponents.

b. $y^{-\frac{3}{4}}$

$y^{-\frac{3}{4}} = \frac{1}{y^{\frac{3}{4}}}$ $b^{-n} = \frac{1}{b^n}$

$= \frac{1}{y^{\frac{3}{4}}} \cdot \frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}}$ Why use $\frac{y^{\frac{1}{4}}}{y^{\frac{1}{4}}}$?

$= \frac{y^{\frac{1}{4}}}{y^{\frac{4}{4}}}$ $y^{\frac{3}{4}} \cdot y^{\frac{1}{4}} = y^{\frac{3}{4}+\frac{1}{4}}$

$= \frac{y^{\frac{1}{4}}}{y}$ $y^{\frac{4}{4}} = y^1$ or y

CHECK Your Progress

4A. $a^{\frac{1}{4}} \cdot a^{\frac{9}{4}}$

4B. $r^{-\frac{4}{5}}$

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Study Tip

Indices

When simplifying a radical expression, always use the smallest index possible.

EXAMPLE Simplify Radical Expressions

5 Simplify each expression.

a. $\dfrac{\sqrt[8]{81}}{\sqrt[6]{3}}$

$$\frac{\sqrt[8]{81}}{\sqrt[6]{3}} = \frac{81^{\frac{1}{8}}}{3^{\frac{1}{6}}} \quad \text{Rational exponents}$$

$$= \frac{(3^4)^{\frac{1}{8}}}{3^{\frac{1}{6}}} \quad 81 = 3^4$$

$$= \frac{3^{\frac{1}{2}}}{3^{\frac{1}{6}}} \quad \text{Power of a Power}$$

$$= 3^{\frac{1}{2}-\frac{1}{6}} \quad \text{Quotient of Powers}$$

$$= 3^{\frac{1}{3}} \quad \text{Simplify.}$$

$$= \sqrt[3]{3} \quad \text{Rewrite in radical form.}$$

b. $\sqrt[4]{9z^2}$

$$\sqrt[4]{9z^2} = (9z^2)^{\frac{1}{4}} \quad \text{Rational exponents}$$

$$= (3^2 \cdot z^2)^{\frac{1}{4}} \quad 9 = 3^2$$

$$= 3^{2\left(\frac{1}{4}\right)} \cdot z^{2\left(\frac{1}{4}\right)} \quad \text{Power of a Power}$$

$$= 3^{\frac{1}{2}} \cdot z^{\frac{1}{2}} \quad \text{Multiply.}$$

$$= \sqrt{3} \cdot \sqrt{z} \quad 3^{\frac{1}{2}} = \sqrt{3},\ z^{\frac{1}{2}} = \sqrt{z}$$

$$= \sqrt{3z} \quad \text{Simplify.}$$

c. $\dfrac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1}$

$$\frac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} = \frac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} + 1} \cdot \frac{m^{\frac{1}{2}} - 1}{m^{\frac{1}{2}} - 1} \quad m^{\frac{1}{2}} - 1 \text{ is the conjugate of } m^{\frac{1}{2}} + 1.$$

$$= \frac{m - 2m^{\frac{1}{2}} + 1}{m - 1} \quad \text{Multiply.}$$

CHECK Your Progress

5A. $\dfrac{\sqrt[4]{32}}{\sqrt[3]{2}}$ **5B.** $\sqrt[3]{16x^4}$ **5C.** $\dfrac{y^{\frac{1}{2}} + 2}{y^{\frac{1}{2}} - 2}$

Test-Taking Tip

When working with rational exponents, the rules are the same as when you add, subtract, or multiply fractions.

STANDARDIZED TEST EXAMPLE

6 If x is a positive number, then $\dfrac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = ?$

A $x^{\frac{2}{3}}$ **B** $x^{\frac{3}{2}}$ **C** $\sqrt[3]{x^5}$ **D** $\sqrt[5]{x^3}$

$$\frac{x^{\frac{4}{3}} \cdot x^{\frac{2}{3}}}{x^{\frac{1}{3}}} = \frac{x^{\frac{6}{3}}}{x^{\frac{1}{3}}} \quad \text{Add the exponents in the numerator.}$$

$$= \frac{x^2}{x^{\frac{1}{3}}} \quad \text{Simplify.}$$

$$= x^2 \cdot x^{-\frac{1}{3}} \quad \text{Quotient of Powers}$$

$$= x^{\frac{5}{3}} \text{ or } \sqrt[3]{x^5} \quad \text{The answer is C.}$$

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CHECK Your Progress

6. If y is positive, then $\dfrac{y \cdot y^2}{y^{\frac{1}{2}}} = ?$

F $y^{\frac{3}{2}}$ G $y^{\frac{5}{2}}$ H $\sqrt[2]{y^3}$ J $\sqrt[5]{y^2}$

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CONCEPT SUMMARY *Expressions with Rational Exponents*

An expression with rational exponents is simplified when all of the following conditions are met.

- It has no negative exponents.
- It has no fractional exponents in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

CHECK Your Understanding

Example 1 (p. 415)

Write each expression in radical form.

1. $7^{\frac{1}{3}}$ **2.** $x^{\frac{2}{3}}$

Write each radical using rational exponents.

3. $\sqrt[4]{26}$ **4.** $\sqrt[3]{6x^5y^7}$

Example 2 (p. 416)

Evaluate each expression.

5. $125^{\frac{1}{3}}$ **6.** $81^{-\frac{1}{4}}$ **7.** $27^{\frac{2}{3}}$ **8.** $\dfrac{54}{9^{\frac{3}{2}}}$

Example 3 (p. 417)

9. ECONOMICS When inflation causes the price of an item to increase, the new cost C and the original cost c are related by the formula $C = c(1 + r)^n$, where r is the rate of inflation per year as a decimal and n is the number of years. What would be the price of a \$4.99 item after six months of 5% inflation?

Examples 4, 5 (pp. 417–418)

Simplify each expression.

10. $a^{\frac{2}{3}} \cdot a^{\frac{1}{4}}$ **11.** $\dfrac{x^{\frac{5}{6}}}{x^{\frac{1}{6}}}$ **12.** $\dfrac{a^2}{b^{\frac{1}{3}}} \cdot \dfrac{b}{a^{\frac{1}{2}}}$

13. $\sqrt[6]{27x^3}$ **14.** $\dfrac{\sqrt[4]{27}}{\sqrt[4]{3}}$ **15.** $\dfrac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}} - 1}$

Example 6 (pp. 418–419)

16. STANDARDIZED TEST PRACTICE If a is positive, then $\dfrac{a^5 \cdot a^{\frac{2}{3}}}{a^{\frac{4}{3}}} = ?$

A a B a^2 C $\sqrt[3]{a^{13}}$ D $\sqrt[13]{a^3}$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
17–24	1
25–32	2
33, 34	3
35–42, 50	4
43–49	5
51, 52	6

Write each expression in radical form.

17. $6^{\frac{1}{5}}$ **18.** $4^{\frac{1}{3}}$ **19.** $c^{\frac{2}{5}}$ **20.** $(x^2)^{\frac{4}{3}}$

Write each radical using rational exponents.

21. $\sqrt{23}$ **22.** $\sqrt[3]{62}$ **23.** $\sqrt[4]{16z^2}$ **24.** $\sqrt[3]{5x^2y}$

Evaluate each expression.

25. $16^{\frac{1}{4}}$ **26.** $216^{\frac{1}{3}}$ **27.** $25^{-\frac{1}{2}}$ **28.** $(-27)^{-\frac{2}{3}}$

29. $81^{-\frac{1}{2}} \cdot 81^{\frac{3}{2}}$ **30.** $8^{\frac{3}{2}} \cdot 8^{\frac{5}{2}}$ **31.** $\frac{16^{\frac{1}{2}}}{9^{\frac{1}{2}}}$ **32.** $\frac{8^{\frac{1}{3}}}{64^{\frac{1}{3}}}$

BASKETBALL **For Exercises 33 and 34, use the following information.**
A women's regulation-sized basketball is slightly smaller than a men's basketball. The radius r of the ball that holds V volume of air is $r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$.

33. Find the radius of a women's basketball if it will hold 413 cubic inches of air.

34. Find the radius of a men's basketball if it will hold 455 cubic inches of air.

Simplify each expression.

35. $y^{\frac{5}{3}} \cdot y^{\frac{7}{3}}$ **36.** $x^{\frac{3}{4}} \cdot x^{\frac{9}{4}}$ **37.** $\left(b^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **38.** $\left(a^{-\frac{2}{3}}\right)^{-\frac{1}{6}}$

39. $w^{-\frac{4}{5}}$ **40.** $\frac{r^{\frac{2}{3}}}{r^{\frac{1}{6}}}$ **41.** $\frac{a^{-\frac{1}{2}}}{6a^{\frac{1}{3}} \cdot a^{-\frac{1}{4}}}$ **42.** $\frac{y^{\frac{3}{2}}}{y^{\frac{1}{2}} + 2}$

43. $\sqrt[4]{25}$ **44.** $\sqrt[6]{27}$ **45.** $\sqrt{17} \cdot \sqrt[3]{17^2}$ **46.** $\sqrt[8]{25x^4y^4}$

47. $\frac{xy}{\sqrt{z}}$ **48.** $\sqrt[3]{\sqrt{8}}$ **49.** $\frac{8^{\frac{1}{6}} - 9^{\frac{1}{4}}}{\sqrt{3} + \sqrt{2}}$ **50.** $\frac{x^{\frac{5}{3}} - x^{\frac{1}{3}}z^{\frac{4}{3}}}{x^{\frac{2}{3}} + z^{\frac{2}{3}}}$

51. **GEOMETRY** A triangle has a base of $3r^{\frac{1}{2}}s^{\frac{1}{4}}$ units and a height of $4r^{\frac{1}{4}}s^{\frac{1}{2}}$ units. Find the area of the triangle.

52. **GEOMETRY** Find the area of a circle whose radius is $3x^{\frac{2}{3}}y^{\frac{1}{5}}z^2$ centimeters.

53. Find the simplified form of $32^{\frac{1}{2}} + 3^{\frac{1}{2}} - 8^{\frac{1}{2}}$.

54. What is the simplified form of $81^{\frac{1}{3}} - 24^{\frac{1}{3}} + 3^{\frac{1}{3}}$?

55. **BIOLOGY** Suppose a culture has 100 bacteria to begin with and the number of bacteria doubles every 2 hours. Then the number N of bacteria after t hours is given by $N = 100 \cdot 2^{\frac{t}{2}}$. How many bacteria will be present after $3\frac{1}{2}$ hours?

EXTRA PRACTICE
See pages 907 and 932.
Math Online
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H.O.T. Problems

56. **OPEN ENDED** Determine a value of b for which $b^{\frac{1}{6}}$ is an integer.

57. **REASONING** Explain why $(-16)^{\frac{1}{2}}$ is not a real number.

58. **CHALLENGE** Explain how to solve $9^x = 3^{x + \frac{1}{2}}$ for x.

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59. REASONING Determine whether $\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$ is *always*, *sometimes*, or *never* true. Explain.

60. *Writing in Math* Refer to the information on page 415 to explain how rational exponents can be applied to astronomy. Explain how to write the formula $r = D\left(\frac{M_p}{M_S}\right)^{\frac{2}{5}}$ in radical form and simplify it.

STANDARDIZED TEST PRACTICE

61. ACT/SAT If $3^5 \cdot p = 3^3$, then $p =$

A -3^2

B 3^{-2}

C $\frac{1}{3}$

D $3^{\frac{1}{3}}$

62. REVIEW Which of the following sentences is true about the graphs of $y = 2(x-3)^2 + 1$ and $y = 2(x+3)^2 + 1$?

F Their vertices are maximums

G The graphs have the same shape with different vertices.

H The graphs have different shapes with different vertices.

J One graph has a vertex that is a maximum while the other graph has a vertex that is a minimum.

Spiral Review

Simplify. (Lessons 7-4 and 7-5)

63. $\sqrt{4x^3y^2}$

64. $\left(2\sqrt{6}\right)\left(3\sqrt{12}\right)$

65. $\sqrt{32} + \sqrt{18} - \sqrt{50}$

66. $\sqrt[4]{(-8)^4}$

67. $\sqrt[4]{(x-5)^2}$

68. $\sqrt{\frac{9}{36}x^4}$

TEMPERATURE **For Exercises 69 and 70, use the following information.**
There are three temperature scales: Fahrenheit (°F), Celsius (°C), and Kelvin (K). The function $K(C) = C + 273$ can be used to convert Celsius temperatures to Kelvin. The function $C(F) = \frac{5}{9}(F - 32)$ can be used to convert Fahrenheit temperatures to Celsius. (Lesson 7-1)

69. Write a composition of functions that could be used to convert Fahrenheit temperatures to Kelvin.

70. Find the temperature in Kelvin for the boiling point of water and the freezing point of water if water boils at 212°F and freezes at 32°F.

71. PHYSICS A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-6)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each power.** (Lesson 7-5)

72. $\left(\sqrt{x-2}\right)^2$

73. $\left(\sqrt[3]{2x-3}\right)^3$

74. $\left(\sqrt{x}+1\right)^2$

75. $\left(2\sqrt{x}-3\right)^2$

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7-7 Solving Radical Equations and Inequalities

Main Ideas

- Solve equations containing radicals.
- Solve inequalities containing radicals.

New Vocabulary

radical equation
extraneous solution
radical inequality

GET READY for the Lesson

Computer chips are made from the element silicon, which is found in sand. Suppose a company that manufactures computer chips uses the formula $C = 10n^{\frac{2}{3}} + 1500$ to estimate the cost C in dollars of producing n chips. This can be rewritten as a radical equation.

Solve Radical Equations Equations with radicals that have variables in the radicands are called **radical equations**. To solve this type of equation, raise each side of the equation to a power equal to the index of the radical to eliminate the radical.

It is very important that you check your solution. Sometimes you will obtain a number that does not satisfy the original equation. Such a number is called an **extraneous solution.**

EXAMPLE **Solve Radical Equations**

1 **Solve each equation.**

a. $\sqrt{x+1} + 2 = 4$

$\sqrt{x+1} + 2 = 4$ Original equation

$\sqrt{x+1} = 2$ Subtract 2 from each side to isolate the radical.

$(\sqrt{x+1})^2 = 2^2$ Square each side to eliminate the radical.

$x + 1 = 4$ Find the squares.

$x = 3$ Subtract 1 from each side.

CHECK $\sqrt{x+1} + 2 = 4$ Original equation

$\sqrt{3+1} + 2 \stackrel{?}{=} 4$ Replace x with 3.

$4 = 4$ ✓ Simplify.

The solution checks. The solution is 3.

b. $\sqrt{x-15} = 3 - \sqrt{x}$

$\sqrt{x-15} = 3 - \sqrt{x}$ Original equation

$(\sqrt{x-15})^2 = (3 - \sqrt{x})^2$ Square each side.

$x - 15 = 9 - 6\sqrt{x} + x$ Find the squares.

$-24 = -6\sqrt{x}$ Isolate the radical.

$4 = \sqrt{x}$ Divide each side by -6.

$4^2 = (\sqrt{x})^2$ Square each side again.

$16 = x$ Evaluate the squares.

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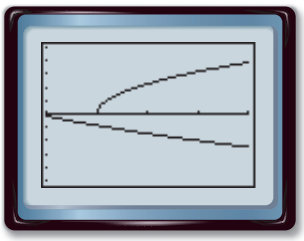
CHECK $\sqrt{x-15} = 3 - \sqrt{x}$

$\sqrt{16-15} \stackrel{?}{=} 3 - \sqrt{16}$

$\sqrt{1} \stackrel{?}{=} 3 - 4$

$1 \neq -1$

The solution does not check, so the equation has no real solution. The graphs of $y = \sqrt{x-15}$ and $y = 3 - \sqrt{x}$ are shown. The graphs do not intersect, which confirms that there is no solution.



[10, 30] scl: 5 by [−5, 5] scl: 1

CHECK Your Progress

1A. $5 = \sqrt{x-2} - 1$ **1B.** $\sqrt{x+15} = 5 + \sqrt{x}$

To undo a square root, you square the expression. To undo an nth root, you must raise the expression to the nth power.

Study Tip

Alternative Method

To solve a radical equation, you can substitute a variable for the radical expression. In Example 2, let $A = 5n - 1$.

$3A^{\frac{1}{3}} - 2 = 0$

$3A^{\frac{1}{3}} = 2$

$A^{\frac{1}{3}} = \frac{2}{3}$

$A = \frac{8}{27}$

$5n - 1 = \frac{8}{27}$

$n = \frac{7}{27}$

EXAMPLE 2 Solve a Cube Root Equation

Solve $3(5n-1)^{\frac{1}{3}} - 2 = 0$.

In order to remove the $\frac{1}{3}$ power, or cube root, you must first isolate it and then raise each side of the equation to the third power.

$3(5n-1)^{\frac{1}{3}} - 2 = 0$ Original equation

$3(5n-1)^{\frac{1}{3}} = 2$ Add 2 to each side.

$(5n-1)^{\frac{1}{3}} = \frac{2}{3}$ Divide each side by 3.

$\left[(5n-1)^{\frac{1}{3}}\right]^3 = \left(\frac{2}{3}\right)^3$ Cube each side.

$5n - 1 = \frac{8}{27}$ Evaluate the cubes.

$5n = \frac{35}{27}$ Add 1 to each side.

$n = \frac{7}{27}$ Divide each side by 5.

CHECK

$3(5n-1)^{\frac{1}{3}} - 2 = 0$ Original equation

$3\left(5 \cdot \frac{7}{27} - 1\right)^{\frac{1}{3}} - 2 \stackrel{?}{=} 0$ Replace n with $\frac{7}{27}$.

$3\left(\frac{8}{27}\right)^{\frac{1}{3}} - 2 \stackrel{?}{=} 0$ Simplify.

$3\left(\frac{2}{3}\right) - 2 \stackrel{?}{=} 0$ The cube root of $\frac{8}{27}$ is $\frac{2}{3}$.

$0 = 0$ ✓ Subtract.

CHECK Your Progress **Solve each equation.**

2A. $(3n+2)^{\frac{1}{3}} + 1 = 0$ **2B.** $(2y+6)^{\frac{1}{4}} - 2 = 0$

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Solve Radical Inequalities A **radical inequality** is an inequality that has a variable in a radicand.

Study Tip

Radical Inequalities

Since a principal square root is never negative, inequalities that simplify to the form $\sqrt{ax + b} \leq c$, where c is a negative number, have no solutions.

EXAMPLE **Solve a Radical Inequality**

3 **Solve $2 + \sqrt{4x - 4} \leq 6$.**

Since the radicand of a square root must be greater than or equal to zero, first solve $4x - 4 \geq 0$ to identify the values of x for which the left side of the given inequality is defined.

$$\begin{aligned} 4x - 4 &\geq 0 \\ 4x &\geq 4 \\ x &\geq 1 \end{aligned}$$

Now solve $2 + \sqrt{4x - 4} \leq 6$.

$$\begin{aligned} 2 + \sqrt{4x - 4} &\leq 6 && \text{Original inequality} \\ \sqrt{4x - 4} &\leq 4 && \text{Isolate the radical.} \\ 4x - 4 &\leq 16 && \text{Eliminate the radical.} \\ 4x &\leq 20 && \text{Add 4 to each side.} \\ x &\leq 5 && \text{Divide each side by 4.} \end{aligned}$$

It appears that $1 \leq x \leq 5$. You can test some x-values to confirm the solution. Let $f(x) = 2 + \sqrt{4x - 4}$. Use three test values: one less than 1, one between 1 and 5, and one greater than 5. Organize the test values in a table.

$x = 0$	$x = 2$	$x = 7$
$f(0) = 2 + \sqrt{4(0) - 4}$ $= 2 + \sqrt{-4}$ Since $\sqrt{-4}$ is not a real number, the inequality is not satisfied.	$f(2) = 2 + \sqrt{4(2) - 4}$ $= 4$ Since $4 \leq 6$, the inequality is satisfied.	$f(7) = 2 + \sqrt{4(7) - 4}$ ≈ 6.90 Since $6.90 \not\leq 6$, the inequality is not satisfied.

The solution checks. Only values in the interval $1 \leq x \leq 5$ satisfy the inequality. You can summarize the solution with a number line.

CHECK Your Progress

Solve each inequality.

3A. $\sqrt{2x + 2} + 1 \geq 5$ **3B.** $\sqrt{4x - 4} - 2 < 4$

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CONCEPT SUMMARY ***Solving Radical Inequalities***

To solve radical inequalities, complete the following steps.

Step 1 If the index of the root is even, identify the values of the variable for which the radicand is nonnegative.

Step 2 Solve the inequality algebraically.

Step 3 Test values to check your solution.

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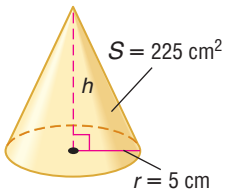
CHECK Your Understanding

Example 1 (pp. 422–423)

Solve each equation.

1. $\sqrt{4x+1} = 3$ **2.** $4 - (7-y)^{\frac{1}{2}} = 0$ **3.** $1 + \sqrt{x+2} = 0$

4. GEOMETRY The surface area S of a cone can be found by using $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone. Find the height of the cone.



Example 2 (p. 423)

Solve each equation.

5. $\frac{1}{6}(12a)^{\frac{1}{3}} = 1$ **6.** $\sqrt[3]{x-4} = 3$ **7.** $(3y)^{\frac{1}{3}} + 2 = 5$

Example 3 (p. 424)

Solve each inequality.

8. $\sqrt{2x+3} - 4 \leq 5$ **9.** $\sqrt{b+12} - \sqrt{b} > 2$ **10.** $\sqrt{y-7} + 5 \geq 10$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–22	1
23–30	2
31–38	3

Solve each equation.

11. $\sqrt{x} = 4$ **12.** $\sqrt{y} - 7 = 0$

13. $a^{\frac{1}{2}} + 9 = 0$ **14.** $2 + 4z^{\frac{1}{2}} = 0$

15. $7 + \sqrt{4x+8} = 9$ **16.** $5 + \sqrt{4y-5} = 12$

17. $\sqrt{x-5} = \sqrt{2x-4}$ **18.** $\sqrt{2t-7} = \sqrt{t+2}$

19. $\sqrt{x-6} - \sqrt{x} = 3$ **20.** $\sqrt{y+21} - 1 = \sqrt{y+12}$

21. $\sqrt{b+1} = \sqrt{b+6} - 1$ **22.** $\sqrt{4z+1} = 3 + \sqrt{4z-2}$

23. $\sqrt[3]{c-1} = 2$ **24.** $\sqrt[3]{5m+2} = 3$

25. $(6n-5)^{\frac{1}{3}} + 3 = -2$ **26.** $(5x+7)^{\frac{1}{5}} + 3 = 5$

27. $(3x-2)^{\frac{1}{5}} + 6 = 5$ **28.** $(7x-1)^{\frac{1}{3}} + 4 = 2$

Real-World Link

A ponderal index p is a measure of a person's body based on height h in meters and mass m in kilograms. One such formula is $p = \frac{\sqrt[3]{m}}{h}$.

Source: *A Dictionary of Food and Nutrition*

29. The formula $s = 2\pi\sqrt{\frac{\ell}{32}}$ represents the swing of a pendulum, where s is the time in seconds to swing back and forth, and ℓ is the length of the pendulum in feet. Find the length of a pendulum that makes one swing in 1.5 seconds.

30. HEALTH **Refer to the information at the left.**
A 70-kilogram person who is 1.8 meters tall has a ponderal index of about 2.29. How much weight could such a person gain and still have an index of at most 2.5?

Lori Adamski Peek/Getty Images
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Solve each inequality.

31. $1 + \sqrt{7x - 3} > 3$

32. $\sqrt{3x + 6} + 2 \leq 5$

33. $-2 + \sqrt{9 - 5x} \geq 6$

34. $6 - \sqrt{2y + 1} < 3$

35. $\sqrt{2} - \sqrt{x + 6} \leq -\sqrt{x}$

36. $\sqrt{a + 9} - \sqrt{a} > \sqrt{3}$

37. $\sqrt{b - 5} - \sqrt{b + 7} \leq 4$

38. $\sqrt{c + 5} + \sqrt{c + 10} > 2$

39. PHYSICS When an object is dropped from the top of a 50-foot tall building, the object will be h feet above the ground after t seconds, where $\frac{1}{4}\sqrt{50 - h} = t$. How far above the ground will the object be after 1 second?

EXTRA PRACTICE
See pages 907, 932.
Math Online
Self-Check Quiz at algebra2.com

40. FISH The relationship between the length and mass of certain fish can be approximated by the equation $L = 0.46\sqrt[3]{M}$, where L is the length in meters and M is the mass in kilograms. Solve this equation for M.

H.O.T. Problems

41. REASONING Determine whether the equation $\frac{\sqrt{(x^2)^2}}{-x} = x$ is *sometimes*, *always*, or *never* true when x is a real number. Explain your reasoning.

42. Which One Doesn't Belong? Which equation does *not* have a solution?

$\sqrt{x - 1} + 3 = 4$ | $\sqrt{x + 1} + 3 = 4$ | $\sqrt{x - 2} + 7 = 10$ | $\sqrt{x + 2} - 7 = -10$

43. OPEN ENDED Write an equation containing two radicals for which 1 is a solution.

44. CHALLENGE Explain how you know that $\sqrt{x + 2} + \sqrt{2x - 3} = -1$ has no real solution without actually solving it.

45. *Writing in Math* Refer to the information on page 422 to describe how the cost and the number of units manufactured are related. Rewrite the manufacturing equation $C = 10n^{\frac{2}{3}} + 1500$ as a radical equation, and write a step-by-step explanation of how to determine the maximum number of chips the company could make for $10,000.

STANDARDIZED TEST PRACTICE

46. ACT/SAT If $\sqrt{x + 5} + 1 = 4$, what is the value of x?

A 4 **B** 10 **C** 11 **D** 20

47. REVIEW Which set of points describes a function?

F {(3, 0), (−2, 5), (2, −1), (2, 9)}

G {(−3, 5), (−2, 3), (−1, 5), (0, 7)}

H {(2, 5), (2, 4), (2, 3), (2, 2)}

J {(3, 1), (−3, 2), (3, 3), (−3, 4)}

48. REVIEW What is an equivalent form of $\frac{4}{5 + i}$?

A $\frac{10 - 2i}{13}$

B $\frac{5 - i}{6}$

C $\frac{6 - i}{6}$

D $\frac{6 - i}{13}$

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Spiral Review

Write each radical using rational exponents. (Lesson 7-6)

49. $\sqrt[7]{5^3}$ **50.** $\sqrt{x+7}$ **51.** $\left(\sqrt[3]{x^2+1}\right)^2$

Simplify. (Lesson 7-5)

52. $\sqrt{72x^6y^3}$ **53.** $\frac{1}{\sqrt[3]{10}}$ **54.** $\left(5-\sqrt{3}\right)^2$

55. SALES Sales associates at Electronics Unlimited earn $8 an hour plus a 4% commission on the merchandise they sell. Write a function to describe their income, and find how much merchandise they must sell in order to earn $500 in a 40-hour week. (Lesson 7-2)

Find $(f+g)(x)$, $(f-g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$. (Lesson 7-1)

56. $f(x) = x + 5$, $g(x) = x - 3$

57. $f(x) = 10x - 20$, $g(x) = x - 2$

58. $f(x) = 4x^2 - 9$, $g(x) = \frac{1}{2x+3}$

59. ENTERTAINMENT A magician asked a member of his audience to choose any number. He said, "Multiply your number by 3. Add the sum of your number and 8 to that result. Now divide by the sum of your number and 2." The magician announced the final answer without asking the original number. What was the final answer? How did he know what it was? (Lesson 6-4)

Simplify. (Lesson 6-2)

60. $(x+2)(2x-8)$ **61.** $(3p+5)(2p-4)$ **62.** $(a^2+a+1)(a-1)$

CONSTRUCTION For Exercises 63 and 64, use the graph at right that shows the amount of money awarded for construction in Texas. (Lesson 2-5)

63. Let the independent variable be years since 1999. Write a prediction equation from the data for 1999, 2000, 2001, and 2002.

64. Use your prediction equation to predict the amount for 2010.

Source: *Texas Almanac*

Cross-Curricular Project

Algebra and Social Studies

Population Explosion It is time to complete your project. Use the information and data you have gathered about the population to prepare a Web page. Be sure to include graphs, tables, and equations in the presentation.

MathOnline **Cross-Curricular Project at algebra2.com**

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EXTEND 7-7

Graphing Calculator Lab

Solving Radical Equations and Inequalities

You can use a TI-83/84 Plus graphing calculator to solve radical equations and inequalities. One way to do this is by rewriting the equation or inequality so that one side is 0 and then using the zero feature on the calculator.

ACTIVITY 1 **Solve $\sqrt{x} + \sqrt{x+2} = 3$.**

Step 1 Rewrite the equation.

- Subtract 3 from each side of the equation to obtain $\sqrt{x} + \sqrt{x+2} - 3 = 0$.
- Enter the function $y = \sqrt{x} + \sqrt{x+2} - 3$ in the Y= list.

KEYSTROKES: *Review entering a function on page 399.*

Step 2 Use a table.

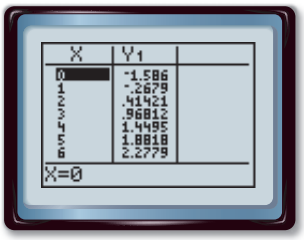
- You can use the TABLE function to locate intervals where the solution(s) lie. First, enter the starting value and the interval for the table.

KEYSTROKES: 2nd [TBLSET] 0 ENTER 1 ENTER

Step 3 Estimate the solution.

- Complete the table and estimate the solution(s).

KEYSTROKES: 2nd [TABLE]

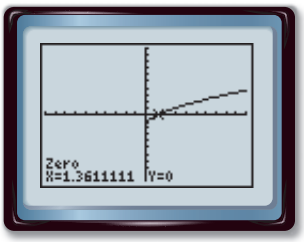


Since the function changes sign from negative to positive between $x = 1$ and $x = 2$, there is a solution between 1 and 2.

Step 4 Use the zero feature.

- Graph, then select zero from CALC menu.

KEYSTROKES: GRAPH 2nd [CALC] 2



[−10, 10] scl: 1 by [−10, 10] scl: 1

Place the cursor to the left of the zero and press ENTER for the Left Bound. Then place the cursor to the right of the zero and press ENTER for the Right Bound. Press ENTER to solve. The solution is about 1.36. This agrees with the estimate made by using the TABLE.

Online **Other Calculator Keystrokes** at algebra2.com

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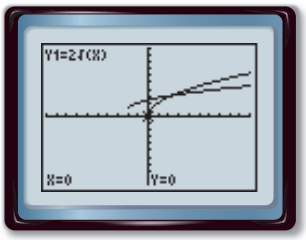
ACTIVITY 2 Solve $2\sqrt{x} > \sqrt{x+2} + 1$.

Step 1 Graph each side of the inequality and use the trace feature.

- In the Y= list, enter $y_1 = 2\sqrt{x}$ and $y_2 = \sqrt{x+2} + 1$. Then press GRAPH.



[−10, 10] scl: 1 by [−10, 10] scl: 1

- Press TRACE. You can use ▲ or ▼ to switch the cursor between the two curves.

The calculator screen above shows that, for points to the left of where the curves cross, Y1 < Y2 or $2\sqrt{x} < \sqrt{x+2} + 1$. To solve the original inequality, you must find points for which Y1 > Y2. These are the points to the right of where the curves cross.

Step 2 Use the intersect feature.

- You can use the INTERSECT feature on the CALC menu to approximate the x-coordinate of the point at which the curves cross.

KEYSTROKES: 2nd [CALC] 5

- Press ENTER for each of First curve?, Second curve?, and Guess?.

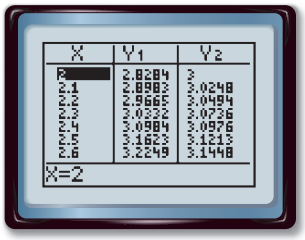
[−10, 10] scl: 1 by [−10, 10] scl: 1

The calculator screen shows that the x-coordinate of the point at which the curves cross is about 2.40. Therefore, the solution of the inequality is about $x > 2.40$. *Use the symbol > in the solution because the symbol in the original inequality is >.*

Step 3 **Use the table feature to check your solution.**

Start the table at 2 and show x-values in increments of 0.1. Scroll through the table.

KEYSTROKES: 2nd [TBLSET] 2 ENTER .1 ENTER 2nd [TABLE]



Notice that when x is less than or equal to 2.4, Y1 < Y2. This verifies the solution $\{x \mid x > 2.40\}$.

EXERCISES

Solve each equation or inequality.

1. $\sqrt{x+4} = 3$
2. $\sqrt{3x-5} = 1$
3. $\sqrt{x+5} = \sqrt{3x+4}$
4. $\sqrt{x+3} + \sqrt{x-2} = 4$
5. $\sqrt{3x-7} = \sqrt{2x-2} - 1$
6. $\sqrt{x+8} - 1 = \sqrt{x+2}$
7. $\sqrt{x-3} \geq 2$
8. $\sqrt{x+3} > 2\sqrt{x}$
9. $\sqrt{x} + \sqrt{x-1} < 4$
10. Explain how you could apply the technique in the first example to solving an inequality.

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CHAPTER 7

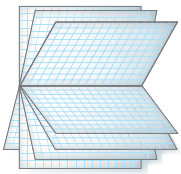
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Operations on Functions (Lesson 7-1)

Operation	Definition
Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$
Composition	$[f \circ g](x) = f[g(x)]$

Inverse and Square Root Functions

(Lesson 7-2 and 7-3)

- Reverse the coordinates of ordered pairs to find the inverse of a relation.
- Two functions are inverses if and only if both of their compositions are the identity function.

Roots of Real Numbers (Lesson 7-4)

Real nth roots of b, $\sqrt[n]{b}$, or $-\sqrt[n]{b}$			
n	$\sqrt[n]{b}$ if $b > 0$	$\sqrt[n]{b}$ if $b < 0$	$\sqrt[n]{b}$ if $b = 0$
even	one positive root one negative root	no real roots	one real root, 0
odd	one positive root no negative roots	no positive roots one negative root	

Radicals (Lessons 7-5 through 7-7)

For any real numbers a and b and any integer $n > 1$,

- Product Property: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- Quotient Property: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- For any nonzero real number b, and any integers m and n, with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$.
- To solve a radical equation, isolate the radical. Then raise each side of the equation to a power equal to the index of the radical.

Key Vocabulary

composition of functions (p. 385)
conjugates (p. 411)
extraneous solution (p. 422)
identity function (p. 393)
inverse function (p. 392)
inverse relation (p. 391)
like radical expressions (p. 411)
nth root (p. 402)
one-to-one (p. 394)
principal root (p. 402)
radical equation (p. 422)
radical inequality (p. 424)
rationalizing the denominator (p. 409)
square root function (p. 397)
square root inequality (p. 399)

Vocabulary Check

Choose a word or term from the list above that best completes each statement or phrase.

1. A(n)_____ is an equation with radicals that have variables in the radicands.
2. A solution of a transformed equation that is not a solution of the original equation is a(n) _____.
3. _____ have the same index and the same radicand.
4. When a number has more than one real root, the _____ is the nonnegative root.
5. $f(x) = 6x - 2$ and $g(x) = \frac{x + 2}{6}$ are _____ since $[f \circ g](x) = x$ and $[g \circ f](x) = x$.
6. A(n) _____ is when a function is performed, and then a second function is performed on the result of the first function.
7. A(n) _____ function is a function whose inverse is a function.
8. The process of eliminating radicals from a denominator or fractions from a radicand is called _____.
9. Two relations are _____ if and only if whenever one relation contains the element (a, b), the other relation contains the element (b, a).

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Lesson-by-Lesson Review

7–1 Operations on Functions (pp. 384–390)

Find $[g \circ h](x)$ and $[h \circ g](x)$.

10. $h(x) = 2x - 1$, $g(x) = 3x + 4$

11. $h(x) = x^2 + 2$, $g(x) = x - 3$

12. $h(x) = x^2 + 1$, $g(x) = -2x + 1$

13. $h(x) = -5x$, $g(x) = 3x - 5$

14. $h(x) = x^3$, $g(x) = x - 2$

15. $h(x) = x + 4$, $g(x) = |x|$

16. TIME The formula $h = \frac{m}{60}$ converts minutes m to hours h, and $d = \frac{h}{24}$ converts hours to days d. Write a composition of functions that converts minutes to days.

Example 1 If $f(x) = x^2 - 2$ and $g(x) = 8x - 1$, find $g[f(x)]$ and $f[g(x)]$.

$g[f(x)] = 8(x^2 - 2) - 1$ Replace $f(x)$ with $x^2 - 2$.

$= 8x^2 - 16 - 1$ Multiply.

$= 8x^2 - 17$ Simplify.

$f[g(x)] = (8x - 1)^2 - 2$ Replace $g(x)$ with $8x - 1$.

$= 64x^2 - 16x + 1 - 2$ Expand.

$= 64x^2 - 16x - 1$ Simplify.

7–2 Inverse Functions and Relations (pp. 391–396)

Find the inverse of each function. Then graph the function and its inverse.

17. $f(x) = 3x - 4$

18. $f(x) = -2x - 3$

19. $g(x) = \frac{1}{3}x + 2$

20. $f(x) = \frac{-3x + 1}{2}$

21. $y = x^2$

22. $y = (2x + 3)^2$

23. SALES Jim earns \$10 an hour plus a 10% commission on sales. Write a function to describe Jim's income. If Jim wants to earn \$1000 in a 40-hour week, what should his sales be?

24. BANKING During the last month, Jonathan has made two deposits of \$45, made a deposit of double his original balance, and withdrawn \$35 five times. His balance is now \$189. Write an equation that models this problem. How much money did Jonathan have in his account at the beginning of the month?

Example 2 Find the inverse of $f(x) = -3x + 1$.

Rewrite $f(x)$ as $y = -3x + 1$. Then interchange the variables and solve for y.

$x = -3y + 1$ Interchange the variables.

$3y = -x + 1$ Solve for y.

$y = \frac{-x + 1}{3}$ Divide each side by 3.

$f^{-1}(x) = \frac{-x + 1}{3}$ Rewrite in function notation.

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CHAPTER 7

Study Guide and Review

7–3 Square Root Functions and Inequalities (pp. 397–401)

Graph each function.

25. $y = \frac{1}{3}\sqrt{x+2}$

26. $y = \sqrt{5x-3}$

27. $y = 4 + 2\sqrt{x-3}$

Graph each inequality.

28. $y \geq \sqrt{x} - 2$

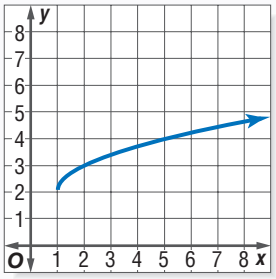
29. $y < \sqrt{4x-5}$

30. OCEAN The speed a tsunami, or tidal wave, can travel is modeled by the equation $s = 356\sqrt{d}$, where s is the speed in kilometers per hour and d is the average depth of the water in kilometers. A tsunami is found to be traveling at 145 kilometers per hour. What is the average depth of the water? Round to the nearest hundredth.

Example 3 Graph $y = 2 + \sqrt{x-1}$.

Make a table of values and graph the function.

x	1	2	3	4	5
y	2	3	$2 + \sqrt{2}$ or 3.4	$2 + \sqrt{3}$ or 3.7	7



7–4 *n*th Roots (pp. 402–406)

Simplify.

31. $\pm\sqrt{256}$

32. $\sqrt[3]{-216}$

33. $\sqrt{(-8)^2}$

34. $\sqrt[5]{c^5d^{15}}$

35. $\sqrt{(x^4-3)^2}$

36. $\sqrt[3]{(512+x^2)^3}$

37. $\sqrt[4]{16m^8}$

38. $\sqrt{a^2 - 10a + 25}$

39. PHYSICS The velocity v of an object can be defined as $v = \sqrt{\frac{2K}{m}}$, where m is the mass of an object and K is the kinetic energy. Find the velocity of an object with a mass of 15 grams and a kinetic energy of 750.

Example 4 Simplify $\sqrt{81x^6}$.

$\sqrt{81x^6} = \sqrt{(9x^3)^2}$ $81x^6 = (9x^3)^2$

$= 9|x^3|$ Use absolute value since x could be negative.

Example 5 Simplify $\sqrt[7]{2187x^{14}y^{35}}$.

$\sqrt[7]{2187x^{14}y^{35}}$

$= \sqrt[7]{(3x^2y^5)^7}$ $2187x^{14}y^{35} = (3x^2y^5)^7$

$= 3x^2y^5$ Evaluate.

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Mixed Problem Solving
For mixed problem-solving practice, see page 932.

7–5 Operations with Radical Expressions (pp. 408–414)

Simplify.

40. $\sqrt[6]{128}$

41. $5\sqrt{12} - 3\sqrt{75}$

42. $6\sqrt[5]{11} - 8\sqrt[5]{11}$

43. $(\sqrt{8} + \sqrt{12})^2$

44. $\sqrt{8} \cdot \sqrt{15} \cdot \sqrt{21}$

45. $\dfrac{\sqrt{243}}{\sqrt{3}}$

46. $\dfrac{1}{3 + \sqrt{5}}$

47. $\dfrac{\sqrt{10}}{4 + \sqrt{2}}$

48. GEOMETRY The measures of the legs of a right triangle can be represented by the expressions $4x^2y^2$ and $8x^2y^2$. Use the Pythagorean Theorem to find a simplified expression for the measure of the hypotenuse.

Example 6 Simplify $6\sqrt[5]{32m^3} \cdot 5\sqrt[5]{1024m^2}$.

$6\sqrt[5]{32m^3} \cdot 5\sqrt[5]{1024m^2}$

$= 6 \cdot 5\sqrt[5]{(32m^3 \cdot 1024m^2)}$ Product Property of Radicals

$= 30\sqrt[5]{2^5 \cdot 4^5 \cdot m^5}$ Factor into exponents of 5 if possible.

$= 30\sqrt[5]{2^5} \cdot \sqrt[5]{4^5} \cdot \sqrt[5]{m^5}$ Product Property of Radicals

$= 30 \cdot 2 \cdot 4 \cdot m$ or $240m$ Write the fifth roots.

7–6 Rational Exponents (pp. 415–421)

Evaluate.

49. $27^{-\frac{2}{3}}$

50. $9^{\frac{1}{3}} \cdot 9^{\frac{5}{3}}$

51. $\left(\dfrac{8}{27}\right)^{-\frac{2}{3}}$

Simplify.

52. $\dfrac{1}{y^{\frac{2}{5}}}$

53. $\dfrac{xy}{\sqrt[3]{z}}$

54. $\dfrac{3x + 4x^2}{x^{-\frac{2}{3}}}$

55. ELECTRICITY The amount of current in amperes I that an appliance uses can be calculated using the formula $I = \left(\dfrac{P}{R}\right)^{\frac{1}{2}}$, where P is the power in watts and R is the resistance in ohms. How much current does an appliance use if $P =$ 120 watts and $R = 3$ ohms? Round your answer to the nearest tenth.

Example 7 Write $32^{\frac{4}{5}} \cdot 32^{\frac{2}{5}}$ in radical form.

$32^{\frac{4}{5}} \cdot 32^{\frac{2}{5}} = 32^{\frac{4}{5} + \frac{2}{5}}$ Product of powers

$= 32^{\frac{6}{5}}$ Add.

$= (2^5)^{\frac{6}{5}}$ $32 = 2^5$

$= 2^6$ or 64 Power of a power

Example 8 Simplify $\dfrac{3x}{\sqrt[3]{z}}$.

$\dfrac{3x}{\sqrt[3]{z}} = \dfrac{3x}{z^{\frac{1}{3}}}$ Rational exponents

$= \dfrac{3x}{z^{\frac{1}{3}}} \cdot \dfrac{z^{\frac{2}{3}}}{z^{\frac{2}{3}}}$ Rationalize the denominator.

$= \dfrac{3xz^{\frac{2}{3}}}{z}$ or $\dfrac{3x^3\sqrt{z^2}}{z}$ Rewrite in radical form.

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7–7 Solving Radical Equations and Inequalities (pp. 422–427)

Solve each equation or inequality.

56. $\sqrt{x} = 6$

57. $y^{\frac{1}{3}} - 7 = 0$

58. $(x-2)^{\frac{3}{2}} = -8$

59. $\sqrt{x+5} - 3 = 0$

60. $\sqrt{3t-5} - 3 = 4$

61. $\sqrt{2x-1} = 3$

62. $\sqrt[4]{2x-1} = 2$

63. $\sqrt{y+5} = \sqrt{2y-3}$

64. $\sqrt{y+1} + \sqrt{y-4} = 5$

65. $1 + \sqrt{5x-2} > 4$

66. $\sqrt{-2x+14} - 6 \geq -4$

67. $10 - \sqrt{2x+7} \leq 3$

68. $6 + \sqrt{3y+4} < 6$

69. $\sqrt{d+3} + \sqrt{d+7} > 4$

70. $\sqrt{2x+5} - \sqrt{9+x} > 0$

71. GRAVITY Hugo drops his keys from the top of a Ferris wheel. The formula $t = \frac{1}{4}\sqrt{65-h}$ describes the time t in seconds when the keys are h feet above the boardwalk. If Hugo was 65 meters high when he dropped the keys, how many meters above the boardwalk will the keys be after 2 seconds?

Example 9 Solve $\sqrt{3x-8} + 1 = 3$.

$\sqrt{3x-8} + 1 = 3$ Original equation

$\sqrt{3x-8} = 2$ Subtract 1 from each side.

$(\sqrt{3x-8})^2 = 2^2$ Square each side.

$3x - 8 = 4$ Evaluate the squares.

$3x = 12$ Add 8 to each side.

$x = 4$ Divide each side by 3. Check this solution.

Example 10 Solve $\sqrt{4x-3} - 2 > 3$.

$\sqrt{4x-3} - 2 > 3$ Original inequality

$\sqrt{4x-3} > 5$ Add 2 to each side.

$(\sqrt{4x-3})^2 > 5^2$ Square each side.

$4x - 3 > 25$ Evaluate the squares.

$4x > 28$ Add 3 to each side.

$x > 7$ Divide each side by 4.

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CHAPTER 7

Practice Test

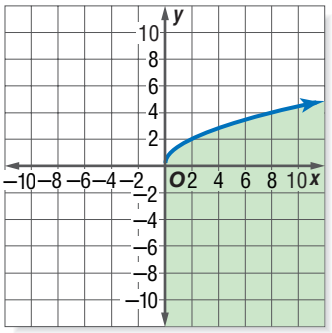
Determine whether each pair of functions are inverse functions.

1. $f(x) = 4x - 9, g(x) = \frac{x - 9}{4}$
2. $f(x) = \frac{1}{x + 2}, g(x) = \frac{1}{x} - 2$

If $f(x) = 2x - 4$ and $g(x) = x^2 + 3$, find each value.

3. $(f + g)(x)$
4. $(f - g)(x)$
5. $(f \cdot g)(x)$
6. $\left(\frac{f}{g}\right)(x)$

7. **MULTIPLE CHOICE** Which inequality represents the graph below?



A $y \geq \sqrt{2x}$

B $y \leq \sqrt{2x}$

C $y < 2\sqrt{x}$

D none of these

Solve each equation.

8. $\sqrt{b + 15} = \sqrt{3b + 1}$
9. $\sqrt{2x} = \sqrt{x - 4}$
10. $\sqrt[4]{y + 2} + 9 = 14$
11. $\sqrt[3]{2w - 1} + 11 = 18$
12. $\sqrt{4x + 28} = \sqrt{6x + 38}$
13. $1 + \sqrt{x + 5} = \sqrt{x + 12}$

Simplify.

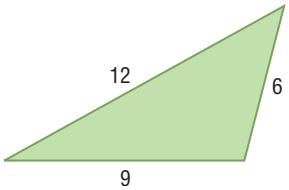
14. $\sqrt{175}$
15. $(5 + \sqrt{3})(7 - 2\sqrt{3})$
16. $(6 - 4\sqrt{2})(-5 + \sqrt{2})$
17. $3\sqrt{6} + 5\sqrt{54}$
18. $\frac{9}{5 - \sqrt{3}}$
19. $\frac{16}{-2 + \sqrt{5}}$
20. $\left(9^{\frac{1}{2}} \cdot 9^{\frac{2}{3}}\right)^{\frac{1}{6}}$
21. $11^{\frac{1}{2}} \cdot 11^{\frac{7}{3}} \cdot 11^{\frac{1}{6}}$
22. $\sqrt[6]{256s^{11}t^{18}}$
23. $\frac{b^{\frac{1}{2}}}{b^{\frac{3}{2}} - b^{\frac{1}{2}}}$

Solve each inequality.

24. $\sqrt{3x + 1} \geq 5$
25. $3 + \sqrt{5x - 1} < 11$
26. $1 - \sqrt{2y + 1} < -6$

27. **SKYDIVING** The approximate time t in seconds that it takes an object to fall a distance of d feet is given by $t = \sqrt{\frac{d}{16}}$. Suppose a parachutist falls 11 seconds before the parachute opens. How far does the parachutist fall during this time period?

28. **GEOMETRY** The area of a triangle with sides of length a, b, and c is given by $\sqrt{s(s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$. If the lengths of the sides of a triangle are 6, 9, and 12 feet, what is the area of the triangle expressed in radical form?



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CHAPTER 7

Standardized Test Practice

Cumulative, Chapters 1–7

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Marilyn bought a pair of jeans and a sweater at her favorite clothing store. She spent $120 not including tax. If the price of the sweater s was $12 less than twice the cost of the jeans j which system of linear equations could be used to determine the price of each item?

A $j + s = 120a$
$s = 2j - 12$

B $j + s = 120$
$j = 2s - 12$

C $j + 120 = s$
$s = 2j - 12$

D $j + s = 12$
$s = 2j - 120$

2. GRIDDABLE If $f(x) = 3x$ and $g(x) = x^2 - 1$, what is the value of $f(g(-3))$?

3. What is the effect on the graph of the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?

F The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.

G The graph is rotated 90 degrees about the origin.

H The graph is narrower.

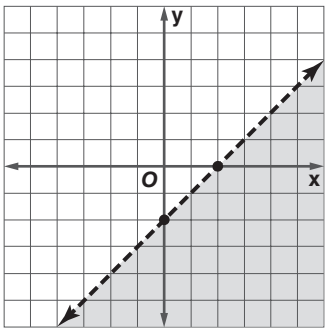
J The graph is wider.

TEST-TAKING TIP

QUESTION 3 If the question involves a graph but does not include the graph, draw one. A diagram can help you see relationships among the given values that will help you answer the question.

4. Which graph best represents all the pairs of numbers (x, y) such that $x - y > 2$?

A



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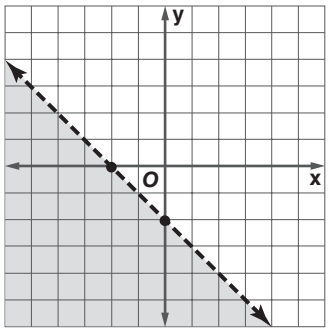

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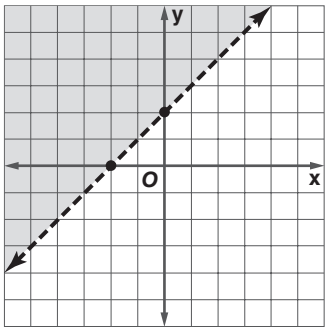

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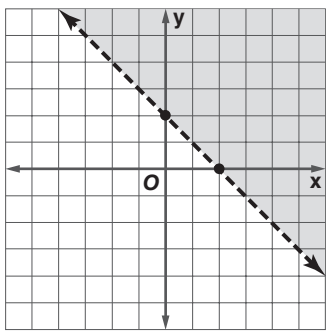

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5. The table below shows the cost of a pizza depending on the diameter of the pizza.

Diameter (inches)	Cost ($)
6	5.00
10	8.10
15	11.70
20	15.00

Which conclusion can be made based on the information in the table?

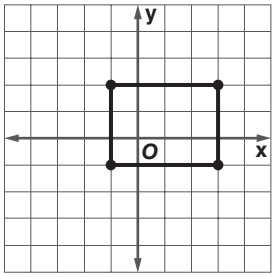
F The cost of a 12-inch would be less than $9.00.

G The cost of a 24-inch would be less than $18.00.

H The cost of an 18-inch would be more than $13.70.

J The cost of an 8-inch would be less than $6.00.

6. A rectangle is graphed on the coordinate grid.



Which two points lie on the same line of symmetry of the square?

A (−1, −1) and (−1, 2)

B (0, 2) and (0, −1)

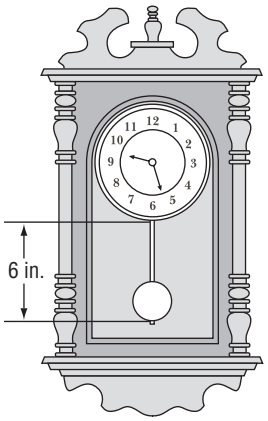
C (1, 2) and (1, −1)

D (3, 2) and (−1, −1)

Pre-AP

Record your answers on a sheet of paper. Show your work.

7. The period of a pendulum is the time it takes for the pendulum to make one complete swing back and forth. The formula $T = 2\pi\sqrt{\frac{L}{32}}$ gives the period T in seconds for a pendulum L feet long.



a. What is the period of the pendulum in the wall clock shown? Round to the nearest hundredth of a second.

b. Solve the formula for the length of the pendulum L in terms of the time T. Show each step of the process.

c. If you are building a grandfather clock and you want the pendulum to have a period of 2 seconds, how long should you make the pendulum? Round to the nearest tenth of a foot.

NEED EXTRA HELP?							
If You Missed Question...	1	2	3	4	5	6	7
Go to Lesson or Page...	2-4	7-1	5-7	2-7	1-3	5-1	5-6

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UNIT 3
Advanced Functions and Relations

Focus

Use a variety of representations, tools, and technology to model mathematical situations to solve meaningful problems.

CHAPTER 8
Rational Expressions and Equations

BIG Idea Formulate equations and inequalities based on rational functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

BIG Idea Connect algebraic and geometric representations of functions.

CHAPTER 9
Exponential and Logarithmic Relations

BIG Idea Formulates equations and inequalities based on exponential and logarithmic functions, use a variety of methods to solve them, and analyze the solutions in terms of the situation.

CHAPTER 10
Conic Sections

BIG Idea Explore the relationship between the geometric and algebraic descriptions of conic sections.

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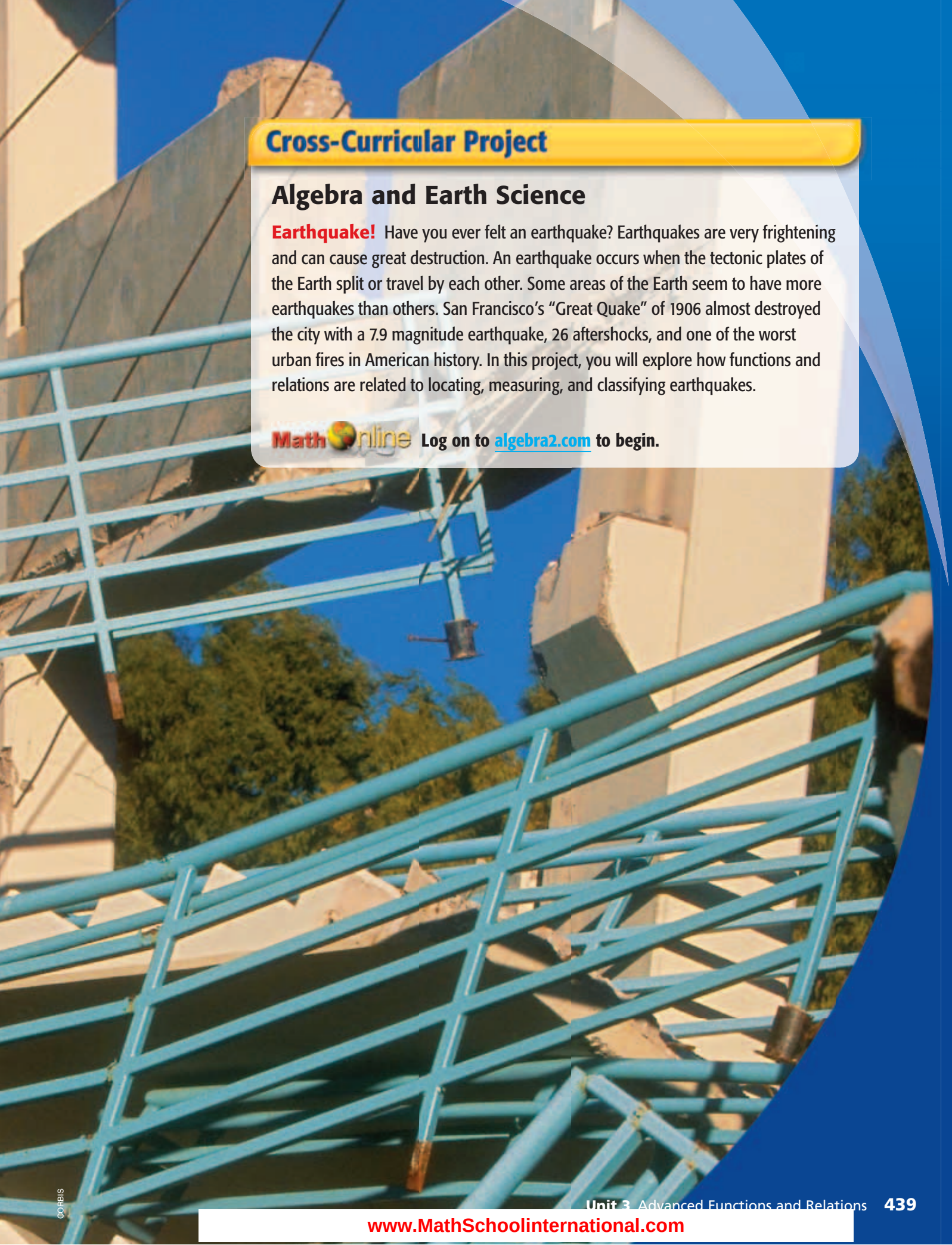

Cross-Curricular Project

Algebra and Earth Science

Earthquake! Have you ever felt an earthquake? Earthquakes are very frightening and can cause great destruction. An earthquake occurs when the tectonic plates of the Earth split or travel by each other. Some areas of the Earth seem to have more earthquakes than others. San Francisco's "Great Quake" of 1906 almost destroyed the city with a 7.9 magnitude earthquake, 26 aftershocks, and one of the worst urban fires in American history. In this project, you will explore how functions and relations are related to locating, measuring, and classifying earthquakes.

MathOnline **Log on to algebra2.com to begin.**

CORBIS

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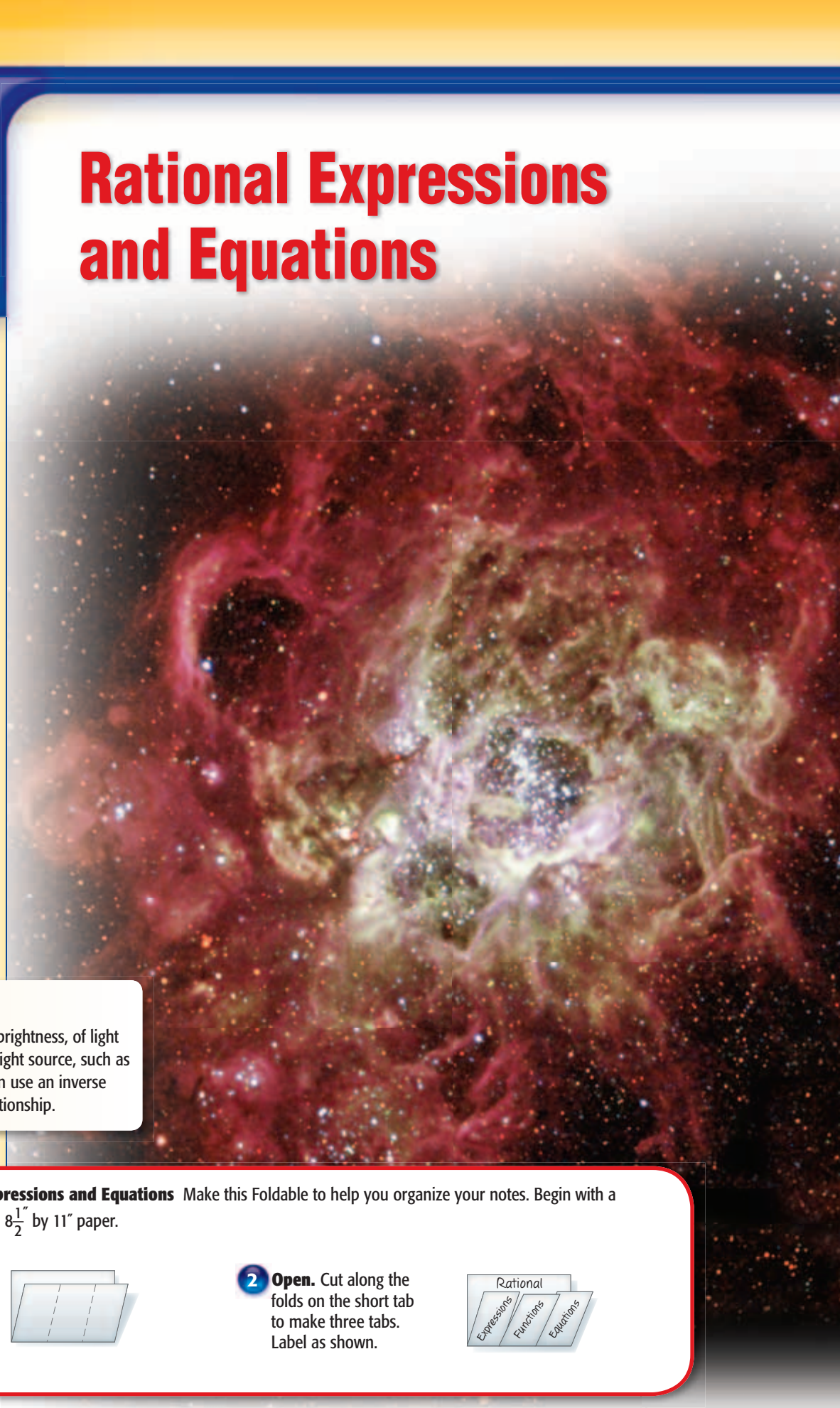
CHAPTER 8

Rational Expressions and Equations



BIG Ideas

- Simplify rational expressions.
- Graph rational functions.
- Solve direct, joint, and inverse variation problems.
- Identify graphs and equations as different types of functions.
- Solve rational equations and inequalities.

Key Vocabulary

continuity (p. 457)
direct variation (p. 465)
inverse variation (p. 467)
rational expression (p. 442)

Real-World Link

Intensity of Light The intensity, or brightness, of light decreases as the distance between a light source, such as a star, and a viewer increases. You can use an inverse variation equation to express this relationship.

FOLDABLES Study Organizer

Rational Expressions and Equations Make this Foldable to help you organize your notes. Begin with a sheet of plain $8\frac{1}{2}''$ by 11″ paper.

1 **Fold** in half lengthwise leaving a $1\frac{1}{2}''$ margin at the top. Fold again in thirds.

2 **Open.** Cut along the folds on the short tab to make three tabs. Label as shown.

NASA

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GET READY for Chapter 8

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICK Check

Solve each equation. Write in simplest form. (Lesson 1-3)

1. $\frac{8}{5}x = \frac{4}{15}$
2. $\frac{27}{14}t = \frac{6}{7}$
3. $\frac{3}{10} = \frac{12}{25}a$
4. $\frac{6}{7} = 9m$
5. $\frac{9}{8}b = 18$
6. $\frac{6}{7}s = \frac{3}{4}$
7. $\frac{1}{3}r = \frac{5}{6}$
8. $\frac{2}{3}n = 7$
9. **INCOME** Jamal's allowance is $\frac{7}{8}$ of Syretta's allowance. If Syretta's allowance is \$20, how much is Jamal's allowance? (Lesson 1-3)
10. **BAKING** Marc gave away $\frac{2}{3}$ of the cookies he baked. If he gave away 24 cookies, how many cookies did he bake? (Lesson 1-3)

Solve each proportion. (Prerequisite Skill)

11. $\frac{3}{4} = \frac{r}{16}$
12. $\frac{8}{16} = \frac{5}{y}$
13. $\frac{6}{8} = \frac{m}{20}$
14. $\frac{t}{3} = \frac{5}{24}$
15. $\frac{5}{a} = \frac{6}{18}$
16. $\frac{3}{4} = \frac{b}{6}$
17. $\frac{v}{9} = \frac{12}{18}$
18. $\frac{7}{p} = \frac{1}{4}$
19. $\frac{2}{5} = \frac{3}{z}$
20. $\frac{9}{10} = \frac{r}{12}$
21. **REAL ESTATE** A house which is assessed for \$200,000 pays \$3000 in taxes. What should the taxes be on a house in the same area that is assessed at \$350,000? (Prerequisite Skill)

QUICK Review

EXAMPLE 1

Solve $\frac{13}{17} = \frac{41}{43}k$. Write in simplest form.

$(43)\frac{13}{17} = 41k$ Multiply each side by 43.

$\frac{559}{17} = 41k$ Simplify.

$\frac{559}{(41)17} = k$ Divide each side by 41.

$\frac{559}{697} = k$ Simplify.

Since the GCF of 559 and 697 is 1, the solution is in simplest form.

EXAMPLE 2

Solve the proportion $\frac{4}{7} = \frac{u}{15}$.

$\frac{4}{7} = \frac{u}{15}$ Write the equation.

$4(15) = 7u$ Find the cross products.

$60 = 7u$ Simplify.

$\frac{60}{7} = u$ Divide each side by 7.

Since the GCF of 60 and 7 is 1, the answer is in simplified form. $u = \frac{60}{7}$ or $8\frac{4}{7}$.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

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8-1 Multiplying and Dividing Rational Expressions

Main Ideas

- Simplify rational expressions.
- Simplify complex fractions.

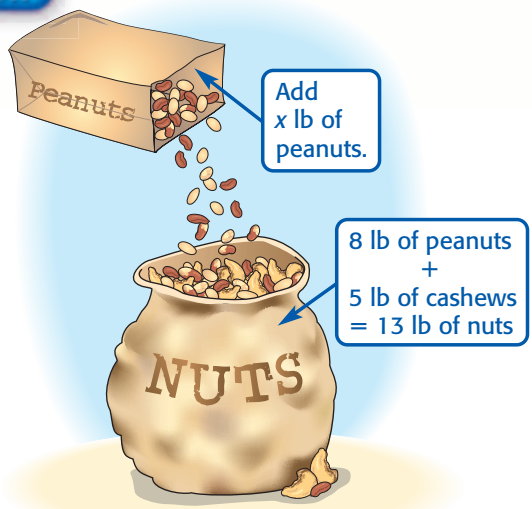
New Vocabulary

rational expression

complex fraction

GET READY for the Lesson

The Goodie Shoppe sells candy and nuts by the pound. One item is a mixture made with 8 pounds of peanuts and 5 pounds of cashews. Therefore, $\frac{8}{8+5}$ or $\frac{8}{13}$ of the mixture is peanuts. If the store manager adds an additional x pounds of peanuts to the mixture, then $\frac{8+x}{13+x}$ of the mixture will be peanuts.



Simplify Rational Expressions A ratio of two polynomial expressions such as $\frac{8+x}{13+x}$ is called a **rational expression**. Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar.

To write a fraction in simplest form, you divide both the numerator and denominator by their greatest common factor (GCF). To simplify a rational expression, you use similar techniques.

EXAMPLE 1 Simplify a Rational Expression

a. Simplify $\frac{2x(x-5)}{(x-5)(x^2-1)}$.

Look for common factors.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x}{x^2-1} \cdot \frac{\overset{1}{\cancel{x-5}}}{\underset{1}{\cancel{x-5}}}$$ How is this similar to simplifying $\frac{10}{15}$?

$$= \frac{2x}{x^2-1}$$ Simplify.

b. Under what conditions is this expression undefined?

Just as with a fraction, a rational expression is undefined if the denominator is equal to 0. To find when this expression is undefined, completely factor the original denominator.

$$\frac{2x(x-5)}{(x-5)(x^2-1)} = \frac{2x(x-5)}{(x-5)(x-1)(x+1)}$$ $x^2-1 = (x-1)(x+1)$

The values that would make the denominator equal to 0 are 5, 1, or -1. So the expression is undefined when $x = 5$, $x = 1$, or $x = -1$.

Study Tip

Excluded Values

Numbers that would cause the expression to be undefined are called **excluded values**.

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CHECK Your Progress

Simplify each expression. Under what conditions is the expression undefined?

1A. $\dfrac{3y(y+6)}{(y+6)(y^2-8y+12)}$

1B. $\dfrac{4x^3(x^2-7x-8)}{12x(x^2-64)}$

STANDARDIZED TEST PRACTICE **Use the Process of Elimination**

2 **For what value(s) of x is $\dfrac{x^2+x-12}{x^2+7x+12}$ undefined?**

A $-4, -3$ **B** -4 **C** 0 **D** $-4, 3$

Test-Taking Tip

Eliminating Choices
Sometimes you can save time by looking at the possible answers and eliminating choices, rather than actually evaluating an expression or solving an equation.

Read the Test Item

You want to determine which values of x make the denominator equal to 0.

Solve the Test Item

Notice that if x equals 0 or a positive number, $x^2 + 7x + 12$ must be greater than 0. Therefore, you can eliminate choices C and D. Since both A and B contain -4, determine whether the denominator equals 0 when $x = -3$.

$x^2 + 7x + 12 = (-3)^2 + 7(-3) + 12$ $x = -3$

$= 9 - 21 + 12$ or 0 Multiply and simplify.

Since the denominator equals 0 when $x = -3$, the answer is A.

CHECK Your Progress

2. For what values of x is $\dfrac{x^2+9}{x^2+15x-34}$ undefined?

F $-17, -2$ **G** $-17, 2$ **H** $-2, 17$ **J** $2, 17$

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Sometimes you can factor out -1 in the numerator or denominator to help simplify rational expressions.

EXAMPLE **Simplify by Factoring Out –1**

3 **Simplify $\dfrac{z^2w - z^2}{z^3 - z^3w}$.**

$\dfrac{z^2w - z^2}{z^3 - z^3w} = \dfrac{z^2(w-1)}{z^3(1-w)}$ Factor the numerator and the denominator.

$= \dfrac{1\,z^2(-1)(1-w)}{z^3(1-w)}$ $w - 1 = -(-w + 1)$ or $-1(1 - w)$

$= \dfrac{-1}{z}$ or $-\dfrac{1}{z}$ Simplify.

CHECK Your Progress

Simplify each expression.

3A. $\dfrac{xy - 3x}{3x^2 - x^2y}$

3B. $\dfrac{2x - x^2}{x^2y - 4y}$

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Remember that to multiply two fractions, you multiply the numerators and multiply the denominators. To divide two fractions, you multiply by the multiplicative inverse, or reciprocal, of the divisor.

Multiplication

$$\frac{5}{6} \cdot \frac{4}{15} = \frac{\overset{1}{\cancel{5}} \cdot 2 \cdot \overset{1}{\cancel{2}}}{2 \cdot 3 \cdot 3 \cdot \underset{1}{\cancel{5}}} = \frac{2}{3 \cdot 3} \text{ or } \frac{2}{9}$$

Division

$$\frac{3}{7} \div \frac{9}{14} = \frac{3}{7} \cdot \frac{14}{9} = \frac{\overset{1}{\cancel{3}} \cdot 2 \cdot \overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}} \cdot \underset{1}{\cancel{3}} \cdot 3} \text{ or } \frac{2}{3}$$

The same procedures are used for multiplying and dividing rational expressions.

KEY CONCEPT — ***Rational Expressions***

Multiplying Rational Expressions

Words To multiply two rational expressions, multiply the numerators and the denominators.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, if $b \neq 0$ and $d \neq 0$.

Dividing Rational Expressions

Words To divide two rational expressions, multiply by the reciprocal of the divisor.

Symbols For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, if $b \neq 0$, $c \neq 0$ and $d \neq 0$.

The following examples show how these rules are used with rational expressions.

EXAMPLE Multiply and Divide Rational Expressions

Study Tip

Alternative Method

When multiplying rational expressions, you can multiply first and then divide by the common factors. For instance, in Example 4, $\frac{4a}{5b} \cdot \frac{15b^2}{16a^3} = \frac{60ab^2}{80a^3b}$. Now divide the numerator and denominator by the common factors.

$$\frac{\overset{3}{\cancel{60}}\overset{1}{\cancel{a}}\overset{b}{\cancel{b^2}}}{\underset{4}{\cancel{80}}\underset{a^2}{\cancel{a^3}}\underset{1}{\cancel{b}}} = \frac{3b}{4a^2}$$

4 Simplify each expression.

a. $\frac{4a}{5b} \cdot \frac{15b^2}{16a^3}$

$$\frac{4a}{5b} \cdot \frac{15b^2}{16a^3} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{a}} \cdot 3 \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{b}} \cdot b}{\underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot \underset{1}{\cancel{a}} \cdot a \cdot a} \quad \text{Factor.}$$

$$= \frac{3 \cdot b}{2 \cdot 2 \cdot a \cdot a} \quad \text{Simplify.}$$

$$= \frac{3b}{4a^2} \quad \text{Simplify.}$$

b. $\frac{4x^2y}{15a^3b^3} \div \frac{2xy^2}{5ab^3}$

$$\frac{4x^2y}{15a^3b^3} \div \frac{2xy^2}{5ab^3} = \frac{4x^2y}{15a^3b^3} \cdot \frac{5ab^3}{2xy^2} \quad \text{Multiply by the reciprocal of the divisor.}$$

$$= \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{x}} \cdot x \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{5}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}} \cdot \overset{1}{\cancel{b}}}{3 \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{a}} \cdot a \cdot a \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{y}} \cdot y} \quad \text{Factor.}$$

$$= \frac{2 \cdot x}{3 \cdot a \cdot a \cdot y} \quad \text{Simplify.}$$

$$= \frac{2x}{3a^2y} \quad \text{Simplify.}$$

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CHECK Your Progress

4A. $\frac{8t^2s}{5r^2} \cdot \frac{15sr}{12t^3s^2}$

4B. $\frac{9m^2n^3}{16ab^4} \cdot \frac{8a^2b}{27m^5n}$

4C. $\frac{18ab^2}{25x^2y^3} \div \frac{9b}{10xy}$

4D. $\frac{14pq^2}{15w^7z^3} \div \frac{21p^3q}{35w^3z^8}$

Sometimes you must factor the numerator and/or the denominator first before you can simplify a product or a quotient of rational expressions.

EXAMPLE Polynomials in the Numerator and Denominator

5 **Simplify each expression.**

a. $\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2}$

$$\frac{x^2 + 2x - 8}{x^2 + 4x + 3} \cdot \frac{3x + 3}{x - 2} = \frac{(x + 4)\cancel{(x - 2)}^{1}}{(x + 3)\cancel{(x + 1)}^{1}} \cdot \frac{3\cancel{(x + 1)}^{1}}{\cancel{(x - 2)}^{1}}$$ Factor.

$$= \frac{3(x + 4)}{(x + 3)}$$ Simplify.

$$= \frac{3x + 12}{x + 3}$$ Distributive Property

b. $\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9}$

$$\frac{a + 2}{a + 3} \div \frac{a^2 + a - 12}{a^2 - 9} = \frac{a + 2}{a + 3} \cdot \frac{a^2 - 9}{a^2 + a - 12}$$ Multiply by the reciprocal of the divisor.

$$= \frac{(a + 2)\cancel{(a + 3)}^{1}\cancel{(a - 3)}^{1}}{\cancel{(a + 3)}^{1}(a + 4)\cancel{(a - 3)}^{1}}$$ Factor.

$$= \frac{a + 2}{a + 4}$$ Simplify.

CHECK Your Progress

5A. $\frac{y - 1}{5y + 15} \cdot \frac{y^2 + 5y + 6}{y^2 + 4y - 5}$

5B. $\frac{b^2 + 2b - 35}{b^2 - 4} \div \frac{b - 5}{b + 2}$

Simplify Complex Fractions A **complex fraction** is a rational expression whose numerator and/or denominator contains a rational expression. The expressions below are complex fractions.

$$\frac{\frac{a}{5}}{3b} \qquad \frac{\frac{3}{t}}{t + 5} \qquad \frac{\frac{m^2 - 9}{8}}{\frac{3 - m}{12}} \qquad \frac{\frac{1}{p} + 2}{\frac{3}{p} - 4}$$

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To simplify a complex fraction, rewrite it as a division expression, and use the rules for division.

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EXAMPLE Simplify a Complex Fraction

6 Simplify $\dfrac{\dfrac{r^2}{r^2 - 25s^2}}{\dfrac{r}{5s - r}}$.

$$\frac{\dfrac{r^2}{r^2 - 25s^2}}{\dfrac{r}{5s - r}} = \frac{r^2}{r^2 - 25s^2} \div \frac{r}{5s - r}$$ Express as a division expression.

$$= \frac{r^2}{r^2 - 25s^2} \cdot \frac{5s - r}{r}$$ Multiply by the reciprocal of the divisor.

$$= \frac{\overset{1}{\cancel{r}} \cdot r(-1)(\overset{1}{\cancel{r - 5s}})}{(r + 5s)(\underset{1}{\cancel{r - 5s}})\overset{1}{\cancel{r}}}$$ Factor.

$$= \frac{-r}{r + 5s} \text{ or } -\frac{r}{r + 5s}$$ Simplify.

CHECK Your Progress

Simplify each expression.

6A. $\dfrac{\dfrac{(x + 3)^2}{x^2 - 16}}{\dfrac{x + 3}{x + 4}}$

6B. $\dfrac{\dfrac{y - 7}{y - 3}}{\dfrac{y^2 - 49}{y^2 + 4y - 21}}$

CHECK Your Understanding

Example 1 (pp. 442–443)

Simplify each expression.

1. $\dfrac{45mn^3}{20n^7}$

2. $\dfrac{a + b}{a^2 - b^2}$

3. $\dfrac{x^2 + 6x + 9}{x + 3}$

4. $\dfrac{36c^3d^2}{54cd^5}$

Example 2 (p. 443)

5. STANDARDIZED TEST PRACTICE Identify all values of y for which $\dfrac{y - 4}{y^2 - 4y - 12}$ is undefined.

A $-2, 4, 6$ **B** $-6, -4, 2$ **C** $-2, 0, 6$ **D** $-2, 6$

Simplify each expression.

Example 3 (p. 443)

6. $\dfrac{9y^2 - 6y^3}{2y^2 + 5y - 12}$

7. $\dfrac{b^3 - a^3}{a^2 - b^2}$

Example 4 (pp. 444–445)

8. $\dfrac{2a^2}{5b^2c} \cdot \dfrac{3bc^3}{8a^2}$

9. $\dfrac{3t + 6}{7t - 7} \cdot \dfrac{14t - 14}{5t + 10}$

10. $\dfrac{35}{16x^2} \div \dfrac{21}{4x}$

11. $\dfrac{20xy^3}{21} \div \dfrac{15x^3y^2}{14}$

Example 5 (p. 445)

12. $\dfrac{12p^2 + 6p - 6}{4(p + 1)^2} \div \dfrac{6p - 3}{2p + 10}$

13. $\dfrac{x^2 + 6x + 9}{x^2 + 7x + 6} \div \dfrac{4x + 12}{3x + 3}$

Example 6 (p. 446)

14. $\dfrac{\dfrac{c^3d^3}{a}}{\dfrac{xc^2d}{ax^2}}$

15. $\dfrac{\dfrac{2y}{y^2 - 4}}{\dfrac{3}{y^2 - 4y + 4}}$

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Exercises

HOMEWORK HELP

For Exercises	See Examples
16–19	1
20, 21	3
22–25	4
26–29	5
30–33	6
34, 35	2

Simplify each expression.

16. $\frac{30bc}{12b^3}$

17. $\frac{-3mn^3}{21m^2n^2}$

18. $\frac{5t-5}{t^2-1}$

19. $\frac{c+5}{2c+10}$

20. $\frac{3t-6}{2-t}$

21. $\frac{9-t^2}{t^2+t-12}$

22. $\frac{3xyz}{4xz} \cdot \frac{6x^2}{3y^2}$

23. $\frac{-4ab}{21c} \cdot \frac{14c^2}{18a^2}$

24. $\frac{3}{5d} \div \left(-\frac{9}{15df}\right)$

25. $\frac{p^3}{2q} \div \frac{-p}{4q}$

26. $\frac{3t^2}{t+2} \cdot \frac{t+2}{t^2}$

27. $\frac{4w+4}{3} \cdot \frac{1}{w+1}$

28. $\frac{4t^2-4}{9(t+1)^2} \cdot \frac{3t+3}{2t-2}$

29. $\frac{3p-21}{p^2-49} \cdot \frac{p^2-7p}{3p}$

30. $\frac{\frac{m^3}{3n}}{-\frac{m^4}{9n^2}}$

31. $\frac{\frac{p^3}{2q}}{-\frac{p^2}{4q}}$

32. $\frac{\frac{m+n}{5}}{\frac{m^2+n^2}{5}}$

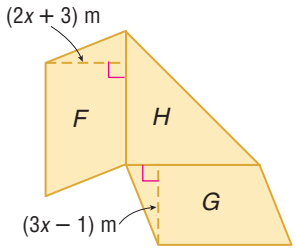
33. $\frac{\frac{x+y}{2x-y}}{\frac{x+y}{2x+y}}$

34. Under what conditions is $\frac{x-4}{(x+5)(x-1)}$ undefined?

35. For what values is $\frac{2d(d+1)}{(d+1)(d^2-4)}$ undefined?

36. GEOMETRY A parallelogram with an area of $6x^2 - 7x - 5$ square units has a base of $3x - 5$ units. Determine the height of the parallelogram.

37. GEOMETRY Parallelogram F has an area of $8x^2 + 10x - 3$ square meters and a height of $2x + 3$ meters. Parallelogram G has an area of $6x^2 + 13x - 5$ square meters and a height of $3x - 1$ meters. Find the area of right triangle H.



Simplify each expression.

38. $\frac{(-3x^2y)^3}{9x^2y^2}$

39. $\frac{(-2rs^2)^2}{12r^2s^3}$

40. $\frac{(-5mn^2)^3}{5m^2n^4}$

41. $\frac{y^2+4y+4}{3y^2+5y-2}$

42. $\frac{a^2+2a+1}{2a^2+3a+1}$

43. $\frac{3x^2-2x-8}{3x^2-12}$

44. $\frac{a^2-4}{6-3a}$

45. $\frac{b^2-4b+3}{3-2b-b^2}$

46. $\frac{6x^2-6}{14x^2-28x+14}$

47. $\frac{25a^2b^3}{6x^2y} \cdot \frac{8xy^2}{20a^3b^2}$

48. $\frac{-9cd}{8xw} \cdot \frac{(-4w)^2}{15c}$

49. $\frac{2x^3y}{z^5} \div \left(\frac{4xy}{z^3}\right)^2$

50. $\frac{w^2-11w+24}{w^2-18w+80} \cdot \frac{w^2-15w+50}{w^2-9w+20}$

51. $\frac{r^2+2r-8}{r^2+4r+3} \div \frac{r-2}{3r+3}$

52. $\frac{\frac{5x^2-5x-30}{45-15x}}{\frac{6+x-x^2}{4x-12}}$

EXTRA PRACTICE
See pages 907, 933.

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Real-World Link

Ray Allen is a five-time All Star and member of team USA for the 2000 Olympics.

Source: NBA

53. Under what conditions is $\frac{a^2 + ab + b^2}{a^2 - b^2}$ undefined?

BASKETBALL **For Exercises 54 and 55, use the following information.**
At the end of the 2005-2006 season, the Seattle Sonics' Ray Allen had made 5422 field goals out of 12,138 attempts during his NBA career.

54. Write a ratio to represent the ratio of the number of career field goals made to career field goals attempted by Ray Allen at the end of the 2005-2006 season.

55. Suppose Ray Allen attempted a field goals and made m field goals during the 2006-2007 season. Write a rational expression to represent the ratio of the number of career field goals made to the number of career field goals attempted at the end of the 2006-2007 season.

AIRPLANES **For Exercises 56–58, use the formula $d = rt$ and the following information.**
An airplane is traveling at the rate r of 500 miles per hour for a time t of $(6 + x)$ hours. A second airplane travels at the rate r of $(540 + 90x)$ miles per hour for a time t of 6 hours.

56. Write a rational expression to represent the ratio of the distance d traveled by the first airplane to the distance d traveled by the second airplane.

57. Simplify the rational expression. What does this expression tell you about the distances traveled of the two airplanes?

58. Under what condition is the rational expression undefined? Describe what this condition would tell you about the two airplanes.

Graphing Calculator

For Exercises 59–62, consider $f(x) = \frac{-15x^2 + 10x}{5x}$ and $g(x) = -3x + 2$.

59. Simplify $\frac{-15x^2 + 10x}{5x}$. What do you observe about the expression?

60. Graph $f(x)$ and $g(x)$ on a graphing calculator. How do the graphs appear?

61. Use the table feature to examine the function values for $f(x)$ and $g(x)$. How do the tables compare?

62. How can you use what you have observed with $f(x)$ and $g(x)$ to verify that expressions are equivalent or to identify excluded values?

H.O.T. Problems

63. **OPEN ENDED** Write two rational expressions that are equivalent.

64. **CHALLENGE** Rewrite $\frac{a + \sqrt{b}}{-a^2 + b}$ so it has a numerator of 1.

65. **Which One Doesn't Belong?** Identify the expression that does not belong with the other three. Explain your reasoning.

$\frac{1}{x - 1}$ $\frac{x^2 + 3x + 2}{x - 5}$ $\frac{x + 1}{\sqrt{x + 3}}$ $\frac{x^2 + 1}{3}$

66. **REASONING** Determine whether $\frac{2d + 5}{3d + 5} = \frac{2}{3}$ is *sometimes*, *always*, or *never* true. Explain.

67. ***Writing in Math*** Use the information about rational expressions on page 442 to explain how rational expressions are used in mixtures. Include an example of a mixture problem that could be represented by $\frac{8 + x}{13 + x + y}$.

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STANDARDIZED TEST PRACTICE

68. ACT/SAT For what value(s) of x is $\frac{4x}{x^2 - x}$ undefined?

A $-1, 1$

B $-1, 0, 1$

C $0, 1$

D 0

69. REVIEW Which is the simplified form of $\frac{4x^3y^2z^{-1}}{(x^{-2}y^3z^2)^2}$?

F $\frac{4x^7}{y^4z^5}$

G $\frac{4xy}{z^5}$

H $\frac{4}{y^3z^5}$

J $\frac{4}{xy^4z^5}$

Spiral Review

Graph each function. State the domain and range. (Lesson 7-3)

70. $y = \sqrt{x - 2}$

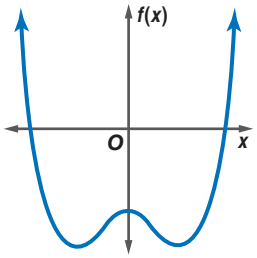
71. $y = \sqrt{x} - 1$

72. $y = 2\sqrt{x} + 1$

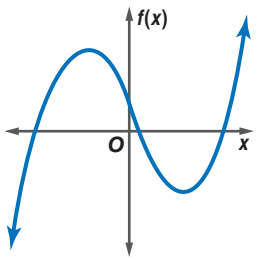
73. Determine whether $f(x) = x - 2$ and $g(x) = 2x$ are inverse functions. (Lesson 7-2)

Determine whether each graph represents an odd-degree or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-3)

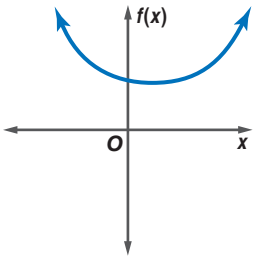
74.



75.



76.



77. ASTRONOMY Earth is an average of 1.496×10^8 kilometers from the Sun. If light travels 3×10^5 kilometers per second, how long does it take sunlight to reach Earth? (Lesson 6-1)

Solve each equation by factoring. (Lesson 5-3)

78. $r^2 - 3r = 4$

79. $18u^2 - 3u = 1$

80. $d^2 - 5d = 0$

Solve each equation. (Lesson 1-4)

81. $|2x + 7| + 5 = 0$

82. $5|3x - 4| = x + 1$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 1-3)

83. $\frac{2}{3} + x = -\frac{4}{9}$

84. $x + \frac{5}{8} = -\frac{5}{6}$

85. $x - \frac{3}{5} = \frac{2}{3}$

86. $x + \frac{3}{16} = -\frac{1}{2}$

87. $x - \frac{1}{6} = -\frac{7}{9}$

88. $x - \frac{3}{8} = -\frac{5}{24}$

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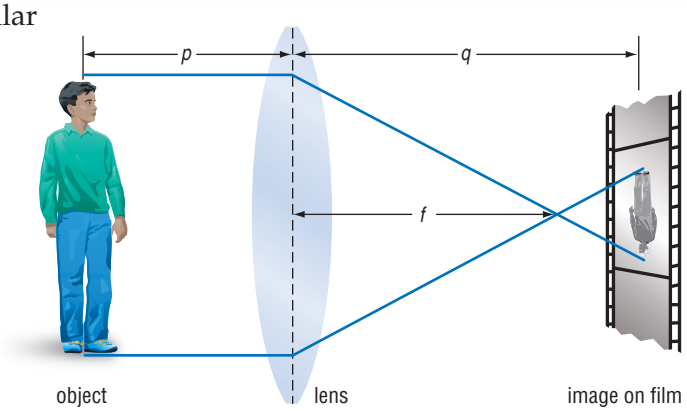
8-2 Adding and Subtracting Rational Expressions

Main Ideas

- Determine the LCM of polynomials.
- Add and subtract rational expressions.

GET READY for the Lesson

In order to produce a picture that is "in focus," the distance between the camera lens and the film q must be controlled so that it satisfies a particular relationship. If the distance from the subject to the lens is p and the focal length of the lens is f, then the formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine the correct distance between the lens and the film.



LCM of Polynomials To find $\frac{5}{6} - \frac{1}{4}$ or $\frac{1}{f} - \frac{1}{p}$, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the denominators.

To find the LCM of two or more numbers or polynomials, factor each number or polynomial. The LCM contains each factor the greatest number of times it appears as a factor.

LCM of 6 and 4	**LCM of $a^2 - 6a + 9$ and $a^2 + a - 12$**
$6 = 2 \cdot 3$	$a^2 - 6a + 9 = (a - 3)^2$
$4 = 2^2$	$a^2 + a - 12 = (a - 3)(a + 4)$
LCM $= 2^2 \cdot 3$ or 12	LCM $= (a - 3)^2(a + 4)$

EXAMPLE **LCM of Monomials**

1 **Find the LCM of $18r^2s^5$, $24r^3st^2$, and $15s^3t$.**

$18r^2s^5 = 2 \cdot 3^2 \cdot r^2 \cdot s^5$	Factor the first monomial.
$24r^3st^2 = 2^3 \cdot 3 \cdot r^3 \cdot s \cdot t^2$	Factor the second monomial.
$15s^3t = 3 \cdot 5 \cdot s^3 \cdot t$	Factor the third monomial.
LCM $= 2^3 \cdot 3^2 \cdot 5 \cdot r^3 \cdot s^5 \cdot t^2$ $= 360r^3s^5t^2$	Use each factor the greatest number of times it appears as a factor and simplify.

CHECK Your Progress

Find the LCM of each set of monomials.

1A. $12a^2b^4$, $27ac^3$, $18a^5b^2c$

1B. $6m^3n^5$, $42mnp^2$, $36m^3n^4p$

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EXAMPLE 2 LCM of Polynomials

Find the LCM of $p^3 + 5p^2 + 6p$ and $p^2 + 6p + 9$.

$p^3 + 5p^2 + 6p = p(p + 2)(p + 3)$ Factor the first polynomial.

$p^2 + 6p + 9 = (p + 3)^2$ Factor the second polynomial.

$\text{LCM} = p(p + 2)(p + 3)^2$ Use each factor the greatest number of times it appears as a factor.

CHECK Your Progress

Find the LCM of each set of polynomials.

2A. $q^2 - 4q + 4$ and $q^3 - 3q^2 + 2q$

2B. $2k^3 - 5k^2 - 12k$ and $k^3 - 8k^2 + 16k$

Add and Subtract Rational Expressions As with fractions, to add or subtract rational expressions, you must have common denominators.

Specific Case		General Case
$\frac{2}{3} + \frac{3}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{3 \cdot 3}{5 \cdot 3}$	Find equivalent fractions that have a common denominator.	$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d}{c \cdot d} + \frac{b \cdot c}{d \cdot c}$
$= \frac{10}{15} + \frac{9}{15}$	Simplify each numerator and denominator.	$= \frac{ad}{cd} + \frac{bc}{cd}$
$= \frac{19}{15}$	Add the numerators.	$= \frac{ad + bc}{cd}$

As with fractions, you can use the least common multiple of the denominators to find the least common denominator for two rational expressions.

EXAMPLE 3 Monomial Denominators

Simplify $\frac{7x}{15y^2} + \frac{y}{18xy}$.

$\frac{7x}{15y^2} + \frac{y}{18xy} = \frac{7x \cdot 6x}{15y^2 \cdot 6x} + \frac{y \cdot 5y}{18xy \cdot 5y}$ The LCD is $90xy^2$. Find the equivalent fractions that have this denominator.

$= \frac{42x^2}{90xy^2} + \frac{5y^2}{90xy^2}$ Simplify each numerator and denominator.

$= \frac{42x^2 + 5y^2}{90xy^2}$ Add the numerators.

CHECK Your Progress

Simplify each expression.

3A. $\frac{8a}{9b} - \frac{1}{7ab^2}$

3B. $\frac{1}{8m^2n} + \frac{2}{mn^2}$

3C. $\frac{2}{3xy} - \frac{3x}{5y}$

3D. $\frac{6c}{7b^2} + \frac{2d}{3ab}$

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EXAMPLE Polynomial Denominators

4 Simplify $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$.

$$\frac{w+12}{4w-16} - \frac{w+4}{2w-8} = \frac{w+12}{4(w-4)} - \frac{w+4}{2(w-4)}$$ Factor the denominators.

$$= \frac{w+12}{4(w-4)} - \frac{(w+4)(2)}{2(w-4)(2)}$$ The LCD is $4(w-4)$.

$$= \frac{(w+12)-(2)(w+4)}{4(w-4)}$$ Subtract the numerators.

$$= \frac{w+12-2w-8}{4(w-4)}$$ Distributive Property

$$= \frac{-w+4}{4(w-4)}$$ Combine like terms.

$$= \frac{-1(w-4)}{4(w-4)} \text{ or } -\frac{1}{4}$$ Simplify.

Study Tip

Common Factors

Sometimes when you simplify the numerator, the polynomial contains a factor common to the denominator. Thus the rational expression can be further simplified.

CHECK Your Progress Simplify each expression.

4A. $\frac{x+6}{6x-18} + \frac{x-6}{2x-6}$

4B. $\frac{x-1}{3x^2+8x+5} - \frac{x-1}{12x+20}$

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One way to simplify a complex fraction is to simplify the numerator and the denominator separately, and then simplify the resulting expressions.

EXAMPLE Simplify Complex Fractions

5 Simplify $\frac{\frac{1}{x}-\frac{1}{y}}{1+\frac{1}{x}}$.

$$\frac{\frac{1}{x}-\frac{1}{y}}{1+\frac{1}{x}} = \frac{\frac{y}{xy}-\frac{x}{xy}}{\frac{x}{x}+\frac{1}{x}}$$ The LCD of the numerator is xy. The LCD of the denominator is x.

$$= \frac{\frac{y-x}{xy}}{\frac{x+1}{x}}$$ Simplify the numerator and denominator.

$$= \frac{y-x}{xy} \div \frac{x+1}{x}$$ Write as a division expression.

$$= \frac{y-x}{xy} \cdot \frac{1}{x+1}$$ Multiply by the reciprocal of the divisor.

$$= \frac{y-x}{y(x+1)} \text{ or } \frac{y-x}{xy+y}$$ Simplify.

CHECK Your Progress Simplify each expression.

5A. $\frac{\frac{1}{y}+\frac{1}{x}}{\frac{1}{y}-\frac{1}{x}}$

5B. $\frac{\frac{a}{b}+1}{1-\frac{b}{a}}$

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EXAMPLE 6 Use a Complex Fraction to Solve a Problem

COORDINATE GEOMETRY Find the slope of the line that passes through $A\left(\frac{2}{p}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{3}{p}\right)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$ Definition of slope

$$= \frac{\frac{3}{p} - \frac{1}{2}}{\frac{1}{3} - \frac{2}{p}}$$ $y_2 = \frac{3}{p}$, $y_1 = \frac{1}{2}$, $x_2 = \frac{1}{3}$, and $x_1 = \frac{2}{p}$

$$= \frac{\frac{6 - p}{2p}}{\frac{p - 6}{3p}}$$ The LCD of the numerator is $2p$. The LCD of the denominator is $3p$.

$$= \frac{6 - p}{2p} \div \frac{p - 6}{3p}$$ Write as a division expression.

$$= \frac{6 - p}{2p} \cdot \frac{3p}{p - 6} \text{ or } -\frac{3}{2}$$ The slope is $-\frac{3}{2}$.

Study Tip

Check Your Solution

You can check your answer by letting p equal any nonzero number, say 1. Use the definition of slope to find the slope of the line through the points.

CHECK Your Progress

Find the slope of the line that passes through each pair of points.

6A. $C\left(\frac{1}{4}, \frac{4}{q}\right)$ and $D\left(\frac{5}{q}, \frac{1}{5}\right)$

6B. $E\left(\frac{7}{w}, \frac{1}{7}\right)$ and $F\left(\frac{1}{7}, \frac{7}{w}\right)$

CHECK Your Understanding

Examples 1, 2 (pp. 450–451)

Find the LCM of each set of polynomials.

1. $12y^2, 6x^2$

2. $16ab^3, 5b^2a^2, 20ac$

3. $x^2 - 2x, x^2 - 4$

4. $x^3 - 4x^2 - 5x, x^2 + 6x + 5$

Simplify each expression.

Example 3 (p. 451)

5. $\frac{2}{x^2y} - \frac{x}{y}$

6. $\frac{7a}{15b^2} - \frac{b}{18ab}$

7. $\frac{5}{3m} - \frac{2}{7m} - \frac{1}{2m}$

8. $\frac{3x}{5} - \frac{1}{2x^2} + \frac{3}{4x}$

Example 4 (p. 452)

9. $\frac{6}{d^2 + 4d + 4} + \frac{5}{d + 2}$

10. $\frac{a}{a^2 - a - 20} + \frac{2}{a + 4}$

11. $\frac{1}{x^2 - 4} + \frac{x}{x + 2}$

12. $\frac{x}{x + 1} + \frac{3}{x^2 - 4x - 5}$

Example 5 (p. 452)

13. $\frac{x + \frac{x}{3}}{x - \frac{x}{6}}$

14. $\frac{1 - \frac{1}{x}}{x - \frac{1}{x}}$

15. $\frac{2 - \frac{4}{x}}{x - \frac{4}{x}}$

16. $\frac{x - \frac{x}{2}}{x + \frac{x}{8}}$

Example 6 (p. 453)

17. GEOMETRY An expression for the area of a rectangle is $4x + 16$. Find the width of the rectangle. Express in simplest form.

$\frac{x + 4}{x + 2}$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
18, 19	1
20, 21	2
22–25	3
26–31	4
32, 33	5
34, 35	6

Find the LCM of each set of polynomials.

18. $10s^2, 35s^2t^2$

19. $36x^2y, 20xyz$

20. $4w - 12, 2w - 6$

21. $x^2 - y^2, x^3 + x^2y$

Simplify each expression.

22. $\frac{6}{ab} + \frac{8}{a}$

23. $\frac{5}{6v} + \frac{7}{4v}$

24. $\frac{3x}{4y^2} - \frac{y}{6x}$

25. $\frac{5}{a^2b} - \frac{7a}{5a^2}$

26. $\frac{7}{y - 8} - \frac{6}{8 - y}$

27. $\frac{a}{a - 4} - \frac{3}{4 - a}$

28. $\frac{m}{m^2 - 4} + \frac{2}{3m + 6}$

29. $\frac{y}{y + 3} - \frac{6y}{y^2 - 9}$

30. $\frac{5}{x^2 - 3x - 28} + \frac{7}{2x - 14}$

31. $\frac{d - 4}{d^2 + 2d - 8} - \frac{d + 2}{d^2 - 16}$

32. $\frac{\frac{1}{b + 2} + \frac{1}{b - 5}}{\frac{2b^2 - b - 3}{b^2 - 3b - 10}}$

33. $\frac{(x + y)\left(\frac{1}{x} - \frac{1}{y}\right)}{(x - y)\left(\frac{1}{x} + \frac{1}{y}\right)}$

34. GEOMETRY An expression for the length of one rectangle is $\frac{x^2 - 9}{x - 2}$. The length of a similar rectangle is expressed as $\frac{x + 3}{x^2 - 4}$. What is the scale factor of the two rectangles? Write in simplest form.

35. GEOMETRY Find the slope of a line that contains the points $A\left(\frac{1}{p}, \frac{1}{q}\right)$ and $B\left(\frac{1}{q}, \frac{1}{p}\right)$. Write in simplest form.

Find the LCM of each set of polynomials.

36. $14a^3, 15bc^3, 12b^3$

37. $9p^2q^3, 6pq^4, 4p^3$

38. $2t^2 + t - 3, 2t^2 + 5t + 3$

39. $n^2 - 7n + 12, n^2 - 2n - 8$

Simplify each expression.

40. $\frac{5}{r} + 7$

41. $\frac{2x}{3y} + 5$

42. $\frac{3}{4q} - \frac{2}{5q} - \frac{1}{2q}$

43. $\frac{11}{9} - \frac{7}{2w} - \frac{6}{5w}$

44. $\frac{1}{h^2 - 9h + 20} - \frac{5}{h^2 - 10h + 25}$

45. $\frac{x}{x^2 + 5x + 6} - \frac{2}{x^2 + 4x + 4}$

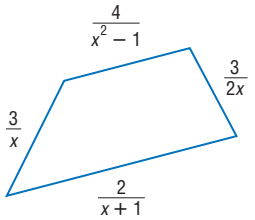
46. $\frac{m^2 + n^2}{m^2 - n^2} + \frac{m}{n - m} + \frac{n}{m + n}$

47. $\frac{y + 1}{y - 1} + \frac{y + 2}{y - 2} + \frac{y}{y^2 - 3y + 2}$

48. Write $\left(\frac{2s}{2s + 1} - 1\right) \div \left(1 + \frac{2s}{1 - 2s}\right)$ in simplest form.

49. What is the simplest form of $\left(3 + \frac{5}{a + 2}\right) \div \left(3 - \frac{10}{a + 7}\right)$?

50. GEOMETRY Find the perimeter of the quadrilateral. Express in simplest form.



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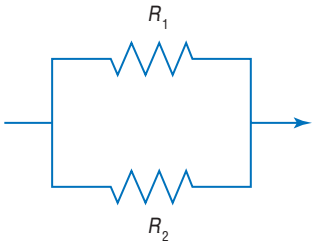
Real-World Link

The Tour de France is the most popular bicycle road race. It lasts 21 days and covers 2500 miles.

Source: *World Book Encyclopedia*

ELECTRICITY **For Exercises 51 and 52, use the following information.**
In an electrical circuit, if two resistors with resistance R_1 and R_2 are connected in parallel as shown, the relationship between these resistances and the resulting combination resistance R is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



51. If R_1 is x ohms and R_2 is 4 ohms less than twice x ohms, write an expression for $\frac{1}{R}$.

52. A circuit with two resistors connected in parallel has an effective resistance of 25 ohms. One of the resistors has a resistance of 30 ohms. Find the resistance of the other resistor.

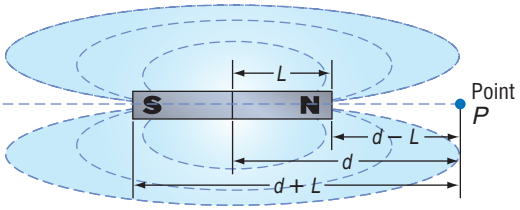
BICYCLING **For Exercises 53–55, use the following information.**
Jalisa is competing in a 48-mile bicycle race. She travels half the distance at one rate. The rest of the distance, she travels 4 miles per hour slower.

53. If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.

54. Write an expression for the time spent at the slower pace.

55. Write an expression for the time Jalisa needed to complete the race.

56. **MAGNETS** For a bar magnet, the magnetic field strength H at a point P along the axis of the magnet is $H = \frac{m}{2L(d-L)^2} - \frac{m}{2L(d+L)^2}$. Write a simpler expression for H.



EXTRA PRACTICE
See pages 908, 933.
MathOnline
Self-Check Quiz at
algebra2.com

H.O.T. Problems

57. **OPEN ENDED** Write two polynomials that have a LCM of $d^3 - d$.

58. **FIND THE ERROR** Lorena and Yong-Chan are simplifying $\frac{x}{a} - \frac{x}{b}$. Who is correct? Explain your reasoning.

Lorena

$$\frac{x}{a} - \frac{x}{b} = \frac{bx}{ab} - \frac{ax}{ab}$$
$$= \frac{bx - ax}{ab}$$

Yong-Chan

$$\frac{x}{a} - \frac{x}{b} = \frac{x}{a - b}$$

59. **CHALLENGE** Find two rational expressions whose sum is $\frac{2x-1}{(x+1)(x-2)}$.

60. **REASONING** In the expression $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, a, b, and c are nonzero real numbers. Determine whether each statement is *sometimes*, *always*, or *never* true. Explain your answer.

a. abc is a common denominator.

b. abc is the LCD.

c. ab is the LCD.

d. b is the LCD.

e. The sum is $\frac{bc + ac + ab}{abc}$.

Pascal Rondeau/Allsport/Getty Images
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61. ***Writing in Math*** Use the information on page 450 to explain how subtraction of rational expressions is used in photography. Include an equation that could be used to find the distance between the lens and the film if the focal length of the lens is 50 millimeters and the distance between the lens and the object is 1000 millimeters.

STANDARDIZED TEST PRACTICE

62. ACT/SAT What is the sum of $\frac{x-y}{5}$ and $\frac{x+y}{4}$?

A $\frac{x+9y}{20}$

B $\frac{9x+y}{20}$

C $\frac{9x-y}{20}$

D $\frac{x-9y}{20}$

63. REVIEW

Given: Two angles are complementary. The measure of one angle is 15° more than the measure of the other angle.
Conclusion: The measures of the angles are 30° and 45°.

This conclusion —

F is contradicted by the first statement given.

G is verified by the first statement given.

H invalidates itself because a 45° angle cannot be complementary to another.

J verifies itself because 30° is 15° less than 45°.

Spiral Review

Simplify each expression. (Lesson 8-1)

64. $\frac{9x^2y^3}{(5xyz)^2} \div \frac{(3xy)^3}{20x^2y}$

65. $\frac{5a^2-20}{2a+2} \cdot \frac{4a}{10a-20}$

66. Graph $y \leq \sqrt{x+1}$. (Lesson 7-7)

Find all of the zeros of each function. (Lesson 6-9)

67. $g(x) = x^4 - 8x^2 - 9$

68. $h(x) = 3x^3 - 5x^2 + 13x - 5$

69. GARDENS Helene Jonson has a rectangular garden 25 feet by 50 feet. She wants to increase the garden on all sides by an equal amount. If the area of the garden is to be increased by 400 square feet, by how much should each dimension be increased? (Lesson 5-5)

70. Three times a number added to four times a second number is 22. The second number is two more than the first number. Find the numbers. (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.** (Lesson 5-3)

71. $x^2 + 3x + 2$

72. $x^2 - 6x + 5$

73. $x^2 + 11x - 12$

74. $x^2 - 16$

75. $3x^2 - 75$

76. $x^3 - 3x^2 + 4x - 12$

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8-3 Graphing Rational Functions

Main Ideas

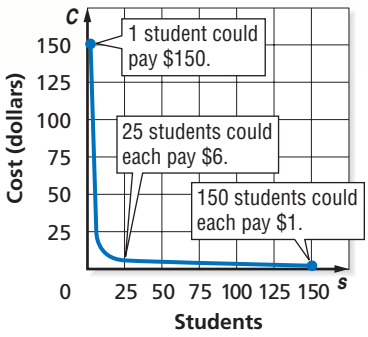
- Determine the limitations on the domains and ranges of the graphs of rational functions.
- Graph rational functions.

New Vocabulary

rational function
continuity
asymptote
point discontinuity

GET READY for the Lesson

A group of students want to get their favorite teacher, Mr. Salgado, a retirement gift. They plan to get him a gift certificate for a weekend package at a lodge in a state park. The certificate costs \$150. If c represents the cost for each student and s represents the number of students, then $c = \frac{150}{s}$.



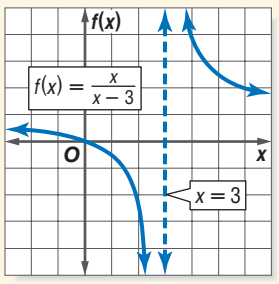
Domain and Range The function $c = \frac{150}{s}$ is a rational function. A **rational function** has an equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. Here are other rational functions.

$$f(x) = \frac{x}{x+3} \qquad g(x) = \frac{5}{x-6} \qquad h(x) = \frac{x+4}{(x-1)(x+4)}$$

No denominator in a rational function can be zero because division by zero is not defined. The functions above are not defined at $x = -3$, $x = 6$, and $x = 1$ and $x = -4$, respectively. The domain of a rational function is limited to values for which the function is defined.

The graphs of rational functions may have breaks in **continuity**. This means that, unlike polynomial functions, which can be traced with a pencil never leaving the paper, not all rational functions are traceable. Breaks in continuity occur at values that are excluded from the domain. They can appear as vertical asymptotes or as point discontinuity. An **asymptote** is a line that the graph of the function approaches, but never touches. **Point discontinuity** is like a hole in a graph.

KEY CONCEPT — *Vertical Asymptotes*

Property	Words	Example	Model
Vertical Asymptote	If the rational expression of a function is written in simplest form and the function is undefined for $x = a$, then the line $x = a$ is a vertical asymptote.	For $f(x) = \frac{x}{x-3}$, the line $x = 3$ is a vertical asymptote.	

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Concepts in Motion
Animation
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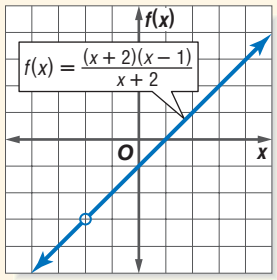
KEY CONCEPT — **Point Discontinuity**

Property	Words	Example	Model
Point Discontinuity	If the original function is undefined for $x = a$ but the rational expression of the function in simplest form is defined for $x = a$, then there is a hole in the graph at $x = a$.	$f(x) = \frac{(x + 2)(x - 1)}{x + 2}$ can be simplified to $f(x) = x - 1$. So, $x = -2$ represents a hole in the graph.	

EXAMPLE Limitations on Domain

1 **Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 - 1}{x^2 - 6x + 5}$.**

First factor the numerator and denominator of the rational expression.

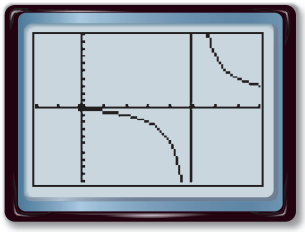
$$\frac{x^2 - 1}{x^2 - 6x + 5} = \frac{(x - 1)(x + 1)}{(x - 1)(x - 5)}$$

The function is undefined for $x = 1$ and $x = 5$. Since $\frac{(x - 1)(x + 1)}{(x - 1)(x - 5)} = \frac{x + 1}{x - 5}$,

$x = 5$ is a vertical asymptote, and $x = 1$ represents a hole in the graph.

Study Tip
Parent Function
The parent function for the family of rational, or reciprocal, functions is $y = \frac{1}{x}$.

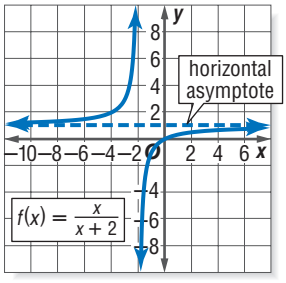
CHECK You can use a graphing calculator to check this solution. The graphing calculator screen at the right shows the graph of $f(x)$. The graph shows the vertical asymptote at $x = 5$. It is not clear from the graph that the function is not defined at $x = 1$. However, if you use the value function of the CALC menu and enter 1 at the X= prompt, you will see that no value is returned for Y=. This shows that $f(x)$ is not defined at $x = 1$.


[−2, 8] scl: 1 by [−10, 10] scl: 1

CHECK Your Progress

1. Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of $f(x) = \frac{x^2 + 6x + 8}{x^2 - 16}$.

For some rational functions, the values of the range are limited. Often a horizontal asymptote occurs where a value is excluded from the range. For example, 1 is excluded from the range of $f(x) = \frac{x}{x + 2}$. The graph of $f(x)$ gets increasingly close to a horizontal asymptote as x increases or decreases.



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Graph Rational Functions You can use what you know about vertical asymptotes and point discontinuity to graph rational functions.

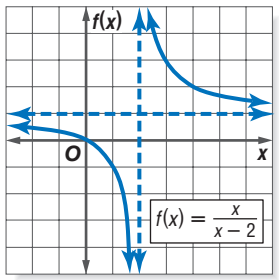
EXAMPLE 2 Graph with Vertical and Horizontal Asymptotes

2 Graph $f(x) = \frac{x}{x-2}$.

The function is undefined for $x = 2$. Since $\frac{x}{x-2}$ is in simplest form, $x = 2$ is a vertical asymptote. Draw the vertical asymptote. Make a table of values. Plot the points and draw the graph.

As $|x|$ increases, it appears that the y-values of the function get closer and closer to 1. The line with the equation $f(x) = 1$ is a horizontal asymptote of the function.

x	$f(x)$
−50	0.96154
−20	0.90909
−10	0.83333
−2	0.5
−1	0.33333
0	0
1	−1
3	3
4	2
5	1.6667
10	1.25
20	1.1111
50	1.0417



Study Tip

Graphing Rational Functions

Finding the x- and y-intercepts is often useful when graphing rational functions.

CHECK Your Progress

2. Graph $f(x) = \frac{x+1}{x-1}$.

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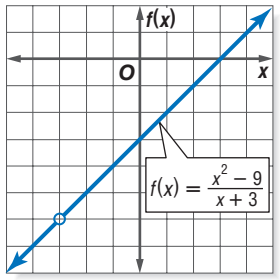
As you have learned, graphs of rational functions may have point discontinuity rather than vertical asymptotes. The graphs of these functions appear to have holes. These holes are usually shown as circles on graphs.

EXAMPLE 3 Graph with Point Discontinuity

3 Graph $f(x) = \frac{x^2 - 9}{x+3}$.

Notice that $\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3}$ or $x - 3$.

Therefore, the graph of $f(x) = \frac{x^2-9}{x+3}$ is the graph of $f(x) = x - 3$ with a hole at $x = -3$.



CHECK Your Progress

3. Graph $f(x) = \frac{x^2+4x-5}{x+5}$.

In the real world, sometimes values on the graph of a rational function are not meaningful.

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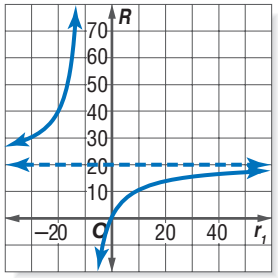
Real-World EXAMPLE Use Graphs of Rational Functions

4 **AVERAGE SPEED** A boat traveled upstream at r_1 miles per hour. During the return trip to its original starting point, the boat traveled at r_2 miles per hour. The average speed for the entire trip R is given by the formula $R = \frac{2r_1r_2}{r_1 + r_2}$.

a. Let r_1 be the independent variable and let R be the dependent variable. Draw the graph if $r_2 = 10$ miles per hour.

The function is $R = \frac{2r_1(10)}{r_1 + 10}$ or $R = \frac{20r_1}{r_1 + 10}$. The vertical asymptote is $r_1 = -10$. Graph the vertical asymptote and the function. Notice that the horizontal asymptote is $R = 20$.



b. What is the R-intercept of the graph? The R-intercept is 0.

c. What domain and range values are meaningful in the context of the problem?

In the problem context, the speeds are nonnegative values. Therefore, only values of r_1 greater than or equal to 0 and values of R between 0 and 20 are meaningful.

CHECK Your Progress

4. SALARIES A company uses the formula $S(x) = \frac{45x + 25}{x + 1}$ to determine the salary in thousands of dollars of an employee during his xth year. Draw the graph of $S(x)$. What domain and range values are meaningful in the context of the problem? What is the meaning of the horizontal asymptote for the graph?

CHECK Your Understanding

Example 1 (p. 458)

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

1. $f(x) = \frac{3}{x^2 - 4x + 4}$

2. $f(x) = \frac{x - 1}{x^2 + 4x - 5}$

Graph each rational function.

Example 2 (p. 459)

3. $f(x) = \frac{x}{x + 1}$

4. $f(x) = \frac{6}{(x - 2)(x + 3)}$

5. $f(x) = \frac{4}{(x - 1)^2}$

6. $f(x) = \frac{x - 5}{x + 1}$

Example 3 (p. 459)

7. $f(x) = \frac{x^2 - 25}{x - 5}$

8. $f(x) = \frac{x + 2}{x^2 - x - 6}$

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Example 4 (p. 460)

ELECTRICITY **For Exercises 9–12, use the following information.**
The current I in amperes in an electrical circuit with three resistors in series is given by the equation $I = \frac{V}{R_1 + R_2 + R_3}$, where V is the voltage in volts in the circuit and R_1, R_2, and R_3 are the resistances in ohms of the three resistors.

9. Let R_1 be the independent variable, and let I be the dependent variable. Graph the function if $V = 120$ volts, $R_2 = 25$ ohms, and $R_3 = 75$ ohms.

10. Give the equation of the vertical asymptote and the R_1- and I-intercepts of the graph.

11. Find the value of I when the value of R_1 is 140 ohms.

12. What domain and range values are meaningful in the context of the problem?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
13–16	1
17–26	2
27, 28	3
29–36	4

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

13. $f(x) = \frac{2}{x^2 - 5x + 6}$

14. $f(x) = \frac{4}{x^2 + 2x - 8}$

15. $f(x) = \frac{x + 3}{x^2 + 7x + 12}$

16. $f(x) = \frac{x - 5}{x^2 - 4x - 5}$

Graph each rational function.

17. $f(x) = \frac{1}{x}$

18. $f(x) = \frac{3}{x}$

19. $f(x) = \frac{1}{x + 2}$

20. $f(x) = \frac{-5}{x + 1}$

21. $f(x) = \frac{x}{x - 3}$

22. $f(x) = \frac{5x}{x + 1}$

23. $f(x) = \frac{-3}{(x - 2)^2}$

24. $f(x) = \frac{1}{(x + 3)^2}$

25. $f(x) = \frac{x + 4}{x - 1}$

26. $f(x) = \frac{x - 1}{x - 3}$

27. $f(x) = \frac{x^2 - 36}{x + 6}$

28. $f(x) = \frac{x^2 - 1}{x - 1}$

PHYSICS **For Exercises 29–32, use the following information.**
Under certain conditions, when two objects collide, the objects are repelled from each other with velocity given by the equation $V_f = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2}$.
In this equation m_1 and m_2 are the masses of the two objects, v_1 and v_2 are the initial speeds of the two objects, and V_f is the final speed of the second object.

29. Let m_2 be the independent variable, and let V_f be the dependent variable. Graph the function if $m_1 = 5$ kilograms and $v_1 = 15$ meters per second, and $v_2 = 20$ meters per second.

30. Use the equation and the values in Exercise 29 to determine the final speed if $m_2 = 20$ kilograms.

31. Give the equation of any asymptotes and the m_2- and V_f-intercepts of the graph.

32. What domain and range values are meaningful in the context of the problem?

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BASKETBALL **For Exercises 33–36, use the following information.**
Zonta plays basketball for Centerville High School. So far this season, she has made 6 out of 10 free throws. She is determined to improve her free-throw percentage. If she can make x consecutive free throws, her free-throw percentage can be determined using $P(x) = \frac{6 + x}{10 + x}$.

33. Graph the function.

34. What part of the graph is meaningful in the context of the problem?

35. Describe the meaning of the y-intercept.

36. What is the equation of the horizontal asymptote? Explain its meaning with respect to Zonta's shooting percentage.

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

37. $f(x) = \frac{x^2 - 8x + 16}{x - 4}$

38. $f(x) = \frac{x^2 - 3x + 2}{x - 1}$

Graph each rational function.

39. $f(x) = \frac{3}{(x - 1)(x + 5)}$

40. $f(x) = \frac{-1}{(x + 2)(x - 3)}$

41. $f(x) = \frac{x}{x^2 - 1}$

42. $f(x) = \frac{x - 1}{x^2 - 4}$

43. $f(x) = \frac{6}{(x - 6)^2}$

44. $f(x) = \frac{1}{(x + 2)^2}$

45. $f(x) = \frac{x^2 + 6x + 5}{x + 1}$

46. $f(x) = \frac{x^2 - 4x}{x - 4}$

HISTORY **For Exercises 47–49, use the following information.**
In Maria Gaetana Agnesi's book *Analytical Institutions*, Agnesi discussed the characteristics of the equation $x^2y = a^2(a - y)$, the graph of which is called the "curve of Agnesi." This equation can be expressed as $y = \frac{a^3}{x^2 + a^2}$.

47. Graph $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = 4$.

48. Describe the graph. What are the limitations on the domain and range?

49. Make a conjecture about the shape of the graph of $f(x) = \frac{a^3}{x^2 + a^2}$ if $a = -4$. Explain your reasoning.

EXTRA PRACTICE
See pages 908, 933.

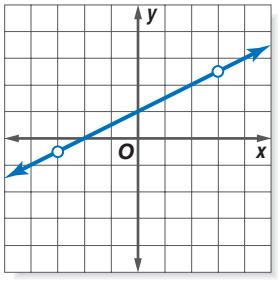
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

50. **OPEN ENDED** Write a function the graph of which has vertical asymptotes located at $x = -5$ and $x = 2$.

51. **REASONING** Compare and contrast the graphs of $f(x) = \frac{(x - 1)(x + 5)}{x - 1}$ and $g(x) = x + 5$.

52. **CHALLENGE** Write a rational function for the graph at the right.



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53. CHALLENGE Write three rational functions that have a vertical asymptote at $x = 3$ and a hole at $x = -2$.

54. *Writing in Math* Use the information on page 457 to explain how rational functions can be used when buying a group gift. Explain why only part of the graph of the rational function is meaningful in the context of the problem.

STANDARDIZED TEST PRACTICE

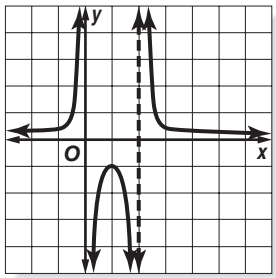
55. ACT/SAT Which set is the domain of the function graphed below?

A $\{x \mid x \neq 0, 2\}$

B $\{x \mid x \neq -2, 0\}$

C $\{x \mid x < 4\}$

D $\{x > -4\}$



56. REVIEW $\dfrac{x+2}{x+3} + \dfrac{4}{x^2+x-6} =$

F $\dfrac{-3x-9}{x^2+x-6}$

H $\dfrac{x^2}{x^2+x-6}$

G $\dfrac{x^2-3x-24}{x^2+x-6}$

J $\dfrac{x^2+x-1}{x^2+x-6}$

Spiral Review

Simplify each expression. (Lessons 8-1 and 8-2)

57. $\dfrac{3m+2}{m+n} + \dfrac{4}{2m+2n}$

58. $\dfrac{5}{x+3} - \dfrac{2}{x-2}$

59. $\dfrac{2w-4}{w+3} \div \dfrac{2w+6}{5}$

Find all of the rational zeros for each function. (Lesson 6-8)

60. $f(x) = x^3 + 5x^2 + 2x - 8$

61. $g(x) = 2x^3 - 9x^2 + 7x + 6$

62. ART Joyce Jackson purchases works of art for an art gallery. Two years ago she bought a painting for $20,000, and last year she bought one for $35,000. If paintings appreciate 14% per year, how much are the two paintings worth now? (Lesson 6-5)

Solve each equation by completing the square. (Lesson 5-5)

63. $x^2 + 8x + 20 = 0$

64. $x^2 + 2x - 120 = 0$

65. $x^2 + 7x - 17 = 0$

66. Write the slope-intercept form of the equation for the line that passes through $(1, -2)$ and is perpendicular to the line with equation $y = -\frac{1}{5}x + 2$. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each proportion.**

67. $\dfrac{16}{v} = \dfrac{32}{9}$

68. $\dfrac{7}{25} = \dfrac{a}{5}$

69. $\dfrac{6}{15} = \dfrac{8}{s}$

70. $\dfrac{b}{9} = \dfrac{40}{30}$

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EXTEND 8-3

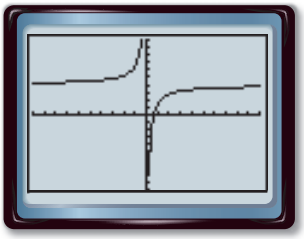
Graphing Calculator Lab

Graphing Rational Functions

A TI-83/84 Plus graphing calculator can be used to explore the graphs of rational functions. These graphs have some features that never appear in the graphs of polynomial functions.

ACTIVITY 1

Graph $y = \frac{8x - 5}{2x}$ in the standard viewing window. Find the equations of any asymptotes.



[−10, 10] scl: 1 by [−10, 10] scl: 1

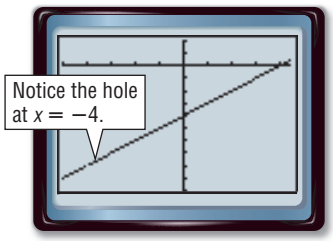
Enter the equation in the Y= list.

KEYSTROKES: Y= (8 X,T,θ,n — 5) ÷ (2 X,T,θ,n) ZOOM 6

By looking at the equation, we can determine that if $x = 0$, the function is undefined. The equation of the vertical asymptote is $x = 0$. Notice what happens to the y-values as x grows larger and as x gets smaller. The y-values approach 4. So, the equation for the horizontal asymptote is $y = 4$.

ACTIVITY 2

Graph $y = \frac{x^2 - 16}{x + 4}$ in the window [−5, 4.4] by [−10, 2] with scale factors of 1.



[−5, 4.4] scl: 1 by [−10, 2] scl: 1

Because the function is not continuous, put the calculator in dot mode.

KEYSTROKES: MODE ▼ ▼ ▼ ▼ ▶ ENTER

This graph looks like a line with a break in continuity at $x = -4$. This happens because the denominator is 0 when $x = -4$. Therefore, the function is undefined when $x = -4$.

If you TRACE along the graph, when you come to $x = -4$, you will see that there is no corresponding y-value.

EXERCISES

Use a graphing calculator to graph each function. Be sure to show a complete graph. Draw the graph on a sheet of paper. Write the x-coordinates of any points of discontinuity and/or the equations of any asymptotes.

1. $f(x) = \frac{1}{x}$ **2.** $f(x) = \frac{x}{x + 2}$ **3.** $f(x) = \frac{2}{x - 4}$

4. $f(x) = \frac{2x}{3x - 6}$ **5.** $f(x) = \frac{4x + 2}{x - 1}$ **6.** $f(x) = \frac{x^2 - 9}{x + 3}$

7. Which graph(s) has point discontinuity?

8. Describe functions that have point discontinuity.

Other Calculator Keystrokes at algebra2.com

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8-4 Direct, Joint, and Inverse Variation

Main Ideas

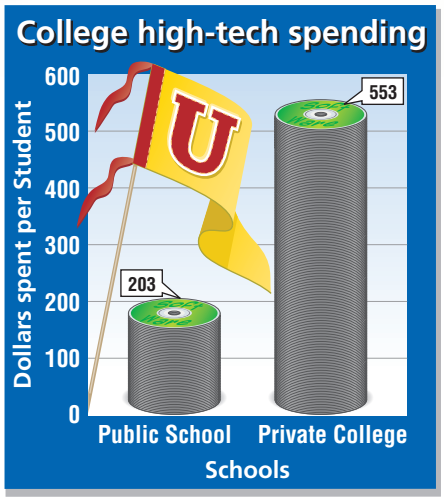
- Recognize and solve direct and joint variation problems.
- Recognize and solve inverse variation problems.

New Vocabulary

direct variation
constant of variation
joint variation
inverse variation

GET READY for the Lesson

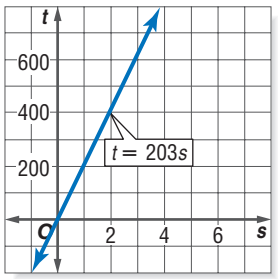
The total high-tech spending t of an average public college can be found by using the equation $t = 203s$, where s is the number of students.



Source: dthonline.com

Direct Variation and Joint Variation The relationship given by $t = 203s$ is an example of direct variation. A **direct variation** can be expressed in the form $y = kx$. The k in this equation is called the **constant of variation**.

Notice that the graph of $t = 203s$ is a straight line through the origin. An equation of a direct variation is a special case of an equation written in slope-intercept form, $y = mx + b$. When $m = k$ and $b = 0$, $y = mx + b$ becomes $y = kx$. So the slope of a direct variation equation is its constant of variation.



To express a direct variation, we say that y varies directly as x. In other words, as x increases, y increases or decreases at a constant rate.

KEY CONCEPT *Direct Variation*

y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the constant of variation.

If you know that y varies directly as x and one set of values, you can use a proportion to find the other set of corresponding values.

$$y_1 = kx_1 \quad \text{and} \quad y_2 = kx_2$$

$$\frac{y_1}{x_1} = k \qquad \frac{y_2}{x_2} = k \qquad \text{Therefore, } \frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Using the properties of equality, you can find many other proportions that relate these same x- and y-values.

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EXAMPLE 1 Direct Variation

If y varies directly as x and $y = 12$ when $x = -3$, find y when $x = 16$.

Use a proportion that relates the values.

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$ Direct proportion

$\frac{12}{-3} = \frac{y_2}{16}$ $y_1 = 12$, $x_1 = -3$, and $x_2 = 16$

$16(12) = -3(y_2)$ Cross multiply.

$192 = -3y_2$ Simplify.

$-64 = y_2$ Divide each side by -3.

When $x = 16$, the value of y is -64.

CHECK Your Progress

1. If r varies directly as s and $r = -20$ when $s = 4$, find r when $s = -6$.

Another type of variation is joint variation. **Joint variation** occurs when one quantity varies directly as the product of two or more other quantities.

KEY CONCEPT *Joint Variation*

y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

If you know that y varies jointly as x and z and one set of values, you can use a proportion to find the other set of corresponding values.

$y_1 = kx_1z_1$ and $y_2 = kx_2z_2$

$\frac{y_1}{x_1z_1} = k$ $\frac{y_2}{x_2z_2} = k$ Therefore, $\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$.

EXAMPLE 2 Joint Variation

Suppose y varies jointly as x and z. Find y when $x = 8$ and $z = 3$, if $y = 16$ when $z = 2$ and $x = 5$.

Use a proportion that relates the values.

$\frac{y_1}{x_1z_1} = \frac{y_2}{x_2z_2}$ Joint variation

$\frac{16}{5(2)} = \frac{y_2}{8(3)}$ $y_1 = 16$, $x_1 = 5$, $z_1 = 2$, $x_2 = 8$, and $z_2 = 3$

$8(3)(16) = 5(2)(y_2)$ Cross multiply.

$384 = 10y_2$ Simplify.

$38.4 = y_2$ Divide each side by 10.

When $x = 8$ and $z = 3$, the value of y is 38.4.

CHECK Your Progress

2. Suppose r varies jointly as s and t. Find r when $s = 2$ and $t = 8$, if $r = 70$ when $s = 10$ and $t = 4$.

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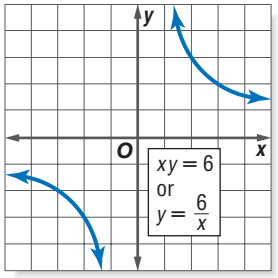
Inverse Variation Another type of variation is inverse variation. For two quantities with **inverse variation**, as one quantity increases, the other quantity decreases. For example, speed and time for a fixed distance vary inversely with each other. When you travel to a particular location, as your speed increases, the time it takes to arrive at that location decreases.

KEY CONCEPT *Direct Variation*

y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Suppose y varies inversely as x such that $xy = 6$ or $y = \frac{6}{x}$. The graph of this equation is shown at the right. Since k is a positive value, as the values of x increase, the values of y decrease.

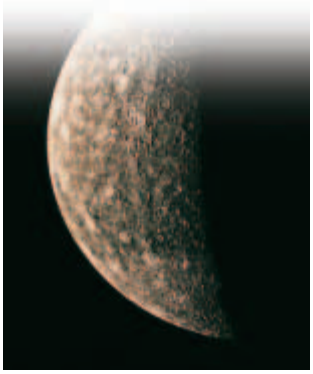


A proportion can be used with inverse variation to solve problems where some quantities are known. The following proportion is only one of several that can be formed.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitution Property of Equality}$$

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \quad \text{Divide each side by } y_1y_2.$$



Real-World Link

Mercury is about 36 million miles from the Sun, making it the closest planet to the Sun. Its proximity to the Sun causes its temperature to be as high as 800°F.

Source: *World Book Encyclopedia*

EXAMPLE Inverse Variation

3 **If r varies inversely as t and $r = 18$ when $t = -3$, find r when $t = -11$.**

$\frac{r_1}{t_2} = \frac{r_2}{t_1}$ Use a proportion that relates the values.

$\frac{18}{-11} = \frac{r_2}{-3}$ $r_1 = 18$, $t_1 = -3$, and $t_2 = -11$

$18(-3) = -11(r_2)$ Cross multiply.

$-54 = -11r_2$ Simplify.

$4\frac{10}{11} = r_2$ Divide each side by -11.

CHECK Your Progress

3. If x varies inversely as y and $x = 24$ when $y = 4$, find x when $y = 12$.

Real-World EXAMPLE

4 **SPACE** **The apparent length of an object is inversely proportional to one's distance from the object. Earth is about 93 million miles from the Sun. Use the information at the left to find how many times as large the diameter of the Sun would appear on Mercury than on Earth.**

Explore The apparent diameter of the Sun varies inversely with the distance from the Sun. You know Mercury's distance from the Sun and Earth's distance from the Sun. You want to know how much larger the diameter of the Sun appears on Mercury than on Earth.

JPL/TSADO/Tom Stack & Associates

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Plan Let the apparent diameter of the Sun from Earth equal 1 unit and the apparent diameter of the Sun from Mercury equal m. Then use a proportion that relates the values.

Solve

$$\frac{\text{distance from Mercury}}{\text{apparent diameter from Earth}} = \frac{\text{distance from Earth}}{\text{apparent diameter from Mercury}}$$ Inverse variation

$$\frac{36 \text{ million miles}}{1 \text{ unit}} = \frac{93 \text{ million miles}}{m \text{ units}}$$ Substitution

$$(36 \text{ million miles})(m \text{ units}) = (93 \text{ million miles})(1 \text{ unit})$$ Cross multiply.

$$m = \frac{(93 \text{ million miles})(1 \text{ unit})}{36 \text{ million miles}}$$ Divide each side by 36 million miles.

$$m \approx 2.58 \text{ units}$$ Simplify.

Check Since the distance between the Sun and Earth is between 2 and 3 times the distance between the Sun and Mercury, the answer seems reasonable. From Mercury, the diameter of the Sun will appear about 2.58 times as large as it appears from Earth.

CHECK Your Progress

4. SPACE Jupiter is about 483.6 million miles from the Sun. Use the information above to find how many times as large the diameter of the Sun would appear on Earth as on Jupiter.

Online **Personal Tutor at** algebra2.com

CHECK Your Understanding

Examples 1–3 (pp. 466–467)

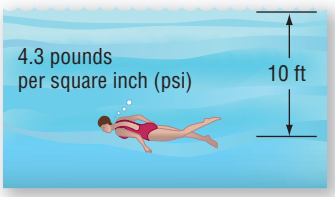
1. If y varies directly as x and $y = 18$ when $x = 15$, find y when $x = 20$.

2. Suppose y varies jointly as x and z. Find y when $x = 9$ and $z = -5$, if $y = -90$ when $z = 15$ and $x = -6$.

3. If y varies inversely as x and $y = -14$ when $x = 12$, find x when $y = 21$.

Example 4 (pp. 467–468)

SWIMMING For Exercises 4–7, use the following information.
When a person swims underwater, the pressure in his or her ears varies directly with the depth at which he or she is swimming.



4. Write a direct variation equation that represents this situation.

5. Find the pressure at 60 feet.

6. It is unsafe for amateur divers to swim where the water pressure is more than 65 pounds per square inch. How deep can an amateur diver safely swim?

7. Make a table showing the number of pounds of pressure at various depths of water. Use the data to draw a graph of pressure versus depth.

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
8, 9	1
10, 11	2
12, 13	3
14, 15	4

8. If y varies directly as x and $y = 15$ when $x = 3$, find y when $x = 12$.

9. If y varies directly as x and $y = 8$ when $x = 6$, find y when $x = 15$.

10. Suppose y varies jointly as x and z. Find y when $x = 2$ and $z = 27$, if $y = 192$ when $x = 8$ and $z = 6$.

11. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$.

12. If y varies inversely as x and $y = 5$ when $x = 10$, find y when $x = 2$.

13. If y varies inversely as x and $y = 16$ when $x = 5$, find y when $x = 20$.

14. GEOMETRY How does the circumference of a circle vary with respect to its radius? What is the constant of variation?

15. TRAVEL A map of Alaska is scaled so that 3 inches represents 93 miles. How far apart are Anchorage and Fairbanks if they are 11.6 inches apart on the map?

Real-World Career

Travel Agent

Travel agents give advice and make arrangements for transportation, accommodations, and recreation. For international travel, they also provide information on customs and currency exchange.

Math Online

For more information, go to algebra2.com.

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation.

16. $\frac{n}{m} = 1.5$

17. $3 = \frac{a}{b}$

18. $a = 5bc$

19. $V = \frac{1}{3}Bh$

20. $p = \frac{12}{q}$

21. $\frac{2.5}{t} = s$

22. $vw = -18$

23. $y = -7x$

24. $V = \pi r^2 h$

25. If y varies directly as x and $y = 9$ when x is -15, find y when $x = 21$.

26. If y varies directly as x and $x = 6$ when $y = 0.5$, find y when $x = 10$.

27. Suppose y varies jointly as x and z. Find y when $x = \frac{1}{2}$ and $z = 6$, if $y = 45$ when $x = 6$ and $z = 10$.

28. If y varies jointly as x and z and $y = \frac{1}{8}$ when $x = \frac{1}{2}$ and $z = 3$, find y when $x = 6$ and $z = \frac{1}{3}$.

29. If y varies inversely as x and $y = 2$ when $x = 25$, find x when $y = 40$.

30. If y varies inversely as x and $y = 4$ when $x = 12$, find y when $x = 5$.

31. CHEMISTRY Boyle's Law states that when a sample of gas is kept at a constant temperature, the volume varies inversely with the pressure exerted on it. Write an equation for Boyle's Law that expresses the variation in volume V as a function of pressure P.

32. CHEMISTRY Charles' Law states that when a sample of gas is kept at a constant pressure, its volume V will increase directly as the temperature t. Write an equation for Charles' Law that expresses volume as a function.

LAUGHTER For Exercises 33–35, use the following information.

A newspaper reported that the average American laughs 15 times per day.

33. Write an equation to represent the average number of laughs produced by m household members during a period of d days.

34. Is your equation in Exercise 33 a *direct*, *joint*, or *inverse* variation?

35. Assume that members of your household laugh the same number of times each day as the average American. How many times would the members of your household laugh in a week?

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Lynn M. Stone/Bruce Coleman, Inc.

Real-World Link

In order to sustain itself in its cold habitat, a Siberian tiger requires 20 pounds of meat per day.

Source: *Wildlife Fact File*

BIOLOGY **For Exercises 36–38, use the information at the left.**

36. Write an equation to represent the amount of meat needed to sustain s Siberian tigers for d days.

37. Is your equation in Exercise 36 a *direct*, *joint*, or *inverse* variation?

38. How much meat do three Siberian tigers need for the month of January?

39. **WORK** Paul drove from his house to work at an average speed of 40 miles per hour. The drive took him 15 minutes. If the drive home took him 20 minutes and he used the same route in reverse, what was his average speed going home?

40. **WATER SUPPLY** Many areas of Northern California depend on the snowpack of the Sierra Nevada Mountains for their water supply. If 250 cubic centimeters of snow will melt to 28 cubic centimeters of water, how much water does 900 cubic centimeters of snow produce?

41. **RESEARCH** According to Johannes Kepler's third law of planetary motion, the ratio of the square of a planet's period of revolution around the Sun to the cube of its mean distance from the Sun is constant for all planets. Verify that this is true for at least three planets.

ASTRONOMY **For Exercises 42–44, use the following information.**

Astronomers can use the brightness of two light sources, such as stars, to compare the distances from the light sources. The intensity, or brightness, of light I is inversely proportional to the square of the distance from the light source d.

42. Write an equation that represents this situation.

43. If d is the independent variable and I is the dependent variable, graph the equation from Exercise 42 when $k = 16$.

44. If two people are viewing the same light source, and one person is three times the distance from the light source as the other person, compare the light intensities that the two people observe.

GRAVITY **For Exercises 45–47, use the following information.**

According to the Law of Universal Gravitation, the attractive force F in Newtons between any two bodies in the universe is directly proportional to the product of the masses m_1 and m_2 in kilograms of the two bodies and inversely proportional to the square of the distance d in meters between the bodies. That is, $F = G\frac{m_1 m_2}{d_2}$. G is the universal gravitational constant. Its value is 6.67×10^{-11} Nm^2/kg^2.

45. The distance between Earth and the Moon is about 3.84×10^8 meters. The mass of the Moon is 7.36×10^{22} kilograms. The mass of Earth is 5.97×10^{24} kilograms. What is the gravitational force that the Moon and Earth exert upon each other?

46. The distance between Earth and the Sun is about 1.5×10^{11} meters. The mass of the Sun is about 1.99×10^{30} kilograms. What is the gravitational force that the Sun and Earth exert upon each other?

47. Find the gravitational force exerted on each other by two 1000-kilogram iron balls a distance of 0.1 meter apart.

EXTRA PRACTICE
See pages 908, 933.
MathOnline
Self-Check Quiz at algebra2.com

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H.O.T. Problems

48. OPEN ENDED Describe two real life quantities that vary directly with each other and two quantities that vary inversely with each other.

49. CHALLENGE Write a real-world problem that involves a joint variation. Solve the problem.

50. *Writing in Math* Use the information about variation on page 465 to explain how variation is used to determine the total cost if you know the unit cost.

STANDARDIZED TEST PRACTICE

51. ACT/SAT Suppose b varies inversely as the square of a. If a is multiplied by 9, which of the following is true for the value of b?

A It is multiplied by $\frac{1}{3}$.

B It is multiplied by $\frac{1}{9}$.

C It is multiplied by $\frac{1}{81}$.

D It is multiplied by 3.

52. REVIEW If $ab = 1$ and a is less than 0, which of the following statements cannot be true?

F b is negative.

G b is less than a.

H As a increases, b decreases.

J As a increases, b increases.

Spiral Review

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function. (Lesson 8-3)

53. $f(x) = \dfrac{x+1}{x^2-1}$ **54.** $f(x) = \dfrac{x+3}{x^2+x-12}$ **55.** $f(x) = \dfrac{x^2+4x+3}{x+3}$

Simplify each expression. (Lesson 8-2)

56. $\dfrac{3x}{x-y} + \dfrac{4x}{y-x}$ **57.** $\dfrac{t}{t+2} - \dfrac{2}{t^2-4}$ **58.** $\dfrac{m-\frac{1}{m}}{1+\frac{4}{m}-\frac{5}{m^2}}$

59. BIOLOGY One estimate for the number of cells in the human body is 100,000,000,000,000. Write this number in scientific notation. (Lesson 6-1)

State the slope and the y-intercept of the graph of each equation. (Lesson 2-4)

60. $y = 0.4x + 1.2$ **61.** $2y = 6x + 14$ **62.** $3x + 5y = 15$

GET READY for the Next Lesson

PREREQUISITE SKILL Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

63. $h(x) = \dfrac{2}{3}$ **64.** $g(x) = 3|x|$ **65.** $f(x) = [\![2x]\!]$

66. $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x \le 0 \end{cases}$ **67.** $h(x) = |x-2|$ **68.** $g(x) = -3$

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CHAPTER 8

Mid-Chapter Quiz

Lessons 8-1 through 8-4

Simplify each expression. (Lesson 8-1)

1. $\frac{t^2 - t - 6}{t^2 - 6t + 9}$

2. $\frac{3ab^3}{8a^2b} \cdot \frac{4ac}{9b^4}$

3. $\frac{-4}{8x} \div \frac{16}{2xy^2}$

4. $\frac{48}{6a + 42} \cdot \frac{7a + 49}{16}$

5. $\frac{w^2 + 5w + 4}{6} \div \frac{w + 1}{18w + 24}$

6. $\frac{\frac{x^2 + x}{x + 1}}{\frac{x}{x - 1}}$

7. MULTIPLE CHOICE For all $t \neq 5$, $\frac{t^2 - 25}{3t - 15} =$ (Lesson 8-2)

A $\frac{t - 5}{3}$.

B $\frac{t + 5}{3}$.

C $t - 5$.

D $t + 5$.

Simplify each expression. (Lesson 8-2)

8. $\frac{4a + 2}{a + b} + \frac{1}{-b - a}$

9. $\frac{2x}{5ab^3} + \frac{4y}{3a^2b^2}$

10. $\frac{5}{n + 6} - \frac{4}{n - 1}$

11. $\frac{x - 5}{2x - 6} - \frac{x - 7}{4x - 12}$

For Exercises 12–14, use the following information.

Lucita is going to a beach 100 miles away. She travels half the distance at one rate. The rest of the distance, she travels 15 miles per hour slower. (Lesson 8-2)

12. If x represents the faster pace in miles per hour, write an expression that represents the time spent at that pace.

13. Write an expression for the amount of time spent at the slower pace.

14. Write an expression for the amount of time Lucita needed to complete the trip.

Graph each rational function. (Lesson 8-3)

15. $f(x) = \frac{x - 1}{x - 4}$

16. $f(x) = \frac{-2}{x^2 - 6x + 9}$

17. MULTIPLE CHOICE What is the range of the function $y = \frac{x^2 + 8}{2}$? (Lesson 8-3)

F $\{y | y \neq \pm 2\sqrt{2}\}$

G $\{y | y \geq 4\}$

H $\{y | y \geq 0\}$

J $\{y | y \leq 0\}$

WORK For Exercises 18 and 19, use the following information. (Lesson 8-3)

Andy is a new employee at Quick Oil Change. The company's goal is to change every customer's oil in 10 minutes. So far, he has changed 13 out of 20 customers' oil in 10 minutes. Suppose Andy changes the next x customers' oil in 10 minutes. His 10-minute oil changing percentage can be determined using $P(x) = \frac{13 + x}{20 + x}$.

18. Graph the function.

19. What domain and range values are meaningful in the context of the problem?

Find each value. (Lesson 8-4)

20. If y varies inversely as x and $x = 14$ when $y = 7$, find x when $y = 2$.

21. If y varies directly as x and $y = 1$ when $x = 5$, find y when $x = 22$.

22. If y varies jointly as x and z and $y = 80$ when $x = 25$ and $z = 4$, find y when $x = 20$ and $z = 7$.

For Exercises 23–25, use the following information.

In order to remain healthy, a horse requires 10 pounds of hay a day. (Lesson 8-4)

23. Write an equation to represent the amount of hay needed to sustain x horses for d days.

24. Is your equation a *direct*, *joint*, or *inverse* variation? Explain.

25. How much hay do three horses need for the month of July?

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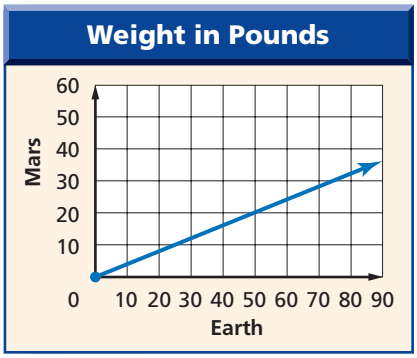
8-5 Classes of Functions

Main Ideas

- Identify graphs as different types of functions.
- Identify equations as different types of functions.

GET READY for the Lesson

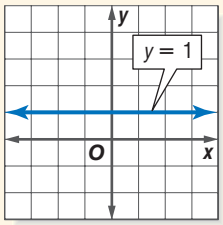
The purpose of the Mars Exploration Program is to study conditions on Mars. The findings will help NASA prepare for a possible mission with human explorers. The graph at the right compares a person's weight on Earth with his or her weight on Mars. This graph represents a direct variation, which you studied in the previous lesson.



Identify Graphs In this book, you have studied several types of graphs representing special functions. The following is a summary of these graphs.

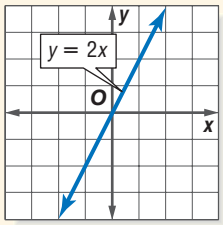
CONCEPT SUMMARY *Special Functions*

Constant Function



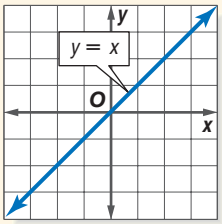
The general equation of a constant function is $y = a$, where a is any number. Its graph is a horizontal line that crosses the y-axis at a.

Direct Variation Function



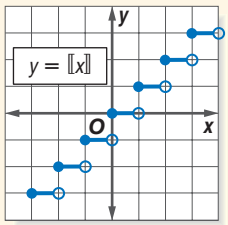
The general equation of a direct variation function is $y = ax$, where a is a nonzero constant. Its graph is a line that passes through the origin and is neither horizontal nor vertical.

Identity Function



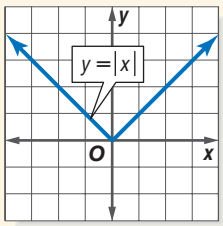
The identity function $y = x$ is a special case of the direct variation function in which the constant is 1. Its graph passes through all points with coordinates (a, a).

Greatest Integer Function



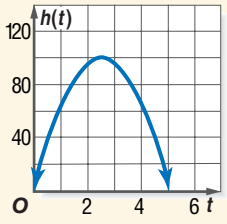
If an equation includes an expression inside the greatest integer symbol, the function is a greatest integer function. Its graph looks like steps.

Absolute Value Function



An equation with the independent variable inside absolute value symbols is an absolute value function. Its graph is in the shape of a V.

Quadratic Function



The general equation of a quadratic function is $y = ax^2 + bx + c$, where $a \neq 0$. Its graph is a parabola.

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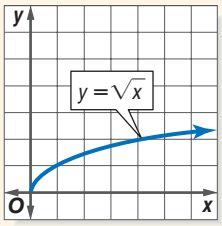
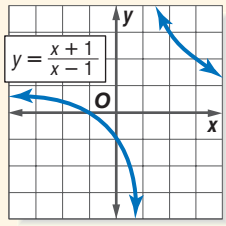
CONCEPT SUMMARY — Special Functions

Square Root Function



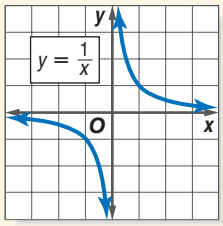
If an equation includes the independent variable inside the radical sign, the function is a square root function. Its graph is a curve that starts at a point and continues in only one direction.

Rational Function



The general equation for a rational function is $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions. Its graph may have one or more asymptotes and/or holes.

Inverse Variation Function



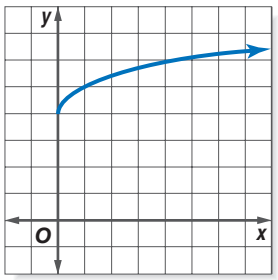
The inverse variation function $y = \frac{a}{x}$ is a special case of the rational function where $p(x)$ is a constant and $q(x) = x$. Its graph has two asymptotes, $x = 0$ and $y = 0$.

You can use the shape of the graphs of each type of function to identify the type of function that is represented by a given graph. To do so, keep in mind the graph of the parent function of each function type.

EXAMPLE 1 Identify a Function Given the Graph

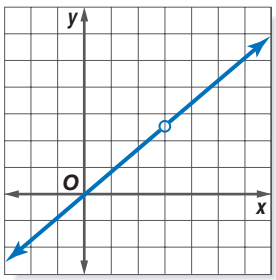
Identify the type of function represented by each graph.

a.



The graph has a starting point and curves in one direction. The graph represents a square root function.

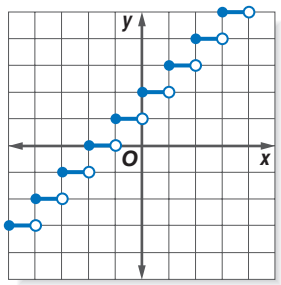
b.



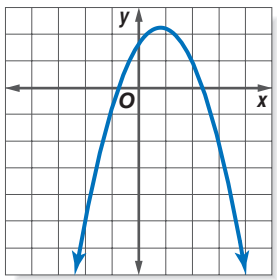
The graph appears to be a direct variation since it is a straight line passing through the origin. However, the hole indicates that it represents a rational function.

CHECK Your Progress

1A.



1B.



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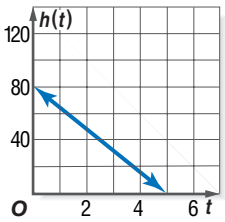
Identify Equations If you can identify an equation as a type of function, you can determine the shape of the graph.

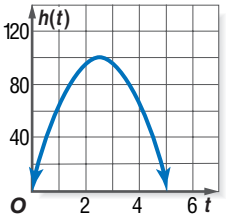
EXAMPLE Match Equation with Graph

2 **ROCKETRY** **Emily launched a toy rocket from ground level. The height above the ground level h, in feet, after t seconds is given by the formula $h(t) = -16t^2 + 80t$. Which graph depicts the height of the rocket during its flight?**

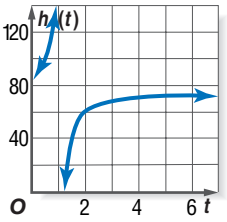
a.



b.



c.



The function includes a second-degree polynomial. Therefore, it is a quadratic function, and its graph is a parabola. Graph **b** is on the only parabola. Therefore, the answer is graph **b**.

CHECK Your Progress

2. Which graph above could represent an elevator moving from a height of 80 feet to ground level in 5 seconds?

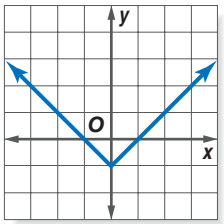
Sometimes recognizing an equation as a specific type of function can help you graph the function.

EXAMPLE Identify a Function Given its Equation

3 **Identify the type of function represented by each equation. Then graph the equation.**

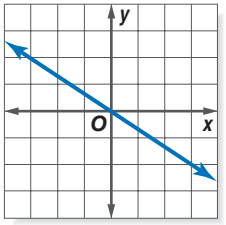
a. $y = |x| - 1$

Since the equation includes an expression inside absolute value symbols, it is an absolute value function. Therefore, the graph will be in the shape of a V. Plot some points and graph the absolute value function.



b. $y = -\frac{2}{3}x$

The function is in the form $y = ax$, where $a = -\frac{2}{3}$. Therefore, it is a direct variation function. The graph passes through the origin and has a slope of $-\frac{2}{3}$.



CHECK Your Progress

3A. $y = [\![x - 1]\!]$

3B. $y = \frac{-1}{x + 1}$

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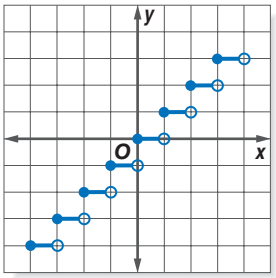
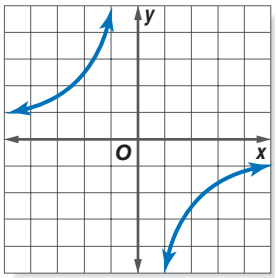
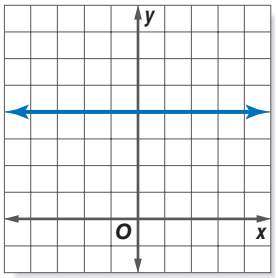
CHECK Your Understanding

Example 1 (p. 474)

Identify the type of function represented by each graph.

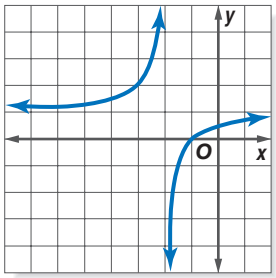
1. 

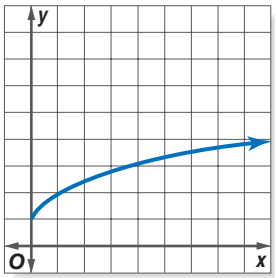
2. 

3. 

Example 2 (p. 475)

Match each graph with an equation at the right.

4. 

5. 

a. $y = x^2 + 2x + 3$
b. $y = \sqrt{x} + 1$
c. $y = \dfrac{x+1}{x+2}$
d. $y = [\![2x]\!]$

6. **GEOMETRY** Write the equation for the area of a circle. Identify the equation as a type of function. Describe the graph of the function.

Example 3 (p. 475)

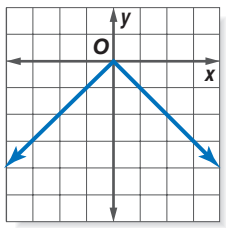
Identify the type of function represented by each equation. Then graph the equation.

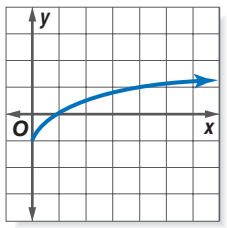
7. $y = x$
8. $y = -x^2 + 2$
9. $y = |x + 2|$

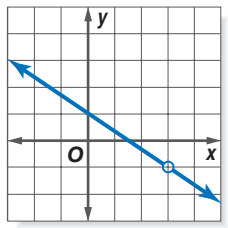
Exercises

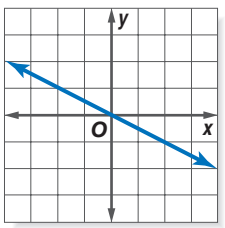
HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16–23	3
24–31	2

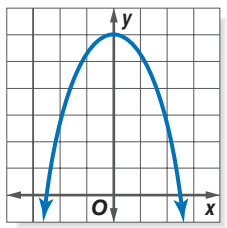
Identify the function represented by each graph.

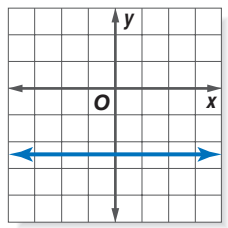
10. 

11. 

12. 

13. 

14. 

15. 

Identify the type of function represented by each equation. Then graph the equation.

16. $y = -1.5$
17. $y = 2.5x$
18. $y = \sqrt{9x}$
19. $y = \dfrac{4}{x}$
20. $y = \dfrac{x^2 - 1}{x - 1}$
21. $y = 3[\![x]\!]$
22. $y = |2x|$
23. $y = 2x^2$

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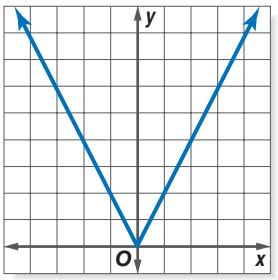
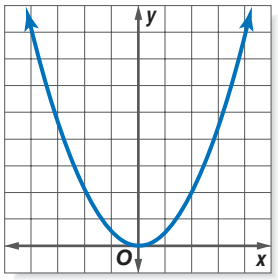
Match each graph with an equation at the right.

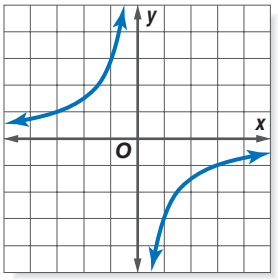
24.



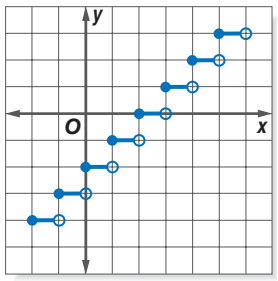
25.



26.



27.



a. $y = [\![x]\!] - 2$
b. $y = 2|x|$
c. $y = 2\sqrt{x}$
d. $y = -3x$
e. $y = 0.5x^2$
f. $y = -\frac{3}{x+1}$
g. $y = -\frac{3}{x}$

Real-World Link

When the Hope Diamond was shipped from New York to the Smithsonian Institution in Washington, D.C., it was mailed in a plain brown paper package.

Source: usps.com

HEALTH **For Exercises 28–30, use the following information.**
A woman painting a room will burn an average of 4.5 Calories per minute.

28. Write an equation for the number of Calories burned in m minutes.

29. Identify the equation in Exercise 28 as a type of function.

30. Describe the graph of the function.

31. ARCHITECTURE The shape of the Gateway Arch of the Jefferson National Expansion Memorial in St. Louis, Missouri, resembles the graph of the function $f(x) = -0.00635x^2 + 4.0005x - 0.07875$, where x is in feet. Describe the shape of the Gateway Arch.

MAIL **For Exercises 32 and 33, use the following information.**
In 2006, the cost to mail a first-class letter was 39¢ for any weight up to and including 1 ounce. Each additional ounce or part of an ounce added 24¢ to the cost.

32. Make a graph showing the postal rates to mail any letter from 0 to 8 ounces.

33. Compare your graph in Exercise 32 to the graph of the greatest integer function.

EXTRA PRACTICE
See pages 909, 933.

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H.O.T. Problems

34. OPEN ENDED Find a counterexample to the statement *All functions are continuous*. Describe your function.

35. CHALLENGE Identify each table of values as a type of function.

a.

x	$f(x)$
−5	7
−3	5
−1	3
0	2
1	3
3	5
5	7
7	9

b.

x	$f(x)$
−5	24
−3	8
−1	0
0	−1
1	0
3	8
5	24
7	48

c.

x	$f(x)$
−1.3	−1
−1.7	−1
0	1
0.8	1
0.9	1
1	2
1.5	2
2.3	3

d.

x	$f(x)$
−5	undefined
−3	undefined
−1	undefined
0	0
1	1
4	2
9	3
16	4

Terry Smith Images/Alamy Images
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36. CHALLENGE Without graphing either function, explain how the graph of $y = [\![x + 2]\!] - 3$ is related to the graph of $y = [\![x + 1]\!] - 1$.

37. *Writing in Math* Use the information on page 473 to explain how the graph of a function can be used to determine the type of relationship that exists between the quantities represented by the domain and the range.

STANDARDIZED TEST PRACTICE

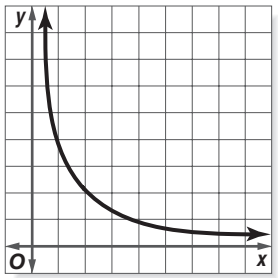
38. ACT/SAT The curve below could be part of the graph of which function?

A $y = \sqrt{x}$

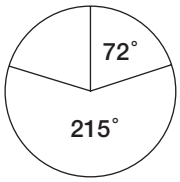
B $y = x^2 - 5x + 4$

C $xy = 4$

D $y = -x + 20$



39. REVIEW A paper plate with a 12-inch diameter is divided into 3 sections.



What is the approximate length of the arc of the largest section?

F 20.3 inches

G 22.5 inches

H 24.2 inches

J 26.5 inches

Spiral Review

40. If x varies directly as y and $y = \frac{1}{5}$ when $x = 11$, find x when $y = \frac{2}{5}$. (Lesson 8-4)

Graph each rational function. (Lesson 8-3)

41. $f(x) = \frac{3}{x + 2}$

42. $f(x) = \frac{8}{(x - 1)(x + 3)}$

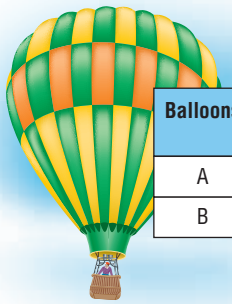
43. $f(x) = \frac{x^2 - 5x + 4}{x - 4}$

Solve each equation by factoring. (Lesson 5-2)

44. $x^2 + 6x + 8 = 0$

45. $2q^2 + 11q = 21$

HOT-AIR BALLOONS **For Exercises 46 and 47, use the table.** (Lesson 3-2)



Balloons	Distance from Ground (m)	Rate of Ascension (m/min)
A	60	15
B	40	20

46. If both balloons are launched at the same time, how long will it take for them to be the same distance from the ground?

47. What is the distance of the balloons from the ground at that time?

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the LCM of each set of polynomials.** (Lesson 8-2)

48. $15ab^2c, 6a^3, 4bc^2$

49. $9x^3, 5xy^2, 15x^2y^3$

50. $5d - 10, 3d - 6$

51. $x^2 - y^2, 3x + 3y$

52. $a^2 - 2a - 3, a^2 - a - 6$

53. $2t^2 - 9t - 5, t^2 + t - 30$

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8-6 Solving Rational Equations and Inequalities

Main Ideas

- Solve rational equations.
- Solve rational inequalities.

New Vocabulary

rational equation

rational inequality

GET READY for the Lesson

A music download service advertises downloads for \$1 per song. The service also charges a monthly access fee of \$15. If a customer downloads x songs in one month, the bill in dollars will be $15 + x$. The actual cost per song is $\frac{15 + x}{x}$. To find how many songs a person would need to download to make the actual cost per song \$1.25, you would need to solve the equation $\frac{15 + x}{x} = 1.25$.

Solve Rational Equations The equation $\frac{15 + x}{x} = 6$ is an example of a rational equation. In general, any equation that contains one or more rational expressions is called a **rational equation**.

Rational equations are easier to solve if the fractions are eliminated. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD). Remember that when you multiply each side by the LCD, each term on each side must be multiplied by the LCD.

EXAMPLE **Solve a Rational Equation**

1 **Solve $\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$. Check your solution.**

The LCD for the terms is $28(z + 2)$.

$$\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$$ Original equation

$$28(z + 2)\left(\frac{9}{28} + \frac{3}{z + 2}\right) = 28(z + 2)\left(\frac{3}{4}\right)$$ Multiply each side by $28(z + 2)$.

$$\overset{1}{\cancel{28}}(z + 2)\left(\frac{9}{\underset{1}{\cancel{28}}}\right) + 28(\cancel{z + 2})\left(\frac{3}{\underset{1}{\cancel{z + 2}}}\right) = \overset{7}{\cancel{28}}(z + 2)\left(\frac{3}{\underset{1}{\cancel{4}}}\right)$$ Distributive Property

$$(9z + 18) + 84 = 21z + 42$$ Simplify.

$$9z + 102 = 21z + 42$$ Simplify.

$$60 = 12z$$ Subtract $9z$ and 42 from each side.

$$5 = z$$ Divide each side by 12.

(continued on the next page)

Hugh Threlfall/Alamy Images

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CHECK

$$\frac{9}{28}+\frac{3}{z+2}=\frac{3}{4}$$ Original equation

$$\frac{9}{28}+\frac{3}{5+2}\stackrel{?}{=}\frac{3}{4}$$ $z = 5$

$$\frac{9}{28}+\frac{3}{7}\stackrel{?}{=}\frac{3}{4}$$ Simplify.

$$\frac{9}{28}+\frac{12}{28}\stackrel{?}{=}\frac{3}{4}$$ Simplify.

$$\frac{3}{4}=\frac{3}{4}\ \checkmark$$ The solution is correct.

CHECK Your Progress

Solve each equation. Check your solution.

1A. $\frac{5}{6}+\frac{2}{x-6}=\frac{1}{2}$ **1B.** $\frac{7}{12}+\frac{9}{x-4}=\frac{55}{48}$

When solving a rational equation, any possible solution that results in a zero in the denominator must be excluded from your list of solutions.

EXAMPLE Elimination of a Possible Solution

2 **Solve $r+\frac{r^2-5}{r^2-1}=\frac{r^2+r+2}{r+1}$. Check your solution.**

$$r+\frac{r^2-5}{r^2-1}=\frac{r^2+r+2}{r+1}$$ Original equation; the LCD is (r^2-1).

$$(r^2-1)\left(r+\frac{r^2-5}{r^2-1}\right)=(r^2-1)\left(\frac{r^2+r+2}{r+1}\right)$$ Multiply each side by the LCD.

$$(r^2-1)r+(\cancel{r^2}^{\,1}-1)\left(\frac{r^2-5}{\cancel{r^2-1}_{\,1}}\right)=(\cancel{r^2}-1)\left(\frac{r^2+r+2}{\cancel{r+1}_{\,1}}\right)$$ Distributive Property

$$(r^3-r)+(r^2-5)=(r-1)(r^2+r+2)$$ Simplify.

$$r^3+r^2-r-5=r^3+r-2$$ Simplify.

$$r^2-2r-3=0$$ Subtract (r^3+r-2) from each side.

$$(r-3)(r+1)=0$$ Factor.

$$r-3=0 \quad \text{or} \quad r+1=0$$ Zero Product Property

$$r=3 \qquad r=-1$$

CHECK

$$r+\frac{r^2-5}{r^2-1}=\frac{r^2+r+2}{r+1}$$

$$3+\frac{3^2-5}{3^2-1}\stackrel{?}{=}\frac{3^2+3+2}{3+1}$$

$$3+\frac{4}{8}\stackrel{?}{=}\frac{14}{4}$$

$$\frac{7}{2}=\frac{7}{2}\ \checkmark$$

$$r+\frac{r^2-5}{r^2-1}=\frac{r^2+r+2}{r+1}$$

$$-1+\frac{(-1)^2-5}{(-1)^2-1}\stackrel{?}{=}\frac{(-1)^2+(-1)+2}{-1+1}$$

$$-1+\frac{-4}{0}\stackrel{?}{=}\frac{2}{0}$$

Since $r = -1$ results in a zero in the denominator, eliminate -1 from the list of solutions. The solution is 3.

Study Tip

Extraneous Solutions

Multiplying each side of an equation by the LCD of rational expressions can yield results that are not solutions of the original equation. These solutions are called *extraneous solutions*.

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CHECK Your Progress **Solve each equation. Check your solution.**

2A. $\frac{2}{r+1} - \frac{1}{r-1} = \frac{-2}{r^2-1}$ **2B.** $\frac{7n}{3n+3} - \frac{5}{4n-4} = \frac{3n}{2n+2}$

Real-World Link

The Loetschberg Tunnel is 21 miles long. It was created for train travel and cut travel time between the locations in half.

Source: usatoday.com

Real-World EXAMPLE

3 **TUNNELS** **The Loetschberg tunnel was built to connect Bern, Switzerland, with the ski resorts in the southern Swiss Alps. The Swiss used one company that started at the north end and another company that started at the south end. Suppose the company at the north end could drill the entire tunnel in 22.2 years and the south company could do it in 21.8 years. How long would it have taken the two companies to drill the tunnel?**

In 1 year, the north company could complete $\frac{1}{22.2}$ of the tunnel.

In 2 years, the north company could complete $\frac{1}{22.2} \cdot 2$ or $\frac{2}{22.2}$ of the tunnel.

In t years, the north company could complete $\frac{1}{22.2} \cdot t$ or $\frac{t}{22.2}$ of the tunnel.

Likewise, in t years, the south company could complete $\frac{1}{21.8} \cdot t$ or $\frac{t}{21.8}$ of the tunnel.

Together, they completed the whole tunnel.

Part completed by the north company	plus	part completed by the south company	equals	entire tunnel.
$\frac{t}{22.2}$	$+$	$\frac{t}{21.8}$	$=$	1

$$\frac{t}{22.2} + \frac{t}{21.8} = 1 \quad \text{Original equation}$$

$$483.96\left(\frac{t}{22.2} + \frac{t}{21.8}\right) = 483.96(1) \quad \text{Multiply each side by 483.96.}$$

$$21.8t + 22.2t = 483.96 \quad \text{Simplify.}$$

$$44t = 483.96 \quad \text{Simplify.}$$

$$t \approx 11 \quad \text{Divide each side by 44.}$$

It would have taken about 11 years to build the tunnel.

This answer is reasonable. Working alone, either company could have drilled the tunnel in about 22 years. Working together, they must be able to do it in about half that time.

CHECK Your Progress

3. WORK Breanne and Owen paint houses together. If Breanne can paint a particular house in 6 days and Owen can paint the same house in 5 days, how long would it take the two of them if they work together?

Online Personal Tutor at algebra2.com

Yoshiko Kusano/AP/Wide World Photos
www.MathSchoolinternational.com

Rate problems frequently involve rational equations.

Real-World EXAMPLE

4 **NAVIGATION** **The speed of the current in the Puget sound is 5 miles per hour. A barge travels 26 miles with the current and returns in $10\frac{2}{3}$ hours. What is the speed of the barge in still water?**

Words The formula that relates distance, time, and rate is $d = rt$ or $\frac{d}{r} = t$.

Variables Let r = the speed of the barge in still water. Then the speed of the barge with the current is $r + 5$, and the speed of the barge against current is $r - 5$.

	Time going with the current	plus	time going against the current	equals	total time.
Equation	$\frac{26}{r+5}$	$+$	$\frac{26}{r-5}$	$=$	$10\frac{2}{3}$

$$\frac{26}{r+5} + \frac{26}{r-5} = 10\frac{2}{3}$$ Original equation

$$3(r^2 - 25)\left(\frac{26}{r+5} + \frac{26}{r-5}\right) = 3(r^2 - 25)10\frac{2}{3}$$ Multiply each side by $3(r^2 - 25)$.

$$3(r^2 - 25)\left(\frac{26}{r+5}\right) + 3(r^2 - 25)\left(\frac{26}{r-5}\right) = 3(r^2 - 25)\left(\frac{32}{3}\right)$$ Distributive Property

$$(78r - 390) + (78r + 390) = 32r^2 - 800$$ Simplify.

$$156r = 32r^2 - 800$$ Simplify.

$$0 = 32r^2 - 156r - 800$$ Subtract $156r$ from each side.

$$0 = 8r^2 - 39r - 200$$ Divide each side by 4.

Use the Quadratic Formula to solve for r.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$r = \frac{-(-39) \pm \sqrt{(-39)^2 - 4(8)(-200)}}{2(8)}$$ $x = r$, $a = 8$, $b = -39$, and $c = -200$

$$r = \frac{39 \pm \sqrt{7921}}{16}$$ Simplify.

$$r = \frac{39 \pm 89}{16}$$ Simplify.

$r = 8$ or -3.125 Simplify.

Since speed must be positive, it is 8 miles per hour. Is this answer reasonable?

Study Tip

Look Back

To review the **Quadratic Formula**, see Lesson 5-6.

CHECK Your Progress

4. SWIMMING The speed of the current in a body of water is 1 mile per hour. Juan swims 2 miles against the current and 2 miles with the current in a total time of $2\frac{2}{3}$ hours. How fast can Juan swim in still water?

Concepts in Motion

Interactive Lab

algebra2.com

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Solve Rational Inequalities Inequalities that contain one or more rational expressions are called **rational inequalities**. To solve rational inequalities, complete the following steps.

Step 1 State the excluded values.

Step 2 Solve the related equation.

Step 3 Use the values determined in Steps 1 and 2 to divide a number line into intervals. Test a value in each interval to determine which intervals contain values that satisfy the original inequality.

EXAMPLE Solve a Rational Inequality

5 Solve $\frac{1}{4a} + \frac{5}{8a} > \frac{1}{2}$.

Step 1 Values that make a denominator equal to 0 are excluded from the domain. For this inequality, the excluded value is 0.

Step 2 Solve the related equation.

$$\frac{1}{4a} + \frac{5}{8a} = \frac{1}{2} \quad \text{Related equation}$$

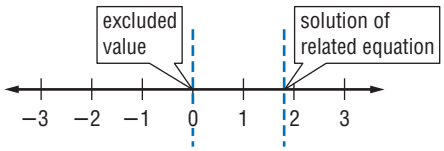
$$8a\left(\frac{1}{4a} + \frac{5}{8a}\right) = 8a\left(\frac{1}{2}\right) \quad \text{Multiply each side by } 8a.$$

$$2 + 5 = 4a \quad \text{Simplify.}$$

$$7 = 4a \quad \text{Add.}$$

$$1\frac{3}{4} = a \quad \text{Divide each side by 4.}$$

Step 3 Draw vertical lines at the excluded value and at the solution to separate the number line into intervals.



Now test a sample value in each interval to determine if the values in the interval satisfy the inequality.

Test $a = -1$.

$$\frac{1}{4(-1)} + \frac{5}{8(-1)} \stackrel{?}{>} \frac{1}{2}$$

$$-\frac{1}{4} - \frac{5}{8} \stackrel{?}{>} \frac{1}{2}$$

$$-\frac{7}{8} \not> \frac{1}{2}$$

Test $a = 1$.

$$\frac{1}{4(1)} + \frac{5}{8(1)} \stackrel{?}{>} \frac{1}{2}$$

$$\frac{1}{4} + \frac{5}{8} \stackrel{?}{>} \frac{1}{2}$$

$$\frac{7}{8} > \frac{1}{2} \checkmark$$

Test $a = 2$.

$$\frac{1}{4(2)} + \frac{5}{8(2)} \stackrel{?}{>} \frac{1}{2}$$

$$\frac{1}{8} + \frac{5}{16} \stackrel{?}{>} \frac{1}{2}$$

$$\frac{7}{16} \not> \frac{1}{2}$$

$a < 0$ is *not* a solution. $0 < a < 1\frac{3}{4}$ is a solution. $a > 1\frac{3}{4}$ is *not* a solution.

The solution is $0 < a < 1\frac{3}{4}$.

CHECK Your Progress

Solve each inequality.

5A. $\frac{1}{3b} - \frac{2}{5b} < \frac{1}{15}$

5B. $1 + \frac{5}{x-1} \leq \frac{7}{6}$

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CHECK Your Understanding

Example 1 (pp. 479–480)

Solve each equation. Check your solutions.

1. $\frac{2}{d} + \frac{1}{4} = \frac{11}{12}$
2. $t + \frac{12}{t} - 8 = 0$
3. $\frac{1}{x-1} + \frac{2}{x} = 0$
4. $\frac{12}{v^2 - 16} - \frac{24}{v-4} = 3$

Example 2 (p. 480)

5. $\frac{w}{w-1} + w = \frac{4w-3}{w-1}$
6. $\frac{4n^2}{n^2-9} - \frac{2n}{n+3} = \frac{3}{n-3}$

Examples 3, 4 (pp. 481, 482)

7. **WORK** A worker can powerwash a wall of a certain size in 5 hours. Another worker can do the same job in 4 hours. If the workers work together, how long would it take to do the job? Determine whether your answer is reasonable.

Example 5 (p. 483)

Solve each inequality.

8. $\frac{4}{c+2} > 1$
9. $\frac{1}{3v} + \frac{1}{4v} < \frac{1}{2}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16, 17	2
18–21	5
22, 23	3, 4

Solve each equation or inequality. Check your solutions.

10. $\frac{y}{y+1} = \frac{2}{3}$
11. $\frac{p}{p-2} = \frac{2}{5}$
12. $s + 5 = \frac{6}{s}$
13. $a + 1 = \frac{6}{a}$
14. $\frac{9}{t-3} = \frac{t-4}{t-3} + \frac{1}{4}$
15. $\frac{5}{x+1} - \frac{1}{3} = \frac{x+2}{x+1}$
16. $\frac{2}{y+2} - \frac{y}{2-y} = \frac{y^2+4}{y^2-4}$
17. $\frac{1}{d+4} = \frac{2}{d^2+3d-4} - \frac{1}{1-d}$
18. $\frac{7}{a+1} > 7$
19. $\frac{10}{m+1} > 5$
20. $5 + \frac{1}{t} > \frac{16}{t}$
21. $7 - \frac{2}{b} < \frac{5}{b}$

22. **NUMBER THEORY** The ratio of 16 more than a number to 12 less than that number is 1 to 3. What is the number?

23. **NUMBER THEORY** The sum of a number and 8 times its reciprocal is 6. Find the number(s).

Solve each equation or inequality. Check your solutions.

24. $\frac{b-4}{b-2} = \frac{b-2}{b+2} + \frac{1}{b-2}$
25. $\frac{1}{n-2} = \frac{2n+1}{n^2+2n-8} + \frac{2}{n+4}$
26. $\frac{2q}{2q+3} - \frac{2q}{2q-3} = 1$
27. $\frac{4}{z-2} - \frac{z+6}{z+1} = 1$
28. $\frac{2}{3y} + \frac{5}{6y} > \frac{3}{4}$
29. $\frac{1}{2p} + \frac{3}{4p} < \frac{1}{2}$

30. **ACTIVITIES** The band has 30 more members than the school chorale. If each group had 10 more members, the ratio of their membership would be 3:2. How many members are in each group?

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Real-World Career

Chemist

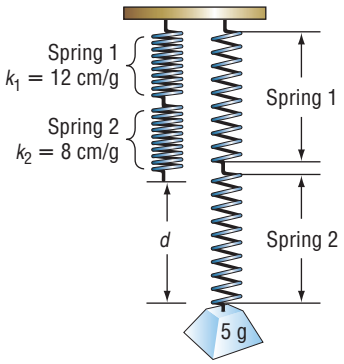
Many chemists work for manufacturers developing products or doing quality control to ensure the products meet industry and government standards.

Math Online

For more information, go to algebra2.com.

PHYSICS **For Exercises 31 and 32, use the following information.**

The distance a spring stretches is related to the mass attached to the spring. This is represented by $d = km$, where d is the distance, m is the mass, and k is the spring constant. When two springs with spring constants k_1 and k_2 are attached in a series, the resulting spring constant k is found by the equation $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$.



31. If one spring with constant of 12 centimeters per gram is attached in a series with another spring with constant of 8 centimeters per gram, find the resultant spring constant.

32. If a 5-gram object is hung from the series of springs, how far will the springs stretch? Is this answer reasonable in this context?

33. **CYCLING** On a particular day, the wind added 3 kilometers per hour to Alfonso's rate when he was cycling with the wind and subtracted 3 kilometers per hour from his rate on his return trip. Alfonso found that in the same amount of time he could cycle 36 kilometers with the wind, he could go only 24 kilometers against the wind. What is his normal bicycling speed with no wind? Determine whether your answer is reasonable.

34. **CHEMISTRY** Kiara adds an 80% acid solution to 5 milliliters of solution that is 20% acid. The function that represents the percent of acid in the resulting solution is $f(x) = \frac{5(0.20) + x(0.80)}{5 + x}$, where x is the amount of 80% solution added. How much 80% solution should be added to create a solution that is 50% acid?

35. **NUMBER THEORY** The ratio of 3 more than a number to the square of 1 more than that number is less than 1. Find the numbers which satisfy this statement.

EXTRA PRACTICE
See pages 909, 933.
Math Online
Self-Check Quiz at algebra2.com

STATISTICS **For Exercises 36 and 37, use the following information.**

A number x is the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.

36. Eight is the harmonic mean of 20 and what number?

37. What is the harmonic mean of 5 and 8?

H.O.T. Problems

38. **OPEN ENDED** Write a rational equation that can be solved by first multiplying each side by $5(a + 2)$.

39. **FIND THE ERROR** Jeff and Dustin are solving $2 - \frac{3}{a} = \frac{2}{3}$. Who is correct? Explain your reasoning.

Jeff
$2 - \frac{3}{a} = \frac{2}{3}$
$6a - 9 = 2a$
$4a = 9$
$a = 2.25$

Dustin
$2 - \frac{3}{a} = \frac{2}{3}$
$2 - 9 = 2a$
$-7 = 2a$
$-3.5 = a$

40. **CHALLENGE** Solve for a if $\frac{1}{a} - \frac{1}{b} = c$.

Keith Wood/Getty Images
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41. ***Writing in Math*** Use the information about music downloads on page 479 to explain how rational equations are used to solve problems involving unit price. Include an explanation of why the actual price per download could never be $1.00.

STANDARDIZED TEST PRACTICE

42. **ACT/SAT** Amanda wanted to determine the average of her 6 test scores. She added the scores correctly to get T, but divided by 7 instead of 6. The result was 12 less than her actual average. Which equation could be used to determine the value of T?

A $6T + 12 = 7T$

B $\frac{T}{7} = \frac{T - 12}{6}$

C $\frac{T}{7} + 12 = \frac{T}{6}$

D $\frac{T}{6} = \frac{T - 12}{7}$

43. **REVIEW**

What is $\frac{10a^{-3}}{29b^4} \div \frac{5a^{-5}}{16b^{-7}}$?

F $\frac{25b^3}{232a^8}$

G $\frac{25}{232a^2b^3}$

H $\frac{32b^3}{29a^8}$

J $\frac{32a^2}{29b^{11}}$

Spiral Review

Identify the type of function represented by each equation. Then graph the equation. (Lesson 8-5)

44. $y = 2x^2 + 1$ **45.** $y = 2\sqrt{x}$ **46.** $y = 0.8x$

47. If y varies inversely as x and $y = 24$ when $x = 9$, find y when $x = 6$. (Lesson 8-4)

Solve each inequality. (Lesson 5-8)

48. $(x + 11)(x - 3) > 0$ **49.** $x^2 - 4x \leq 0$ **50.** $2b^2 - b < 6$

Find each product, if possible. (Lesson 4-3)

51. $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 & -3 \\ 8 & -4 & 9 \end{bmatrix}$ **52.** $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

53. **HEALTH** The prediction equation $y = 205 - 0.5x$ relates a person's maximum heart rate for exercise y and age x. Use the equation to find the maximum heart rate for an 18-year-old. (Lesson 2-5)

Determine the value of r so that a line through the points with the given coordinates has the given slope. (Lesson 2-3)

54. $(r, 2), (4, -6)$; slope $= -\frac{8}{3}$ **55.** $(r, 6), (8, 4)$; slope $= \frac{1}{2}$

56. Evaluate $[(-7 + 4) \times 5 - 2] \div 6$. (Lesson 1-1)

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EXTEND 8-6

Graphing Calculator Lab

Solving Rational Equations and Inequalities with Graphs and Tables

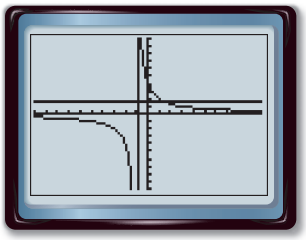
You can use a TI-83/84 graphing calculator to solve rational equations by graphing or by using the table feature. Graph both sides of the equation and locate the point(s) of intersection.

ACTIVITY 1 Solve $\frac{4}{x+1} = \frac{3}{2}$.

Step 1 **Graph each side of the equation.**

Graph each side of the equation as a separate function. Enter $\frac{4}{x+1}$ as Y1 and $\frac{3}{2}$ as Y2. Then graph the two equations.

KEYSTROKES: [Y=] 4 [÷] [(] [X,T,θ,n] [+] 1 [)] [ENTER] 3 [÷] 2 [ZOOM] 6



[−10, 10] scl: 1 by [−10, 10] scl: 1

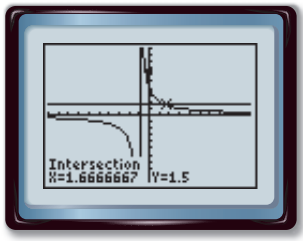
Because the calculator is in connected mode, a vertical line may appear connecting the two branches of the graph. This is not part of the graph.

Step 2 **Use the intersect feature.**

The intersect feature on the [CALC] menu allows you to approximate the ordered pair of the point at which the graphs cross.

KEYSTROKES: [2nd] [CALC] 5

Select one graph and press [ENTER]. Select the other graph, press [ENTER], and press [ENTER] again.



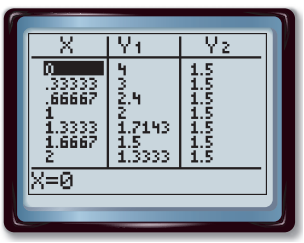
[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution is $1\frac{2}{3}$.

Step 3 **Use the table feature.**

Verify the solution using the table feature. Set up the table to show x-values in increments of $\frac{1}{3}$.

KEYSTROKES: [2nd] [TBLSET] 0 [ENTER] 1 [÷] 3 [ENTER] [2nd] [TABLE]



The table displays x-values and corresponding y-values for each graph. At $x = 1\frac{2}{3}$, both functions have a y-value of 1.5. Thus, the solution of the equation is $1\frac{2}{3}$.

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You can use a similar procedure to solve rational inequalities using a graphing calculator.

ACTIVITY 2 Solve $\frac{3}{x} + \frac{7}{x} > 9$.

Step 1 **Enter the inequalities.**

Rewrite the problem as a system of inequalities.

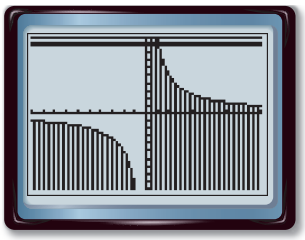
The first inequality is $\frac{3}{x} + \frac{7}{x} > y$ or $y < \frac{3}{x} + \frac{7}{x}$. Since this inequality includes the *less than* symbol, shade below the curve. First, enter the boundary and then use the arrow and ENTER keys to choose the shade below icon, ◣.

The second inequality is $y > 9$. Shade above the curve since this inequality contains *greater than*.

KEYSTROKES: ◀ ◀ ENTER ENTER ENTER ▶ ▶ 3 ÷ X,T,θ,n + 7 ÷ X,T,θ,n ENTER
◀ ◀ ENTER ENTER ▶ ▶ 9 GRAPH

Step 2 **Graph the system.**

KEYSTROKES: GRAPH



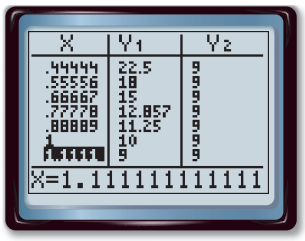
[−10, 10] scl: 1 by [−10, 10] scl: 1

The solution set of the original inequality is the set of x-values of the points in the region where the shadings overlap. Using the calculator's intersect feature, you can conclude that the solution set is $\left\{x \middle| 0 > x > 1\frac{1}{9}\right\}$.

Step 3 **Use the table feature.**

Verify using the table feature. Set up the table to show x-values in increments of $\frac{1}{9}$.

KEYSTROKES: 2nd [TBLSET] 0 ENTER 1 ÷ 9
ENTER 2nd [TABLE]



Scroll through the table. Notice that for x-values greater than 0 and less than $1\frac{1}{9}$, Y1 > Y2. This confirms that the solution of the inequality is $\left\{x \middle| 0 > x > 1\frac{1}{9}\right\}$.

EXERCISES

Solve each equation or inequality.

1. $\frac{1}{x} + \frac{1}{2} = \frac{2}{x}$
2. $\frac{1}{x-4} = \frac{2}{x-2}$
3. $\frac{4}{x} = \frac{6}{x^2}$
4. $\frac{1}{1-x} = 1 - \frac{x}{x-1}$
5. $\frac{1}{x+4} = \frac{2}{x^2+3x-4} - \frac{1}{1-x}$
6. $\frac{1}{x} + \frac{1}{2x} > 5$
7. $\frac{1}{x-1} + \frac{2}{x} < 0$
8. $1 + \frac{5}{x-1} \leq 0$
9. $2 + \frac{1}{x-1} \geq 0$

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CHAPTER 8

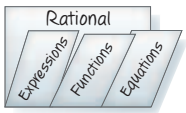
Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Rational Expressions (Lessons 8-1 and 8-2)

- Multiplying and dividing rational expressions is similar to multiplying and dividing fractions.
- To simplify complex fractions, simplify the numerator and the denominator separately, and then simplify the resulting expression.

Direct, Joint, and Inverse Variation

(Lesson 8-4)

- Direct Variation: There is a nonzero constant k such that $y = kx$.
- Joint Variation: There is a number k such that $y = kxz$, where $x \neq 0$ and $z \neq 0$.
- Inverse Variation: There is a nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$.

Classes of Functions (Lesson 8-5)

- The following functions can be classified as special functions: constant function, direct variation function, identity function, greatest integer function, absolute value function, quadratic function, square root function, rational function, inverse variation function.

Rational Equations and Inequalities

(Lesson 8-6)

- Eliminate fractions in rational equations by multiplying each side of the equation by the LCD.
- Possible solutions of a rational equation must exclude values that result in zero in the denominator.

Key Vocabulary

asymptote (p. 457)
complex fraction (p. 445)
constant of variation (p. 465)
continuity (p. 457)
direct variation (p. 465)
inverse variation (p. 467)
joint variation (p. 466)
point discontinuity (p. 457)
rational equation (p. 479)
rational expression (p. 442)
rational function (p. 457)
rational inequality (p. 483)

Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined word or number to make a true sentence.

1. The equation $y = \frac{x^2 - 1}{x + 1}$ has a(n) asymptote at $x = -1$.
2. The equation $y = 3x$ is an example of a(n) direct variation equation.
3. The equation $y = \frac{x^2}{x + 1}$ is a(n) polynomial equation.
4. The graph of $y = \frac{4}{x - 4}$ has a(n) variation at $x = 4$.
5. The equation $b = \frac{2}{a}$ is a(n) inverse variation equation.
6. On the graph of $y = \frac{x - 5}{x + 2}$, there is a break in continuity at $x = 2$.
7. The expression $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ is an example of a complex fraction.
8. In the direct variation $y = 6x$, 6 is the degree.

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CHAPTER 8

Study Guide and Review

Lesson-by-Lesson Review

8–1 Multiplying and Dividing Rational Expressions (pp. 442–449)

Simplify each expression.

9. $\frac{-4ab}{21c} \cdot \frac{14c^2}{22a^2}$

10. $\frac{a^2 - b^2}{6b} \div \frac{a + b}{36b^2}$

11. $\frac{\frac{x^2 + 7x + 10}{x + 2}}{\frac{x^2 + 2x - 15}{x + 2}}$

12. $\frac{\frac{1}{n^2 - 6n + 9}}{\frac{n + 3}{2n^2 - 18}}$

13. $\frac{y^2 - y - 12}{y + 2} \div \frac{y - 4}{y^2 - 4y - 12}$

14. $\frac{x^2 + 3x - 10}{x^2 + 8x + 15} \cdot \frac{x^2 + 5x + 6}{x^2 + 4x + 4}$

15. GEOMETRY A triangle has an area of $2x^2 + 4x - 16$ square meters. If the base is $x - 2$ meters, find the height.

Example 1 Simplify $\frac{3x}{2y} \cdot \frac{8y^3}{6x^2}$.

$$\frac{3x}{2y} \cdot \frac{8y^3}{6x^2} = \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{y}} \cdot y \cdot y}{\underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{y}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{x}} \cdot x}$$

$$= \frac{2y^2}{x}$$

Example 2 Simplify $\frac{p^2 + 7p}{3p} \div \frac{49 - p^2}{3p - 21}$.

$$\frac{p^2 + 7p}{3p} \div \frac{49 - p^2}{3p - 21} = \frac{p^2 + 7p}{3p} \cdot \frac{3p - 21}{49 - p^2}$$

$$= \frac{\overset{1}{\cancel{p}}(7 + p)}{\underset{1}{\cancel{3p}}} \cdot \frac{\overset{1}{\cancel{-3}}(7 - p)}{(\underset{1}{\cancel{7 + p}})(\underset{1}{\cancel{7 - p}})}$$

$$= -1$$

8–2 Adding and Subtracting Rational Expressions (pp. 450–456)

Simplify each expression.

16. $\frac{x + 2}{x - 5} + 6$

17. $\frac{x - 1}{x^2 - 1} + \frac{2}{5x + 5}$

18. $\frac{7}{y} - \frac{2}{3y}$

19. $\frac{7}{y - 2} - \frac{11}{2 - y}$

20. $\frac{3}{4b} - \frac{2}{5b} - \frac{1}{2b}$

21. $\frac{m + 3}{m^2 - 6m + 9} - \frac{8m - 24}{9 - m^2}$

BIOLOGY **For Exercises 22 and 23, use the following information.**

After a person eats something, the pH or acid level A of their mouth can be determined by the formula $A = -\frac{20.4t}{t^2 + 36} + 6.5$, where t is the number of minutes that have elapsed since the food was eaten.

22. Simplify the equation.

23. What would the acid level be after 30 minutes?

Example 3 Simplify $\frac{14}{x + y} - \frac{9x}{x^2 - y^2}$.

$$\frac{14}{x + y} - \frac{9x}{x^2 - y^2} = \frac{14}{x + y} - \frac{9x}{(x + y)(x - y)}$$

$$= \frac{14(x - y)}{(x + y)(x - y)} - \frac{9x}{(x + y)(x - y)}$$

$$= \frac{14(x - y) - 9x}{(x + y)(x - y)}$$ Subtract the numerators.

$$= \frac{14x - 14y - 9x}{(x + y)(x - y)}$$ Distributive Property

$$= \frac{5x - 14y}{(x + y)(x - y)}$$ Simplify.

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Mixed Problem Solving
For mixed problem-solving practice, see page 801.

8–3 Graphing Rational Functions (pp. 457–463)

Graph each rational function.

24. $f(x) = \frac{4}{x-2}$

25. $f(x) = \frac{x}{x+3}$

26. $f(x) = \frac{2}{x}$

27. $f(x) = \frac{x^2+2x+1}{x+1}$

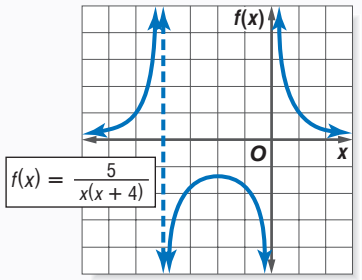
28. $f(x) = \frac{x-4}{x+3}$

29. $f(x) = \frac{5}{(x+1)(x-3)}$

30. SANDWICHES A group makes 45 sandwiches to take on a picnic. The number of sandwiches a person can eat depends on how many people go on the trip. Write and graph a function to illustrate this situation.

Example 4 Graph $f(x) = \frac{5}{x(x+4)}$.

The function is undefined for $x = 0$ and $x = -4$. Since $\frac{5}{x(x+4)}$ is in simplest form, $x = 0$ and $x = -4$ are vertical asymptotes. Draw the two asymptotes and sketch the graph.



8–4 Direct, Joint, and Inverse Variation (pp. 465–471)

31. If y varies directly as x and $y = 21$ when $x = 7$, find x when $y = -5$.

32. If y varies inversely as x and $y = 9$ when $x = 2.5$, find y when $x = -0.6$.

33. If y varies inversely as x and $y = -4$ when $x = 8$, find y when $x = -121$.

34. If y varies jointly as x and z and $x = 2$ and $z = 4$ when $y = 16$, find y when $x = 5$ and $z = 8$.

35. If y varies jointly as x and z and $y = 14$ when $x = 10$ and $z = 7$, find y when $x = 11$ and $z = 8$.

36. EMPLOYMENT Chris's pay varies directly with how many lawns he mows. If his pay is \$65 for 5 yards, find his pay after he has mowed 13 yards.

Example 5 **If y varies inversely as x and $x = 14$ when $y = -6$, find x when $y = -11$.**

$\frac{x_1}{y_2} = \frac{x_2}{y_1}$ Inverse variation

$\frac{14}{-11} = \frac{x_2}{-6}$ $x_1 = 14, y_1 = -6, y_2 = -11$

$14(-6) = -11(x_2)$ Cross multiply.

$-84 = -11x_2$ Simplify.

$7\frac{7}{11} = x_2$ Divide each side by -11.

When $y = -11$, the value of x is $7\frac{7}{11}$.

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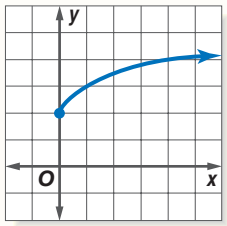
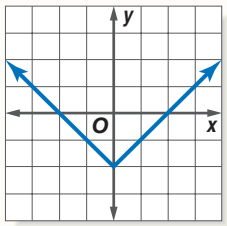
8–5 Classes of Functions (pp. 473–478)

Identify the type of function represented by each graph.

37.

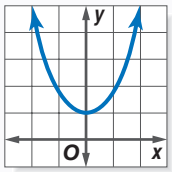


38.



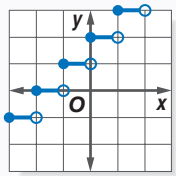
Example 6 **Identify the type of function represented by each graph.**

a.



The graph has a parabolic shape; therefore, it is a quadratic function.

b.



The graph has a stair-step pattern; therefore, it is a greatest integer function.

8–6 Solving Rational Equations and Inequalities (pp. 479–486)

Solve each equation or inequality. Check your solutions.

39. $\frac{3}{y} + \frac{7}{y} = 9$

40. $\frac{3x + 2}{4} = \frac{9}{4} - \frac{3 - 2x}{6}$

41. $\frac{1}{r^2 - 1} = \frac{2}{r^2 + r - 2}$

42. $\frac{x}{x^2 - 1} + \frac{2}{x + 1} = 1 + \frac{1}{2x - 2}$

43. $\frac{1}{3b} - \frac{3}{4b} > \frac{1}{6}$

44. PUZZLES Danielle can put a puzzle together in three hours. Aidan can put the same puzzle together in five hours. How long will it take them if they work together?

Example 7 **Solve** $\frac{1}{x - 1} + \frac{2}{x} = 0$.

The LCD is $x(x - 1)$.

$$\frac{1}{x - 1} + \frac{2}{x} = 0$$

$$x(x - 1)\left(\frac{1}{x - 1} + \frac{2}{x}\right) = x(x - 1)(0)$$

$$x(x - 1)\left(\frac{1}{x - 1}\right) + x(x - 1)\left(\frac{2}{x}\right) = x(x - 1)(0)$$

$$1(x) + 2(x - 1) = 0$$

$$x + 2x - 2 = 0$$

$$3x - 2 = 0$$

$$3x = 2$$

$$x = \frac{2}{3}$$

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CHAPTER 8

Practice Test

Simplify each expression.

1. $\frac{a^2 - ab}{3a} \div \frac{a - b}{15b^2}$

2. $\frac{x^2 - y^2}{y^2} \cdot \frac{y^3}{y - x}$

3. $\frac{x^2 - 2x + 1}{y - 5} \div \frac{x - 1}{y^2 - 25}$

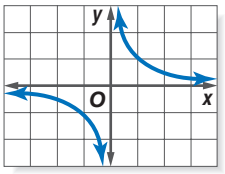
4. $\frac{\frac{x^2 - 1}{x^2 - 3x - 10}}{\frac{x^2 + 3x + 2}{x^2 - 12x + 35}}$

5. $\frac{x - 2}{x - 1} + \frac{6}{7x - 7}$

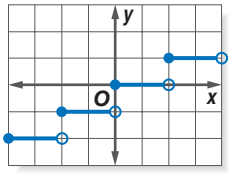
6. $\frac{x}{x^2 - 9} + \frac{1}{2x + 6}$

Identify the type of function represented by each graph.

7.



8.



Graph each rational function.

9. $f(x) = \frac{-4}{x - 3}$

10. $f(x) = \frac{2}{(x - 2)(x + 1)}$

Solve each equation or inequality.

11. $\frac{2}{x - 1} = 4 - \frac{x}{x - 1}$

12. $\frac{9}{28} + \frac{3}{z + 2} = \frac{3}{4}$

13. $5 + \frac{3}{t} > -\frac{2}{t}$

14. $x + \frac{12}{x} - 8 = 0$

15. $\frac{5}{6} - \frac{2m}{2m + 3} = \frac{19}{6}$

16. $\frac{x - 3}{2x} = \frac{x - 2}{2x + 1} - \frac{1}{2}$

17. If y varies inversely as x and $y = 9$ when $x = -\frac{2}{3}$, find x when $y = -7$.

18. If g varies directly as w and $g = 10$ when $w = -3$, find w when $g = 4$.

19. Suppose y varies jointly as x and z. If $x = 10$ when $y = 250$ and $z = 5$, find x when $y = 2.5$ and $z = 4.5$.

20. **AUTO MAINTENANCE** When air is pumped into a tire, the pressure required varies inversely as the volume of the air. If the pressure is 30 pounds per square inch when the volume is 140 cubic inches, find the pressure when the volume is 100 cubic inches.

21. **WORK** Sofia and Julie must pick up all of the apples in the yard so the lawn can be mowed. Working alone, Julie could complete the job in 1.7 hours. Sofia could complete it alone in 2.3 hours. How long will it take them to complete the job when they work together?

ELECTRICITY For Exercises 22 and 23, use the following information.
The current I in a circuit varies inversely with the resistance R.

22. Use the table below to write an equation relating the current and the resistance.

I	0.5	1.0	1.5	2.0	2.5	3.0	5.0
R	12.0	6.0	4.0	3.0	2.4	2.0	1.2

23. What is the constant of variation?

24. **MULTIPLE CHOICE** If $m = \frac{1}{x}$, $n = 7m$, $p = \frac{1}{n}$, $q = 14p$, and $r = \frac{1}{\frac{1}{2}q}$, find x.

A r B q C p D $\frac{1}{r}$

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CHAPTER 8

Standardized Test Practice

Cumulative, Chapters 1–8

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Jamal is putting a stone walkway around a circular pond. He has enough stones to make a walkway 144 feet long. If he uses all of the stones to surround the pond, what is the radius of the pond?

A $\frac{144}{\pi}$ ft

B $\frac{72}{\pi}$ ft

C 144π ft

D 72π ft

2. Hooke's Law states that the force needed to keep a spring stretched x units is directly proportional to x. If a spring is stretched 5 centimeters, and a force of 40 N is required to maintain the spring stretched to 5 centimeters, what force is needed to keep the spring stretched 14 centimeters?

F 8 N

G 19 N

H 112 N

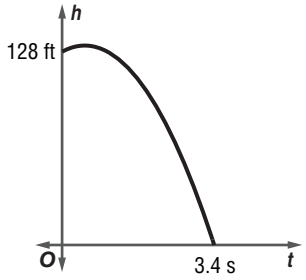
J 1600 N

3. GRIDDABLE Perry drove to the gym at an average rate of 30 miles per hour. It took him 45 minutes. Going home, he took the same route, but drove at a rate of 45 miles per hour. How many miles is it to his house from the gym?

TEST-TAKING TIP

Question 3 When answering questions, make sure you know exactly what the question is asking you to find. For example, if you find the time that it takes him to drive home from the gym in question 3, you have not solved the problem. You need to find the distance the gym is from his home.

4. A ball was thrown upward with an initial velocity of 16 feet per second from the top of a building 128 feet high. Its height h in feet above the ground t seconds later will be $h = 128 + 16t - 16t^2$.



Which is the best conclusion about the ball's action?

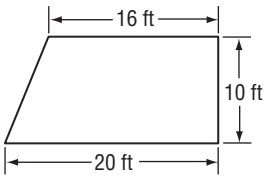
A The ball stayed above 128 feet for more than 3 seconds.

B The ball returned to the ground in less than 4 seconds.

C The ball traveled more slowly going up than it did going down.

D The ball traveled less than 128 feet in 3.4 seconds.

5. Martha is putting a stone walkway around the garden pictured below.



About how many feet of stone are needed?

F 36.0 ft

G 46.0 ft

H 46.8 ft

J 56.8 ft

6. Which of these equations describes a relationship in which every negative real number x corresponds to a nonnegative real number y?

A $y = -x$

B $y = x$

C $y = x^2$

D $y = x^3$

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7. Miller's General Store needs at least 120 of their employees to oppose the building of a new cafeteria in order for the cafeteria not to be built. Miller's employs 1532 people. Jay surveyed a random sample of employees and asked which facility the employees want built.

Survey Results

Facility	Employees
Gym	50
Cafeteria	80
Park	65
Parking Garage	140

Based on the data in the survey, about how many employees are likely to choose building a new cafeteria?

F 77 **H** 230

G 80 **J** 366

8. Mario purchased a pair of shoes that were on sale for $85. The shoes were originally $110. Which expression can be used to determine the percent of the original price that Mario saved on the purchase of his shoes?

A $\frac{85}{100} \times 100$ **C** $\frac{110}{85} \times 100$

B $\frac{110 - 85}{110} \times 100$ **D** $\frac{110 - 85}{85} \times 100$

9. Lisa is 6 years younger than Petra. Stella is twice as old as Petra. The total of their ages is 54. Which equation can be used to find Petra's age?

F $x + (x - 6) + 2(x - 6) = 54$

G $x - 6x + (x + 2) = 54$

H $x - 6 + 2x = 54$

J $x + (x - 6) + 2x = 54$

10. $\angle M$ and $\angle N$ are supplementary angles. If $m\angle M$ is x, which equation can be used to find y, the measure of $\angle N$?

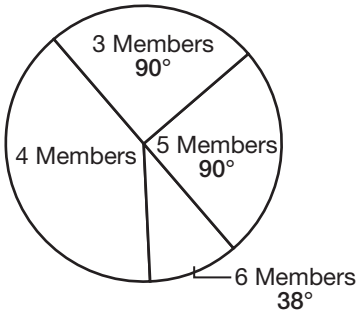
A $y = 90 + x$

B $y = 90 - x$

C $y = 180 - x$

D $y = x + 180$

11. GRIDDABLE The graph that shows the sizes of families in Gretchen's class is shown below. The diameter of the circle is 2 inches.



What is the approximate length in inches of the arc with the section that contains 4 family members? Round to the nearest hundredth.

Pre-AP

Record your answers on a sheet of paper. Show your work.

12. A gear that is 8 inches in diameter turns a smaller gear that is 3 inches in diameter.

a. Does this situation represent a *direct* or *inverse* variation? Explain your reasoning.

b. If the larger gear makes 36 revolutions, how many revolutions does the smaller gear make in that time?

NEED EXTRA HELP?												
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12
Go to Lesson...	8-4	8-4	8-4	5-7	1-4	2-4	12-1	1-3	2-4	1-3	10-3	8-4

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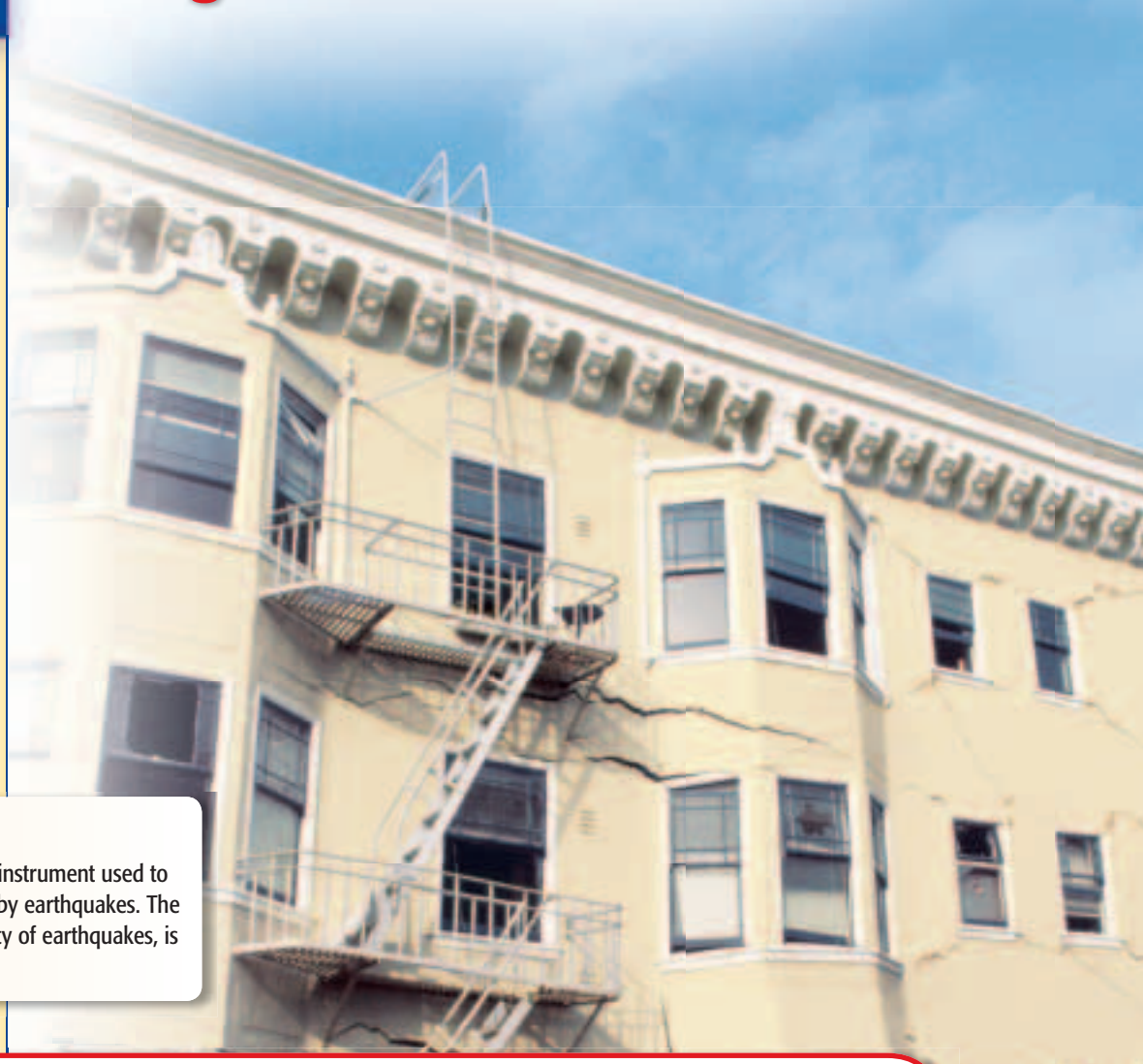
CHAPTER 9

Exponential and Logarithmic Relations



BIG Ideas

- Simplify exponential and logarithmic expressions.
- Solve exponential and logarithmic equations and inequalitites.
- Solve problems involving exponential growth and decay.

Key Vocabulary

common logarithm (p. 528)
exponential function (p. 499)
logarithm (p. 510)
natural base, *e* (p. 536)
natural logarithm (p. 537)

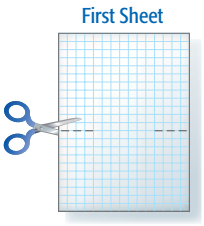
Real-World Link

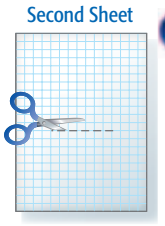
Seismograph A seismograph is an instrument used to detect and record the forces caused by earthquakes. The Richter Scale, which rates the intensity of earthquakes, is a logarithmic scale.

FOLDABLES Study Organizer

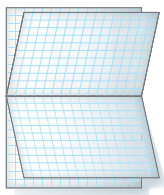
Exponential and Logarithmic Relations Make this Foldable to help you organize your notes. Begin with two sheets of grid paper.

1 **Fold** in half along the width. On the first sheet, cut 5 cm along the fold at the ends. On the second sheet, cut in the center, stopping 5 cm from the ends.





2 **Insert** the first sheet through the second sheet and align the folds. Label the pages with lesson numbers.



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GET READY for Chapter 9

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICK Check

Simplify. Assume that no variable equals zero. (Lesson 6-1)

1. $x^5 \cdot x \cdot x^6$

2. $(3ab^4c^2)^3$

3. $\dfrac{-36x^7y^4z^3}{21x^4y^9z^4}$

4. $\left(\dfrac{4ab^2}{64b^3c}\right)^2$

5. CHEMISTRY The density D of an object in grams per milliliter is found by dividing the mass m of the substance by the volume V of the object. A sample of gold has a mass of 4.2×10^{-2} kilograms and a volume of 2.2×10^{-6} cubic meters. Find the density of gold. (Lesson 6-1)

Find the inverse of each function. Then graph the function and its inverse. (Lesson 7-2)

6. $f(x) = -2x$

7. $f(x) = 3x - 2$

8. $f(x) = -x + 1$

9. $f(x) = \dfrac{x - 4}{3}$

REMODELING For Exercises 10 and 11, use the following information.

Marc is wallpapering a 23-foot by 9-foot wall. The wallpaper costs $11.99 per square yard. The formula $f(x) = 9x$ converts square yards to square feet. (Lesson 7-2)

10. Find the inverse $f^{-1}(x)$. What is the significance of $f^{-1}(x)$?

11. What will the wallpaper cost?

QUICK Review

EXAMPLE 1 **Simplify $\dfrac{(x^2y^3z^4)^2}{x^2x^3y^3y^4z^4z^5}$. Assume that no variable equals zero.**

$$\frac{(x^2y^3z^4)^2}{x^2x^3y^3y^4z^4z^5}$$

$= \dfrac{x^4y^6z^8}{x^5y^7z^9}$ Simplify the numerator by using the Power of a Power Rule and the denominator by using the Product of Powers Rule.

$= \dfrac{1}{xyz}$ or $x^{-1}y^{-1}z^{-1}$ Simplify by using the Quotient of Powers Rule.

EXAMPLE 2 **Find the inverse of $f(x) = 2x + 3$.**

Step 1 Replace $f(x)$ with y in the original equation. $f(x) = 2x + 3 \rightarrow y = 2x + 3$

Step 2 Interchange x and y: $x = 2y + 3$.

Step 3 Solve for y.

$x = 2y + 3$ Inverse

$x - 3 = 2y$ Subtract 3 from each side.

$\dfrac{x - 3}{2} = y$ Divide each side by 2.

$\dfrac{1}{2}x - \dfrac{3}{2} = y$ Simplify.

Step 4 Replace y with $f^{-1}(x)$.

$$y = \frac{1}{2}x - \frac{3}{2} \rightarrow f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$$

Option 2

Math Online Take the Online Readiness Quiz at algebra2.com.

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9-1 Exponential Functions

Main Ideas

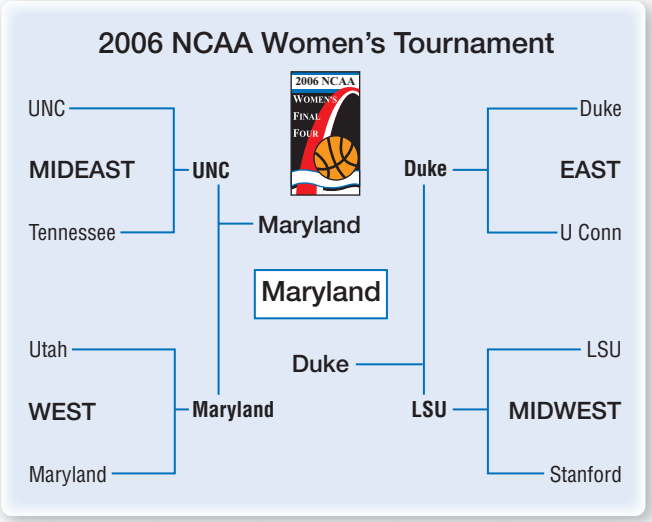
- Graph exponential functions.
- Solve exponential equations and inequalities.

New Vocabulary

exponential function
exponential growth
exponential decay
exponential equation
exponential inequality

GET READY for the Lesson

The NCAA women's basketball tournament begins with 64 teams and consists of 6 rounds of play. The winners of the first round play against each other in the second round. The winners then move from the Sweet Sixteen to the Elite Eight to the Final Four and finally to the Championship Game. The number of teams y that compete in a tournament of x rounds is $y = 2^x$.



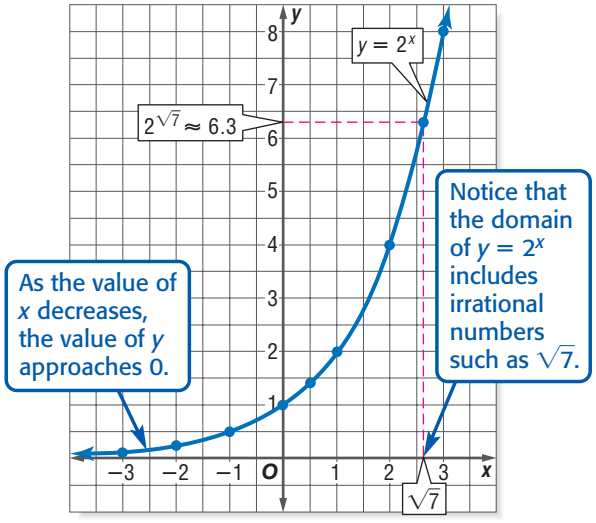
Exponential Functions In an exponential function like $y = 2^x$, the base is a constant, and the exponent is a variable. Let's examine the graph of $y = 2^x$.

EXAMPLE 1 Graph an Exponential Function

Sketch the graph of $y = 2^x$. Then state the function's domain and range.

Make a table of values. Connect the points to sketch a smooth curve.

x	$y = 2^x$
-3	$2^{-3} = \frac{1}{8}$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
$\frac{1}{2}$	$2^{\frac{1}{2}} = \sqrt{2}$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$



The domain is all real numbers, and the range is all positive numbers.

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CHECK Your Progress

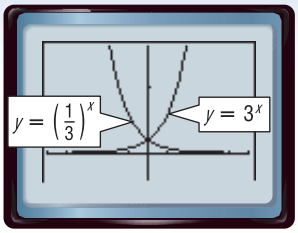
1. Sketch the graph of $y = \left(\frac{1}{2}\right)^x$. Then state the function's domain and range.

You can use a TI-83/84 Plus graphing calculator to look at the graphs of two other exponential functions, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.

GRAPHING CALCULATOR LAB

Families of Exponential Functions

The calculator screen shows the graphs of parent functions $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.



[−5, 5] scl: 1 by [−2, 8] scl: 1

THINK AND DISCUSS

1. How do the shapes of the graphs compare?
2. How do the asymptotes and y-intercepts of the graphs compare?
3. Describe the relationship between the graphs.
4. Graph each group of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.
   a. $y = 2^x$, $y = 3^x$, and $y = 4^x$
   b. $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, and $y = \left(\frac{1}{4}\right)^x$
   c. $y = -3(2)^x$ and $y = 3(2)^x$; $y = -1(2)^x$ and $y = 2^x$.
5. Describe the relationship between the graphs of $y = -1(2)^x$ and $y = 2^x$. Then graph the functions on a graphing calculator to verify your conjecture.

Study Tip

Common Misconception

Be sure not to confuse polynomial functions and exponential functions. While $y = x^3$ and $y = 3^x$ each have an exponent, $y = x^3$ is a polynomial function and $y = 3^x$ is an exponential function.

The Graphing Calculator Lab allowed you to discover many characteristics of the graphs of exponential functions. In general, an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an **exponential function** with base b. Exponential functions have the following characteristics.

Study Tip

Look Back

To review **continuous functions** and **one-to-one functions**, see Lessons 2-1 and 7-2.

1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The x-axis is an asymptote of the graph.
4. The range is the set of all positive numbers if $a > 0$ and all negative numbers if $a < 0$.
5. The graph contains the point $(0, a)$. That is, the y-intercept is a.
6. The graphs of $y = ab^x$ and $y = a\left(\frac{1}{b}\right)^x$ are reflections across the y-axis.

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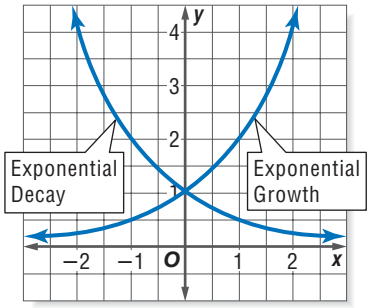
Study Tip

Exponential Growth and Decay

Notice that the graph of an exponential growth function *rises* from left to right. The graph of an exponential decay function *falls* from left to right.

There are two types of exponential functions: **exponential growth** and **exponential decay**. The base of an exponential growth function is a number greater than one. The base of an exponential decay function is a number between 0 and 1.



KEY CONCEPT — ***Exponential Growth and Decay***

Symbols If $a > 0$ and $b > 1$, the function $y = ab^x$ represents exponential growth.

Example If $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents exponential decay.

EXAMPLE Identify Exponential Growth and Decay

2 **Determine whether each function represents exponential *growth* or *decay*.**

a. $y = \left(\frac{1}{5}\right)^x$

The function represents exponential decay, since the base, $\frac{1}{5}$, is between 0 and 1.

b. $y = 7(1.2)^x$

The function represents exponential growth, since the base, 1.2, is greater than 1.

CHECK Your Progress

2A. $y = 2(5)^x$

2B. $y = \left(\frac{2}{3}\right)^x$

Study Tip

Checking Reasonableness

In Example 2, you learned that if $a > 1$ and $b > 1$, then the function represents growth. Here, $a = 1{,}321{,}045$ and $b = 1.002$, and the population representing growth increased.

Exponential functions are frequently used to model the growth or decay of a population. You can use the y-intercept and one other point on the graph to write the equation of an exponential function.

Real-World EXAMPLE Write an Exponential Function

3 **POPULATION** **In 2000, the population of Phoenix was 1,321,045, and it increased to 1,331,391 in 2004.**

a. Write an exponential function of the form $y = ab^x$ that could be used to model the population y of Phoenix. Write the function in terms of x, the number of years since 2000.

For 2000, the time x equals 0, and the initial population y is 1,321,045. Thus, the y-intercept, and value of a, is 1,321,045.

For 2004, the time x equals 2004 – 2000 or 4, and the population y is 1,331,391. Substitute these values and the value of a into an exponential function to approximate the value of b.

$y = ab^x$	Exponential function
$1{,}331{,}391 = 1{,}321{,}045b^4$	Replace x with 4, y with 1,331,391, and a with 1,321,045.
$1.008 \approx b^4$	Divide each side by 1,321,045.
$\sqrt[4]{1.008} \approx b$	Take the 4th root of each side.

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Real-World Link

The first virus that spread via cell phone networks was discovered in June 2004.

Source: internetnews.com

To find the 4th root of 1.008, use selection 5: $\sqrt[x]{\ }$ under the MATH menu on the TI–83/84 Plus.

KEYSTROKES: 4 [MATH] 5 1.008 [ENTER] 1.001994028

An equation that models the population growth of Phoenix from 2000 to 2004 is $y = 1{,}321{,}045(1.002)^x$.

b. Suppose the population of Pheonix continues to increase at the same rate. Estimate the population in 2015.

For 2015, the time equals 2015 – 2000 or 15.

$y = 1{,}321{,}045(1.002)^x$ Modeling equation

$= 1{,}321{,}045(1.002)^{15}$ Replace x with 15.

$\approx 1{,}360{,}262$ Use a calculator.

The population in Phoenix will be about 1,360,262 in 2015.

CHECK Your Progress

3. SPAM In 2003, the amount of annual cell phone spam messages totaled about ten million. In 2005, the total grew exponentially to 500 million. Write an exponential function of the form $y = ab^x$ that could be used to model the increase of spam messages y. Write the function in terms of x, the number of years since 2003. If the number of spam messages continues increasing at the same rate, estimate the annual number of spam messages in 2010.

Online Personal Tutor at algebra2.com

Exponential Equations and Inequalities

Exponential equations are equations in which variables occur as exponents.

KEY CONCEPT *Property of Equality for Exponential Functions*

Symbols If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $2^x = 2^8$, then $x = 8$.

EXAMPLE Solve Exponential Equations

4 **Solve each equation.**

a. $3^{2n+1} = 81$

$3^{2n+1} = 81$ Original equation

$3^{2n+1} = 3^4$ Rewrite 81 as 3^4 so each side has the same base.

$2n + 1 = 4$ Property of Equality for Exponential Functions

$2n = 3$ Subtract 1 from each side.

$n = \frac{3}{2}$ Divide each side by 2.

(continued on the next page)

Pierre Arsenault/Masterfile

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CHECK

$3^{2n+1} = 81$ Original equation

$3^{2\left(\frac{3}{2}\right)+1} \stackrel{?}{=} 81$ Substitute $\frac{3}{2}$ for n.

$3^4 \stackrel{?}{=} 81$ Simplify.

$81 = 81$ ✔ Simplify.

b. $4^{2x} = 8^{x-1}$

$4^{2x} = 8^{x-1}$ Original equation

$(2^2)^{2x} = (2^3)^{x-1}$ Rewrite each side with a base of 2.

$2^{4x} = 2^{3(x-1)}$ Power of a Power

$4x = 3(x-1)$ Property of Equality for Exponential Functions

$4x = 3x - 3$ Distributive Property

$x = -3$ Subtract $3x$ from each side.

CHECK

$4^{2x} = 8^{x-1}$ Original equation

$4^{2(-3)} \stackrel{?}{=} 8^{-3-1}$ Substitute -3 for x.

$4^{-6} \stackrel{?}{=} 8^{-4}$ Simplify.

$\frac{1}{4096} = \frac{1}{4096}$ ✔ Simplify.

CHECK Your Progress

Solve each equation.

4A. $4^{2n-1} = 64$

4B. $5^{5x} = 125^{x+2}$

The following property is useful for solving inequalities involving exponential functions or **exponential inequalities**.

KEY CONCEPT ***Property of Inequality for Exponential Functions***

Symbols If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Example If $5^x < 5^4$, then $x < 4$.

This property also holds true for $\leq$ and $\geq$.

EXAMPLE **Solve Exponential Inequalities**

5 Solve $4^{3p-1} > \frac{1}{256}$.

$4^{3p-1} > \frac{1}{256}$ Original inequality

$4^{3p-1} > 4^{-4}$ Rewrite $\frac{1}{256}$ as $\frac{1}{4^4}$ or 4^{-4} so each side has the same base.

$3p - 1 > -4$ Property of Inequality for Exponential Functions

$3p > -3$ Add 1 to each side.

$p > -1$ Divide each side by 3.

Study Tip

Look Back

You can review **negative exponents** in Lesson 6-1.

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CHECK Test a value of p greater than -1; for example, $p = 0$.

$4^{3p-1} > \frac{1}{256}$ Original inequality

$4^{3(0)-1} \stackrel{?}{>} \frac{1}{256}$ Replace p with 0.

$4^{-1} \stackrel{?}{>} \frac{1}{256}$ Simplify.

$\frac{1}{4} > \frac{1}{256}$ ✔ $a^{-1} = \frac{1}{a}$

CHECK Your Progress

Solve each inequality.

5A. $3^{2x-1} \geq \frac{1}{243}$

5B. $2^{x+2} > \frac{1}{32}$

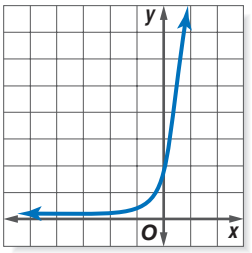
CHECK Your Understanding

Example 1 (p. 498–499)

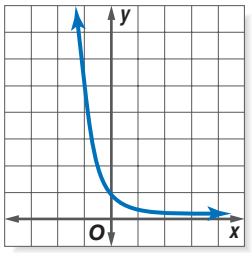
Match each function with its graph.

1. $y = 5^x$ **2.** $y = 2(5)^x$ **3.** $y = \left(\frac{1}{5}\right)^x$

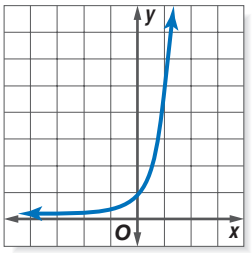
a.



b.



c.



Sketch the graph of each function. Then state the function's domain and range.

4. $y = 3(4)^x$ **5.** $y = 2\left(\frac{1}{3}\right)^x$

Example 2 (p. 500)

Determine whether each function represents exponential *growth* or *decay*.

6. $y = (0.5)^x$ **7.** $y = 0.3(5)^x$

Example 3 (pp. 500–501)

Write an exponential function for the graph that passes through the given points.

8. (0, 3) and (−1, 6) **9.** (0, −18) and (−2, −2)

MONEY **For Exercises 10 and 11, use the following information.**
In 1993, My-Lien inherited $1,000,000 from her grandmother. She invested all of the money, and by 2005, the amount had grown to $1,678,000.

10. Write an exponential function that could be used to model the money y. Write the function in terms of x, the number of years since 1993.

11. Assume that the amount of money continues to grow at the same rate. Estimate the amount of money in 2015. Is this estimate reasonable? Explain your reasoning.

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Example 4 (pp. 501–502)

Solve each equation. Check your solution.

12. $2^{n+4} = \frac{1}{32}$ **13.** $9^{2y-1} = 27^{y}$ **14.** $4^{3x+2} = \frac{1}{256}$

Example 5 (pp. 502–503)

Solve each inequality. Check your solution.

15. $5^{2x+3} \leq 125$ **16.** $3^{3x-2} > 81$ **17.** $4^{4a+6} \leq 16^{a}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
18–21	1
22–27	2
28–38	3
39–44	4
45–48	5

Sketch the graph of each function. Then state the function's domain and range.

18. $y = 2(3)^{x}$ **19.** $y = 5(2)^{x}$

20. $y = 0.5(4)^{x}$ **21.** $y = 4\left(\frac{1}{3}\right)^{x}$

Determine whether each function represents exponential *growth* or *decay*.

22. $y = 10(3.5)^{x}$ **23.** $y = 2(4)^{x}$ **24.** $y = 0.4\left(\frac{1}{3}\right)^{x}$

25. $y = 3\left(\frac{5}{2}\right)^{x}$ **26.** $y = 30^{-x}$ **27.** $y = 0.2(5)^{-x}$

Write an exponential function for the graph that passes through the given points.

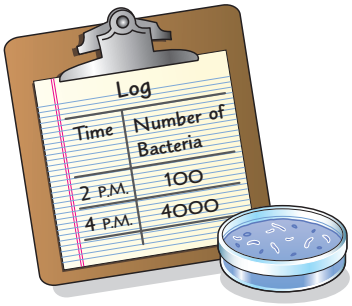
28. (0, −2) and (−2, −32) **29.** (0, 3) and (1, 15)

30. (0, 7) and (2, 63) **31.** (0, −5) and (−3, −135)

32. (0, 0.2) and (4, 51.2) **33.** (0, −0.3) and (5, −9.6)

BIOLOGY **For Exercises 34 and 35, use the following information.**

The number of bacteria in a colony is growing exponentially.



34. Write an exponential function to model the population y of bacteria x hours after 2 P.M.

35. How many bacteria were there at 7 P.M. that day?

Cross-Curricular Project

The magnitude of an earthquake can be represented by an exponential equation. Visit algebra2.com to continue work on your project.

MONEY **For Exercises 36–38, use the following information.**

Suppose you deposit a principal amount of P dollars in a bank account that pays compound interest. If the annual interest rate is r (expressed as a decimal) and the bank makes interest payments n times every year, the amount of money A you would have after t years is given by $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$.

36. If the principal, interest rate, and number of interest payments are known, what type of function is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$? Explain your reasoning.

37. Write an equation giving the amount of money you would have after t years if you deposit \$1000 into an account paying 4% annual interest compounded quarterly (four times per year).

38. Find the account balance after 20 years.

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EXTRA PRACTICE
See pages 909, 934.

MathOnline
Self-Check Quiz at algebra2.com

Solve each equation. Check your solution.

39. $2^{3x+5} = 128$

40. $5^{n-3} = \frac{1}{25}$

41. $\left(\frac{1}{9}\right)^{m} = 81^{m+4}$

42. $\left(\frac{1}{7}\right)^{y-3} = 343$

43. $10^{x-1} = 100^{2x-3}$

44. $36^{2p} = 216^{p-1}$

Solve each inequality. Check your solution.

45. $3^{n-2} > 27$

46. $2^{2n} \leq \frac{1}{16}$

47. $16^{n} < 8^{n+1}$

48. $32^{5p+2} \geq 16^{5p}$

Sketch the graph of each function. Then state the function's domain and range.

49. $y = -\left(\frac{1}{5}\right)^{x}$

50. $y = -2.5(5)^{x}$

Real-World Link

Since computers were invented, computational speed has multiplied by a factor of 4 about every three years.

Source: wired.com

COMPUTERS **For Exercises 51 and 52, use the information at the left.**

51. If a typical computer operates with a computational speed s today, write an expression for the speed at which you can expect an equivalent computer to operate after x three-year periods.

52. Suppose your computer operates with a processor speed of 2.8 gigahertz and you want a computer that can operate at 5.6 gigahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer?

POPULATION **For Exercises 53–55, use the following information.**

Every ten years, the Bureau of the Census counts the number of people living in the United States. In 1790, the population of the U.S. was 3.93 million. By 1800, this number had grown to 5.31 million.

53. Write an exponential function that could be used to model the U.S. population y in millions for 1790 to 1800. Write the equation in terms of x, the number of decades x since 1790.

54. Assume that the U.S. population continued to grow at least that fast. Estimate the population for the years 1820, 1840, and 1860. Then compare your estimates with the actual population for those years, which were 9.64, 17.06, and 31.44 million, respectively.

55. **RESEARCH** Estimate the population of the U.S. in the most recent census. Then use the Internet or other reference to find the actual population of the U.S. in the most recent census. Has the population of the U.S. continued to grow at the same rate at which it was growing in the early 1800s? Explain.

Graphing Calculator

Graph each pair of functions on the same screen. Then compare the graphs, listing both similarities and differences in shape, asymptotes, domain, range, and y-intercepts.

56.

Parent Function	New Function
$y = 2^{x}$	$y = 2^{x} + 3$

57.

Parent Function	New Function
$y = 3^{x}$	$y = 3^{x+1}$

58.

Parent Function	New Function
$y = \left(\frac{1}{5}\right)^{x}$	$y = \left(\frac{1}{5}\right)^{x-2}$

59.

Parent Function	New Function
$y = \left(\frac{1}{4}\right)^{x}$	$y = \left(\frac{1}{4}\right)^{x} - 1$

60. Describe the effect of changing the values of h and k in the equation $y = 2^{x-h} + k$.

H.O.T. Problems

61. **OPEN ENDED** Give an example of a value of b for which $y = b^{x}$ represents exponential decay.

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62. REASONING Identify each function as *linear*, *quadratic*, or *exponential*.

a. $y = 3x^2$ **b.** $y = 4(3)^x$ **c.** $y = 2x + 4$ **d.** $y = 4(0.2)^x + 1$

63. CHALLENGE Decide whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For a positive base b other than 1, $b^x > b^y$ if and only if $x > y$.

64. *Writing in Math* Use the information about women's basketball on page 498 to explain how an exponential function can be used to describe the teams in a tournament. Include an explanation of how you could use the equation $y = 2^x$ to determine the number of rounds of tournament play for 128 teams and an example of an inappropriate number of teams for a tournament.

STANDARDIZED TEST PRACTICE

65. ACT/SAT If $4^{x+2} = 48$, then $4^x =$

A 3.0

B 6.4

C 6.9

D 12.0

66. REVIEW If the equation $y = 3^x$ is graphed, which of the following values of x would produce a point closest to the x-axis?

F $\frac{3}{4}$

G $\frac{1}{4}$

H 0

J $-\frac{3}{4}$

Spiral Review

Solve each equation. Check your solutions. (Lesson 8-6)

67. $\frac{15}{p} + p = 16$ **68.** $\frac{s-3}{s+4} = \frac{6}{s^2-16}$ **69.** $\frac{2a-5}{a-9} + \frac{a}{a+9} = \frac{-6}{a^2-81}$

Identify each equation as a type of function. Then graph the equation. (Lesson 8-5)

70. $y = \sqrt{x-2}$ **71.** $y = -2[\![x]\!]$ **72.** $y = 8$

Find the inverse of each matrix, if it exists. (Lesson 4-7)

73. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **74.** $\begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$ **75.** $\begin{bmatrix} -5 & 6 \\ -11 & 3 \end{bmatrix}$

76. ENERGY A circular cell must deliver 18 watts of energy. If each square centimeter of the cell that is in sunlight produces 0.01 watt of energy, how long must the radius of the cell be? (Lesson 7-4)

GET READY for the Next Lesson

Find $g[h(x)]$ and $h[g(x)]$. (Lesson 7-5)

77. $h(x) = 2x - 1$, $g(x) = x - 5$ **78.** $h(x) = x + 3$, $g(x) = x^2$ **79.** $h(x) = 2x + 5$, $g(x) = -x + 3$

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EXTEND 9-1

Graphing Calculator Lab

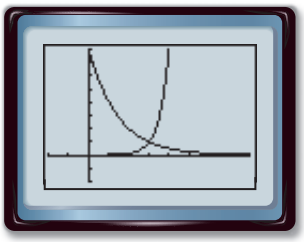
Solving Exponential Equations and Inequalities

You can use a TI-83/84 Plus graphing calculator to solve exponential equations by graphing or by using the table feature. To do this, you will write the equations as systems of equations.

ACTIVITY 1 Solve $2^{3x-9} = \left(\frac{1}{2}\right)^{x-3}$.

Step 1 **Graph each side of the equation.**

Graph each side of the equation as a separate function. Enter $2^{(3x-9)}$ as Y1. Enter $\left(\frac{1}{2}\right)^{(x-3)}$ as Y2. Be sure to include the added parentheses around each exponent. Then graph the two equations.



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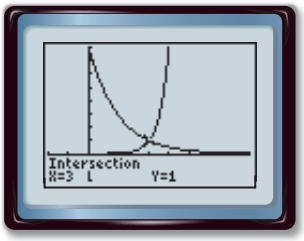
KEYSTROKES: *See pages 92–94 to review graphing equations.*

Step 2 **Use the intersect feature.**

You can use the intersect feature on the CALC menu to approximate the ordered pair of the point at which the curves cross.



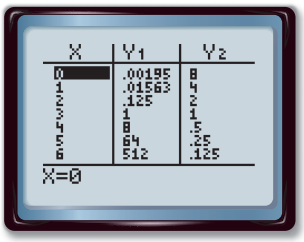
[−2, 8] scl: 1 by [−2, 8] scl: 1

KEYSTROKES: *See page 121 to review how to use the intersect feature.*

The calculator screen shows that the x-coordinate of the point at which the curves cross is 3. Therefore, the solution of the equation is 3.

Step 3 **Use the TABLE feature.**

You can also use the TABLE feature to locate the point at which the curves cross.



KEYSTROKES: 2nd [TABLE]

The table displays x-values and corresponding y-values for each graph. Examine the table to find the x-value for which the y-values for the graphs are equal. At $x = 3$, both functions have a y-value of 1. Thus, the solution of the equation is 3.

CHECK Substitute 3 for x in the original equation.

$2^{3x-9} = \left(\frac{1}{2}\right)^{x-3}$ Original equation

$2^{3(3)-9} \stackrel{?}{=} \left(\frac{1}{2}\right)^{3-3}$ Substitute 3 for x.

$2^0 \stackrel{?}{=} \left(\frac{1}{2}\right)^0$ Simplify.

$1 = 1$ ✓ The solution checks.

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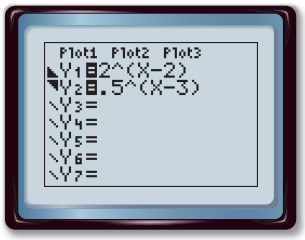
A similar procedure can be used to solve exponential inequalities using a graphing calculator.

ACTIVITY 2 **Solve $2^{x-2} \geq 0.5^{x-3}$.**

Step 1 Enter the related inequalities.

Rewrite the problem as a system of inequalities.

The first inequality is $2^{x-2} \geq y$ or $y \leq 2^{x-2}$. Since this last inequality includes the *less than or equal to* symbol, shade below the curve. First enter the boundary and then use the arrow and ENTER keys to choose the shade below icon, ▤ .

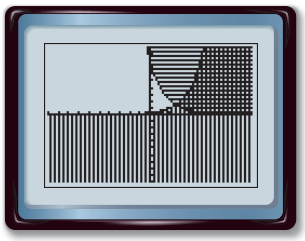


The second inequality is $y \geq 0.5^{x-3}$. Shade above the curve since this inequality contains *greater than or equal to.*

KEYSTROKES: Y= 2 ^ (X,T,θ,*n* − 2) ▼ ◀ ◀ ▲ ENTER ENTER ENTER ▼ ENTER ENTER ▶ ▶ .5 ^ (X,T,θ,*n* − 3)

Step 2 Graph the system.

KEYSTROKES: GRAPH



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The *x*-values of the points in the region where the shadings overlap are the solutions of the original inequality. Using the calculator's **intersect** feature, you can conclude that the solution set is $\{x | x \geq 2.5\}$.

Step 3 Use the **TABLE** feature.

Verify using the **TABLE** feature. Set up the table to show *x*-values in increments of 0.5.

KEYSTROKES: 2nd [TBLSET] 0 ENTER .5 ENTER 2nd [TABLE]

Notice that for *x*-values greater than $x = 2.5$, **Y1** > **Y2**. This confirms the solution of the inequality is $\{x | x \geq 2.5\}$.

EXERCISES

Solve each equation or inequality.

1. $9^{x-1} = \frac{1}{81}$
2. $4^{x+3} = 2^{5x}$
3. $5^{x-1} = 2^{x}$
4. $3.5^{x+2} = 1.75^{x+3}$
5. $-3^{x+4} = -0.5^{2x+3}$
6. $6^{2-x} - 4 > -0.25^{x-2.5}$
7. $16^{x-1} > 2^{2x+2}$
8. $3^{x} - 4 \leq 5^{\frac{x}{2}}$
9. $5^{x+3} \leq 2^{x+4}$
10. $12^{x} - 5 > 6^{1-x}$
11. Explain why this technique of graphing a system of equations or inequalities works to solve exponential equations and inequalities.

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9-2 Logarithms and Logarithmic Functions

Main Ideas

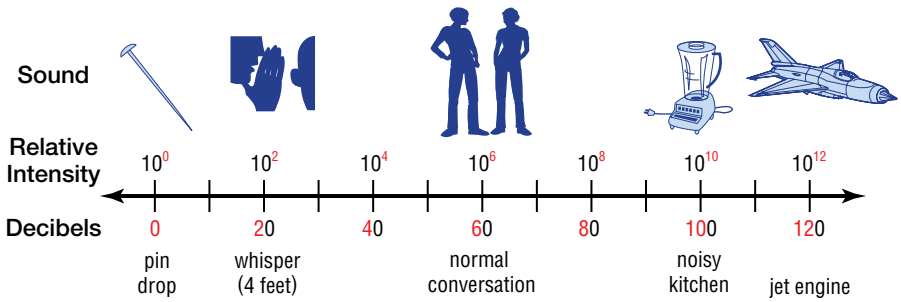
- Evaluate logarithmic expressions.
- Solve logarithmic equations and inequalities.

New Vocabulary

logarithm
logarithmic function
logarithmic equation
logarithmic inequality

GET READY for the Lesson

Many scientific measurements have such an enormous range of possible values that it makes sense to write them as powers of 10 and simply keep track of their exponents. For example, the loudness of sound is measured in units called decibels. The graph shows the relative intensities and decibel measures of common sounds.



The decibel measure of the loudness of a sound is the exponent or logarithm of its relative intensity multiplied by 10.

Review Vocabulary

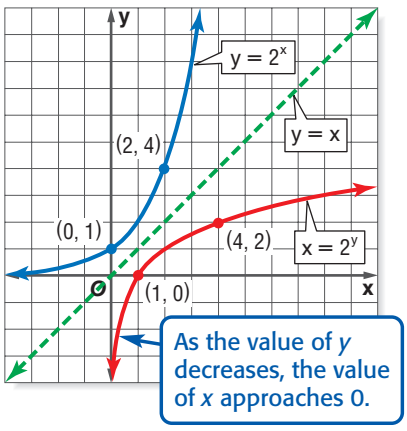
Inverse Relation when one relation contains the element (a, b), the other relation contains the element (b, a) (Lesson 7-6)

Inverse Function The inverse function of $f(x)$ is $f^{-1}(x)$. (Lesson 7-6)

Logarithmic Functions and Expressions To better understand what is meant by a logarithm, consider the graph of $y = 2^x$ and its inverse. Since exponential functions are one-to-one, the inverse of $y = 2^x$ exists and is also a function. Recall that you can graph the inverse of a function by interchanging the x- and y-values in the ordered pairs of the function. Consider the exponential function $y = 2^x$.

$y = 2^x$	
x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$x = 2^y$	
x	y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



Concepts in Motion
Animation
algebra2.com

The inverse of $y = 2^x$ can be defined as $x = 2^y$. Notice that the graphs of these two functions are reflections of each other over the line $y = x$.

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In general, the inverse of $y = b^x$ is $x = b^y$. In $x = b^y$, y is called the **logarithm** of x. It is usually written as $y = \log_b x$ and is read *y equals log base b of x*.

KEY CONCEPT — ***Logarithm with Base b***

Words Let b and x be positive numbers, $b \neq 1$. The *logarithm of x with base b* is denoted $\log_b x$ and is defined as the exponent y that makes the equation $b^y = x$ true.

Symbols Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.

EXAMPLE 1 Logarithmic to Exponential Form

Study Tip

Zero Exponent
Recall that for any $b \neq 0$, $b^0 = 1$.

1 Write each equation in exponential form.

a. $\log_8 1 = 0$

$\log_8 1 = 0 \rightarrow 1 = 8^0$

b. $\log_2 \frac{1}{16} = -4$

$\log_2 \frac{1}{16} = -4 \rightarrow \frac{1}{16} = 2^{-4}$

CHECK Your Progress

1A. $\log_4 16 = 2$

1B. $\log_3 \frac{1}{27} = -3$

EXAMPLE 2 Exponential to Logarithmic Form

2 Write each equation in logarithmic form.

a. $10^3 = 1000$

$10^3 = 1000 \rightarrow \log_{10} 1000 = 3$

b. $9^{\frac{1}{2}} = 3$

$9^{\frac{1}{2}} = 3 \rightarrow \log_9 3 = \frac{1}{2}$

CHECK Your Progress

2A. $4^3 = 64$

2B. $125^{\frac{1}{3}} = 5$

You can use the definition of logarithm to find the value of a logarithmic expression.

EXAMPLE 3 Evaluate Logarithmic Expressions

3 Evaluate $\log_2 64$.

$\log_2 64 = y$ Let the logarithm equal y.

$64 = 2^y$ Definition of logarithm

$2^6 = 2^y$ $64 = 2^6$

$6 = y$ Property of Equality for Exponential Functions

So, $\log_2 64 = 6$.

CHECK Your Progress

Evaluate each expression.

3A. $\log_3 81$

3B. $\log_4 256$

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The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, is called a **logarithmic function.** As shown in the graph on the previous page, this function is the inverse of the exponential function $y = b^x$ and has the following characteristics.

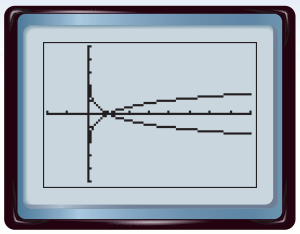
1. The function is continuous and one-to-one.
2. The domain is the set of all positive real numbers.
3. The y-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point (1, 0). That is, the x-intercept is 1.

GEOMETRY SOFTWARE LAB

The calculator screen shows the graphs of $y = \log_4 x$ and $y = \log_{\frac{1}{4}} x$.

KEYSTROKES: Y= LOG X,T,θ,n) ÷ LOG 4) ENTER
LOG X,T,θ,n) ÷ LOG 1 ÷ 4) GRAPH

THINK AND DISCUSS



1. How do the shapes of the graphs compare?
2. How do the asymptotes and the x-intercepts of the graphs compare?
3. Describe the relationship between the graphs.
4. Graph each pair of functions on the same screen. Then compare and contrast the graphs.

a.	$y = \log_4 x$	$y = \log_4 x + 2$
b.	$y = \log_4 x$	$y = \log_4 (x + 2)$
c.	$y = \log_4 x$	$y = 3 \log_4 x$

5. Describe the relationship between $y = \log_4 x$ and $y = -1(\log_4 x)$.
6. What are a reasonable domain and range for each function?
7. What is a reasonable viewing window in order to see the trends of both functions?

Study Tip

Look Back

To review **composition of functions**, see Lesson 7-5.

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ are inverses of each other, their composites are the identity function. That is, $f[g(x)] = x$ and $g[f(x)] = x$.

$$f[g(x)] = x \qquad g[f(x)] = x$$
$$f(\log_b x) = x \qquad g(b^x) = x$$
$$b^{\log_b x} = x \qquad \log_b b^x = x$$

Thus, if their bases are the same, exponential and logarithmic functions "undo" each other. You can use this inverse property of exponents and logarithms to simplify expressions and solve equations. For example, $\log_6 6^8 = 8$ and $3^{\log_3 (4x - 1)} = 4x - 1$.

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Solve Logarithmic Equations and Inequalities A **logarithmic equation** is an equation that contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.

EXAMPLE **Solve a Logarithmic Equation**

4 Solve $\log_4 n = \frac{5}{2}$.

$\log_4 n = \frac{5}{2}$ Original equation

$n = 4^{\frac{5}{2}}$ Definition of logarithm

$n = (2^2)^{\frac{5}{2}}$ $4 = 2^2$

$n = 2^5$ or 32 Power of a Power

CHECK Your Progress

Solve each equation.

4A. $\log_9 x = \frac{3}{2}$ **4B.** $\log_{16} x = \frac{5}{2}$

A **logarithmic inequality** is an inequality that involves logarithms. In the case of inequalities, the following property is helpful.

KEY CONCEPT *Logarithmic to Exponential Inequality*

Symbols If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.

Examples $\log_2 x > 3$ $\quad x > 2^3$ $\qquad$ $\log_3 x < 5$ $\quad 0 < x < 3^5$

Study Tip

Special Values

If $b > 0$ and $b \neq 1$, then the following statements are true.

- $\log_b b = 1$ because $b^1 = b$.
- $\log_b 1 = 0$ because $b^0 = 1$.

EXAMPLE **Solve a Logarithmic Inequality**

5 Solve $\log_5 x < 2$. Check your solution.

$\log_5 x < 2$ Original inequality

$0 < x < 5^2$ Logarithmic to exponential inequality

$0 < x < 25$ Simplify.

The solution set is $\{x \mid 0 < x < 25\}$.

CHECK Try 5 to see if it satisfies the inequality.

$\log_5 x < 2$ Original inequality

$\log_5 5 \stackrel{?}{<} 2$ Substitute 5 for x.

$1 < 2$ ✓ $\log_5 5 = 1$ because $5^1 = 5$.

CHECK Your Progress

Solve each inequality. Check your solution.

5A. $\log_4 x > 3$ **5B.** $\log_2 x < 4$

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Use the following property to solve logarithmic equations that have logarithms with the same base on each side.

KEY CONCEPT *Property of Equality for Logarithmic Functions*

Symbols If b is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_7 x = \log_7 3$, then $x = 3$.

EXAMPLE Solve Equations with Logarithms on Each Side

6 **Solve $\log_5 (p^2 - 2) = \log_5 p$. Check your solution.**

$\log_5 (p^2 - 2) = \log_5 p$	Original equation
$p^2 - 2 = p$	Property of Equality for Logarithmic Functions
$p^2 - p - 2 = 0$	Subtract p from each side.
$(p - 2)(p + 1) = 0$	Factor.
$p - 2 = 0$ or $p + 1 = 0$	Zero Product Property
$p = 2$ $\quad p = -1$	Solve each equation.

CHECK Substitute each value into the original equation.

Check $p = 2$.

$\log_5 (2^2 - 2) \stackrel{?}{=} \log_5 2$ Substitute 2 for p.

$\log_5 2 = \log_5 2$ ✓ Simplify.

Check $p = -1$.

$\log_5 [(-1)^2 - 2] \stackrel{?}{=} \log_5 (-1)$ Substitute -1 for p.

Since $\log_5 (-1)$ is undefined, -1 is an *extraneous* solution and must be eliminated. Thus, the solution is 2.

Study Tip

Extraneous Solutions

The domain of a logarithmic function does not include negative values. For this reason, be sure to check for extraneous solutions of logarithmic equations.

CHECK Your Progress

Solve each equation. Check your solution.

6A. $\log_3 (x^2 - 15) = \log_3 2x$ **6B.** $\log_{14} (m^2 - 30) = \log_{14} m$

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Use the following property to solve logarithmic inequalities that have the same base on each side. Exclude values from your solution set that would result in taking the logarithm of a number less than or equal to zero in the original inequality.

KEY CONCEPT *Property of Inequality for Logarithmic Functions*

Symbols If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

Example If $\log_2 x > \log_2 9$, then $x > 9$.

This property also holds for $\leq$ and $\geq$.

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Study Tip

Look Back

To review **compound inequalities**, see Lesson 1-6.

EXAMPLE 7 **Solve Inequalities with Logarithms on Each Side**

Solve $\log_{10}(3x - 4) < \log_{10}(x + 6)$. Check your solution.

$\log_{10}(3x - 4) < \log_{10}(x + 6)$ Original inequality

$3x - 4 < x + 6$ Property of Inequality for Logarithmic Functions

$2x < 10$ Addition and Subtraction Properties of Inequalities

$x < 5$ Divide each side by 2.

We must exclude from this solution all values of x such that $3x - 4 \leq 0$ or $x + 6 \leq 0$.

Thus, $x > \frac{4}{3}$, $x > -6$, and $x < 5$. This compound inequality simplifies to $\frac{4}{3} < x < 5$. The solution set is $\left\{x \middle| \frac{4}{3} < x < 5\right\}$.

CHECK Your Progress

7. Solve $\log_5(2x + 1) \leq \log_5(x + 4)$. Check your solution.

CHECK Your Understanding

Example 1 (p. 510)

Write each equation in logarithmic form.

1. $5^4 = 625$ **2.** $7^{-2} = \frac{1}{49}$ **3.** $3^5 = 243$

Example 2 (p. 510)

Write each equation in exponential form.

4. $\log_3 81 = 4$ **5.** $\log_{36} 6 = \frac{1}{2}$ **6.** $\log_{125} 5 = \frac{1}{3}$

Example 3 (p. 510)

Evaluate each expression.

7. $\log_4 256$ **8.** $\log_2 \frac{1}{8}$ **9.** $\log_6 216$

Example 4 (p. 512)

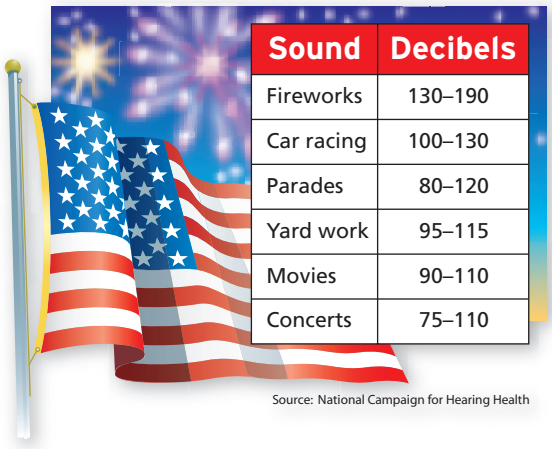
Solve each equation. Check your solutions.

10. $\log_9 x = \frac{3}{2}$ **11.** $\log_{\frac{1}{10}} x = -3$ **12.** $\log_b 9 = 2$

SOUND For Exercises 13–15, use the following information.
An equation for loudness L, in decibels, is $L = 10 \log_{10} R$, where R is the relative intensity of the sound.

13. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels.

14. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels.



Sound	Decibels
Fireworks	130–190
Car racing	100–130
Parades	80–120
Yard work	95–115
Movies	90–110
Concerts	75–110

Source: National Campaign for Hearing Health

15. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities.

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Example 5 (p. 512) **Solve each inequality. Check your solutions.**

16. $\log_4 x < 2$ **17.** $\log_3 (2x - 1) \leq 2$ **18.** $\log_{16} x \geq \frac{1}{4}$

Example 6 (p. 513) **Solve each equation. Check your solutions.**

19. $\log_5 (3x - 1) = \log_5 (2x^2)$ **20.** $\log_{10} (x^2 - 10x) = \log_{10} (-21)$

Example 7 (p. 514) **Solve each inequality. Check your solutions.**

21. $\log_2 (3x - 5) > \log_2 (x + 7)$ **22.** $\log_5 (5x - 7) \leq \log_5 (2x + 5)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
23–28	1
29–34	2
35–43	3
44–51	4
52–55	5
56, 57	6
58, 59	7

Write each equation in exponential form.

23. $\log_5 125 = 3$ **24.** $\log_{13} 169 = 2$ **25.** $\log_4 \frac{1}{4} = -1$

26. $\log_{100} \frac{1}{10} = -\frac{1}{2}$ **27.** $\log_8 4 = \frac{2}{3}$ **28.** $\log_{\frac{1}{5}} 25 = -2$

Write each equation in logarithmic form.

29. $8^3 = 512$ **30.** $3^3 = 27$ **31.** $5^{-3} = \frac{1}{125}$

32. $\left(\frac{1}{3}\right)^{-2} = 9$ **33.** $100^{\frac{1}{2}} = 10$ **34.** $2401^{\frac{1}{4}} = 7$

Evaluate each expression.

35. $\log_2 16$ **36.** $\log_{12} 144$ **37.** $\log_{16} 4$

38. $\log_9 243$ **39.** $\log_2 \frac{1}{32}$ **40.** $\log_3 \frac{1}{81}$

41. $\log_{10} 0.001$ **42.** $\log_4 16^x$ **43.** $\log_3 27^x$

Solve each equation. Check your solutions.

44. $\log_9 x = 2$ **45.** $\log_{25} n = \frac{3}{2}$ **46.** $\log_{\frac{1}{7}} x = -1$

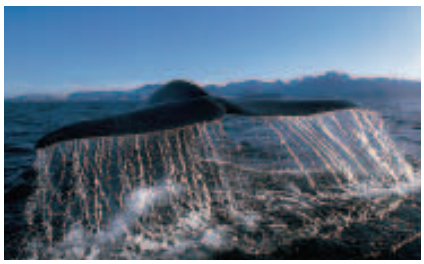
47. $\log_{10} (x^2 + 1) = 1$ **48.** $\log_b 64 = 3$ **49.** $\log_b 121 = 2$

WORLD RECORDS **For Exercises 50 and 51, use the information given for Exercises 13–15 to find the relative intensity of each sound.**

Source: *The Guinness Book of Records*

50. The loudest animal sounds are the low-frequency pulses made by blue whales when they communicate. These pulses have been measured up to 188 decibels.

51. The loudest insect is the African cicada that produces a calling song that measures 106.7 decibels at a distance of 50 centimeters.



EXTRA PRACTICE
See pages 910, 934.

Math Online
Self-Check Quiz at algebra2.com

(l)Mark Jones/Minden Pictures, (r)Jane I
www.MathSchoolinternational.com

Solve each equation or inequality. Check your solutions.

52. $\log_2 c > 8$

53. $\log_{64} y \leq \frac{1}{2}$

54. $\log_{\frac{1}{3}} p < 0$

55. $\log_2 (3x - 8) \geq 6$

56. $\log_6 (2x - 3) = \log_6 (x + 2)$

57. $\log_7 (x^2 + 36) = \log_7 100$

58. $\log_2 (4y - 10) \geq \log_2 (y - 1)$

59. $\log_{10} (a^2 - 6) > \log_{10} a$

Show that each statement is true.

60. $\log_5 25 = 2 \log_5 5$

61. $\log_{16} 2 \cdot \log_2 16 = 1$

62. $\log_7 [\log_3 (\log_2 8)] = 0$

63. Sketch the graphs of $y = \log_{\frac{1}{2}} x$ and $y = \left(\frac{1}{2}\right)^x$ on the same axes. Then describe the relationship between the graphs.

64. Sketch the graphs of $y = \log_3 x$, $y = \log_3 (x + 2)$, $y = \log_3 x - 3$. Then describe the relationship between the graphs.

Real-World Link

The Loma Prieta earthquake measured 7.1 on the Richter scale and interrupted the 1989 World Series in San Francisco.

Source: U.S. Geological Survey

EARTHQUAKES **For Exercises 65 and 66, use the following information.**
The magnitude of an earthquake is measured on a logarithmic scale called the Richter scale. The magnitude M is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion.

65. How many times as great is the amplitude caused by an earthquake with a Richter scale rating of 7 as an aftershock with a Richter scale rating of 4?

66. How many times as great was the motion caused by the 1906 San Francisco earthquake that measured 8.3 on the Richter scale as that caused by the 2001 Bhuj, India, earthquake that measured 6.9?

67. **NOISE ORDINANCE** A proposed city ordinance will make it illegal to create sound in a residential area that exceeds 72 decibels during the day and 55 decibels during the night. How many times as intense is the noise level allowed during the day than at night? (*Hint:* See information on page 514.)

Graphing Calculator

FAMILY OF GRAPHS **For Exercises 68 and 69, use the following information.**
Consider the functions $y = \log_2 x + 3$, $y = \log_2 x - 4$, $y = \log_2 (x - 1)$, and $y = \log_2 (x + 2)$.

68. Use a graphing calculator to sketch the graphs on the same screen. Describe this family of graphs in terms of its parent graph $y = \log_2 x$.

69. What are a reasonable domain and range for each function?

H.O.T. Problems

70. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation.

71. **Which One Doesn't Belong?** Find the expression that does not belong. Explain.

$\log_4 16$	$\log_2 16$	$\log_2 4$	$\log_3 9$

72. **FIND THE ERROR** Paul and Clemente are solving $\log_3 x = 9$. Who is correct? Explain your reasoning.

Paul

$\log_3 x = 9$

$3^x = 9$

$3^x = 3^2$

$x = 2$

Clemente

$\log_3 x = 9$

$x = 3^9$

$x = 19{,}683$

David Weintraub/Photo Researchers

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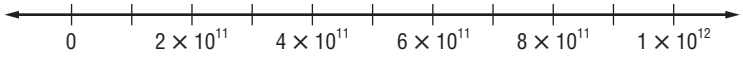
73. CHALLENGE Using the definition of a logarithmic function where $y = \log_b x$, explain why the base b cannot equal 1.

74. *Writing in Math* Use the information about sound on page 509 to explain how a logarithmic scale can be used to measure sound. Include the relative intensities of a pin drop, a whisper, normal conversation, kitchen noise, and a jet engine written in scientific notation. Also include a plot of each of these relative intensities on the scale below and an explanation as to why the logarithmic scale might be preferred over the scale below.



STANDARDIZED TEST PRACTICE

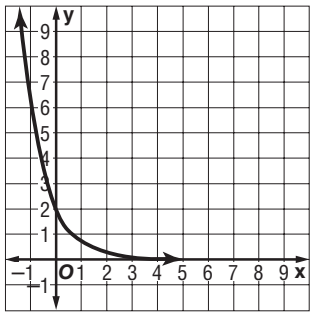
75. ACT/SAT What is the equation of the function?

A $y = 2(3)^x$

B $y = 2\left(\frac{1}{3}\right)^x$

C $y = 3\left(\frac{1}{2}\right)^x$

D $y = 3(2)^x$



76. REVIEW What is the solution to the equation $3^x = 11$?

F $x = 2$

G $x = \log_{10} 2$

H $x = \log_{10} 11 + \log_{10} 3$

J $x = \frac{\log_{10} 11}{\log_{10} 3}$

Spiral Review

Simplify each expression. (Lesson 9-1)

77. $x^{\sqrt{6}} \cdot x^{\sqrt{6}}$

78. $\left(b^{\sqrt{6}}\right)^{\sqrt{24}}$

Solve each equation. Check your solutions. (Lesson 8-6)

79. $\frac{2x+1}{x} - \frac{x+1}{x-4} = \frac{-20}{x^2-4x}$

80. $\frac{2a-5}{a-9} - \frac{a-3}{3a+2} = \frac{5}{3a^2-25a-18}$

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 5-6)

81. $9y^2 = 49$

82. $2p^2 = 5p + 6$

83. BANKING Donna Bowers has $8000 she wants to save in the bank. A 12-month certificate of deposit (CD) earns 4% annual interest, while a regular savings account earns 2% annual interest. Ms. Bowers doesn't want to tie up all her money in a CD, but she has decided she wants to earn $240 in interest for the year. How much money should she put in to each type of account? (Lesson 4-4)

GET READY for the Next Lesson

Simplify. Assume that no variable equals zero. (Lesson 6-1)

84. $x^4 \cdot x^6$

85. $(2a^2b)^3$

86. $\frac{a^4n^7}{a^3n}$

87. $\left(\frac{b^7}{a^4}\right)^0$

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EXTEND 9-2

Graphing Calculator Lab

Modeling Data Using Exponential Functions

We are often confronted with data for which we need to find an equation that best fits the information. We can find exponential and logarithmic functions of best fit using a TI-83/84 Plus graphing calculator.

ACTIVITY

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

U.S. Population Density			
Year	People per square mile	Year	People per square mile
1790	4.5	1900	21.5
1800	6.1	1910	26.0
1810	4.3	1920	29.9
1820	5.5	1930	34.7
1830	7.4	1940	37.2
1840	9.8	1950	42.6
1850	7.9	1960	50.6
1860	10.6	1970	57.5
1870	10.9	1980	64.0
1880	14.2	1990	70.3
1890	17.8	2000	80.0

Source: Northeast-Midwest Institute

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the number of people per square mile is related to the year.

Step 1 Enter the year into L1 and the people per square mile into L2.

KEYSTROKES: *See pages 92 and 93 to review how to enter lists.*

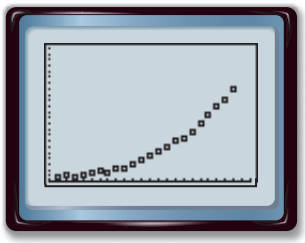
Be sure to clear the Y= list. Use the ▶ key to move the cursor from L1 to L2.

Step 2 Draw the scatter plot.

KEYSTROKES: *See pages 92 and 93 to review how to graph a scatter plot.*

Make sure that Plot 1 is on, the scatter plot is chosen, Xlist is L1, and Ylist is L2. Use the viewing window [1780, 2020] with a scale factor of 10 by [0, 115] with a scale factor of 5.

We see from the graph that the equation that best fits the data is a curve. Based on the shape of the curve, try an exponential model.



[1780, 2020] scl: 10 by [0, 115] scl: 5

Step 3 To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

KEYSTROKES: STAT ▶ 0 2nd [L1] , 2nd [L2] ENTER

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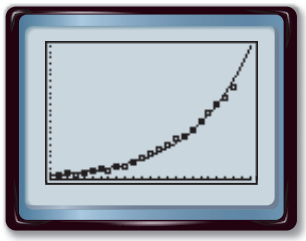
The calculator also reports an r value of 0.991887235. Recall that this number is a correlation coefficient that indicates how well the equation fits the data. A perfect fit would be $r = 1$. Therefore, we can conclude that this equation is a pretty good fit for the data.

To check this equation visually, overlap the graph of the equation with the scatter plot.



[1780, 2020] scl: 10 by [0, 115] scl: 5

KEYSTROKES: Y= VARS 5 ▶ ▶ 1 GRAPH

The *residual* is the difference between actual and predicted data. The predicted population per square mile in 2000 using this model was 86.9. (To calculate, press 2nd [CALC] 1 2000 ENTER.) So, the residual for 2000 was 80.0 − 86.9, or −6.9.

b. If this trend continues, what will be the population per square mile in 2010?

To determine the population per square mile in 2010, from the graphics screen, find the value of y when $x = 2010$.

[1780, 2020] scl: 10 by [0, 115] scl: 5

KEYSTROKES: 2nd [CALC] 1 2010 ENTER

The calculator returns a value of approximately 100.6. If this trend continues, in 2010, there will be approximately 100.6 people per square mile.

Exercises

Jewel received $30 from her aunt and uncle for her seventh birthday. Her father deposited it into a bank account for her. Both Jewel and her father forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years.

Elapsed Time (years)	Balance
0	$30.00
5	$41.10
10	$56.31
15	$77.16
20	$105.71
25	$144.83
30	$198.43

1. Use a graphing calculator to draw a scatter plot for the data.
2. Calculate and graph the curve of best fit that shows how the elapsed time is related to the balance. Use ExpReg for this exercise.
3. Write the equation of best fit.
4. Write a sentence that describes the fit of the graph to the data.
5. Based on the graph, estimate the balance in 41 years. Check this using the CALC value.
6. Do you think there are any other types of equations that would be good models for these data? Why or why not?

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9-3 Properties of Logarithms

Main Ideas

- Simplify and evaluate expressions using the properties of logarithms.
- Solve logarithmic equations using the properties of logarithms.

GET READY for the Lesson

In Lesson 6-1, you learned that the product of powers is the sum of their exponents.

$$9 \cdot 81 = 3^2 \cdot 3^4 \text{ or } 3^{2+4}$$

In Lesson 9-2, you learned that logarithms *are* exponents, so you might expect that a similar property applies to logarithms. Let's consider a specific case. Does $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$? Investigate by simplifying the expression on each side of the equation.

$\log_3 (9 \cdot 81) = \log_3 (3^2 \cdot 3^4)$ Replace 9 with 3^2 and 81 with 3^4.

$= \log_3 3^{(2+4)}$ Product of Powers

$= 2 + 4 \text{ or } 6$ Inverse Property of Exponents and Logarithms

$\log_3 9 + \log_3 81 = \log_3 3^2 + \log_3 3^4$ Replace 9 with 3^2 and 81 with 3^4.

$= 2 + 4 \text{ or } 6$ Inverse Property of Exponents and Logarithms

Both expressions are equal to 6. So, $\log_3 (9 \cdot 81) = \log_3 9 + \log_3 81$.

Properties of Logarithms Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The Product Property of Logarithms can be derived from the Product of Powers Property of Exponents.

KEY CONCEPT ***Product Property of Logarithms***

Words The logarithm of a product is the sum of the logarithms of its factors.

Symbols For all positive numbers m, n, and b, where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$.

Example $\log_3 (4)(7) = \log_3 4 + \log_3 7$

To show that this property is true, let $b^x = m$ and $b^y = n$. Then, using the definition of logarithm, $x = \log_b m$ and $y = \log_b n$.

$b^x b^y = mn$ Substitution

$b^{x+y} = mn$ Product of Powers

$\log_b b^{x+y} = \log_b mn$ Property of Equality for Logarithmic Functions

$x + y = \log_b mn$ Inverse Property of Exponents and Logarithms

$\log_b m + \log_b n = \log_b mn$ Replace x with $\log_b m$ and y with $\log_b n$.

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You can use the Product Property of Logarithms to approximate logarithmic expressions.

EXAMPLE Use the Product Property

1 **Use $\log_2 3 \approx 1.5850$ to approximate the value of $\log_2 48$.**

$$\begin{aligned}
\log_2 48 &= \log_2 (2^4 \cdot 3) && \text{Replace 48 with } 16 \cdot 3 \text{ or } 2^4 \cdot 3. \\
&= \log_2 2^4 + \log_2 3 && \text{Product Property} \\
&= 4 + \log_2 3 && \text{Inverse Property of Exponents and Logarithms} \\
&\approx 4 + 1.5850 \text{ or } 5.5850 && \text{Replace } \log_2 3 \text{ with } 1.5850.
\end{aligned}$$

Thus, $\log_2 48$ is approximately 5.5850.

Study Tip

Answer Check

You can check this answer by evaluating $2^{5.5850}$ on a calculator. The calculator should give a result of about 48, since $\log_2 48 \approx 5.5850$ means $2^{5.5850} \approx 48$.

CHECK Your Progress

1. Use $\log_4 2 = 0.5$ to approximate the value of $\log_4 32$.

Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar.

KEY CONCEPT *Quotient Property of Logarithms*

Words The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.

Symbols For all positive numbers m, n, and b, where $b \neq 1$,
$\log_b \frac{m}{n} = \log_b m - \log_b n$.

You will prove this property in Exercise 51.

EXAMPLE Use the Quotient Property

2 **Use $\log_3 5 \approx 1.4650$ and $\log_3 20 \approx 2.7268$ to approximate $\log_3 4$.**

$$\begin{aligned}
\log_3 4 &= \log_3 \frac{20}{5} && \text{Replace 4 with the quotient } \frac{20}{5}. \\
&= \log_3 20 - \log_3 5 && \text{Quotient Property} \\
&\approx 2.7268 - 1.4650 \text{ or } 1.2618 && \log_3 20 \approx 2.7268 \text{ and } \log_3 5 \approx 1.4650
\end{aligned}$$

Thus, $\log_3 4$ is approximately 1.2618.

CHECK Use the definition of logarithm and a calculator.

3 ^ 1.2618 ENTER 3.999738507

Since $3^{1.2618} \approx 4$, the answer checks. ✔

CHECK Your Progress

2. Use $\log_5 7 \approx 1.2091$ and $\log_5 21 \approx 1.8917$ to approximate $\log_5 3$.

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Real-World Career

Sound Technician

Sound technicians produce movie sound tracks in motion picture production studios, control the sound of live events such as concerts, or record music in a recording studio.

Math Online

For more information, go to algebra2.com.

Real-World EXAMPLE

1 **SOUND** **The loudness L of a sound is measured in decibels and is given by $L = 10 \log_{10} R$, where R is the sound's relative intensity. Suppose one person talks with a relative intensity of 10^6 or 60 decibels. Would the sound of ten people each talking at that same intensity be ten times as loud, or 600 decibels? Explain your reasoning.**

Let L_1 be the loudness of one person talking. $\rightarrow$ $L_1 = 10 \log_{10} 10^6$

Let L_2 be the loudness of ten people talking. $\rightarrow$ $L_2 = 10 \log_{10} (10 \cdot 10^6)$

Then the increase in loudness is $L_2 - L_1$.

$L_2 - L_1 = 10 \log_{10} (10 \cdot 10^6) - 10 \log_{10} 10^6$ Substitute for L_1 and L_2.

$= 10(\log_{10} 10 + \log_{10} 10^6) - 10 \log_{10} 10^6$ Product Property

$= 10 \log_{10} 10 + 10 \log_{10} 10^6 - 10 \log_{10} 10^6$ Distributive Property

$= 10 \log_{10} 10$ Subtract.

$= 10(1)$ or 10 Inverse Property of Exponents and Logarithms

The sound of ten people talking is perceived by the human ear to be only about 10 decibels louder than the sound of one person talking, or 70 decibels.

CHECK Your Progress

3. How much louder would 100 people talking at the same intensity be than just one person?

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Recall that the power of a power is found by multiplying exponents. The property for the logarithm of a power is similar.

KEY CONCEPT ***Power Property of Logarithms***

Words The logarithm of a power is the product of the logarithm and the exponent.

Symbols For any real number p and positive numbers m and b, where $b \neq 1$, $\log_b m^p = p \log_b m$.

You will prove this property in Exercise 45.

EXAMPLE Power Property of Logarithms

4 **Given $\log_4 6 \approx 1.2925$, approximate the value of $\log_4 36$.**

$\log_4 36 = \log_4 6^2$ Replace 36 with 6^2.

$= 2 \log_4 6$ Power Property

$\approx 2(1.2925)$ or 2.585 Replace $\log_4 6$ with 1.2925.

CHECK Your Progress

4. Given $\log_3 7 \approx 1.7712$, approximate the value of $\log_3 49$.

Phil Cantor/SuperStock

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Solve Logarithmic Equations You can use the properties of logarithms to solve equations involving logarithms.

EXAMPLE Solve Equations Using Properties of Logarithms

5 **Solve each equation.**

a. $3 \log_5 x - \log_5 4 = \log_5 16$

$3 \log_5 x - \log_5 4 = \log_5 16$ Original equation

$\log_5 x^3 - \log_5 4 = \log_5 16$ Power Property

$\log_5 \frac{x^3}{4} = \log_5 16$ Quotient Property

$\frac{x^3}{4} = 16$ Property of Equality for Logarithmic Functions

$x^3 = 64$ Multiply each side by 4.

$x = 4$ Take the cube root of each side.

The solution is 4.

b. $\log_4 x + \log_4 (x - 6) = 2$

$\log_4 x + \log_4 (x - 6) = 2$ Original equation

$\log_4 x(x - 6) = 2$ Product Property

$x(x - 6) = 4^2$ Definition of logarithm

$x^2 - 6x - 16 = 0$ Subtract 16 from each side.

$(x - 8)(x + 2) = 0$ Factor.

$x - 8 = 0$ or $x + 2 = 0$ Zero Product Property

$x = 8$ $x = -2$ Solve each equation.

CHECK Substitute each value into the original equation.

$\log_4 8 + \log_4 (8 - 6) \stackrel{?}{=} 2$

$\log_4 8 + \log_4 2 \stackrel{?}{=} 2$

$\log_4 (8 \cdot 2) \stackrel{?}{=} 2$

$\log_4 16 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log_4 (-2) + \log_4 (-2 - 6) \stackrel{?}{=} 2$

$\log_4 (-2) + \log_4 (-8) \stackrel{?}{=} 2$

Since $\log_4 (-2)$ and $\log_4 (-8)$ are undefined, -2 is an extraneous solution and must be eliminated.

The only solution is 8.

Study Tip

Checking Solutions

It is wise to check all solutions to see if they are valid since the domain of a logarithmic function is not the complete set of real numbers.

CHECK Your Progress

5A. $2 \log_7 x = \log_7 27 + \log_7 3$ **5B.** $\log_6 x + \log_6 (x + 5) = 2$

CHECK Your Understanding

Examples 1, 2 (p. 521)

Use $\log_3 2 \approx 0.6309$ and $\log_3 7 \approx 1.7712$ to approximate the value of each expression.

1. $\log_3 18$ **2.** $\log_3 14$ **3.** $\log_3 \frac{7}{2}$ **4.** $\log_3 \frac{2}{3}$

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Example 3 (p. 522)

5. MOUNTAIN CLIMBING As elevation increases, the atmospheric air pressure decreases. The formula for pressure based on elevation is $a = 15{,}500\,(5 - \log_{10} P)$, where a is the altitude in meters and P is the pressure in pascals (1 psi $\approx$ 6900 pascals). What is the air pressure at the summit in pascals for each mountain listed in the table at the right?

Mountain	Country	Height (m)
Everest	Nepal/Tibet	8850
Trisuli	India	7074
Bonete	Argentina/Chile	6872
McKinley	United States	6194
Logan	Canada	5959

Source: infoplease.com

Example 4 (p. 522)

Given $\log_2 7 \approx 2.8074$ and $\log_5 8 \approx 1.2920$ to approximate the value of each expression.

6. $\log_2 49$ **7.** $\log_5 64$

Example 5 (p. 523)

Solve each equation. Check your solutions.

8. $\log_3 42 - \log_3 n = \log_3 7$ **9.** $\log_2(3x) + \log_2 5 = \log_2 30$

10. $2 \log_5 x = \log_5 9$ **11.** $\log_{10} a + \log_{10} (a + 21) = 2$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–14	1
15–17	2
18–20	3
21–24	4
25–30	5

Use $\log_5 2 \approx 0.4307$ and $\log_5 3 \approx 0.6826$ to approximate the value of each expression.

12. $\log_5 50$ **13.** $\log_5 30$ **14.** $\log_5 20$

15. $\log_5 \frac{2}{3}$ **16.** $\log_5 \frac{3}{2}$ **17.** $\log_5 \frac{4}{3}$

18. $\log_5 9$ **19.** $\log_5 8$ **20.** $\log_5 16$

21. EARTHQUAKES The great Alaskan earthquake, in 1964, was about 100 times as intense as the Loma Prieta earthquake in San Francisco, in 1989. Find the difference in the Richter scale magnitudes of the earthquakes.

PROBABILITY **For Exercises 22–24, use the following information.**
In the 1930s, Dr. Frank Benford demonstrated a way to determine whether a set of numbers have been randomly chosen or the numbers have been manually chosen. If the sets of numbers were not randomly chosen, then the Benford formula, $P = \log_{10}\left(1 + \frac{1}{d}\right)$, predicts the probability of a digit d being the first digit of the set. For example, there is a 4.6% probability that the first digit is 9.

22. Rewrite the formula to solve for the digit if given the probability.

23. Find the digit that has a 9.7% probability of being selected.

24. Find the probability that the first digit is 1 ($\log_{10} 2 \approx 0.30103$).

Solve each equation. Check your solutions.

25. $\log_3 5 + \log_3 x = \log_3 10$ **26.** $\log_4 a + \log_4 9 = \log_4 27$

27. $\log_{10} 16 - \log_{10} (2t) = \log_{10} 2$ **28.** $\log_7 24 - \log_7 (y + 5) = \log_7 8$

29. $\log_2 n = \frac{1}{4} \log_2 16 + \frac{1}{2} \log_2 49$ **30.** $2 \log_{10} 6 - \frac{1}{3} \log_{10} 27 = \log_{10} x$

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Real-World Link

The Greek astronomer Hipparchus made the first known catalog of stars. He listed the brightness of each star on a scale of 1 to 6, the brightest being 1. With no telescope, he could only see stars as dim as the 6th magnitude.

Source: NASA

Solve for n.

31. $\log_a (4n) - 2 \log_a x = \log_a x$

32. $\log_b 8 + 3 \log_b n = 3 \log_b (x - 1)$

Solve each equation. Check your solutions.

33. $\log_{10} z + \log_{10} (z + 3) = 1$

34. $\log_6 (a^2 + 2) + \log_6 2 = 2$

35. $\log_2 (12b - 21) - \log_2 (b^2 - 3) = 2$

36. $\log_2 (y + 2) - \log_2 (y - 2) = 1$

37. $\log_3 0.1 + 2 \log_3 x = \log_3 2 + \log_3 5$

38. $\log_5 64 - \log_5 \frac{8}{3} + \log_5 2 = \log_5 (4p)$

SOUND **For Exercises 39–41, use the formula for the loudness of sound in Example 3 on page 546. Use $\log_{10} 2 \approx 0.3010$ and $\log_{10} 3 \approx 0.4771$.**

39. A certain sound has a relative intensity of R. By how many decibels does the sound increase when the intensity is doubled?

40. A certain sound has a relative intensity of R. By how many decibels does the sound decrease when the intensity is halved?

41. A stadium containing 10,000 cheering people can produce a crowd noise of about 90 decibels. If everyone cheers with the same relative intensity, how much noise, in decibels, is a crowd of 30,000 people capable of producing? Explain your reasoning.

STAR LIGHT **For Exercises 42–44, use the following information.**

The brightness, or apparent magnitude, m of a star or planet is given by $m = 6 - 2.5 \log_{10} \frac{L}{L_0}$, where L is the amount of light L coming to Earth from the star or planet and L_0 is the amount of light from a sixth magnitude star.

42. Find the difference in the magnitudes of Sirius and the crescent moon.

43. Find the difference in the magnitudes of Saturn and Neptune.

44. **RESEARCH** Use the Internet or other reference to find the magnitude of the dimmest stars that we can now see with ground-based telescopes.

EXTRA PRACTICE
See pages 910, 934.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

45. **REASONING** Use the properties of exponents to prove the Power Property of Logarithms.

46. **REASONING** Use the properties of Logarithms to prove that $\log_a \frac{1}{x} = -\log_a x$.

47. **CHALLENGE** Simplify $\log_{\sqrt{a}} (a^2)$ to find an exact numerical value.

48. **CHALLENGE** Simplify $x^{3 \log_x 2 - \log_x 5}$ to find an exact numerical value.

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CHALLENGE **Tell whether each statement is *true* or *false*. If true, show that it is true. If false, give a counterexample.**

49. For all positive numbers m, n, and b, where $b \neq 1$, $\log_b (m + n) = \log_b m + \log_b n$.

50. For all positive numbers m, n, x, and b, where $b \neq 1$, $n \log_b x + m \log_b x = (n + m) \log_b x$.

51. REASONING Use the properties of exponents to prove the Quotient Property of Logarithms.

52. *Writing in Math* Use the information given regarding exponents and logarithms on page 520 to explain how the properties of exponents and logarithms are related. Include examples like the one shown at the beginning of the lesson illustrating the Quotient Property and Power Property of Logarithms, and an explanation of the similarity between one property of exponents and its related property of logarithms in your answer.

STANDARDIZED TEST PRACTICE

53. ACT/SAT To what is $2 \log_5 12 - \log_5 8 - 2 \log_5 3$ equal?

A $\log_5 2$

B $\log_5 3$

C $\log_5 0.5$

D 1

54. REVIEW In a movie theater, 2 boys and 3 girls are seated randomly together. What is the probability that the 2 boys are seated next to each other?

F $\frac{1}{5}$ **G** $\frac{2}{5}$ **H** $\frac{1}{2}$ **J** $\frac{2}{3}$

Spiral Review

Evaluate each expression. (Lesson 9-2)

55. $\log_3 81$ **56.** $\log_9 \frac{1}{729}$ **57.** $\log_7 7^{2x}$

Solve each equation or inequality. Check your solutions. (Lesson 9-1)

58. $3^{5n+3} = 3^{33}$ **59.** $7^a = 49^{-4}$ **60.** $3^{d+4} > 9^d$

61. PHYSICS If a stone is dropped from a cliff, the equation $t = \frac{1}{4}\sqrt{d}$ represents the time t in seconds that it takes for the stone to reach the ground. If d represents the distance in feet that the stone falls, find how long it would take for a stone to hit the ground after falling from a 150-foot cliff. (Lesson 7-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation or inequality. Check your solutions.** (Lesson 9-2)

62. $\log_3 x = \log_3 (2x - 1)$ **63.** $\log_{10} 2^x = \log_{10} 32$

64. $\log_2 3x > \log_2 5$ **65.** $\log_5 (4x + 3) < \log_5 11$

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CHAPTER 9

Mid-Chapter Quiz

Lessons 9-1 through 9-3

RABBIT POPULATION **For Exercises 1 and 2, use the following information.** (Lesson 9-1)

Rabbits reproduce at a tremendous rate and their population increases exponentially in the absence of natural enemies. Suppose there were originally 65,000 rabbits in a region and two years later there are 2,500,000.

1. Write an exponential function that could be used to model the rabbit population y in that region. Write the function in terms of x, the number of years since the original year.
2. Assume that the rabbit population continued to grow at that rate. Estimate the rabbit population in that region seven years later.
3. Determine whether $5(1.2)^x$ represents exponential *growth* or *decay*. Explain. (Lesson 9-1)
4. **SAVINGS** Suppose you deposit \$500 in an account paying 4.5% interest compounded semiannually. Find the dollar value of the account rounded to the nearest penny after 10 years. (Lesson 9-1)

Evaluate each expression. (Lesson 9-2)

5. $\log_8 16$
6. $\log_4 4^{15}$
7. **MULTIPLE CHOICE** What is the value of n if $\log_3 3^{4n-1} = 11$? (Lesson 9-2)

   A 3

   B 4

   C 6

   D 12

Solve each equation or inequality. Check your solution. (Lessons 9-1 through 9-3)

8. $3^{4x} = 3^{3-x}$
9. $3^{2n} \leq \frac{1}{9}$
10. $3^{5x} \cdot 81^{1-x} = 9^{x-3}$
11. $49^x = 7^{x^2-15}$
12. $\log_2 (x + 6) > 5$
13. $\log_5 (4x - 1) = \log_5 (3x + 2)$
14. **MULTIPLE CHOICE** Find the value of x for $\log_2 (9x + 5) = 2 + \log_2 (x^2 - 1)$. (Lesson 9-3)

    F -0.4 **H** 1

    G 0 **J** 3

HEALTH **For Exercises 15–17, use the following information.** (Lesson 9-3)

The pH of a person's blood is given by the function $\text{pH} = 6.1 + \log_{10} B - \log_{10} C$, where B is the concentration of bicarbonate, which is a base, in the blood, and C is the concentration of carbonic acid in the blood.

Substance	pH
Lemon juice	2.3
Milk	6.4
Baking soda	8.4
Ammonia	11.9
Drain cleaner	14.0

15. Use the Quotient Property of Logarithms to simplify the formula for blood pH.
16. Most people have a blood pH of 7.4. What is the approximate ratio of bicarbonate to carbonic acid for blood with this pH?
17. If a person's ratio of bicarbonate to carbonic acid is 17.5:2.25, determine which substance has a pH closest to this person's blood.

ENERGY **For Exercises 18–20, use the following information.** (Lesson 9-3)

The energy E (in kilocalories per gram molecule) needed to transport a substance from the outside to the inside of a living cell is given by $E = 1.4(\log_{10} C_2 - \log_{10} C_1)$, where C_1 is the concentration of the substance outside the cell and C_2 is the concentration inside the cell.

18. Express the value of E as one logarithm.
19. Suppose the concentration of a substance inside the cell is twice the concentration outside the cell. How much energy is needed to transport the substance on the outside of the cell to the inside? (Use $\log_{10} 2 \approx 0.3010$.)
20. Suppose the concentration of a substance inside the cell is four times the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside?

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9-4 Common Logarithms

Main Ideas

- Solve exponential equations and inequalities using common logarithms.
- Evaluate logarithmic expressions using the Change of Base Formula.

New Vocabulary

common logarithm

Change of Base Formula

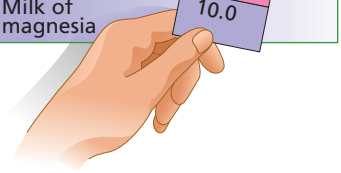
GET READY for the Lesson

The pH level of a substance measures its acidity. A low pH indicates an acid solution while a high pH indicates a basic solution. The pH levels of some common substances are shown.

The pH level of a substance is given by $pH = -\log_{10} [H^+]$, where H^+ is the substance's hydrogen ion concentration in moles per liter. Another way of writing this formula is $pH = -\log [H^+]$.

Acidity of Common Substances

Substance	pH Level
Battery acid	1.0
Sauerkraut	3.5
Tomatoes	4.2
Black coffee	5.0
Milk	6.4
Distilled water	7.0
Eggs	7.8
Milk of magnesia	10.0



Common Logarithms You have seen that the base 10 logarithm function, $y = \log_{10} x$, is used in many applications. Base 10 logarithms are called **common logarithms**. Common logarithms are usually written without the subscript 10.

$$\log_{10} x = \log x, x > 0$$

Most scientific calculators have a LOG key for evaluating common logarithms.

EXAMPLE **Find Common Logarithms**

1 **Use a calculator to evaluate each expression to four decimal places.**

a. log 3

KEYSTROKES: LOG 3 ENTER .4771212547

log 3 is about 0.4771.

b. log 0.2

KEYSTROKES: LOG 0.2 ENTER −.6989700043

log 0.2 is about −0.6990.

Study Tip

Technology

Nongraphing scientific calculators often require entering the number followed by the function, for example, 3 LOG.

CHECK Your Progress

1A. log 5 **1B.** log 0.5

Sometimes an application of logarithms requires that you use the inverse of logarithms, or exponentiation.

$$10^{\log x} = x$$

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Real-World EXAMPLE Solve Logarithmic Equations

2 **EARTHQUAKES** **The amount of energy E, in ergs, that an earthquake releases is related to its Richter scale magnitude M by the equation $\log E = 11.8 + 1.5M$. The Chilean earthquake of 1960 measured 8.5 on the Richter scale. How much energy was released?**

$\log E = 11.8 + 1.5M$	Write the formula.
$\log E = 11.8 + 1.5(8.5)$	Replace M with 8.5.
$\log E = 24.55$	Simplify.
$10^{\log E} = 10^{24.55}$	Write each side using exponents and base 10.
$E = 10^{24.55}$	Inverse Property of Exponents and Logarithms
$E \approx 3.55 \times 10^{24}$	Use a calculator.

The amount of energy released by this earthquake was about 3.55×10^{24} ergs.

CHECK Your Progress

2. Use the equation above to find the energy released by the 2004 Sumatran earthquake, which measured 9.0 on the Richter scale and led to a tsunami.

Personal Tutor at algebra2.com

If both sides of an exponential equation cannot easily be written as powers of the same base, you can solve by taking the logarithm of each side.

EXAMPLE Solve Exponential Equations Using Logarithms

3 **Solve $3^x = 11$.**

$3^x = 11$	Original equation
$\log 3^x = \log 11$	Property of Equality for Logarithmic Functions
$x \log 3 = \log 11$	Power Property of Logarithms
$x = \frac{\log 11}{\log 3}$	Divide each side by log 3.
$x \approx \frac{1.0414}{0.4771}$	Use a calculator.
$x \approx 2.1828$	

The solution is approximately 2.1828.

CHECK You can check this answer using a calculator or by using estimation. Since $3^2 = 9$ and $3^3 = 27$, the value of x is between 2 and 3. In addition, the value of x should be closer to 2 than 3, since 11 is closer to 9 than 27. Thus, 2.1828 is a reasonable solution. ✔

Study Tip

Using Logarithms

When you use the Property of Equality for Logarithmic Functions, this is sometimes referred to as *taking the logarithm of each side.*

CHECK Your Progress

Solve each equation.

3A. $4^x = 15$ **3B.** $6^x = 42$

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EXAMPLE Solve Exponential Inequalities Using Logarithms

4 Solve $5^{3y} < 8^{y-1}$.

$$5^{3y} < 8^{y-1}$$ Original inequality

$$\log 5^{3y} < \log 8^{y-1}$$ Property of Inequality for Logarithmic Functions

$$3y \log 5 < (y-1) \log 8$$ Power Property of Logarithms

$$3y \log 5 < y \log 8 - \log 8$$ Distributive Property

$$3y \log 5 - y \log 8 < -\log 8$$ Subtract $y \log 8$ from each side.

$$y(3 \log 5 - \log 8) < -\log 8$$ Distributive Property

$$y < \frac{-\log 8}{3 \log 5 - \log 8}$$ Divide each side by $3 \log 5 - \log 8$.

$$y < -0.7564$$ Use a calculator.

The solution set is $\{y \mid y < -0.7564\}$.

CHECK Test $y = -1$.

$$5^{3y} < 8^{y-1}$$ Original inequality

$$5^{3(-1)} < 8^{(-1)-1}$$ Replace y with -1.

$$5^{-3} < 8^{-2}$$ Simplify.

$$\frac{1}{125} < \frac{1}{64}$$ ✔ Negative Exponent Property

Study Tip

Solving Inequalities

Remember that the direction of an inequality must be switched if both sides are multiplied or divided by a negative number. Since $3 \log 5 - \log 8 > 0$, the inequality does not change.

CHECK Your Progress Solve each inequality.

4A. $3^{2x} \geq 6^{x+1}$ **4B.** $4^{y} < 5^{2y+1}$

Change of Base Formula The **Change of Base Formula** allows you to write equivalent logarithmic expressions that have different bases.

KEY CONCEPT *Change of Base Formula*

Symbols For all positive numbers, a, b and n, where $a \neq 1$ and $b \neq 1$,

$$\log_a n = \frac{\log_b n}{\log_b a}.$$ ← log base b of original number; ← log base b of old base

Example $\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5}$

To prove this formula, let $\log_a n = x$.

$$a^x = n$$ Definition of logarithm

$$\log_b a^x = \log_b n$$ Property of Equality for Logarithms

$$x \log_b a = \log_b n$$ Power Property of Logarithms

$$x = \frac{\log_b n}{\log_b a}$$ Divide each side by $\log_b a$.

$$\log_a n = \frac{\log_b n}{\log_b a}$$ Replace x with $\log_a n$.

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The Change of Base Formula makes it possible to evaluate a logarithmic expression of any base by translating the expression into one that involves common logarithms.

EXAMPLE Change of Base Formula

5 **Express $\log_4 25$ in terms of common logarithms. Then approximate its value to four decimal places.**

$$\log_4 25 = \frac{\log_{10} 25}{\log_{10} 4} \quad \text{Change of Base Formula}$$

$$\approx 2.3219 \quad \text{Use a calculator.}$$

The value of $\log_4 25$ is approximately 2.3219.

CHECK Your Progress

5. Express $\log_6 8$ in terms of common logarithms. Then approximate its value to four decimal places.

CHECK Your Understanding

Example 1 (p. 528) **Use a calculator to evaluate each expression to four decimal places.**

1. $\log 4$ **2.** $\log 23$ **3.** $\log 0.5$

Example 2 (p. 529)

4. NUTRITION For health reasons, Sandra's doctor has told her to avoid foods that have a pH of less than 4.5. What is the hydrogen ion concentration of foods Sandra is allowed to eat? Use the information at the beginning of the lesson.

Example 3 (p. 529) **Solve each equation. Round to four decimal places.**

5. $9^x = 45$ **6.** $3.1^{a-3} = 9.42$

7. $11^{x^2} = 25.4$ **8.** $7^{t-2} = 5^t$

Example 4 (p. 530) **Solve each inequality. Round to four decimal places.**

9. $4^{5n} > 30$ **10.** $4^{p-1} \leq 3^p$

Example 5 (p. 531) **Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.**

11. $\log_7 5$ **12.** $\log_3 42$ **13.** $\log_2 9$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–19	1
20, 21	2
22–27	3
28–33	4
34–39	5

Use a calculator to evaluate each expression to four decimal places.

14. $\log 5$ **15.** $\log 12$ **16.** $\log 7.2$

17. $\log 2.3$ **18.** $\log 0.8$ **19.** $\log 0.03$

20. POLLUTION The acidity of water determines the toxic effects of runoff into streams from industrial or agricultural areas. A pH range of 6.0 to 9.0 appears to provide protection for freshwater fish. What is this range in terms of the water's hydrogen ion concentration?

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21. BUILDING DESIGN The 1971 Sylmar earthquake in Los Angeles had a Richter scale magnitude of 6.3. Suppose an architect has designed a building strong enough to withstand an earthquake 50 times as intense as the Sylmar quake. Find the magnitude of the strongest quake this building can withstand.

Solve each equation or inequality. Round to four decimal places.

22. $5^x = 52$ **23.** $4^{3p} = 10$ **24.** $3^{n+2} = 14.5$

25. $9^{z-4} = 6.28$ **26.** $8.2^{n-3} = 42.5$ **27.** $2.1^{t-5} = 9.32$

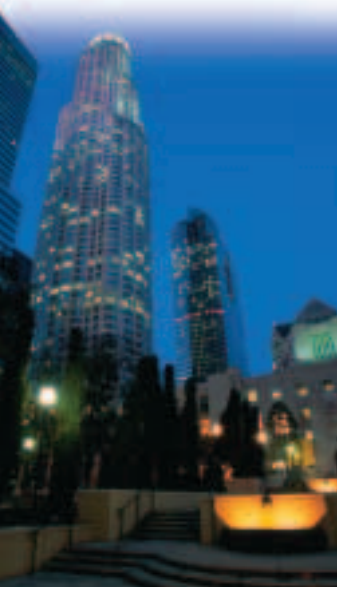
28. $6^x \geq 42$ **29.** $8^{2a} < 124$ **30.** $4^{3x} \leq 72$

31. $8^{2n} > 52^{4n+3}$ **32.** $7^{p+2} \leq 13^{5-p}$ **33.** $3^{y+2} \geq 8^{3y}$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

34. $\log_2 13$ **35.** $\log_5 20$ **36.** $\log_7 3$

37. $\log_3 8$ **38.** $\log_4 (1.6)^2$ **39.** $\log_6 \sqrt{5}$



Real-World Link

There are an estimated 500,000 detectable earthquakes in the world each year. Of these earthquakes, 100,000 can be felt and 100 cause damage.

Source: earthquake.usgs.gov

ACIDITY For Exercises 40–43, use the information at the beginning of the lesson to find each pH given the concentration of hydrogen ions.

40. ammonia: $[H^+] = 1 \times 10^{-11}$ mole per liter

41. vinegar: $[H^+] = 6.3 \times 10^{-3}$ mole per liter

42. lemon juice: $[H^+] = 7.9 \times 10^{-3}$ mole per liter

43. orange juice: $[H^+] = 3.16 \times 10^{-4}$ mole per liter

Solve each equation. Round to four decimal places.

44. $20^{x^2} = 70$ **45.** $2^{x^2-3} = 15$ **46.** $2^{2x+3} = 3^{3x}$

47. $16^{d-4} = 3^{3-d}$ **48.** $5^{5y-2} = 2^{2y+1}$ **49.** $8^{2x-5} = 5^{x+1}$

50. $2^n = \sqrt{3^{n-2}}$ **51.** $4^x = \sqrt{5^{x+2}}$ **52.** $3^y = \sqrt{2^{y-1}}$

MUSIC For Exercises 53 and 54, use the following information.

The musical cent is a unit in a logarithmic scale of relative pitch or intervals. One octave is equal to 1200 cents. The formula to determine the difference in cents between two notes with frequencies a and b is $n = 1200\left(\log_2 \frac{a}{b}\right)$.

53. Find the interval in cents when the frequency changes from 443 Hertz (Hz) to 415 Hz.

54. If the interval is 55 cents and the beginning frequency is 225 Hz, find the final frequency.

MONEY For Exercises 55 and 56, use the following information.

If you deposit P dollars into a bank account paying an annual interest rate r (expressed as a decimal), with n interest payments each year, the amount A you would have after t years is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Marta places \$100 in a savings account earning 2% annual interest, compounded quarterly.

55. If Marta adds no more money to the account, how long will it take the money in the account to reach \$125?

56. How long will it take for Marta's money to double?

EXTRA PRACTICE
See pages 910, 934.
MathOnline
Self-Check Quiz at algebra2.com

Richard Cummins/CORBIS
www.MathSchoolinternational.com

H.O.T. Problems

57. CHALLENGE Solve $\log_{\sqrt{a}} 3 = \log_a x$ for x and explain each step.

58. Write $\dfrac{\log_5 9}{\log_5 3}$ as a single logarithm.

59. CHALLENGE

a. Find the values of $\log_2 8$ and $\log_8 2$.

b. Find the values of $\log_9 27$ and $\log_{27} 9$.

c. Make and prove a conjecture about the relationship between $\log_a b$ and $\log_b a$.

60. *Writing in Math* Use the information about acidity of common substances on page 528 to explain why a logarithmic scale is used to measure acidity. Include the hydrogen ion concentration of three substances listed in the table, and an explanation as to why it is important to be able to distinguish between a hydrogen ion concentration of 0.00001 mole per liter and 0.0001 mole per liter in your answer.

STANDARDIZED TEST PRACTICE

61. ACT/SAT If $2^4 = 3^x$, then what is the approximate value of x?

A 0.63

B 2.34

C 2.52

D 4

62. REVIEW Which equation is equivalent to $\log_4 \dfrac{1}{16} = x$?

F $\dfrac{1^4}{16} = x^4$

G $\left(\dfrac{1}{16}\right)^4 = x$

H $4^x = \dfrac{1}{16}$

J $4^{\frac{1}{16}} = x$

Spiral Review

Use $\log_7 2 \approx 0.3562$ and $\log_7 3 \approx 0.5646$ to approximate the value of each expression. (Lesson 9-3)

63. $\log_7 16$ **64.** $\log_7 27$ **65.** $\log_7 36$

Solve each equation or inequality. Check your solutions. (Lesson 9-2)

66. $\log_4 r = 3$ **67.** $\log_8 z \leq -2$ **68.** $\log_3 (4x - 5) = 5$

69. Use synthetic substitution to find $f(-2)$ for $f(x) = x^3 + 6x - 2$. (Lesson 6-7)

70. MONEY Viviana has two dollars worth of nickels, dimes, and quarters. She has 18 total coins, and the number of nickels equals 25 minus twice the number of dimes. How many nickels, dimes, and quarters does she have? (Lesson 3-5)

GET READY for the Next Lesson

PREREQUISITE SKILL Write an equivalent exponential equation. (Lesson 9-2)

71. $\log_2 3 = x$ **72.** $\log_3 x = 2$ **73.** $\log_5 125 = 3$

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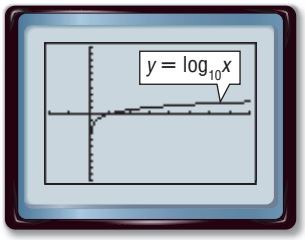
EXTEND 9-4

Graphing Calculator Lab

Solving Logarithmic Equations and Inequalities

You have solved logarithmic equations algebraically. You can also solve logarithmic equations by graphing or by using a table. The calculator has $y = \log_{10} x$ as a built-in function. Enter [Y=] [LOG] [X,T,θ,n] [GRAPH] to view this graph. To graph logarithmic functions with bases other than 10, you must use the Change of Base Formula, $\log_a n = \frac{\log_b n}{\log_b a}$.



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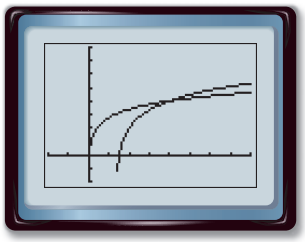
ACTIVITY 1 **Solve $\log_2 (6x - 8) = \log_3 (20x + 1)$.**

Step 1 **Graph each side of the equation.**

Graph each side of the equation as a separate function. Enter $\log_2 (6x - 8)$ as Y1 and $\log_3 (20x + 1)$ as Y2. Then graph the two equations.

KEYSTROKES: [Y=] [LOG] 6 [X,T,θ,n] [−] 8 [)] [÷] [LOG] 2 [)] [ENTER] [LOG] 20 [X,T,θ,n] [+] 1 [)] [÷] [LOG] 3 [)] [GRAPH]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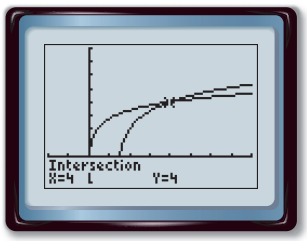
Step 2 **Use the intersect feature.**

Use the intersect feature on the CALC menu to approximate the ordered pair of the point at which the curves cross.

KEYSTROKES: *See page 121 to review how to use the intersect feature.*

The calculator screen shows that the x-coordinate of the point at which the curves cross is 4. Therefore, the solution of the equation is 4.



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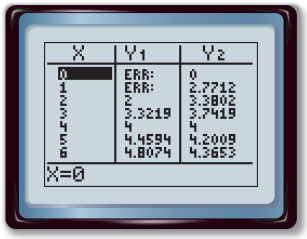
Step 3 **Use the TABLE feature.**

KEYSTROKES: *See page 508.*

Examine the table to find the x-value for which the y-values for the graphs are equal. At $x = 4$, both functions have a y-value of 4. Thus, the solution of the equation is 4.



You can use a similar procedure to solve logarithmic inequalities using a graphing calculator.

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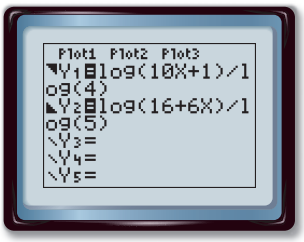
ACTIVITY 2 Solve $\log_4 (10x + 1) < \log_5 (16 + 6x)$.

Step 1 **Enter the inequalities.**

Rewrite the problem as a system of inequalities.

The first inequality is $\log_4 (10x + 1) < y$, which can be written as $y > \log_4 (10x + 1)$. Since this inequality includes the *greater than* symbol, shade above the curve. First enter the boundary and then use the arrow and ENTER keys to choose the shade above icon, ▀.



The second inequality is $y < \log_5 (16 + 6x)$. Shade below the curve since this inequality contains *less than*.

KEYSTROKES: Y= ◀ ◀ ENTER ENTER ▶ ▶ LOG 10 X,T,θ,*n* + 1) ÷ LOG 4) ENTER ◀ ◀ ENTER ENTER ENTER ▶ ▶ LOG 16 + 6 X,T,θ,*n*) ÷ LOG 5)

Step 2 **Graph the system.**

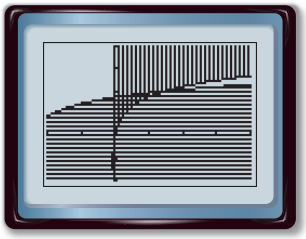
KEYSTROKES: GRAPH

The left boundary of the solution set is where the first inequality is undefined. It is undefined for $10x + 1 \leq 0$.

$10x + 1 \leq 0$

$10x \leq -1$

$x \leq -\frac{1}{10}$



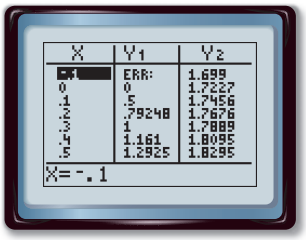
[−2, 8] scl: 1 by [−2, 8] scl: 1

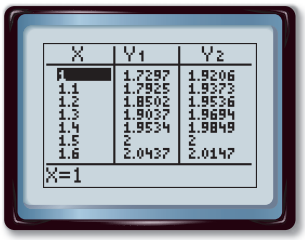
Use the calculator's intersect feature to find the right boundary. You can conclude that the solution set is $\{x \mid -0.1 < x < 1.5\}$.

Step 3 **Use the TABLE feature to check your solution.**

Start the table at −0.1 and show *x*-values in increments of 0.1. Scroll through the table.

KEYSTROKES: 2nd [TBLSET] −0.1 ENTER .5 ENTER 2nd [TABLE]





The table confirms the solution of the inequality is $\{x \mid -0.1 < x < 1.5\}$.

EXERCISES

Solve each equation or inequality. Check your solution.

1. $\log_2 (3x + 2) = \log_3 (12x + 3)$
2. $\log_6 (7x + 1) = \log_4 (4x - 4)$
3. $\log_2 3x = \log_3 (2x + 2)$
4. $\log_{10} (1 - x) = \log_5 (2x + 5)$
5. $\log_4 (9x + 1) > \log_3 (18x - 1)$
6. $\log_3 (3x - 5) \geq \log_3 (x + 7)$
7. $\log_5 (2x + 1) < \log_4 (3x - 2)$
8. $\log_2 2x \leq \log_4 (x + 3)$

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9-5 Base *e* and Natural Logarithms

Main Ideas

- Evaluate expressions involving the natural base and natural logarithms.
- Solve exponential equations and inequalities using natural logarithms.

New Vocabulary

natural base, *e*

natural base exponential function

natural logarithm

natural logarithmic function

GET READY for the Lesson

Suppose a bank compounds interest on accounts *continuously*, that is, with no waiting time between interest payments.

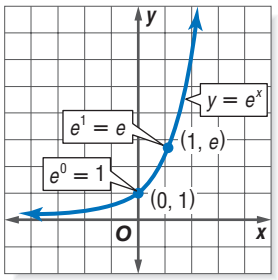
To develop an equation to determine continuously compounded interest, examine what happens to the value A of an account for increasingly larger numbers of compounding periods n. Use a principal P of \$1, an interest rate r of 100% or 1, and time t of 1 year.

Continuously Compounded Interest

n	$A=P\left(1+\frac{r}{n}\right)^{nt}$	A
1 (yearly)	$1\left(1+\frac{1}{1}\right)^{1(1)}$	2
4 (quarterly)	$1\left(1+\frac{1}{4}\right)^{4(1)}$	2.4414...
12 (monthly)	$1\left(1+\frac{1}{12}\right)^{12(1)}$	2.6130...
365 (daily)	$1\left(1+\frac{1}{365}\right)^{365(1)}$	2.7145...
8760 (hourly)	$1\left(1+\frac{1}{8760}\right)^{8760(1)}$	2.7181...

Base *e* and Natural Logarithms In the table above, as n increases, the expression $1\left(1+\frac{1}{n}\right)^{n(1)}$ or $\left(1+\frac{1}{n}\right)^{n}$ approaches the irrational number 2.71828.... This number is referred to as the **natural base, *e*.**

An exponential function with base e is called a **natural base exponential function**. The graph of $y = e^x$ is shown at the right. Natural base exponential functions are used extensively in science to model quantities that grow and decay continuously.



Most calculators have an e^x function for evaluating natural base expressions.

Study Tip

Simplifying Expressions with *e*

You can simplify expressions involving e in the same manner in which you simplify expressions involving π.

Examples:

- $\pi^2 \cdot \pi^3 = \pi^5$
- $e^2 \cdot e^3 = e^5$

EXAMPLE 1 Evaluate Natural Base Expressions

Use a calculator to evaluate each expression to four decimal places.

a. e^2 **KEYSTROKES:** 2nd [e^x] 2 ENTER 7.389056099

$e^2 \approx 7.3891$

b. $e^{-1.3}$ **KEYSTROKES:** 2nd [e^x] −1.3 ENTER .272531793

$e^{-1.3} \approx 0.2725$

CHECK Your Progress

1A. e^5 **1B.** $e^{-2.2}$

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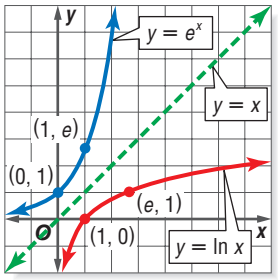
The logarithm with base e is called the **natural logarithm**, sometimes denoted by $\log_e x$, but more often abbreviated $\ln x$. The **natural logarithmic function**, $y = \ln x$, is the inverse of the natural base exponential function, $y = e^x$. The graph of these two functions shows that $\ln 1 = 0$ and $\ln e = 1$.



Study Tip

Calculator Keystrokes

On graphing calculators, you press the LN key before the number. On other calculators, usually you must type the number before pressing the LN key.

Most calculators have an LN key for evaluating natural logarithms.

EXAMPLE Evaluate Natural Logarithmic Expressions

2 **Use a calculator to evaluate each expression to four decimal places.**

a. $\ln 4$ KEYSTROKES: LN 4 ENTER 1.386294361

$\ln 4 \approx 1.3863$

b. $\ln 0.05$ KEYSTROKES: LN 0.05 ENTER −2.995732274

$\ln 0.05 \approx -2.9957$

CHECK Your Progress

2A. $\ln 7$ **2B.** $\ln 0.25$

You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that $\ln x = \log_e x$.

EXAMPLE Write Equivalent Expressions

3 **Write an equivalent exponential or logarithmic equation.**

a. $e^x = 5$

$e^x = 5 \longrightarrow \log_e 5 = x$

$\ln 5 = x$

b. $\ln x \approx 0.6931$

$\ln x \approx 0.6931 \longrightarrow \log_e x \approx 0.6931$

$x \approx e^{0.6931}$

CHECK Your Progress

3A. $e^x = 6$ **3B.** $\ln x \approx 0.5352$

Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to "undo" each other.

$$e^{\ln x} = x \qquad \ln e^x = x$$

For example, $e^{\ln 7} = 7$ and $\ln e^{4x+3} = 4x + 3$.

Equations and Inequalities with *e* and ln Equations and inequalities involving base e are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

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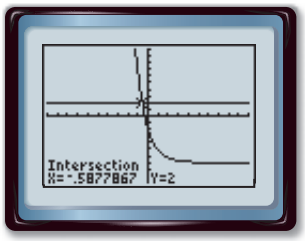
EXAMPLE Solve Base *e* Equations

4 **Solve $5e^{-x} - 7 = 2$. Round to the nearest ten-thousandth.**

$5e^{-x} - 7 = 2$	Original equation
$5e^{-x} = 9$	Add 7 to each side.
$e^{-x} = \frac{9}{5}$	Divide each side by 5.
$\ln e^{-x} = \ln \frac{9}{5}$	Property of Equality for Logarithms
$-x = \ln \frac{9}{5}$	Inverse Property of Exponents and Logarithms
$x = -\ln \frac{9}{5}$	Divide each side by -1.
$x \approx -0.5878$	Use a calculator.

The solution is about -0.5878.

CHECK You can check this value by substituting -0.5878 into the original equation and evaluating, or by finding the intersection of the graphs of $y = 5e^{-x} - 7$ and $y = 2$.



CHECK Your Progress

Solve each equation. Round to the nearest ten-thousandth.

4A. $3e^{x} + 2 = 4$ **4B.** $4e^{-x} - 9 = -2$

Study Tip

Continuously Compounded Interest

Although no banks actually pay interest compounded continuously, the equation $A = Pe^{rt}$ is so accurate in computing the amount of money for quarterly compounding, or daily compounding, that it is often used for this purpose.

When interest is compounded continuously, the amount A in an account after t years is found using the formula $A = Pe^{rt}$, where P is the amount of principal and r is the annual interest rate.

Real-World EXAMPLE Solve Base *e* Inequalities

5 **SAVINGS** **Suppose you deposit \$1000 in an account paying 2.5% annual interest, compounded continuously.**

a. What is the balance after 10 years?

$A = Pe^{rt}$	Continuous compounding formula
$= 1000e^{(0.025)(10)}$	Replace P with 1000, r with 0.025, and t with 10.
$= 1000e^{0.25}$	Simplify.
≈ 1284.03	Use a calculator.

The balance after 10 years would be \$1284.03.

CHECK If the account was earning simple interest, the formula for the interest, would be $I = prt$. In that case, the interest would be $I = (1000)(0.025)(10)$ or \$250. Continuously compounded interest should be greater than simple interest at the same rate. Thus, the solution \$1284.03 is reasonable.

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b. How long will it take for the balance in your account to reach at least $1500?

Words	The balance is at least $1500.
Variable	Let A represent the amount in the account.
Inequality	$A \geq 1500$

$\ln e^{(0.025)t} \geq 1500$ Replace A with $1000e^{(0.025)t}$.

$\ln e^{(0.025)t} \geq 1.5$ Divide each side by 1000.

$\ln e^{(0.025)t} \geq \ln 1.5$ Property of Equality for Logarithms

$0.025t \geq \ln 1.5$ Inverse Property of Exponents and Logarithms

$t \geq \frac{\ln 1.5}{0.025}$ Divide each side by 0.025.

$t \geq 16.22$ Use a calculator.

It will take at least 16.22 years for the balance to reach $1500.

CHECK Your Progress

Suppose you deposit $5000 in an account paying 3% annual interest, compounded continuously.

5A. What is the balance after 5 years?

5B. How long will it take for the balance in your account to reach at least $7000?

Personal Tutor at algebra2.com

EXAMPLE 6 Solve Natural Log Equations and Inequalities

Solve each equation or inequality. Round to the nearest ten-thousandth.

a. $\ln 5x = 4$

$\ln 5x = 4$ Original equation

$e^{\ln 5x} = e^4$ Write each side using exponents and base e.

$5x = e^4$ Inverse Property of Exponents and Logarithms

$x = \frac{e^4}{5}$ Divide each side by 5.

$x \approx 10.9196$ Use a calculator. Check using substitution or graphing.

b. $\ln (x - 1) > -2$

$\ln (x - 1) > -2$ Original inequality

$e^{\ln (x - 1)} > e^{-2}$ Write each side using exponents and base e.

$x - 1 > e^{-2}$ Inverse Property of Exponents and Logarithms

$x > e^{-2} + 1$ Add 1 to each side.

$x > 1.1353$ Use a calculator. Check using substitution.

Study Tip

Equations with ln

As with other logarithmic equations, remember to check for extraneous solutions.

CHECK Your Progress

6A. $\ln 3x = 7$ **6B.** $\ln (3x + 2) < 5$

Extra Examples at algebra2.com

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CHECK Your Understanding

Examples 1, 2 (pp. 536, 537)

Use a calculator to evaluate each expression to four decimal places.

1. e^6 **2.** $e^{-3.4}$ **3.** $e^{0.35}$

4. $\ln 1.2$ **5.** $\ln 0.1$ **6.** $\ln 3.25$

Example 3 (p. 537)

Write an equivalent exponential or logarithmic equation.

7. $e^x = 4$ **8.** $\ln 1 = 0$

Example 4 (p. 538)

Solve each equation. Round to the nearest ten-thousandth.

9. $2e^x - 5 = 1$ **10.** $3 + e^{-2x} = 8$

Example 5 (pp. 538–539)

ALTITUDE **For Exercises 11 and 12, use the following information.**
The altimeter in an airplane gives the altitude or height h (in feet) of a plane above sea level by measuring the outside air pressure P (in kilopascals).

The height and air pressure are related by the model $P = 101.3\, e^{-\frac{h}{26{,}200}}$.

11. Find a formula for the height in terms of the outside air pressure.

12. Use the formula you found in Exercise 11 to approximate the height of a plane above sea level when the outside air pressure is 57 kilopascals.

Example 6 (p. 539)

Solve each equation or inequality. Round to the nearest ten-thousandth.

13. $e^x > 30$ **14.** $\ln x < 6$

15. $2 \ln 3x + 1 = 5$ **16.** $\ln x^2 = 9$

Exercises

HOMEWORK HELP

For Exercises	See Examples
17–20	1
21–24	2
25–32	3
33–40	4
41–46	5
47–54	6

Use a calculator to evaluate each expression to four decimal places.

17. e^4 **18.** e^5 **19.** $e^{-1.2}$ **20.** $e^{0.5}$

21. $\ln 3$ **22.** $\ln 10$ **23.** $\ln 5.42$ **24.** $\ln 0.03$

Write an equivalent exponential or logarithmic equation.

25. $e^{-x} = 5$ **26.** $e^2 = 6x$ **27.** $\ln e = 1$ **28.** $\ln 5.2 = x$

29. $e^{x+1} = 9$ **30.** $e^{-1} = x^2$ **31.** $\ln \frac{7}{3} = 2x$ **32.** $\ln e^x = 3$

Solve each equation. Round to the nearest ten-thousandth.

33. $3e^x + 1 = 5$ **34.** $2e^x - 1 = 0$ **35.** $-3e^{4x} + 11 = 2$ **36.** $8 + 3e^{3x} = 26$

37. $2e^x - 3 = -1$ **38.** $-2e^x + 3 = 0$ **39.** $-2 + 3e^{3x} = 7$ **40.** $1 - \frac{1}{3}e^{5x} = -5$

POPULATION **For Exercises 41 and 42, use the following information.**
In 2005, the world's population was about 6.5 billion. If the world's population continues to grow at a constant rate, the future population P, in billions, can be predicted by $P = 6.5e^{0.02t}$, where t is the time in years since 2005.

41. According to this model, what will the world's population be in 2015?

42. Some experts have estimated that the world's food supply can support a population of at most 18 billion. According to this model, for how many more years will the food supply be able to support the trend in world population growth?

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Real-World Link

To determine the doubling time on an account paying an interest rate r that is compounded annually, investors use the "Rule of 72." Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $\frac{72}{6}$ or 12 years.

Source: datachimp.com

MONEY **For Exercises 43–46, use the formula for continuously compounded interest found in Example 5.**

43. If you deposit \$100 in an account paying 3.5% interest compounded continuously, how long will it take for your money to double?

44. Suppose you deposit A dollars in an account paying an interest rate of r, compounded continuously. Write an equation giving the time t needed for your money to double, or the *doubling time*.

45. Explain why the equation you found in Exercise 44 might be referred to as the "Rule of 70."

46. MAKE A CONJECTURE State a rule that could be used to approximate the amount of time t needed to triple the amount of money in a savings account paying r percent interest compounded continuously.

Solve each equation or inequality. Round to the nearest ten-thousandth.

47. $\ln 2x = 4$ **48.** $\ln 3x = 5$ **49.** $\ln (x + 1) = 1$ **50.** $\ln (x - 7) = 2$

51. $e^x < 4.5$ **52.** $e^x > 1.6$ **53.** $e^{5x} \geq 25$ **54.** $e^{-2x} \leq 7$

E-MAIL **For Exercises 55 and 56, use the following information.**
The number of people N who will receive a forwarded e-mail can be approximated by $N = \frac{P}{1 + (P - S)e^{-0.35t}}$, where P is the total number of people online, S is the number of people who start the e-mail, and t is the time in minutes. Suppose four people want to send an e-mail to all those who are online at that time.

55. If there are 156,000 people online, how many people will have received the e-mail after 25 minutes?

56. How much time will pass before half of the people will receive the e-mail?

EXTRA PRACTICE
See pages 911, 934.
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Self-Check Quiz at algebra2.com

Solve each equation. Round to the nearest ten-thousandth.

57. $\ln x + \ln 3x = 12$ **58.** $\ln 4x + \ln x = 9$

59. $\ln (x^2 + 12) = \ln x + \ln 8$ **60.** $\ln x + \ln (x + 4) = \ln 5$

H.O.T. Problems

61. OPEN ENDED Give an example of an exponential equation that requires using natural logarithms instead of common logarithms to solve.

62. FIND THE ERROR Colby and Elsu are solving $\ln 4x = 5$. Who is correct? Explain your reasoning.

Colby
$\ln 4x = 5$
$10^{\ln 4x} = 10^5$
$4x = 100{,}000$
$x = 25{,}000$

Elsu
$\ln 4x = 5$
$e^{\ln 4x} = e^5$
$4x = e^5$
$x = \frac{e^5}{4}$
$x \approx 37.1033$

63. CHALLENGE Determine whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

For all positive numbers x and y, $\frac{\log x}{\log y} = \frac{\ln x}{\ln y}$.

Jim Craigmyle/Masterfile
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64. ***Writing in Math*** Use the information about banking on page 536 to explain how the natural base e is used in banking. Include an explanation of how to calculate the value of an account whose interest is compounded continuously, and an explanation of how to use natural logarithms to find the time at which the account will have a specified value in your answer.

STANDARDIZED TEST PRACTICE

65. ACT/SAT A recent study showed that the number of Australian homes with a computer doubles every 8 months. Assuming that the number is increasing continuously, at approximately what monthly rate must the number of Australian computer owners be increasing for this to be true?

A 68%

B 8.66%

C 0.0866%

D 0.002%

66. REVIEW Which is the first *incorrect* step in simplifying $\log_3 \frac{3}{48}$?

Step 1: $\log_3 \frac{3}{48} = \log_3 3 - \log_3 48$

Step 2: $= 1 - 16$

Step 3: $= -15$

F Step 1

G Step 2

H Step 3

J Each step is correct.

Spiral Review

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

67. $\log_4 68$ **68.** $\log_6 0.047$ **69.** $\log_{50} 23$

Solve each equation. Check your solutions. (Lesson 9-3)

70. $\log_3 (a + 3) + \log_3 (a - 3) = \log_3 16$ **71.** $\log_{11} 2 + 2 \log_{11} x = \log_{11} 32$

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation. (Lesson 8-4)

72. $mn = 4$ **73.** $\frac{a}{b} = c$ **74.** $y = -7x$

75. BASKETBALL Alexis has never scored a 3-point field goal, but she has scored a total of 59 points so far this season. She has made a total of 42 shots including free throws and 2-point field goals. How many free throws and 2-point field goals has Alexis scored? (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest hundredth.** (Lesson 9-1)

76. $2^x = 10$ **77.** $5^x = 12$ **78.** $6^x = 13$

79. $2(1 + 0.1)^x = 50$ **80.** $10(1 + 0.25)^x = 200$ **81.** $400(1 - 0.2)^x = 50$

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READING MATH

Double Meanings

In mathematics, many words have specific definitions. However, when these words are used in everyday language, they frequently have a different meaning. Study each pair of sentences. How does the meaning of the word in boldface differ?

A. The number of boards that can be cut from a **log** depends on the size of the log.
B. The **log** of a number with base b represents the exponent to which b must be raised to produce that number.

A. Tinted paints are produced by adding small amounts of color to a **base** of white paint.
B. In the expression $\log_b x$, b is referred to as the **base** of the logarithm.

A. When a plant dies, it will **decay**, changing in form and substance, until it appears that the plant has disappeared.
B. If a quantity y satisfies a relationship of the form $y = ae^{-kt}$, the quantity y is described by an exponential **decay** model.

Read the following property and paragraph below. Which words are mathematical words? Which words are ordinary words? Which mathematical words have another meaning in everyday language?

KEY CONCEPT ***Product Property of Radicals***

For any nonnegative real numbers a and b and any integer n greater than 1, $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Simplifying a square root means finding the square root of the greatest perfect square factor of the radicand. You can use the product property of radicals to simplify square roots.

Exercises

Write two sentences for each word. First, use the word in everyday language. Then use the word in a mathematical context.

1. index	**2.** negative	**3.** even	**4.** rational
5. irrational	**6.** like	**7.** rationalize	**8.** coordinates
9. real	**10.** degree	**11.** absolute	**12.** identity

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9-6 Exponential Growth and Decay

Main Ideas

- Use logarithms to solve problems involving exponential decay.
- Use logarithms to solve problems involving exponential growth.

New Vocabulary

rate of decay
rate of growth

GET READY for the Lesson

Certain assets, like homes, can *appreciate* or increase in value over time. Others, like cars, *depreciate* or decrease in value with time. Suppose you buy a car for $22,000 and the value of the car decreases by 16% each year. The table shows the value of the car each year for up to 5 years after it was purchased.

Years after Purchase	Value of Car ($)
0	22,000.00
1	18,480.00
2	15,523.20
3	13,039.49
4	10,953.17
5	9200.66

Exponential Decay The depreciation of the value of a car is an example of exponential decay. When a quantity *decreases* by a fixed percent each year, or other period of time, the amount y of that quantity after t years is given by $y = a(1 - r)^t$, where a is the initial amount and r is the percent of decrease expressed as a decimal. The percent of decrease r is also referred to as the **rate of decay**.

EXAMPLE Exponential Decay of the Form $y = a(1 - r)^t$

1 **CAFFEINE** **A cup of coffee contains 130 milligrams of caffeine. If caffeine is eliminated from the body at a rate of 11% per hour, how long will it take for half of this caffeine to be eliminated?**

Explore The problem gives the amount of caffeine consumed and the rate at which the caffeine is eliminated. It asks you to find the time it will take for half of the caffeine to be eliminated.

Plan Use the formula $y = a(1 - r)^t$. Let t be the number of hours since drinking the coffee. The amount remaining y is half of 130 or 65.

Solve

$y = a(1 - r)^t$ Exponential decay formula

$65 = 130(1 - 0.11)^t$ Replace y with 65, a with 130, and r with 11% or 0.11.

$0.5 = (0.89)^t$ Divide each side by 130.

$\log 0.5 = \log (0.89)^t$ Property of Equality for Logarithms

$\log 0.5 = t \log (0.89)$ Power Property for Logarithms

$\frac{\log 0.5}{\log 0.89} = t$ Divide each side by log 0.89.

$5.9480 \approx t$ Use a calculator.

It will take approximately 6 hours.

Study Tip

Rate of Change
Remember to rewrite the rate of change as a decimal before using it in the formula.

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Check Use the formula to find how much of the original 130 milligrams of caffeine would remain after 6 hours.

$$y = a(1 - r)^t \qquad \text{Exponential decay formula}$$
$$= 130(1 - 0.11)^6 \qquad \text{Replace } a \text{ with 130, } r \text{ with 0.11, and } t \text{ with 6.}$$
$$\approx 64.6 \qquad \text{Use a calculator.}$$

Half of 130 is 65, so the answer seems reasonable. Half of the caffeine will be eliminated from the body in about 6 hours.

CHECK Your Progress

1. **SHOPPING** A store is offering a clearance sale on a certain type of digital camera. The original price for the camera was $198. The price decreases 10% each week until all of the cameras are sold. How many weeks will it take for the price of the cameras to drop below half of the original price?

Another model for exponential decay is given by $y = ae^{-kt}$, where k is a constant. This is the model preferred by scientists. Use this model to solve problems involving radioactive decay. Radioactive decay is the decrease in the intensity of a radioactive material over time. Being able to solve problems involving radioactive decay allows scientists to use carbon dating methods.

EXAMPLE 2 Exponential Decay of the Form $y = ae^{-kt}$

Real-World Career
Paleontologist
Paleontologists study fossils found in geological formations. They use these fossils to trace the evolution of plant and animal life and the geologic history of Earth.

MathOnline
For more information, go to algebra2.com.

PALEONTOLOGY **The *half-life* of a radioactive substance is the time it takes for half of the atoms of the substance to disintegrate. All life on Earth contains Carbon-14, which decays continuously at a fixed rate. The half-life of Carbon-14 is 5760 years. That is, every 5760 years half of a mass of Carbon-14 decays away.**

a. What is the value of k and the equation of decay for Carbon-14?

Let a be the initial amount of the substance. The amount y that remains after 5760 years is then represented by $\frac{1}{2}a$ or $0.5a$.

$$y = ae^{-kt} \qquad \text{Exponential decay formula}$$
$$0.5a = ae^{-k(5760)} \qquad \text{Replace } y \text{ with } 0.5a \text{ and } t \text{ with 5760.}$$
$$0.5 = e^{-5760k} \qquad \text{Divide each side by } a.$$
$$\ln 0.5 = \ln e^{-5760k} \qquad \text{Property of Equality for Logarithmic Functions}$$
$$\ln 0.5 = -5760k \qquad \text{Inverse Property of Exponents and Logarithms}$$
$$\frac{\ln 0.5}{-5760} = k \qquad \text{Divide each side by } -5760.$$
$$\frac{-0.6931472}{-5760} \approx k \qquad \text{Use a calculator.}$$
$$0.00012 \approx k \qquad \text{Simplify.}$$

The value of k for Carbon-14 is 0.00012. Thus, the equation for the decay of Carbon-14 is $y = ae^{-0.00012t}$, where t is given in years.

(continued on the next page)

Richard T. Nowitz/Photo Researchers
www.MathSchoolinternational.com

CHECK Use the formula to find the amount of a sample remaining after 5760 years. Use an original amount of 1.

$y = ae^{-0.00012t}$ Original equation

$= 1e^{-0.00012(5760)}$ $a = 1$ and $t = 5760$

≈ 0.501 Use a calculator.

About half of the amount remains. The answer checks.

b. A paleontologist examining the bones of a woolly mammoth estimates that they contain only 3% as much Carbon-14 as they would have contained when the animal was alive. How long ago did the mammoth die?

Let a be the initial amount of Carbon-14 in the animal's body. Then the amount y that remains after t years is 3% of a or $0.03a$.

$y = ae^{-0.00012t}$ Formula for the decay of Carbon-14

$0.03a = ae^{-0.00012t}$ Replace y with $0.03a$.

$0.03 = e^{-0.00012t}$ Divide each side by a.

$\ln 0.03 = \ln e^{-0.00012t}$ Property of Equality for Logarithms

$\ln 0.03 = -0.00012t$ Inverse Property of Exponents and Logarithms

$\frac{\ln 0.03}{-0.00012} = t$ Divide each side by -0.00012.

$29{,}221 \approx t$ Use a calculator.

The mammoth lived about 29,000 years ago.

CHECK Your Progress

2. A specimen that originally contained 150 milligrams of Carbon-14 now contains 130 milligrams. How old is the fossil?

Exponential Growth When a quantity *increases* by a fixed percent each time period, the amount y of that quantity after t time periods is given by $y = a(1 + r)^t$, where a is the initial amount and r is the percent of increase expressed as a decimal. The percent of increase r is also referred to as the **rate of growth**.

Test-Taking Tip

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left.

1.5% = 0.015

STANDARDIZED TEST EXAMPLE

3 In 1910, the population of a city was 120,000. Since then, the population has increased by 1.5% per year. If the population continues to grow at this rate, what will the population be in 2010?

A 138,000 **B** 531,845 **C** 1,063,690 **D** 1.4×10^{11}

Read the Test Item

You need to find the population of the city 2010 − 1910, or 100, years later. Since the population is growing at a fixed percent each year, use the formula $y = a(1 + r)^t$.

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Real-World Link

The Indian city of Varanasi is the world's oldest continuously inhabited city.

Source: tourismofindia.com

Solve the Test Item

$y = a(1 + r)^t$ Exponential growth formula

$= 120{,}000(1 + 0.015)^{100}$ Replace a with 120,000, r with 0.015, and t with 2010 − 1910, or 100.

$= 120{,}000(1.015)^{100}$ Simplify.

$\approx 531{,}845.48$ Use a calculator.

The answer is B.

CHECK Your Progress

3. Home values in Millersport increase about 4% per year. Mr. Thomas purchased his home eight years ago for $122,000. What is the value of his home now?

F $\$1.36 \times 10^5$ **G** $126,880 **H** $166,965 **J** $175,685

Online Personal Tutor at algebra2.com

Another model for exponential growth, preferred by scientists, is $y = ae^{kt}$, where k is a constant. Use this model to find the constant k.

EXAMPLE Exponential Growth of the Form $y = ae^{kt}$

4 POPULATION **As of 2005, China was the world's most populous country, with an estimated population of 1.31 billion people. The second most populous country was India, with 1.08 billion. The populations of India and China can be modeled by $I(t) = 1.08e^{0.0103t}$ and $C(t) = 1.31e^{0.0038t}$, respectively. According to these models, when will India's population be more than China's?**

You want to find t, the number of years, such that $I(t) > C(t)$.

$I(t) > C(t)$

$1.08e^{0.0103t} > 1.31e^{0.0038t}$ Replace $I(t)$ with $1.08e^{0.0103t}$ and $C(t)$ with $1.31e^{0.0038t}$.

$\ln 1.08e^{0.0103t} > \ln 1.31e^{0.0038t}$ Property of Inequality for Logarithms

$\ln 1.08 + \ln e^{0.0103t} > \ln 1.31 + \ln e^{0.0038t}$ Product Property of Logarithms

$\ln 1.08 + 0.0103t > \ln 1.31 + 0.0038t$ Inverse Property of Exponents and Logarithms

$0.0065t > \ln 1.31 - \ln 1.08$ Subtract $0.0038t$ from each side.

$t > \dfrac{\ln 1.31 - \ln 1.08}{0.0065}$ Divide each side by 0.006.

$t > 29.70$ Use a calculator.

After 30 years, or in 2035, India will be the most populous country.

Concepts in Motion

Interactive Lab
algebra2.com

CHECK Your Progress

4. BACTERIA Two different types of bacteria in two different cultures reproduce exponentially. The first type can be modeled by $B_1(t) = 1200\ e^{0.1532t}$, and the second can be modeled by $B_2(t) = 3000\ e^{0.0466t}$, where t is the number of hours. According to these models, how many hours will it take for the amount of B_1 to exceed the amount of B_2?

Getty Images
www.MathSchoolinternational.com

CHECK Your Understanding

Example 1 (pp. 544–545)

1. **POLICE** Police use blood alcohol content (BAC) to measure the percent concentration of alcohol in a person's bloodstream. In most states, a BAC of 0.08 percent means a person is not allowed to drive. Each hour after drinking, a person's BAC may decrease by 15%. If a person has a BAC of 0.18, how many hours will he need to wait until he can legally drive?

Example 2 (pp. 545–546)

SPACE **For Exercises 2–4, use the following information.**
A radioisotope is used as a power source for a satellite. The power output P (in watts) is given by $P = 50\,e^{-\frac{t}{250}}$, where t is the time in days.

2. Is the formula for power output an example of exponential growth or decay? Explain your reasoning.
3. Find the power available after 100 days.
4. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate?

Example 3 (pp. 546–547)

5. **STANDARDIZED TEST PRACTICE** The weight of a bar of soap decreases by 2.5% each time it is used. If the bar weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses?

A 57.5 g **B** 59.4 g **C** 65 g **D** 93 g

Example 4 (p. 547)

POPULATION GROWTH **For Exercises 6 and 7, use the following information.**
Fayette County, Kentucky, grew from a population of 260,512 in 2000 to a population of 268,080 in 2005.

6. Write an exponential growth equation of the form $y = ae^{kt}$ for Fayette County, where t is the number of years after 2000.
7. Use your equation to predict the population of Fayette County in 2015.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8	1
9–11	2
12–14	3
15, 16	4

8. **COMPUTERS** Zeus Industries bought a computer for $2500. If it depreciates at a rate of 20% per year, what will be its value in 2 years?

9. **HEALTH** A certain medication is eliminated from the bloodstream at a steady rate. It decays according to the equation $y = ae^{-0.1625t}$, where t is in hours. Find the half-life of this substance.

10. **PALENTOLOGY** A paleontologist finds a bone of a human. In the laboratory, she finds that the Carbon-14 found in the bone is $\frac{2}{3}$ of that found in living bone tissue. How old is this bone?

11. **ANTHROPOLOGY** An anthropologist studying the bones of a prehistoric person finds there is so little remaining Carbon-14 in the bones that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. How long ago did the person die?

12. **REAL ESTATE** The Martins bought a condominium for $145,000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years?

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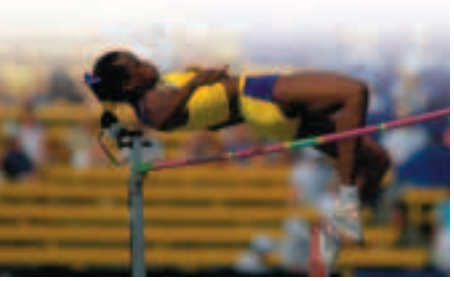


Real-World Link

The women's high jump competition first took place in the USA in 1895, but it did not become an Olympic event until 1926.

Source: www.princeton.edu

ECONOMICS **For Exercises 13 and 14, use the following information.**
The annual Gross Domestic Product (GDP) of a country is the value of all of the goods and services produced in the country during a year. During the period 2001–2004, the Gross Domestic Product of the United States grew about 2.8% per year, measured in 2004 dollars. In 2001, the GDP was $9891 billion.

13. Assuming this rate of growth continues, what will the GDP of the United States be in the year 2015?

14. In what year will the GDP reach $20 trillion?

BIOLOGY **For Exercises 15 and 16, use the following information.**
Bacteria usually reproduce by a process known as *binary fission*. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes.

15. Find the constant k for this type of bacteria under ideal conditions.

16. Write the equation for modeling the exponential growth of this bacterium.

17. **OLYMPICS** In 1928, when the high jump was first introduced as a women's sport at the Olympic Games, the winning women's jump was 62.5 inches, while the winning men's jump was 76.5 inches. Since then, the winning jump for women has increased by about 0.38% per year, while the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women's winning high jump be higher than the men's?

18. **HOME OWNERSHIP** The Mendes family bought a new house 10 years ago for $120,000. The house is now worth $191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation?

FOOD **For Exercises 19 and 20, use the table of suggested times for cooking potatoes in a microwave oven. Assume that the number of minutes is a function of some power of the number of potatoes.**

Number of 8 oz. Potatoes	Cooking Time (min)
2	10
4	15

Source: wholehealthmd.com

EXTRA PRACTICE
See pages 911, 934.
MathOnline
Self-Check Quiz at algebra2.com

19. Write an equation in the form $t = an^b$, where t is the time in minutes, n is the number of potatoes, and a and b are constants. (*Hint:* Use a system of equations to find the constants.)

20. According to the formula, how long should you cook six 8-ounce potatoes in a microwave?

H.O.T. Problems

21. **REASONING** Explain how to solve $y = (1 + r)^t$ for t.

22. **OPEN ENDED** Give an example of a quantity that grows or decays at a fixed rate. Write a real-world problem involving the rate and solve by using logarithms.

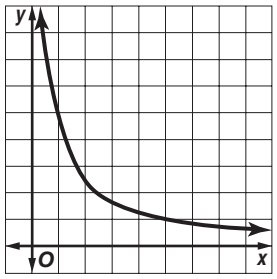
23. **CHALLENGE** The half-life of radium is 1620 years. When will a 20-gram sample of radium be completely gone? Explain your reasoning.

24. ***Writing in Math*** Use the information about car values on page 544 to explain how you can use exponential decay to determine the current value of a car. Include a description of how to find the percent decrease in the value of the car each year and a description of how to find the value of a car for any given year when the rate of depreciation is known.

Karl Weatherly/CORBIS
www.MathSchoolinternational.com

STANDARDIZED TEST PRACTICE

25. ACT/SAT The curve represents a portion of the graph of which function?



A $y = 50 - x$ **C** $y = e^{-x}$

B $y = \log x$ **D** $xy = 5$

26. REVIEW A radioactive element decays over time, according to the equation

$$y = x\left(\frac{1}{4}\right)^{\frac{t}{200}},$$

where x = the number of grams present initially and t = time in years. If 500 grams were present initially, how many grams will remain after 400 years?

F 12.5 grams **H** 62.5 grams

G 31.25 grams **J** 125 grams

Spiral Review

Write an equivalent exponential or logarithmic equation. (Lesson 9-5)

27. $e^3 = y$ **28.** $e^{4n-2} = 29$ **29.** $\ln 4 + 2 \ln x = 8$

Solve each equation or inequality. Round to four decimal places. (Lesson 9-4)

30. $16^x = 70$ **31.** $2^{3p} > 1000$ **32.** $\log_b 81 = 2$

BUSINESS **For Exercises 33–35, use the following information.**
A small corporation decides that 8% of its profits would be divided among its six managers. There are two sales managers and four nonsales managers. Fifty percent would be split equally among all six managers. The other 50% would be split among the four nonsales managers. Let p represent the profits. (Lesson 8-2)

33. Write an expression to represent the share of the profits each nonsales manager will receive.

34. Simplify this expression.

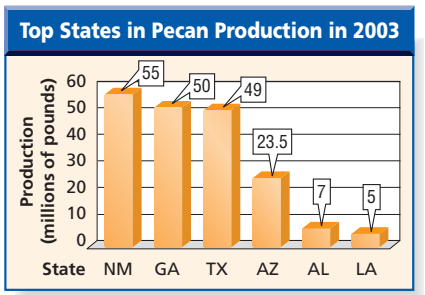
35. Write an expression in simplest form to represent the share of the profits each sales manager will receive.

AGRICULTURE **For Exercises 36–38, use the graph at the right.**
U.S. growers were forecasted to produce 264 million pounds of pecans in 2003. (Lesson 6-1)

36. Write the number of pounds of pecans forecasted by U.S. growers in 2003 in scientific notation.

37. Write the number of pounds of pecans produced by Georgia in 2003 in scientific notation.

38. What percent of the overall pecan production for 2003 can be attributed to Georgia?



Source: www.nass.usda.gov

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EXTEND 9-6

Graphing Calculator Lab

Cooling

In this lab, you will explore the type of equation that models the change in the temperature of water as it cools under various conditions.

SET UP the Lab

- Collect a variety of containers, such as a foam cup, a ceramic coffee mug, and an insulated cup.
- Boil water or collect hot water from a tap.
- Choose a container to test and fill with hot water. Place the temperature probe in the cup.
- Connect the temperature probe to your data collection device.

ACTIVITY

Step 1 Program the device to collect 20 or more samples in 1-minute intervals.

Step 2 Wait a few seconds for the probe to warm to the temperature of the water.

Step 3 Press the button to begin collecting data.

Analyze the Results

1. When the data collection is complete, graph the data in a scatter plot. Use time as the independent variable and temperature as the dependent variable. Write a sentence that describes the points on the graph.
2. Use the STAT menu to find an equation to model the data you collected. Try linear, quadratic, and exponential models. Which model appears to fit the data best? Explain.
3. Would you expect the temperature of the water to drop below the temperature of the room? Explain your reasoning.
4. Use the data collection device to find the temperature of the air in the room. Graph the function $y = t$, where t is the temperature of the room along with the scatter plot and the model equation. Describe the relationship among the graphs. What is the meaning of the relationship in the context of the experiment?

Make a Conjecture

5. Do you think the results of the experiment would change if you used an insulated container? Repeat the experiment to verify your conjecture.
6. How might the results of the experiment change if you added ice to the water? Repeat the experiment to verify your conjecture.

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CHAPTER 9 Study Guide and Review

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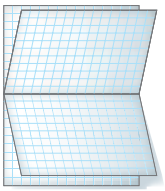
FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Exponential Functions (Lesson 9-1)

- An exponential function is in the form $y = ab^x$, where $a \neq 0$, $b > 0$ and $b \neq 1$.
- Property of Equality for Exponential Functions: If b is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.
- Property of Inequality for Exponential Functions: If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithms and Logarithmic Functions (Lessons 9-2 through 9-4)

- Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number y such that $\log_b x = y$ if and only if $b^y = x$.
- The logarithm of a product is the sum of the logarithms of its factors.
- The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- The logarithm of a power is the product of the logarithm and the exponent.
- The Change of Base Formula: $\log_a n = \dfrac{\log_b n}{\log_b a}$

Natural Logarithms (Lesson 9-5)

- Since the natural base function and the natural logarithmic function are inverses, these two can be used to "undo" each other.

Exponential Growth and Decay (Lesson 9-6)

- Exponential decay: $y = a(1 - r)^t$ or $y = ae^{-kt}$
- Exponential growth: $y = a(1 + r)^t$ or $y = ae^{kt}$

Key Vocabulary

common logarithm (p. 528)
exponential decay (p. 500)
exponential equation (p. 501)
exponential function (p. 499)
exponential growth (p. 500)
exponential inequality (p. 502)
logarithm (p. 510)
logarithmic equation (p. 512)
logarithmic function (p. 511)
logarithmic inequality (p. 512)
natural base, e (p. 536)
natural base exponential function (p. 536)
natural logarithm (p. 537)
natural logarithmic function (p. 537)
rate of decay (p. 544)
rate of growth (p. 546)

Vocabulary Check

State whether each sentence is *true* or *false*. If *false*, replace the underlined word(s) to make a true statement.

1. In $x = b^y$, y is called the logarithm.
2. The change in the number of bacteria in a Petri dish over time is an example of exponential decay.
3. The natural logarithm is the inverse of the exponential function with base 10.
4. The irrational number 2.71828... is referred to as the natural base, e.
5. If a savings account yields 2% interest per year, then 2% is the rate of growth.
6. Radioactive half-life is used to describe the exponential decay of a sample.
7. The inverse of an exponential function is a composite function.
8. If $24^{2y+3} = 24^{y-4}$, then $2y + 3 = y - 4$ by the Property of Equality for Exponential Functions.
9. The Power Property of Logarithms shows that $\ln 9 < \ln 81$.

Vocabulary Review at algebra2.com

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Lesson-by-Lesson Review

9-1 Exponential Functions (pp. 498–506)

Determine whether each function represents exponential *growth* or *decay*.

10. $y = 5(0.7)^x$ **11.** $y = \frac{1}{3}(4)^x$

Write an exponential function for the graph that passes through the given points.

12. (0, −2) and (3, −54)

13. (0, 7) and (1, 1.4)

Solve each equation or inequality. Check your solution.

14. $9^x = \frac{1}{81}$ **15.** $2^{6x} = 4^{5x+2}$

16. $49^{3p+1} = 7^{2p-5}$ **17.** $9^{x^2} \leq 27^{x^2-2}$

18. POPULATION The population of mice in a particular area is growing exponentially. On January 1, there were 50 mice, and by June 1, there were 200 mice. Write an exponential function of the form $y = ab^x$ that could be used to model the mouse population y of the area. Write the function in terms of x, the number of months since January.

Example 1 **Write an exponential function for the graph that passes through (0, 2) and (1, 16).**

$y = ab^x$	Exponential equation
$2 = ab^0$	Substitute (0, 2) into the exponential equation.
$2 = a$	Simplify.
$y = 2b^x$	Intermediate function
$16 = 2b^1$	Substitute (1, 16) into the intermediate function.
$8 = b$	Simplify.
$y = 2(8)^x$	

Example 2 **Solve $64 = 2^{3n+1}$ for n.**

$64 = 2^{3n+1}$	Original equation
$2^6 = 2^{3n+1}$	Rewrite 64 as 2^6 so each side has the same base.
$6 = 3n + 1$	Property of Equality for Exponential Functions
$\frac{5}{3} = n$	The solution is $\frac{5}{3}$.

9-2 Logarithms and Logarithmic Functions (pp. 509–517)

(continued on the next page)

Write each equation in logarithmic form.

19. $7^3 = 343$ **20.** $5^{-2} = \frac{1}{25}$

Write each equation in exponential form.

21. $\log_4 64 = 3$ **22.** $\log_8 2 = \frac{1}{3}$

Evaluate each expression.

23. $4^{\log_4 9}$ **24.** $\log_7 7^{-5}$

25. $\log_{81} 3$ **26.** $\log_{13} 169$

Example 3 **Solve $\log_9 n > \frac{3}{2}$.**

$\log_9 n > \frac{3}{2}$	Original inequality
$n > 9^{\frac{3}{2}}$	Logarithmic to exponential inequality
$n > (3^2)^{\frac{3}{2}}$	$9 = 3^2$
$n > 3^3$	Power of a Power
$n > 27$	Simplify.

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9-2 Logarithms and Logarithmic Functions (pp. 509–517)

Solve each equation or inequality.

27. $\log_4 x = \frac{1}{2}$

28. $\log_{81} 729 = x$

29. $\log_8 (x^2 + x) = \log_8 12$

30. $\log_8 (3y - 1) < \log_8 (y + 5)$

31. CHEMISTRY $\text{pH} = -\log(H^+)$, where H^+ is the hydrogen ion concentration of the substance. How many times as great is the acidity of orange juice with a pH of 3 as battery acid with a pH of 0?

Example 4 **Solve $\log_3 12 = \log_3 2x$.**

$\log_3 12 = \log_3 2x$ Original equation

$12 = 2x$ Property of Equality for Logarithmic Functions

$6 = x$ Divide each side by 2.

9-3 Properties of Logarithms (pp. 520–526)

Use $\log_9 7 \approx 0.8856$ and $\log_9 4 \approx 0.6309$ to approximate the value of each expression.

32. $\log_9 28$ **33.** $\log_9 49$

34. $\log_9 144$ **35.** $\log_9 63$

Solve each equation. Check your solutions.

36. $\log_5 7 + \frac{1}{2}\log_5 4 = \log_5 x$

37. $2\log_2 x - \log_2 (x + 3) = 2$

38. $\log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x$

39. SOUND Use the formula $L = 10 \log_{10} R$, where L is the loudness of a sound and R is the sound's relative intensity, to find out how much louder 10 alarm clocks would be than one alarm clock. Suppose the sound of one alarm clock is 80 decibels.

Example 5 **Use $\log_{12} 9 \approx 0.884$ and $\log_{12} 18 \approx 1.163$ to approximate the value of $\log_{12} 2$.**

$\log_{12} 2 = \log_{12} \frac{18}{9}$ Replace 2 with $\frac{18}{9}$.

$= \log_{12} 18 - \log_{12} 9$ Quotient Property

$\approx 1.163 - 0.884$ or 0.279

Example 6

Solve $\log_3 4 + \log_3 x = 2 \log_3 6$.

$\log_3 4 + \log_3 x = 2 \log_3 6$

$\log_3 4x = 2 \log_3 6$ Product Property of Logarithms

$\log_3 4x = \log_3 6^2$ Power Property of Logarithms

$4x = 36$ Property of Equality for Logarithmic Functions

$x = 9$ Divide each side by 4.

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Mixed Problem Solving
For mixed problem-solving practice, see page 934.

9-4 Common Logarithms (pp. 528–533)

Solve each equation or inequality. Round to four decimal places.

40. $2^x = 53$ **41.** $2.3^{x^2} = 66.6$

42. $3^{4x-7} < 4^{2x+3}$ **43.** $6^{3y} = 8^{y-1}$

44. $12^{x-5} \geq 9.32$ **45.** $2.1^{x-5} = 9.32$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

46. $\log_4 11$ **47.** $\log_2 15$

48. MONEY Diane deposited $500 into a bank account that pays an annual interest rate r of 3% compounded quarterly. Use $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to find how long it will take for Diane's money to double.

Example 7 Solve $5^x = 7$.

$5^x = 7$	Original equation
$\log 5^x = \log 7$	Property of Equality for Logarithmic Functions
$x \log 5 = \log 7$	Power Property of Logarithms
$x = \frac{\log 7}{\log 5}$	Divide each side by log 5.
$x \approx \frac{0.8451}{0.6990}$ or 1.2090	Use a calculator.

9-5 Base *e* and Natural Logarithms (pp. 536–542)

Write an equivalent exponential or logarithmic equation.

49. $e^x = 6$ **50.** $\ln 7.4 = x$

Solve each equation or inequality.

51. $2e^x - 4 = 1$ **52.** $e^x > 3.2$

53. $-4e^{2x} + 15 = 7$ **54.** $\ln 3x \leq 5$

55. $\ln (x - 10) = 0.5$ **56.** $\ln x + \ln 4x = 10$

57. MONEY If you deposit $1200 in an account paying 4.7% interest compounded continuously, how long will it take for your money to triple?

Example 8 Solve $\ln (x + 4) > 5$.

$\ln (x + 4) > 5$	Original inequality
$e^{\ln (x+4)} > e^5$	Write each side using exponents and base *e*.
$x + 4 > e^5$	Inverse Property of Exponents and Logarithms
$x > e^5 - 4$	Subtract 4 from each side.
$x > 144.4132$	Use a calculator.

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9-6 Exponential Growth and Decay (pp. 544–550)

58. BUSINESS Able Industries bought a fax machine for $250. It is expected to depreciate at a rate of 25% per year. What will be the value of the fax machine in 3 years?

59. BIOLOGY For a certain strain of bacteria, k is 0.872 when t is measured in days. Using the formula $y = ae^{kt}$, how long will it take 9 bacteria to increase to 738 bacteria?

60. CHEMISTRY Radium-226 has a half-life of 1800 years. Find the constant k in the decay formula for this compound.

61. POPULATION The population of a city 10 years ago was 45,600. Since then, the population has increased at a steady rate each year. If the population is currently 64,800, find the annual rate of growth for this city.

Example 9 **A certain culture of bacteria will grow from 500 to 4000 bacteria in 1.5 hours. Find the constant k for the growth formula. Use $y = ae^{kt}$.**

$y = ae^{kt}$	Exponential growth formula
$4000 = 500\ e^{k(1.5)}$	Replace y with 4000, a with 500, and t with 1.5.
$8 = e^{1.5k}$	Divide each side by 500.
$\ln 8 = \ln e^{1.5k}$	Property of Equality for Logarithmic Functions
$\ln 8 = 1.5k$	Inverse Property of Exponents and Logarithms
$\frac{\ln 8}{1.5} = k$	Divide each side by 1.5.
$1.3863 \approx k$	Use a calculator.

The constant k for this type of bacteria is about 1.3863.

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CHAPTER 9

Practice Test

1. Write $3^7 = 2187$ in logarithmic form.
2. Write $\log_8 16 = \frac{4}{3}$ in exponential form.
3. Express $\log_3 5$ in terms of common logarithms. Then approximate its value to four decimal places.
4. Evaluate $\log_2 \frac{1}{32}$.

Use $\log_4 7 \approx 1.4037$ and $\log_4 3 \approx 0.7925$ to approximate the value of each expression.

5. $\log_4 21$
6. $\log_4 \frac{7}{12}$

Simplify each expression.

7. $\left(3\sqrt{8}\right)^{\sqrt{2}}$
8. $81^{\sqrt{5}} \div 3^{\sqrt{5}}$

Solve each equation or inequality. Round to four decimal places if necessary.

9. $27^{2p+1} = 3^{4p-1}$
10. $\log_m 144 = -2$
11. $\log_3 3^{(4x-1)} = 15$
12. $4^{2x-3} = 9^{x+3}$
13. $2e^{3x} + 5 = 11$
14. $\log_2 x < 7$
15. $\log_9 (x + 4) + \log_9 (x - 4) = 1$
16. $\log_2 5 + \frac{1}{3} \log_2 27 = \log_2 x$

COINS For Exercises 17 and 18, use the following information.
You buy a commemorative coin for $25. The value of the coin increases at a rate of 3.25% per year.

17. How much will the coin be worth in 15 years?
18. After how many years will the coin have doubled in value?

19. **MULTIPLE CHOICE** The population of a certain country can be modeled by the equation $P(t) = 40\, e^{0.02t}$, where P is the population in millions and t is the number of years since 1900. When will the population be 400 million?

    A 1946 **C** 2015

    B 1980 **D** 2045

STARS For Exercises 20–22, use the following information.
Some stars appear bright only because they are very close to us. Absolute magnitude M is a measure of how bright a star would appear if it were 10 parsecs, about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where d is the star's distance from Earth measured in parsecs and m is its apparent magnitude.

Star	Apparent Magnitude	Distance (parsecs)
Sirius	−1.44	2.64
Vega	0.03	7.76

20. Sirius and Vega are two of the brightest stars. Which star appears brighter?
21. Find the absolute magnitudes of Sirius and Vega.
22. Which star is actually brighter? That is, which has a lower absolute magnitude?

23. **MULTIPLE CHOICE** Humans have about 1,400,000 hairs on their head and lose an average of 75 hairs each day. If a person's body were to *never* replace a hair, approximately how many years would it take for a person to have 1000 hairs left on their head? (Assume that a person can live significantly longer than the average life span.)

    F 85 years **H** 257 years

    G 113 years **J** 511 years

24. **DINOSAURS** A paleontologist finds that the Carbon-14 found in the bone is $\frac{1}{12}$ of that found in living bone tissue. Could this bone have belonged to a dinosaur? Explain your reasoning. (*Hint:* The dinosaurs lived from 220 million to 63 million years ago.)

25. **HEALTH** Radioactive iodine is used to determine the health of the thyroid gland. It decays according to the equation $y = ae^{-0.0856t}$, where t is in days. Find the half-life of this substance.

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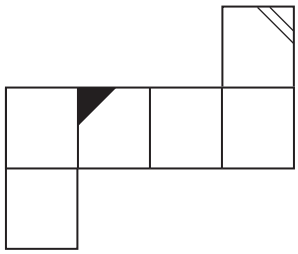
CHAPTER 9

Standardized Test Practice

Cumulative, Chapters 1–9

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. The net below shows the surface of a 3-dimensional figure.



Which 3-dimensional figure does this net represent?

A

B

C

D

TEST-TAKING TIP

Question 1 If you don't know how to solve a problem, eliminate the answer choices you know are incorrect and then guess from the remaining choices. Even eliminating only one answer choice greatly increases your chance of guessing the correct answer.

2. An equation can be used to find the total cost of a pizza with a certain diameter. Using the table below, find the equation that best represents y, the total cost, as a function of x, the diameter in inches.

Diameter, x (in.)	Total Cost, y
9	\$10.80
12	\$14.40
20	\$24.00

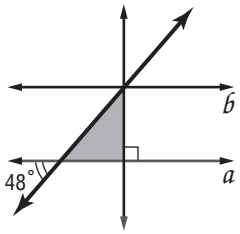
F $y = 1.2x$

G $x = 1.2y$

H $y = 0.83x$

J $x = 0.83y$

3. In the figure below, lines a and b are parallel. What are the measures of the angles in the shaded triangle below?



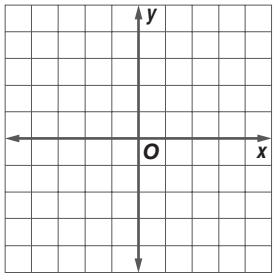
A 42, 48, 90

B 42, 90, 132

C 48, 52, 90

D 48, 90, 132

4. What are the slope and y-intercept of a line that contains the point $(-1, 4)$ and has the same x-intercept as $x + 2y = -3$?



F $m = 2, b = 6$

G $m = -7, b = -3$

H $m = \frac{1}{2}, b = -2$

J $m = -\frac{1}{7}, b = -3$

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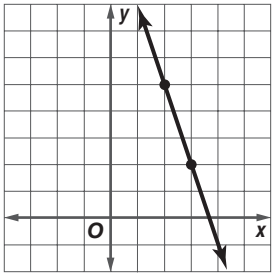
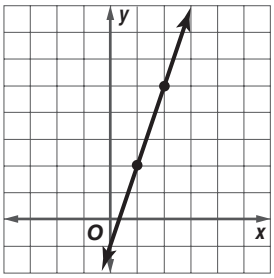
5. Which graph best represents the line passing through the point (2, 5) and perpendicular to $y = 3x$?

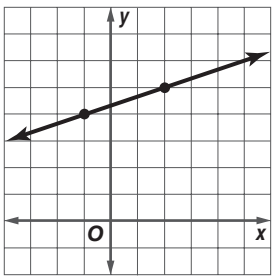
A



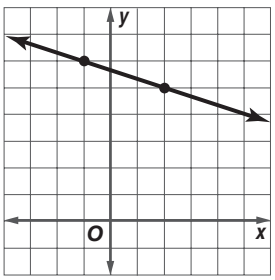
B



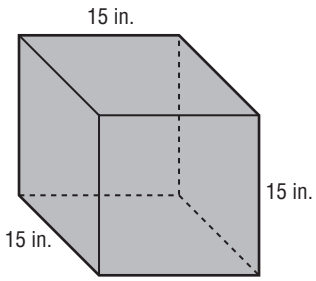
C



D



6. GRIDDABLE Matt has a square trough as shown below. He plans to fill it by emptying cylindrical cans of water with the dimensions as shown.



About how many cylindrical cans will it take to fill the trough?

7. Carson is making a circle graph showing the favorite movie types of customers at his store. The table summarizes the data. What central angle should Carson use for the section representing Comedy?

Type	Customers
Comedy	35
Romance	42
Horror	7
Drama	12
Other	4

F 35 **H** 126
G 63 **J** 150

Pre-AP

Record your answers on a sheet of paper. Show your work.

8. Sarah received $2500 for a graduation gift. She put it into a savings account in which the interest rate was 5.5% per year.

a. How much did she have in her savings account after 5 years?

b. After how many years will the amount in her savings account have doubled?

NEED EXTRA HELP?								
If You Missed Question...	1	2	3	4	5	6	7	8
Go to Lesson	1-3	2-4	3-1	2-4	2-4	6-8	10-3	10-6

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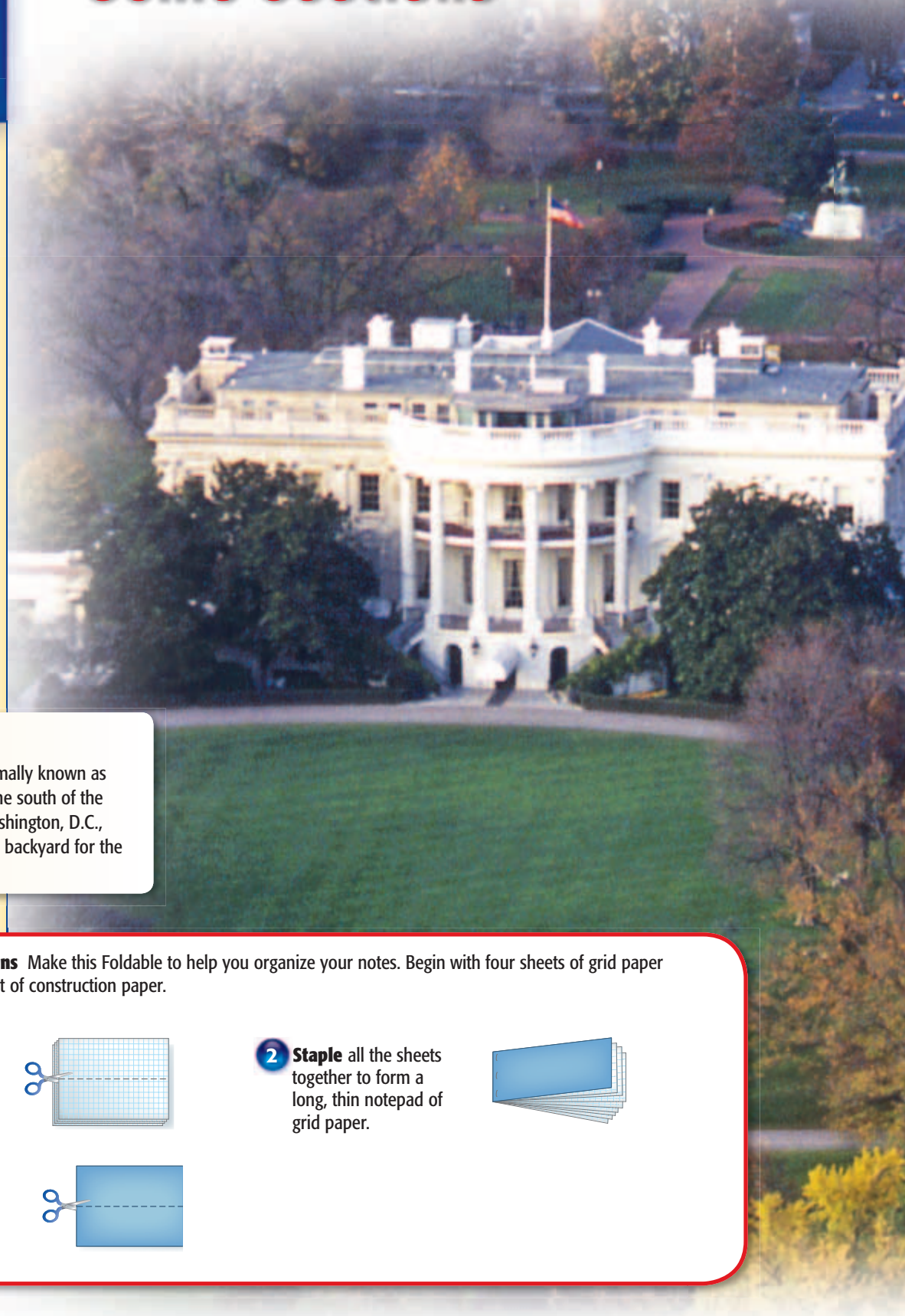
CHAPTER 10

Conic Sections



BIG Ideas

- Use the Midpoint and Distance Formulas.
- Write and graph equations of parabolas, circles, ellipses, and hyperbolas.
- Identify conic sections.
- Solve systems of quadratic equations and inequalities.

Key Vocabulary

circle (p. 574)
conic section (p. 567)
ellipse (p. 581)
hyperbola (p. 590)
parabola (p. 567)

Real-World Link

The Ellipse The Ellipse, which is formally known as President's Park South, is located to the south of the White House. The city planner for Washington, D.C., had intended for The Ellipse to be the backyard for the White House.

FOLDABLES Study Organizer

Conic Sections Make this Foldable to help you organize your notes. Begin with four sheets of grid paper and one sheet of construction paper.

1. **Cut** each sheet of grid paper in half lengthwise. Cut the sheet of construction paper in half lengthwise to form a front and back cover for the booklet of grid paper.

2. **Staple** all the sheets together to form a long, thin notepad of grid paper.

Paul Conklin/PhotoEdit

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GET READY for Chapter 10

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

QUICKCheck

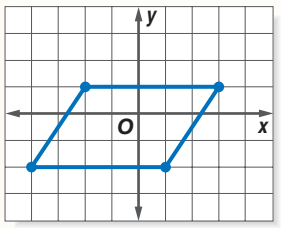
Solve each equation by completing the square. (Lesson 5-5)

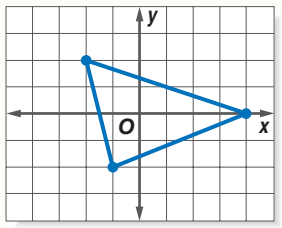
1. $x^2 + 10x + 24 = 0$
2. $x^2 - 2x + 2 = 0$
3. $2x^2 + 5x - 12 = 0$
4. $x^2 + 8x = -15$
5. **FRAMING** Julia is framing a picture in a 12-inch by 12-inch square frame. The frame is twice as wide at the top and bottom as it is at the sides. If the area of the picture is 54 square inches, what are the dimensions? (Lesson 5-5)

A translation is given for each figure.

a. **Write the vertex matrix for the figure.**
b. **Write the translation matrix.**
c. **Find the coordinates in matrix form of the vertices of the translated figure.** (Lesson 4-4)

6. translated 4 units left and 2 units up
7. translated 5 units right and 3 units down





8. **ARCHITECTURE** The Connors plot their deck plans on a grid with each unit equal to 1 foot. They place the corners of a hot tub at (2, 5), (14, 5), (14, 17), and (2, 17). Changes to the plan now require that the hot tub's perimeter be three-fourths that of the original. Determine possible new coordinates for the hot tub. (Lesson 4-4)

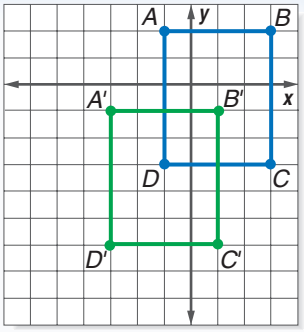
QUICKReview

Example 1 **Solve $x^2 - x - 156 = 0$ by completing the square.**

$$x^2 + (-x) = 156$$
$$x^2 + (-x) + \frac{1}{4} = 156 + \frac{1}{4}$$
$$\left(x - \frac{1}{2}\right)^2 = \frac{625}{4}$$
$$x - \frac{1}{2} = \pm\frac{25}{2}$$
$$x = \frac{1}{2} \pm \frac{25}{2} \quad \rightarrow \quad x = -12 \text{ or } 13$$

Example 2

Translate the blue figure 2 units left and 3 units down.



a. **Write the vertex matrix for the given figure.**
b. **Write the translation matrix.**
c. **Find the coordinates in matrix form of the vertices of the translated figure.**

a. $\begin{bmatrix} -1 & 3 & 3 & -1 \\ 2 & 2 & -3 & -3 \end{bmatrix}$

b. To translate the figure, add the translation matrix $\begin{bmatrix} -2 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 \end{bmatrix}$ to the vertex matrix.

c. $\underbrace{\begin{bmatrix} -1 & 3 & 3 & -1 \\ 2 & 2 & -3 & -3 \end{bmatrix}}_{\text{Original Vertex Matrix}} + \underbrace{\begin{bmatrix} -2 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 \end{bmatrix}}_{\text{Translation Matrix}}$

$$= \begin{bmatrix} -3 & 1 & 1 & -3 \\ -1 & -1 & -6 & -6 \end{bmatrix}$$

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10-1 Midpoint and Distance Formulas

Main Ideas

- Find the midpoint of a segment on the coordinate plane.
- Find the distance between two points on the coordinate plane.

GET READY for the Lesson

A square grid is superimposed on a map of eastern Nebraska where emergency medical assistance by helicopter is available from both Lincoln and Omaha. You can use the formulas in this lesson to determine whether the site of an emergency is closer to Lincoln or to Omaha.

Each side = 10 miles

The Midpoint Formula Recall that point M is the midpoint of segment PQ if M is between P and Q and $PM = MQ$. There is a formula for the coordinates of the midpoint of a segment in terms of the coordinates of the endpoints.

Study Tip

Midpoints

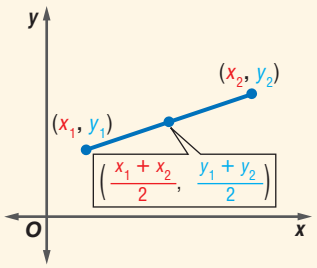
The coordinates of the midpoint are the means of the coordinates of the endpoints.

KEY CONCEPT — **Midpoint Formula**

Words If a line segment has endpoints at (x_1, y_1) and (x_2, y_2), then the midpoint of the segment has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Symbols $\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Model

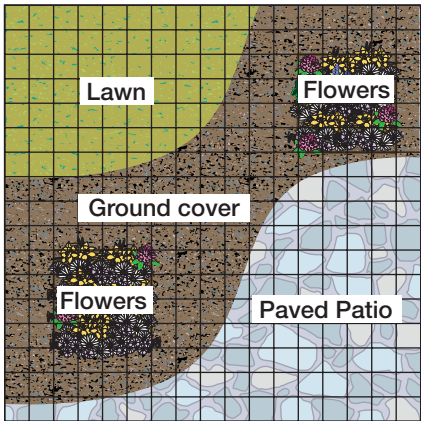


You will show that this formula is correct in Exercise 38.

EXAMPLE 1 **Find a Midpoint**

LANDSCAPING **A landscape design includes two square flower beds and a sprinkler halfway between them. Find the coordinates of the sprinkler if the origin is at the lower left corner of the grid.**

The centers of the flower beds are at (4, 5) and (14, 13). The sprinkler will be at the midpoint of the segment joining these points.



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$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4 + 14}{2}, \frac{5 + 13}{2}\right)$$
$$= \left(\frac{18}{2}, \frac{18}{2}\right) \text{ or } (9, 9)$$

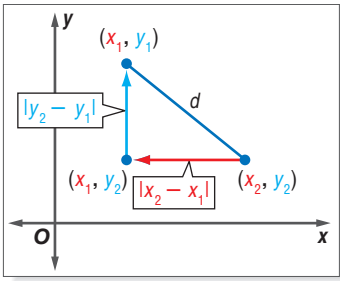
The sprinkler will have coordinates (9, 9).

CHECK Your Progress

1. The landscape architect decides to place a bench in the middle of the lawn area. Find the coordinates of the bench using the endpoints (0, 17) and (7, 11).

The Distance Formula Recall that the distance between two points on a number line whose coordinates are a and b is $|a - b|$ or $|b - a|$. You can use this fact and the Pythagorean Theorem to derive a formula for the distance between two points on a coordinate plane.

Suppose (x_1, y_1) and (x_2, y_2) name two points. Draw a right triangle with vertices at these points and the point (x_1, y_2). The lengths of the legs of the right triangle are $|x_2 - x_1|$ and $|y_2 - y_1|$. Let d represent the distance between (x_1, y_1) and (x_2, y_2).



$c^2 = a^2 + b^2$ Pythagorean Theorem

$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$ Substitute.

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ $|x_2 - x_1|^2 = (x_2 - x_1)^2$; $|y_2 - y_1|^2 = (y_2 - y_1)^2$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Find the nonnegative square root of each side.

Study Tip

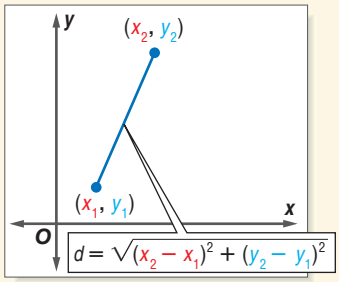
Distance

In mathematics, just as in life, distances are always nonnegative.

KEY CONCEPT *Distance Formula*

Words The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Model



EXAMPLE **Find the Distance Between Two Points**

2 **Find the distance between $A(-3, 6)$ and $B(4, -4)$.**

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{[4 - (-3)]^2 + (-4 - 6)^2}$ Let $(x_1, y_1) = (-3, 6)$ and $(x_2, y_2) = (4, -4)$.

$= \sqrt{7^2 + (-10)^2}$ Subtract.

$= \sqrt{49 + 100}$ or $\sqrt{149}$ units

CHECK Your Progress

2. Find the distance between $R(-6, 5)$ and $S(-3, -2)$.

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STANDARDIZED TEST EXAMPLE

3 A coordinate grid is placed over a California map. Bakersfield is located at (3, −7), and Fresno is located at (−7, 9). If Tulare is halfway between Bakersfield and Fresno, which is the closest to the distance in coordinate units from Bakersfield to Tulare?

A 6.25 **B** 9.5 **C** 12.5 **D** 19

Read the Test Item

The question asks us to find the distance between one city and the midpoint. Find the midpoint and then use the Distance Formula.

Solve the Test Item

Use the Midpoint Formula to find the coordinates of Tulare.

$$\text{midpoint} = \left(\frac{3 + (-7)}{2}, \frac{(-7) + 9}{2}\right) \quad \text{Midpoint Formula}$$

$$= (-2, 1) \quad \text{Simplify.}$$

Test-Taking Tip

In order to check your answer, find the distance between Tulare and Fresno. Since Tulare is at the midpoint, these distances should be equal.

Use the Distance Formula to find the distance between Bakersfield (3, −7) and Tulare (−2, 1).

$$\text{distance} = \sqrt{(-2 - 3)^2 + (1 - (-7))^2} \quad \text{Distance Formula}$$

$$= \sqrt{(-5)^2 + 8^2} \quad \text{Subtract.}$$

$$= \sqrt{89} \text{ or about } 9.4 \quad \text{Simplify.}$$

The answer is B.

CHECK Your Progress

3. The coordinates for points A and B are (−4, −5) and (10, −7), respectively. Find the distance between the midpoint of A and B and point B.

F $\sqrt{10}$ units **H** $\sqrt{50}$ units

G $5\sqrt{10}$ units **J** $10\sqrt{5}$ units

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CHECK Your Understanding

Example 1 (pp. 562–563)

Find the midpoint of the line segment with endpoints at the given coordinates.

1. (−5, 6), (1, 7)

2. (8, 9), (−3, −4.5)

3. (13, −4), (10, 14.6)

4. (−12, −2), (−3.5, −7)

Example 2 (p. 563)

Find the distance between each pair of points with the given coordinates.

5. (2, −4), (10, −10)

6. (7, 8), (−4, 9)

7. (0.5, 1.4), (1.1, 2.9)

8. (−4.3, 2.6), (6.5, −3.4)

Example 3 (p. 564)

9. STANDARDIZED TEST PRACTICE The map of a mall is overlaid with a numeric grid. The kiosk for the cell phone store is halfway between Terry's Ice Cream and the See Clearly eyeglass store. If the ice cream store is at (2, 4) and the eyeglass store is at (78, 46), find the distance the kiosk is from the eyeglass store.

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–15	1
16–21	2, 3

Find the midpoint of the line segment with endpoints at the given coordinates.

10. (8, 3), (16, 7)

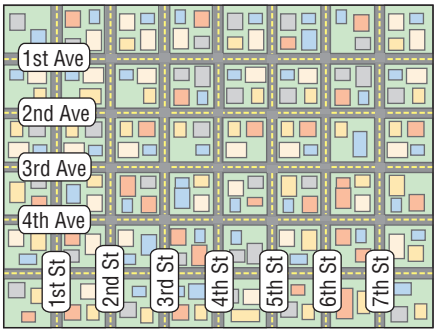
11. (−5, 3), (−3, −7)

12. (6, −5), (−2, −7)

13. (5, 9), (12, 18)

14. GEOMETRY Triangle MNP has vertices $M(3, 5)$, $N(-2, 8)$, and $P(7, -4)$. Find the coordinates of the midpoint of each side.

15. REAL ESTATE In John's town, the numbered streets and avenues form a grid. He belongs to a gym at the corner of 12th Street and 15th Avenue, and the deli where he works is at the corner of 4th Street and 5th Avenue. He wants to rent an apartment halfway between the two. In what area should he look?



Find the distance between each pair of points with the given coordinates.

16. (−4, 9), (1, −3)

17. (1, −14), (−6, 10)

18. (−4, −10), (−3, −11)

19. (9, −2), (12, −14)

20. (0.23, 0.4), (0.68, −0.2)

21. (2.3, −1.2), (−4.5, 3.7)

22. GEOMETRY Quadrilateral $RSTV$ has vertices $R(-4, 6)$, $S(4, 5)$, $T(6, 3)$, and $V(5, -8)$. Find the perimeter of the quadrilateral.

23. GEOMETRY Triangle BCD has vertices $B(4, 9)$, $C(8, -9)$, and $D(-6, 5)$. Find the length of median $\overline{BP}$. (*Hint*: A median connects a vertex of a triangle to the midpoint of the opposite side.)

Find the midpoint of the line segment with endpoints at the given coordinates. Then find the distance between the points.

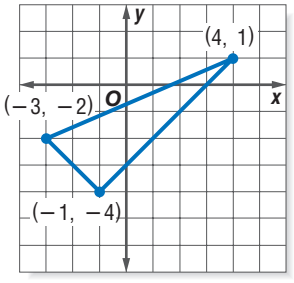
24. $\left(-3, -\frac{2}{11}\right), \left(5, \frac{9}{11}\right)$

25. $\left(0, \frac{1}{5}\right), \left(\frac{3}{5}, -\frac{3}{5}\right)$

26. $\left(2\sqrt{3}, -5\right), \left(-3\sqrt{3}, 9\right)$

27. $\left(\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{4}\right), \left(-\frac{2\sqrt{3}}{3}, \frac{\sqrt{5}}{2}\right)$

28. GEOMETRY Find the perimeter and area of the triangle at the right.



29. GEOMETRY A circle has a radius with endpoints at (2, 5) and (−1, −4). Find the circumference and area of the circle.

30. GEOMETRY Circle Q has a diameter $\overline{AB}$. If A is at (−3, −5) and the center of the circle is at (2, 3), find the coordinates of B.

GEOGRAPHY For Exercises 31 and 32, use the following information.
The U.S. Geological Survey (USGS) has determined the official center of the United States.

31. Approximate the center of the United States. Describe your method.

32. RESEARCH Use the Internet or other reference to look up the USGS geographical center of the United States. How does the location given by USGS compare to the result of your method?

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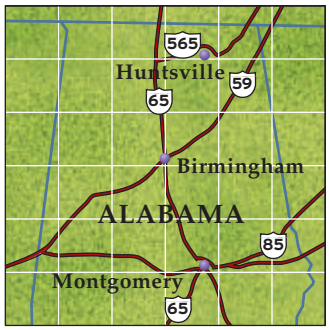
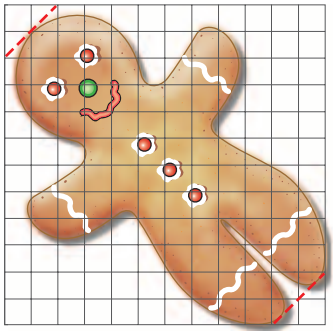
TRAVEL **For Exercises 33 and 34, use the figure at the right, where a grid is superimposed on a map of a portion of the state of Alabama.**



33. How far is it from Birmingham to Montgomery if each unit on the grid represents 40 miles?

34. How long would it take a plane to fly from Huntsville to Montgomery if its average speed is 180 miles per hour?

35. WOODWORKING A stage crew is making the set for a children's play. They want to make some gingerbread shapes out of leftover squares of wood with sides measuring 1 foot. They can make taller shapes by cutting them out of the wood diagonally. To the nearest inch, how tall is the gingerbread shape in the drawing?



EXTRA PRACTICE
See pages 911, 935.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. OPEN ENDED Find two points that are $\sqrt{29}$ units apart.

37. REASONING Identify all of the points that are equidistant from the endpoints of a given segment.

38. CHALLENGE Verify the Midpoint Formula. (*Hint*: You must show that the formula gives the coordinates of a point on the line through the given endpoints and that the point is equidistant from the endpoints.)

39. *Writing in Math* Explain how to use the Distance Formula to approximate the distance between two cities on a map.

STANDARDIZED TEST PRACTICE

40. ACT/SAT Point $D(5, -1)$ is the midpoint of segment $\overline{CE}$. If point C has coordinates (3, 2), what are the coordinates of point E?

A $(8, 1)$

B $(7, -4)$

C $(2, -3)$

D $\left(4, \frac{1}{2}\right)$

41. REVIEW If $\log_{10} x = -3$, what is the value of x?

F $x = 1000$

G $x = \frac{1}{1000}$

H $x = \sqrt{\frac{1}{100}}$

J $x = -\sqrt{\frac{1}{100}}$

Spiral Review

42. COMPUTERS Suppose a computer that costs \$3000 new is only worth \$600 after 3 years. What is the average annual rate of depreciation? (Lesson 9-6)

Solve each equation. Round to the nearest ten-thousandth. (Lesson 9-5)

43. $3e^x - 2 = 0$

44. $e^{3x} = 4$

45. $\ln (x + 2) = 5$

GET READY for the Next Lesson

PREREQUISITE SKILL **Write in the form $y = a(x - h)^2 + k$.** (Lesson 5-5)

46. $y = x^2 + 6x + 9$

47. $y = 2x^2 + 20x + 50$

48. $y = -3x^2 - 18x - 10$

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10-2 Parabolas

Main Ideas

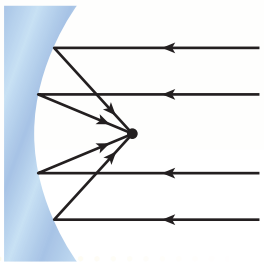
- Write equations of parabolas in standard form.
- Graph parabolas.

New Vocabulary

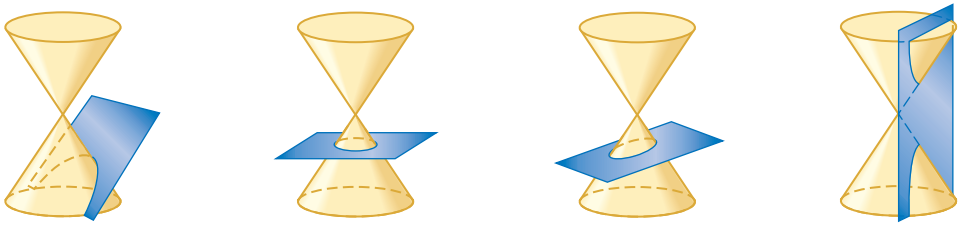
parabola
conic section
focus
directrix
latus rectum

GET READY for the Lesson

A mirror or other reflective object in the shape of a parabola reflects all parallel incoming rays to the same point. Or, if that point is the source of rays, the reflected rays are all parallel.



Equations of Parabolas In Chapter 5, you learned that the graph of an equation of the form $y = ax^2 + bx + c$ is a **parabola.** A parabola can also be obtained by slicing a double cone on a slant as shown below on the left. Any figure that can be obtained by slicing a double cone is called a **conic section.** Other conic sections are also shown below.



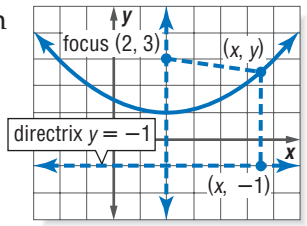
parabola circle ellipse hyperbola

Study Tip

Focus of a Parabola

The focus is the special point referred to at the beginning of the lesson.

A parabola can also be defined as the set of all points in a plane that are the same distance from a given point called the **focus** and a given line called the **directrix.** The parabola at the right has its focus at (2, 3), and the equation of its directrix is $y = -1$. You can use the Distance Formula to find an equation of this parabola.



Let (x, y) be any point on this parabola. The distance from this point to the focus must be the same as the distance from this point to the directrix. The distance from a point to a line is measured along the perpendicular from the point to the line.

distance from (x, y) to $(2, 3)$ = distance from (x, y) to $(x, -1)$

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x-x)^2 + [y-(-1)]^2}$$

$$(x-2)^2 + (y-3)^2 = 0^2 + (y+1)^2 \quad \text{Square each side.}$$

$$(x-2)^2 + y^2 - 6y + 9 = y^2 + 2y + 1 \quad \text{Square } y-3 \text{ and } y+1.$$

$$(x-2)^2 + 8 = 8y \quad \text{Isolate the } y\text{-terms.}$$

$$\frac{1}{8}(x-2)^2 + 1 = y \quad \text{Divide each side by 8.}$$

An equation of the parabola with focus at (2, 3) and directrix with equation $y = -1$ is $y = \frac{1}{8}(x-2)^2 + 1$. The equation of the *axis of symmetry* for this parabola is $x = 2$. The axis of symmetry intersects the parabola at a point called the *vertex*. The vertex is the point where the graph turns. The vertex of this parabola is at (2, 1). Since $\frac{1}{8}$ is positive, the parabola opens upward. Any equation of the form $y = ax^2 + bx + c$ can be written in standard form.

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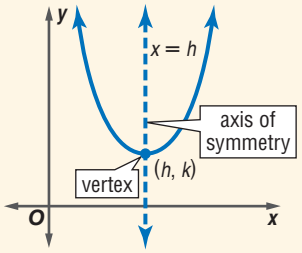
KEY CONCEPT — Equation of a Parabola

The standard form of the equation of a parabola with vertex (h, k) and axis of symmetry $x = h$ is $y = a(x - h)^2 + k$.

- If $a > 0$, k is the minimum value of the related function and the parabola opens upward.
- If $a < 0$, k is the maximum value of the related function and the parabola opens downward.



EXAMPLE 1 Analyze the Equation of a Parabola

Write $y = 3x^2 + 24x + 50$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

$y = 3x^2 + 24x + 50$	Original equation
$= 3(x^2 + 8x) + 50$	Factor 3 from the x-terms.
$= 3(x^2 + 8x + \blacksquare) + 50 - 3(\blacksquare)$	Complete the square on the right side.
$= 3(x^2 + 8x + 16) + 50 - 3(16)$	The 16 added when you complete the square is multiplied by 3.
$= 3(x + 4)^2 + 2$	
$= 3[x - (-4)]^2 + 2$	$(h, k) = (-4, 2)$

The vertex of this parabola is located at $(-4, 2)$, and the equation of the axis of symmetry is $x = -4$. The parabola opens upward.

Study Tip — Look Back
To review **completing the square**, see Lesson 5–5.

CHECK Your Progress

1. Write $y = 4x^2 + 16x + 34$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Study Tip — Translations
If h is positive, translate the graph h units to the right. If h is negative, translate the graph h units to the left. Similarly, if k is positive, translate the graph k units up. If k is negative, translate the graph k units down.

Graph Parabolas You can use symmetry and translations to graph parabolas. The equation $y = a(x - h)^2 + k$ can be obtained from $y = ax^2$ by replacing x with $x - h$ and y with $y - k$. Therefore, the graph of $y = a(x - h)^2 + k$ is the graph of the parent function $y = ax^2$ translated h units to the right or left and k units up or down.

EXAMPLE 2 Graph Parabolas

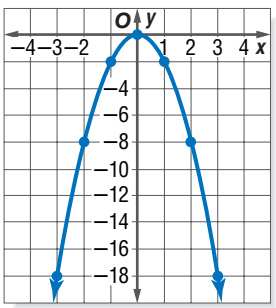
Graph each equation.

a. $y = -2x^2$

For this equation, $h = 0$ and $k = 0$. The vertex is at the origin. Since the equation of the axis of symmetry is $x = 0$, substitute some small positive integers for x and find the corresponding y-values.

x	y
1	−2
2	−8
3	−18

Since the graph is symmetric about the y-axis, the points at $(-1, -2)$, $(-2, -8)$, and $(-3, -18)$ are also on the parabola. Use all of these points to draw the graph.



Notice that each side of the graph is the reflection of the other side about the y-axis.

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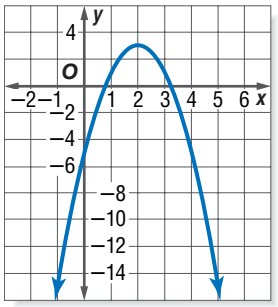
b. $y = -2(x - 2)^2 + 3$

The equation is of the form $y = a(x - h)^2 + k$, where $h = 2$ and $k = 3$. The graph of this equation is the graph of $y = -2x^2$ in part **a** translated 2 units to the right and up 3 units. The vertex is now at (2, 3).



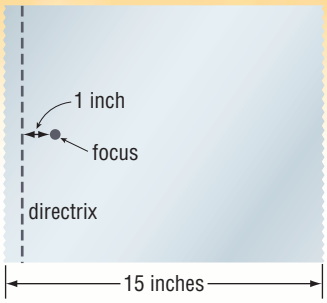
CHECK Your Progress

2A. $y = 3x^2$ **2B.** $y = 3(x - 1)^2 - 4$

ALGEBRA LAB

Parabolas

Step 1 Start with a sheet of wax paper that is about 15 inches long and 12 inches wide. Make a line that is perpendicular to the sides of the sheet by folding the sheet near one end. Open up the paper again. This line is the directrix. Mark a point about midway between the sides of the sheet so that the distance from the directrix is about 1 inch. This is the focus.



Put the focus on top of any point on the directrix and crease the paper. Make about 20 more creases by placing the focus on top of other points on the directrix. The lines form the outline of a parabola.

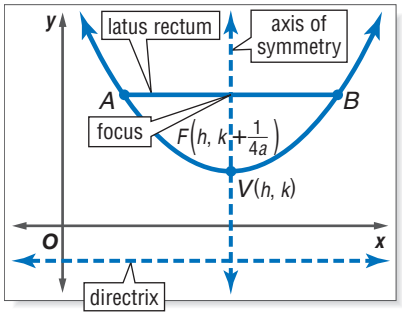
Step 2 Start with a new sheet of wax paper. Form another outline of a parabola with a focus that is about 3 inches from the directrix.

Step 3 On a new sheet of wax paper, form a third outline of a parabola with a focus that is about 5 inches from the directrix.

ANALYZE THE RESULTS

Compare the shapes of the three parabolas. How does the distance between the focus and the directrix affect the shape of a parabola?

The shape of a parabola and the distance between the focus and directrix depend on the value of a in the equation. The line segment through the focus of a parabola and perpendicular to the axis of symmetry is called the **latus rectum.** The endpoints of the latus rectum lie on the parabola.



In the figure, the latus rectum is $\overline{AB}$. The length of the latus rectum of the parabola with equation $y = a(x - h)^2 + k$ is $\left|\frac{1}{a}\right|$ units. The endpoints of the latus rectum are $\left|\frac{1}{2a}\right|$ units from the focus.

Equations of parabolas with vertical axes of symmetry have the parent function $y = x^2$ and are of the form $y = a(x - h)^2 + k$. These are functions. Equations of parabolas with horizontal axes of symmetry are of the form $x = a(y - k)^2 + h$ and are not functions. The parent graph for these equations is $x = y^2$.

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KEY CONCEPT	Information About Parabolas	
Form of Equation	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of Symmetry	$x = h$	$y = k$
Focus	$\left(h, k + \frac{1}{4a}\right)$	$\left(h + \frac{1}{4a}, k\right)$
Directrix	$y = k - \frac{1}{4a}$	$x = h - \frac{1}{4a}$
Direction of Opening	upward if $a > 0$, downward if $a < 0$	right if $a > 0$, left if $a < 0$
Length of Latus Rectum	$\left\|\frac{1}{a}\right\|$ units	$\left\|\frac{1}{a}\right\|$ units

EXAMPLE 3 Graph an Equation Not in Standard Form

Graph $4x - y^2 = 2y + 13$.

First, write the equation in the form $x = a(y - k)^2 + h$.

$4x - y^2 = 2y + 13$	There is a y^2 term, so isolate the y and y^2 terms.
$4x = y^2 + 2y + 13$	Add y^2 to each side.
$4x = (y^2 + 2y + \blacksquare) + 13 - \blacksquare$	Complete the square.
$4x = (y^2 + 2y + 1) + 13 - 1$	Add and subtract 1, since $\left(\frac{2}{2}\right)^2 = 1$.
$4x = (y + 1)^2 + 12$	Write $y^2 + 2y + 1$ as a square.
$x = \frac{1}{4}(y + 1)^2 + 3$	$(h, k) = (3, -1)$

Then use the following information to draw the graph based on the parent graph, $x = y^2$.

vertex: $(3, -1)$

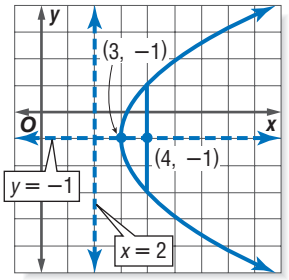
axis of symmetry: $y = -1$

focus: $\left(3 + \frac{1}{4\left(\frac{1}{4}\right)}, -1\right)$ or $(4, -1)$

directrix: $x = 3 - \frac{1}{4\left(\frac{1}{4}\right)}$ or 2

direction of opening: right, since $a > 0$

length of latus rectum: $\left|\frac{1}{\left(\frac{1}{4}\right)}\right|$ or 4 units



The graph is wider than the graph of $x = y^2$ since $a < 1$ and shifted 3 units right and 1 unit down.

Study Tip

When graphing these functions, it may be helpful to sketch the graph of the parent function.

CHECK Your Progress **Graph each equation.**

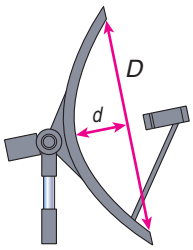
3A. $3x - y^2 = 4y + 25$ **3B.** $y = x^2 + 6x - 4$

EXAMPLE 4 Write and Graph an Equation for a Parabola

SATELLITE TV **Use the information at the left about satellite dishes.**

a. Write an equation that models a cross section of a satellite dish. Assume that the focus is at the origin and the parabola opens to the right.

First, solve for f. Since $\frac{f}{D} = 0.6$, and $D = 60$, $f = 0.6(60)$ or 36.



Real-World Link

The important characteristics of a satellite dish are the diameter D, depth d, and the ratio $\frac{f}{D}$, where f is the distance between the focus and the vertex. A typical dish has the values $D = 60$ cm, $d = 6.25$ cm, and $\frac{f}{D} = 0.6$.

Source: 2000networks.com

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The focus is at (0, 0), and the parabola opens to the right. So the vertex must be at (−36, 0). Thus, $h = -36$ and $k = 0$. Now find a.

$-36 + \frac{1}{4a} = 0$ $h = -36$; The x-coordinate of the focus is 0.

$\frac{1}{4a} = 36$ Add 36 to each side.

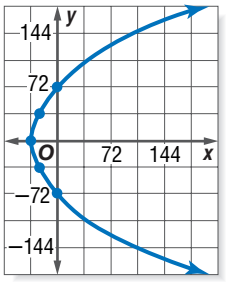
$1 = 144a$ Multiply each side by $4a$.

$\frac{1}{144} = a$ Divide each side by 144.

An equation of the parabola is $x = \frac{1}{144}y^2 - 36$.

b. Graph the equation.

The length of the latus rectum is $\left|\frac{1}{\frac{1}{144}}\right|$ or 144 units, so the graph must pass through (0, 72) and (0, −72). According to the diameter and depth of the dish, the graph must pass through (−29.75, 30) and (−29.75, −30). Use these points and the information from part **a** to draw the graph.



CHECK Your Progress

4. Write and graph an equation for a satellite dish with diameter D of 34 inches and ratio $\frac{f}{D}$ of 0.6.

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Example 1 (p. 568)

1. Write $y = 2x^2 - 12x + 6$ in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

Examples 2, 3 (pp. 568–570)

Graph each equation.

2. $y = (x - 3)^2 - 4$

3. $y = 2(x + 7)^2 + 3$

4. $y = -3x^2 - 8x - 6$

5. $x = \frac{2}{3}y^2 - 6y + 12$

Example 4 (pp. 570–571)

6. COMMUNICATION A microphone is placed at the focus of a parabolic reflector to collect sound for the television broadcast of a football game. Write an equation for the cross section, assuming that the focus is at the origin, the focus is 6 inches from the vertex, and the parabola opens to the right.

Exercises

HOMEWORK HELP

For Exercises	See Examples
7–10	1
11–14	2
15–19	3
20–23	4

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola.

7. $y = x^2 - 6x + 11$

8. $x = y^2 + 14y + 20$

9. $y = \frac{1}{2}x^2 + 12x - 8$

10. $x = 3y^2 + 5y - 9$

Graph each equation.

11. $y = -\frac{1}{6}x^2$

12. $x = \frac{1}{2}y^2$

13. $y = \frac{1}{3}(x + 6)^2 + 3$

14. $y = -\frac{1}{2}(x - 1)^2 + 4$

15. $4(x - 2) = (y + 3)^2$

16. $(y - 8)^2 = -4(x - 4)$

17. $y = x^2 - 12x + 20$

18. $x = y^2 - 14y + 25$

19. $x = 5y^2 + 25y + 60$

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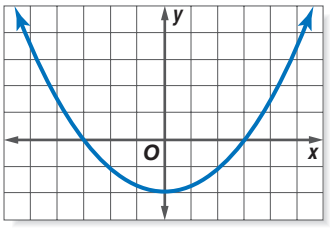
20. Write an equation for the graph at the right.



21. MANUFACTURING The reflective surface in a flashlight has a parabolic shape with a cross section that can be modeled by $y = \frac{1}{3}x^2$, where x and y are in centimeters. How far from the vertex should the filament of the light bulb be located?

22. BRIDGES The Bayonne Bridge connects Staten Island, New York, to New Jersey. It has an arch in the shape of a parabola. Write an equation of a parabola to model the arch, assuming that the origin is at the surface of the water, beneath the vertex of the arch.

23. FOOTBALL When a ball is thrown or kicked, the path it travels is shaped like a parabola. Suppose a football is kicked from ground level, reaches a maximum height of 25 feet, and hits the ground 100 feet from where it was kicked. Assuming that the ball was kicked at the origin, write an equation of the parabola that models the flight of the ball.

For Exercises 24–27, use the equation $x = 3y^2 + 4y + 1$.

24. Draw the graph. Find the x-intercept(s) and y-intercept(s).

25. What is the equation of the axis of symmetry?

26. What are the coordinates of the vertex?

27. How does the graph compare to the graph of the parent function $x = y^2$?

Write an equation for each parabola described below. Then draw the graph.

28. vertex (0, 1), focus (0, 5)

29. vertex (8, 6), focus (2, 6)

30. focus $(-4, -2)$, directrix $x = -8$

31. vertex (1, 7), directrix $y = 3$

32. vertex $(-7, 4)$, axis of symmetry $x = -7$, measure of latus rectum 6, $a < 0$

33. vertex (4, 3), axis of symmetry $y = 3$, measure of latus rectum 4, $a > 0$

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

34. $y = 3x^2 - 24x + 50$

35. $y = -2x^2 + 5x - 10$

36. $x = -4y^2 + 6y + 2$

37. $x = 5y^2 - 10y + 9$

38. $y = \frac{1}{2}x^2 - 3x + \frac{19}{2}$

39. $x = -\frac{1}{3}y^2 - 12y + 15$

40. UMBRELLAS A beach umbrella has an arch in the shape of a parabola that opens downward. The umbrella spans 9 feet across and $1\frac{1}{2}$ feet high. Write an equation of a parabola to model the arch, assuming that the origin is at the point where the pole and umbrella meet, beneath the vertex of the arch.

EXTRA PRACTICE
See pages 912, 935.
MathOnline
Self-Check Quiz at
algebra2.com

(t)James Rooney, (b)Hisham F. Ibrahim
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H.O.T. Problems

41. REASONING How do you change the equation of the parent function $y = x^2$ to shift the graph to the right?

42. OPEN ENDED Write an equation for a parabola that opens to the left. Use the parent graph to sketch the graph of your equation.

43. FIND THE ERROR Yasu is finding the standard form of the equation $y = x^2 + 6x + 4$. What mistake did she make in her work?

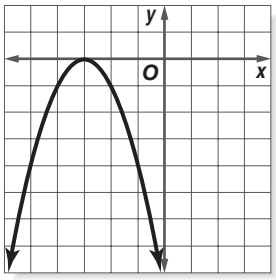
$y = x^2 + 6x + 4$
$y = x^2 + 6x + 9 + 4$
$y = (x + 3)^2 + 4$

44. CHALLENGE The parabola with equation $y = (x - 4)^2 + 3$ has its vertex at (4, 3) and passes through (5, 4). Find an equation of a different parabola with its vertex at (4, 3) and that passes through (5, 4).

45. *Writing in Math* Use the information on page 567 to explain how parabolas can be used in manufacturing. Include why a car headlight with a parabolic reflector is better than one with an unreflected light bulb.

STANDARDIZED TEST PRACTICE

46. ACT/SAT Which is the parent function of the graph shown below?



A $y = -x$
B $y = -\sqrt{x}$
C $y = -|x|$
D $y = -x^2$

47. REVIEW $\log_9 30 =$

F $\log_{10} 9 + \log_{10} 30$
G $\log_{10} 9 - \log_{10} 30$
H $(\log_{10} 9)(\log_{10} 30)$
J $\dfrac{\log_{10} 30}{\log_{10} 9}$

Spiral Review

Find the distance between each pair of points with the given coordinates. (Lesson 10-1)

48. (7, 3), (−5, 8)
49. (4, −1), (−2, 7)
50. (−3, 1), (0, 6)

51. RADIOACTIVITY The decay of Radon-222 can be modeled by the equation $y = ae^{-0.1813t}$, where t is measured in days. What is the half-life of Radon-222? (Lesson 9-6)

52. HEALTH Alisa's heart rate is usually 120 beats per minute when she runs. If she runs for 2 hours every day, about how many times will her heart beat during the amount of time she exercises in two weeks? Express in scientific notation. (Lesson 6-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Simplify each radical expression. (Lessons 7-1 and 7-2)

53. $\sqrt{16}$
54. $\sqrt{25}$
55. $\sqrt{81}$
56. $\sqrt{144}$
57. $\sqrt{12}$
58. $\sqrt{18}$
59. $\sqrt{48}$
60. $\sqrt{72}$

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10-3 Circles

Main Ideas

- Write equations of circles.
- Graph circles.

New Vocabulary

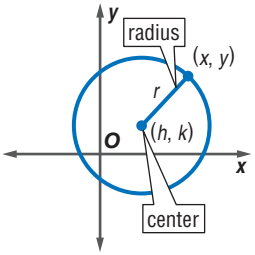
circle
center

GET READY for the Lesson

Radar equipment can be used to detect and locate objects that are too far away to be seen by the human eye. The radar systems at major airports can typically detect and track aircraft up to 45 to 70 miles in any direction from the airport. The boundary of the region that a radar system can monitor can be modeled by a circle.

Equations of Circles A **circle** is the set of all points in a plane that are equidistant from a given point in the plane, called the **center.** Any segment whose endpoints are the center and a point on the circle is a *radius* of the circle.

Assume that (x, y) are the coordinates of a point on the circle at the right. The center is at (h, k), and the radius is r. You can find an equation of the circle by using the Distance Formula.



$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \quad \text{Distance Formula}$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r \quad (x_1, y_1) = (h, k), \ (x_2, y_2) = (x, y), \ d = r$$

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{Square each side.}$$

This is the standard form of the equation of a circle.

KEY CONCEPT *Equation of a Circle*

The equation of a circle with center (h, k) and radius r units is $(x - h)^2 + (y - k)^2 = r^2$.

You can use the standard form of the equation of a circle to write an equation for a circle given its center and the radius or diameter. Recall that a segment that passes through the center of a circle whose endpoints are on the circle is a diameter.

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Real-World EXAMPLE Write an Equation Given the Radius

1 **DELIVERY** **An appliance store offers free delivery within 35 miles of the store. The Jackson store is located 100 miles north and 45 miles east of the corporate office. Write an equation to represent the delivery boundary of the Jackson store if the origin of the coordinate system is the corporate office.**

Words Since the corporate office is at (0, 0), the Jackson store is at (45, 100). The boundary of the delivery region is the circle centered at (45, 100) with radius 35 miles.

Variables

$$(x - h)^2 + (y - k)^2 = r^2$$ Equation of a circle

Equation

$$[x - (-45)]^2 + (y - 100)^2 = 35^2$$ $(h, k) = (45, 100), r = 35$

$$(x - 45)^2 + (y - 100)^2 = 1225$$ Simplify.

Real-World Link

WiFi technology uses radio waves to transmit data. It allows high-speed access to the Internet without the use of cables.

Source: wifiphone.org

CHECK Your Progress

1. WIFI A certain wireless transmitter has a range of thirty miles in any direction. If a WiFi phone is 4 miles south and 3 miles west of the headquarters building, write an equation to represent the area that the phone can communicate via the WiFi system.

EXAMPLE Write an Equation Given a Diameter

2 **Write an equation for a circle if the endpoints of a diameter are at (5, 4) and (−2, −6).**

$$(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$ Midpoint Formula

$$= \left(\frac{5 + (-2)}{2}, \frac{4 + (-6)}{2}\right)$$ $(x_1, y_1) = (5, 4), (x_2, y_2) = (-2, -6)$

$$= \left(\frac{3}{2}, \frac{-2}{2}\right)$$ Add.

$$= \left(\frac{3}{2}, -1\right)$$ Simplify.

Now find the radius.

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Distance Formula

$$= \sqrt{\left(\frac{3}{2} - 5\right)^2 + (-1 - 4)^2}$$ $(x_1, y_1) = (5, 4), (x_2, y_2) = \left(\frac{3}{2}, -1\right)$

$$= \sqrt{\left(-\frac{7}{2}\right)^2 + (-5)^2}$$ Subtract.

$$= \sqrt{\frac{149}{4}}$$ Simplify.

The radius of the circle is $\sqrt{\frac{149}{4}}$ units, so $r^2 = \frac{149}{4}$.

Substitute h, k, and r^2 into the standard form of the equation of a circle.

An equation of the circle is $\left(x - \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{149}{4}$.

CHECK Your Progress

2. Write an equation for a circle if the endpoints of a diameter are at (3, −3) and (1, 5).

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Graph Circles You can use symmetry to help you graph circles.

EXAMPLE Graph an Equation in Standard Form

3 **Find the center and radius of the circle with equation $x^2 + y^2 = 25$. Then graph the circle.**

The center of the circle is at (0, 0), and the radius is 5.

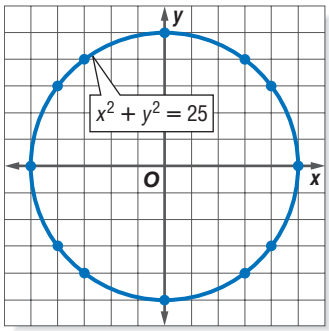
The table lists some integer values for x and y that satisfy the equation.

x	y
0	5
3	4
4	3
5	0

Since the circle is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (−3, 4), (−4, 3), and (−5, 0) lie on the graph.

The circle is also symmetric about the x-axis, so the points at (−4, −3), (−3, −4), (0, −5), (3, −4), and (4, −3) lie on the graph.

Graph all of these points and draw the circle that passes through them.



Cross-Curricular Project

The epicenter of an earthquake can be located by using the equation of a circle. Visit algebra2.com to continue work on your project.

CHECK Your Progress

3. Find the center and radius of the circle with equation $x^2 + y^2 = 81$. Then graph the circle.

Circles with centers that are not at (0, 0) can be graphed using translations. The equation $(x - h)^2 + (y - k)^2 = r^2$ is obtained from the equation $x^2 + y^2 = r^2$ by replacing x with $x - h$ and y with $y - k$. So, the graph of $(x - h)^2 + (y - k)^2 = r^2$ is the graph of $x^2 + y^2 = r^2$ translated h units to the right or left and k units up or down.

EXAMPLE Graph an Equation Not in Standard Form

4 **Find the center and radius of the circle with equation $x^2 + y^2 - 4x + 8y - 5 = 0$. Then graph the circle.**

Complete the squares.

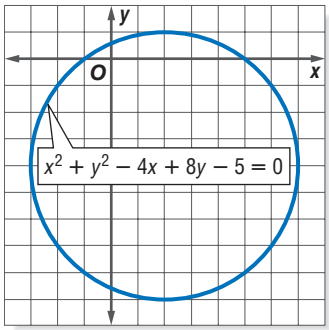
$$x^2 + y^2 - 4x + 8y - 5 = 0$$

$$x^2 - 4x + \blacksquare + y^2 + 8y + \blacksquare = 5 + \blacksquare + \blacksquare$$

$$x^2 - 4x + 4 + y^2 + 8y + 16 = 5 + 4 + 16$$

$$(x - 2)^2 + (y + 4)^2 = 25$$

The center of the circle is at (2, −4), and the radius is 5. In the equation from Example 3, x has been replaced by $x - 2$, and y has been replaced by $y + 4$. The graph is the graph from Example 3 translated 2 units to the right and down 4 units.



CHECK Your Progress

4. Find the center and radius of the circle with equation $x^2 + y^2 + 4x - 10y - 7 = 0$. Then graph the circle.

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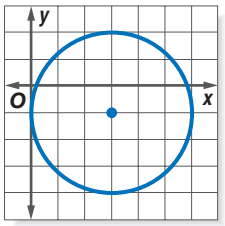
CHECK Your Understanding

Example 1 (p. 575)

1. Write an equation for the graph at the right.



AEROSPACE **For Exercises 2 and 3, use the following information.**
In order for a satellite to remain in a circular orbit above the same spot on Earth, the satellite must be 35,800 kilometers above the equator.

2. Write an equation for the orbit of the satellite. Use the center of Earth as the origin and 6400 kilometers for the radius of Earth.

3. Draw a labeled sketch of Earth and the orbit to scale.

Example 2 (p. 575)

Write an equation for the circle that satisfies each set of conditions.

4. center $(-1, -5)$, radius 2 units

5. endpoints of a diameter at $(-4, 1)$ and $(4, -5)$

6. endpoints of a diameter at $(2, -2)$ and $(-2, -6)$

Examples 3, 4 (p. 576)

Find the center and radius of the circle with the given equation. Then graph the circle.

7. $(x - 4)^2 + (y - 1)^2 = 9$ **8.** $x^2 + (y - 14)^2 = 34$ **9.** $(x - 4)^2 + y^2 = \frac{16}{25}$

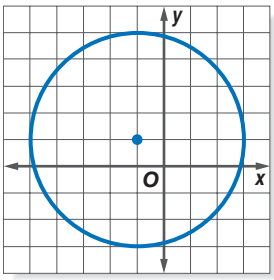
10. $\left(x + \frac{2}{3}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{8}{9}$ **11.** $x^2 + y^2 + 8x - 6y = 0$ **12.** $x^2 + y^2 + 4x - 8 = 0$

Exercises

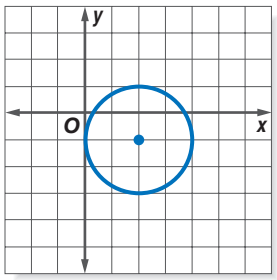
HOMEWORK HELP	
For Exercises	See Examples
13–17	1
18, 19	2
20–25	3
26–31	4

Write an equation for each graph.

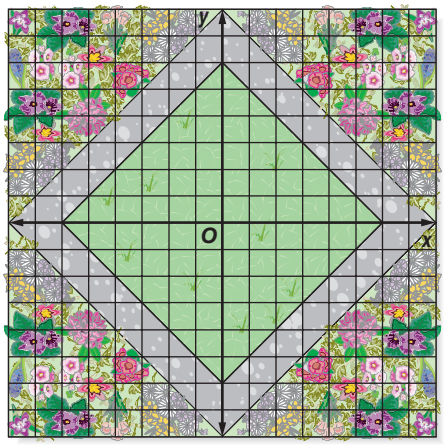
13.



14.



15. LANDSCAPING The design of a garden is shown at the right. A pond is to be built in the center region. What is the equation of the largest circular pond centered at the origin that would fit within the walkways?



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Write an equation for the circle that satisfies each set of conditions.

16. center (0, 3), radius 7 units

17. center (−8, 7), radius $\frac{1}{2}$ unit

18. endpoints of a diameter at (−5, 2) and (3, 6)

19. endpoints of a diameter at (11, 18) and (−13, −19)

Find the center and radius of the circle with the given equation. Then graph the circle.

20. $x^2 + (y + 2)^2 = 4$

21. $x^2 + y^2 = 144$

22. $(x - 3)^2 + (y - 1)^2 = 25$

23. $(x + 3)^2 + (y + 7)^2 = 81$

24. $(x - 3)^2 + y^2 = 16$

25. $(x - 3)^2 + (y + 7)^2 = 50$

26. $x^2 + y^2 + 6y = -50 - 14x$

27. $x^2 + y^2 - 6y - 16 = 0$

28. $x^2 + y^2 + 2x - 10 = 0$

29. $x^2 + y^2 - 18x - 18y + 53 = 0$

30. $x^2 + y^2 + 9x - 8y + 4 = 0$

31. $x^2 + y^2 - 3x + 8y = 20$

Write an equation for the circle that satisfies each set of conditions.

32. center (8, −9), passes through (21, 22)

33. center $\left(-\sqrt{13}, 42\right)$, passes through the origin

34. center at (−8, −7), tangent to y-axis

35. center at (4, 2), tangent to x-axis

36. center in the first quadrant; tangent to $x = -3$, $x = 5$, and the x-axis

37. center in the second quadrant; tangent to $y = -1$, $y = 9$, and the y-axis

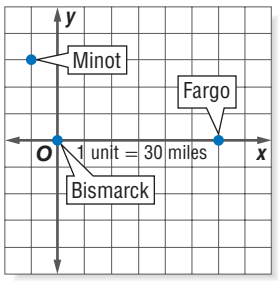
Real-World Link

Southern California has about 10,000 earthquakes per year. Most are too small to be felt.

Source: earthquake.usgs.gov

38. EARTHQUAKES The Rose Bowl is located about 35 miles west and about 40 miles north of downtown Los Angeles. Suppose an earthquake occurs with its epicenter about 55 miles from the stadium. Assume that the origin of a coordinate plane is located at the center of downtown Los Angeles. Write an equation for the set of points that could be the epicenter of the earthquake.

39. RADIO The diagram at the right shows the relative locations of some cities in North Dakota. The x-axis represents Interstate 94. While driving west on the highway, Doralina is listening to a radio station broadcasting from Minot. She estimates the range of the signal to be 120 miles. How far west of Bismarck will she be able to pick up the signal?



Graphing Calculator

For Exercises 40–43, use the following information.

Since a circle is not the graph of a function, you cannot enter its equation directly into a graphing calculator. Instead, you must solve the equation for y. The result will contain a ± symbol, so you will have two functions.

40. Solve $(x + 3)^2 + y^2 = 16$ for y.

41. What two functions should you enter to graph the given equation?

42. Graph $(x + 3)^2 + y^2 = 16$ on a graphing calculator.

43. Solve $(x + 3)^2 + y^2 = 16$ for x. What parts of the circle do the two expressions for x represent?

EXTRA PRACTICE
See pages 912, 935.
MathOnline
Self-Check Quiz at algebra2.com

Long Photography/Getty Images

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H.O.T. Problems

44. OPEN ENDED Write an equation for a circle with center at (6, −2).

45. REASONING Write $x^2 + y^2 + 6x - 2y - 54 = 0$ in standard form by completing the square. Describe the transformation that can be applied to the graph of $x^2 + y^2 = 64$ to obtain the graph of the given equation.

46. FIND THE ERROR Juwan says that the circle with equation $(x - 4)^2 + y^2 = 36$ has radius 36 units. Lucy says that the radius is 6 units. Who is correct? Explain your reasoning.

47. CHALLENGE A circle has its center on the line with equation $y = 2x$. It passes through (1, −3) and has a radius of $\sqrt{5}$ units. Write an equation of the circle.

48. ***Writing in Math*** Use the information about radar equipment on page 574 to explain why circles are important in air traffic control. Include an equation of the circle that determines the boundary of the region where planes can be detected if the range of the radar is 50 miles and the radar is at the origin.

STANDARDIZED TEST PRACTICE

49. ACT/SAT What is the center of the circle with equation $x^2 + y^2 - 10x + 6y + 27 = 0$?

A (−10, 6)

B (1, 1)

C (10, −6)

D (5, −3)

50. REVIEW If the surface area of a cube is increased by a factor of 9, how is the length of the side of the cube changed?

F It is 2 times the original length.

G It is 3 times the original length.

H It is 4 times the original length.

J It is 5 times the original length.

Spiral Review

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola. (Lesson 10-2)

51. $x = -3y^2 + 1$ **52.** $y + 2 = -(x - 3)^2$ **53.** $y = x^2 + 4x$

Find the midpoint of the line segment with endpoints having the given coordinates. (Lesson 10-1)

54. (5, −7), (3, −1) **55.** (2, −9), (−4, 5) **56.** (8, 0), (−5, 12)

Find all of the rational zeros for each function. (Lesson 6-8)

57. $f(x) = x^3 + 5x^2 + 2x - 8$ **58.** $g(x) = 2x^3 - 9x^2 + 7x + 6$

59. PHOTOGRAPHY The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3-2)

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation. Assume that all variables are positive. (Lesson 5-5)

60. $c^2 = 13^2 - 5^2$ **61.** $c^2 = 10^2 - 8^2$ **62.** $(\sqrt{7})^2 = a^2 - 3^2$ **63.** $4^2 = 6^2 - b^2$

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EXPLORE 10-4

Algebra Lab

Investigating Ellipses

ACTIVITY

Follow the steps below to construct another type of conic section.

Step 1 Place two thumbtacks in a piece of cardboard, about 1 foot apart.

Step 2 Tie a knot in a piece of string and loop it around the thumbtacks.

Step 3 Place your pencil in the string. Keep the string tight and draw a curve.

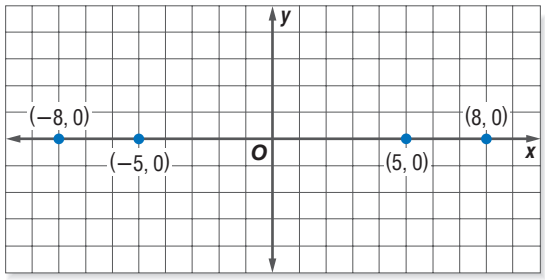
Step 4 Continue drawing until you return to your starting point.

The curve you have drawn is called an **ellipse.** The points where the thumbtacks are located are called the **foci** of the ellipse. *Foci* is the plural of *focus*.

Model and Analyze

Place a large piece of grid paper on a piece of cardboard.

1. Place the thumbtacks at (8, 0) and (−8, 0). Choose a string long enough to loop around both thumbtacks. Draw an ellipse.
2. Repeat Exercise 1, but place the thumbtacks at (5, 0) and (−5, 0). Use the same loop of string and draw an ellipse. How does this ellipse compare to the one in Exercise 1?



Place the thumbtacks at each set of points and draw an ellipse. You may change the length of the loop of string if you like.

3. (12, 0), (−12, 0)

4. (2, 0), (−2, 0)

5. (14, 4), (−10, 4)

Analyze the Results

In Exercises 6–10, describe what happens to the shape of an ellipse when each change is made.

6. The thumbtacks are moved closer together.
7. The thumbtacks are moved farther apart.
8. The length of the loop of string is increased.
9. The thumbtacks are arranged vertically.
10. One thumbtack is removed, and the string is looped around the remaining thumbtack.

Matt Meadows

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10-4 Ellipses

Main Ideas

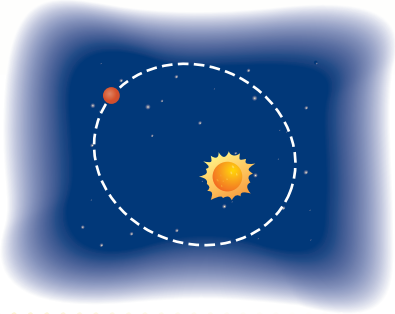
- Write equations of ellipses.
- Graph ellipses.

New Vocabulary

ellipse
foci
major axis
minor axis
center

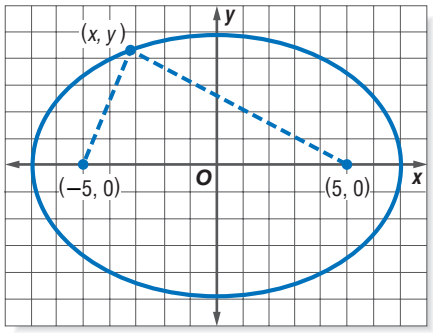
GET READY for the Lesson

Fascination with the sky has caused people to wonder, observe, and make conjectures about the planets since the beginning of history. Since the early 1600s, the orbits of the planets have been known to be ellipses with the Sun at a focus.



Equations of Ellipses As you discovered in the Algebra Lab on page 580, an **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the **foci** of the ellipse.

The ellipse at the right has foci at (5, 0) and (−5, 0). The distances from either of the x-intercepts to the foci are 2 units and 12 units, so the sum of the distances from any point with coordinates (x, y) on the ellipse to the foci is 14 units.



You can use the Distance Formula and the definition of an ellipse to find an equation of this ellipse.

Study Tip

Ellipses
In an ellipse, the constant sum that is the distance from two fixed points must be greater than the distance between the foci.

The distance between (x, y) and $(-5, 0)$ + the distance between (x, y) and $(5, 0)$ = 14.

$$\sqrt{(x+5)^2+y^2}+\sqrt{(x-5)^2+y^2}=14$$

$$\sqrt{(x+5)^2+y^2}=14-\sqrt{(x-5)^2+y^2}$$ Isolate the radicals.

$$(x+5)^2+y^2=196-28\sqrt{(x-5)^2+y^2}+(x-5)^2+y^2$$

$$x^2+10x+25+y^2=196-28\sqrt{(x-5)^2+y^2}+x^2-10x+25+y^2$$

$$20x-196=-28\sqrt{(x-5)^2+y^2}$$ Simplify.

$$5x-49=-7\sqrt{(x-5)^2+y^2}$$ Divide each side by 4.

$$25x^2-490x+2401=49[(x-5)^2+y^2]$$ Square each side.

$$25x^2-490x+2401=49x^2-490x+1225+49y^2$$ Distributive Property

$$-24x^2-49y^2=-1176$$ Simplify.

$$\frac{x^2}{49}+\frac{y^2}{24}=1$$ Divide each side by −1176.

An equation for this ellipse is $\frac{x^2}{49}+\frac{y^2}{24}=1$.

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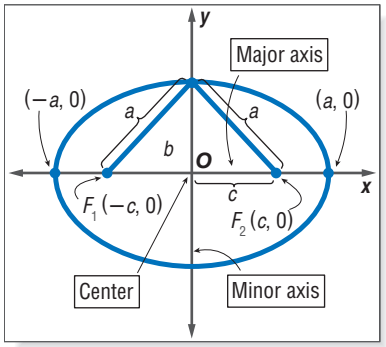
Study Tip

Vertices of Ellipses

The endpoints of each axis are called the *vertices* of the ellipse.

Every ellipse has two axes of symmetry. The points at which the ellipse intersects its axes of symmetry determine two segments with endpoints on the ellipse called the **major axis** and the **minor axis.** The axes intersect at the **center** of the ellipse. The foci of an ellipse always lie on the major axis.

Study the ellipse at the right. The sum of the distances from the foci to any point on the ellipse is the same as the length of the major axis, or $2a$ units. The distance from the center to either focus is c units. By the Pythagorean Theorem, a, b, and c are related by the equation $c^2 = a^2 - b^2$. Notice that the x- and y-intercepts, $(\pm a, 0)$ and $(0, \pm b)$, satisfy the quadratic equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is the standard form of the equation of an ellipse with its center at the origin and a horizontal major axis.



Study Tip

Equations of Ellipses

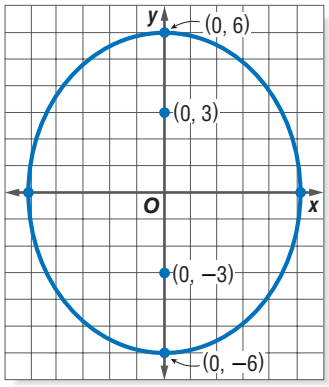
In either case, $a^2 \geq b^2$ and $c^2 = a^2 - b^2$. You can determine if the foci are on the x-axis or the y-axis by looking at the equation. If the x^2 term has the greater denominator, the foci are on the x-axis. If the y^2 term has the greater denominator, the foci are on the y-axis.

KEY CONCEPT — ***Equations of Ellipses with Centers at the Origin***

Standard Form of Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

EXAMPLE Write an Equation for a Graph

1 **Write an equation for the ellipse.**



To write the equation for the ellipse, we need to find the values of a and b for the ellipse. We know that the length of the major axis of any ellipse is $2a$ units. In this ellipse, the length of the major axis is the distance between the points at (0, 6) and (0, −6). This distance is 12 units.

$2a = 12$ Length of major axis = 12

$a = 6$ Divide each side by 2.

The foci are located at (0, 3) and (0, −3), so $c = 3$. We can use the relationship between a, b, and c to determine the value of b.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$9 = 36 - b^2$ $c = 3$ and $a = 6$

$b^2 = 27$ Solve for b^2.

Since the major axis is vertical, substitute 36 for a^2 and 27 for b^2 in the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. An equation of the ellipse is $\frac{y^2}{36} + \frac{x^2}{27} = 1$.

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Real-World Link

The whispering gallery at Chicago's Museum of Science and Industry has a parabolic dish at each focus to help collect sound.

Source: msichicago.org

CHECK Your Progress

1. Write an equation for the ellipse with endpoints of the major axis at $(-5, 0)$ and $(5, 0)$ and endpoints of the minor axis at $(0, -2)$ and $(0, 2)$.

EXAMPLE 2 Write an Equation Given the Lengths of the Axes

MUSEUMS **In an ellipse, sound or light coming from one focus is reflected to the other focus. In a whispering gallery, a person can hear another person whisper from across the room if the two people are standing at the foci. The whispering gallery at the Museum of Science and Industry in Chicago has an elliptical cross section that is 13 feet 6 inches by 47 feet 4 inches.**

a. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

The length of the major axis is $47\frac{1}{3}$ or $\frac{142}{3}$ feet.

$2a = \frac{142}{3}$ Length of major axis $= \frac{142}{3}$

$a = \frac{71}{3}$ Divide each side by 2.

The length of the minor axis is $13\frac{1}{2}$ or $\frac{27}{2}$ feet.

$2b = \frac{27}{2}$ Length of minor axis $= \frac{27}{2}$

$b = \frac{27}{4}$ Divide each side by 2.

Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$ into the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. An equation of the ellipse is $\frac{x^2}{\left(\frac{71}{3}\right)^2} + \frac{y^2}{\left(\frac{27}{4}\right)^2} = 1$.

b. How far apart are the points at which two people should stand to hear each other whisper?

People should stand at the two foci of the ellipse. The distance between the foci is $2c$ units.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$c = \sqrt{a^2 - b^2}$ Take the square root of each side.

$2c = 2\sqrt{a^2 - b^2}$ Multiply each side by 2.

$2c = 2\sqrt{\left(\frac{71}{3}\right)^2 - \left(\frac{27}{4}\right)^2}$ Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$.

$2c \approx 45.37$ Use a calculator.

The points where two people should stand to hear each other whisper are about 45.37 feet or about 45 feet 4 inches apart.

CHECK Your Progress

BILLIARDS Elliptipool is an elliptical pool table with only one pocket that is located on one of the foci. If the ball is placed on the other focus and shot off any edge, it will drop into the pocket located on the other focus. The pool table has axes that are 4 feet 6 inches and 5 feet.

2A. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

2B. How far apart are the two foci?

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Ray F. Hillstrom, Jr.
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Graph Ellipses As with circles, you can use completing the square, symmetry, and transformations to help graph ellipses. An ellipse with its center at the origin is represented by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.

Study Tip

Graphing Calculator

You can graph an ellipse on a graphing calculator by first solving for y. Then graph the two equations that result on the same screen.

EXAMPLE Graph an Equation in Standard Form

1 **Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then graph the ellipse.**

The center of this ellipse is at (0, 0). Since $a^2 = 16$, $a = 4$. Since $b^2 = 4$, $b = 2$.

The length of the major axis is 2(4) or 8 units, and the length of the minor axis is 2(2) or 4 units. Since the x^2 term has the greater denominator, the major axis is horizontal.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$c^2 = 4^2 - 2^2$ or 12 $a = 4, b = 2$

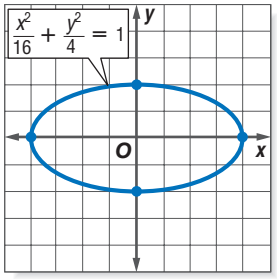
$c = \sqrt{12}$ or $2\sqrt{3}$ Take the square root of each side.

The foci are at $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the ellipse is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (−4, 0), (−3, 1.3), (−2, 1.7), and (−1, 1.9) lie on the graph.

The ellipse is also symmetric about the x-axis, so the points at (−3, −1.3), (−2, −1.7), (−1, −1.9), (0, −2), (1, −1.9), (2, −1.7), and (3, −1.3) lie on the graph.

Graph the intercepts, (−4, 0), (4, 0), (0, 2), and (0, −2), and draw the ellipse that passes through them and the other points.



CHECK Your Progress

3. Find the coordinates of the foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{49} + \frac{y^2}{36} = 1$. Then graph the ellipse.

Suppose an ellipse is translated h units right and k units up, moving the center to the point (h, k). Such a move would be equivalent to replacing x with $x - h$ and replacing y with $y - k$.

KEY CONCEPT *Equations of Ellipses with Centers at (h, k)*

Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

Extra Examples at algebra2.com

www.MathSchoolinternational.com

EXAMPLE Graph an Equation Not in Standard Form

4 **Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 + 4x - 24y + 24 = 0$. Then graph the ellipse.**

Complete the square for each variable to write this equation in standard form.

$$x^2 + 4y^2 + 4x - 24y + 24 = 0$$ Original equation

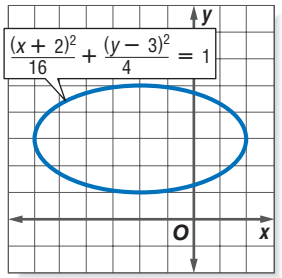
$$(x^2 + 4x + \blacksquare) + 4(y^2 - 6y + \blacksquare) = -24 + \blacksquare + 4(\blacksquare)$$ Complete the squares.

$$(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -24 + 4 + 4(9)$$ $\left(\frac{4}{2}\right)^2 = 4, \left(\frac{-6}{2}\right)^2 = 9$

$$(x + 2)^2 + 4(y - 3)^2 = 16$$ Write the trinomials as perfect squares.

$$\frac{(x + 2)^2}{16} + \frac{(y - 3)^2}{4} = 1$$ Divide each side by 16.

The graph of this ellipse is the graph from Example 3 translated 2 units to the left and up 3 units. The center is at $(-2, 3)$ and the foci are at $(-2 + 2\sqrt{3}, 0)$ and $(-2 - 2\sqrt{3}, 0)$. The length of the major axis is still 8 units, and the length of the minor axis is still 4 units.



Study Tip

Look Back

To review **grouping** and **factoring** common factors of each variable separately, see Lessons 5-3 and 5-5.

CHECK Your Progress

4. Find the coordinates and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 6y^2 + 8x - 12y + 16 = 0$. Then graph the ellipse.

You can use a circle to locate the foci on the graph of a given ellipse.

ALGEBRA LAB

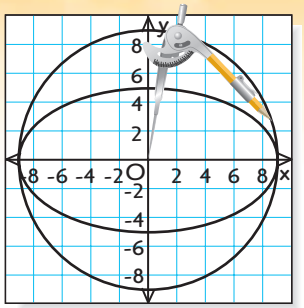
Locating Foci

Step 1 Graph an ellipse so that its center is at the origin. Let the endpoints of the major axis be at $(-9, 0)$ and $(9, 0)$, and let the endpoints of the minor axis be at $(0, -5)$ and $(0, 5)$.

Step 2 Use a compass to draw a circle with center at $(0, 0)$ and radius 9 units.

Step 3 Draw the line with equation $y = 5$ and mark the points at which the line intersects the circle.

Step 4 Draw perpendicular lines from the points of intersection to the x-axis. The foci of the ellipse are located at the points where the perpendicular lines intersect the x-axis.



MAKE A CONJECTURE

Draw another ellipse and locate its foci. Why does this method work?

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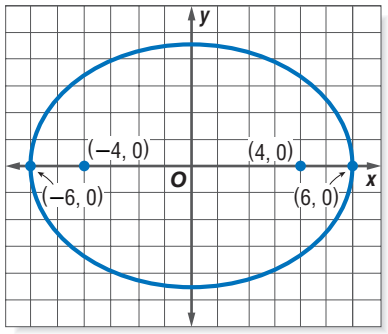
CHECK Your Understanding

Example 1 (pp. 582–583)

1. Write an equation for the ellipse shown at the right.



Write an equation for the ellipse that satisfies each set of conditions.

2. endpoints of major axis at (2, 2) and (2, −10), endpoints of minor axis at (0, −4) and (4, −4)

3. endpoints of major axis at (0, 10) and (0, −10), foci at (0, 8) and (0, −8)

Example 2 (p. 583)

4. ASTRONOMY At its closest point, Earth is 0.99 astronomical units from the center of the Sun. At its farthest point, Earth is 1.021 astronomical units from the center of the Sun. Write an equation for the orbit of Earth, assuming that the center of the orbit is the origin and the Sun lies on the x-axis.

Examples 3, 4 (pp. 584–585)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

5. $\frac{y^2}{18} + \frac{x^2}{9} = 1$

6. $\frac{(x-1)^2}{20} + \frac{(y+2)^2}{4} = 1$

7. $4x^2 + 8y^2 = 32$

8. $x^2 + 25y^2 - 8x + 100y + 91 = 0$

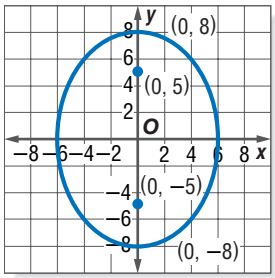
Exercises

HOMEWORK HELP

For Exercises	See Examples
9–15	1
16, 17	2
18–21	3
22–25	4

Write an equation for each ellipse.

9.



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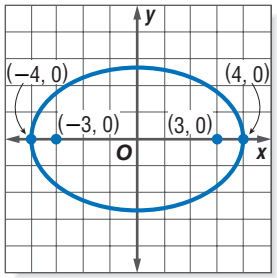

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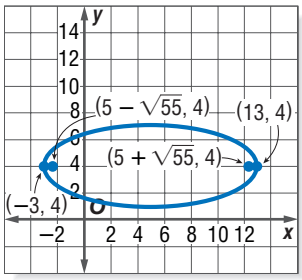
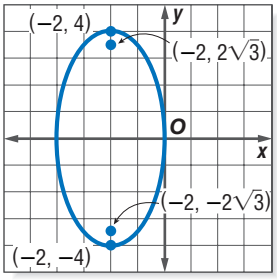


12.



Write an equation for the ellipse that satisfies each set of conditions.

13. endpoints of major axis at (−11, 5) and (7, 5), endpoints of minor axis at (−2, 9) and (−2, 1)

14. endpoints of major axis at (2, 12) and (2, −4), endpoints of minor axis at (4, 4) and (0, 4)

15. major axis 20 units long and parallel to y-axis, minor axis 6 units long, center at (4, 2)

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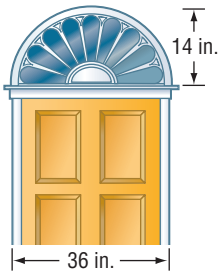
16. ASTRONOMY At its closest point, Venus is 0.719 astronomical units from the Sun. At its farthest point, Venus is 0.728 astronomical units from the Sun. Write an equation for the orbit of Venus. Assume that the center of the orbit is the origin, the Sun lies on the x-axis, and the radius of the Sun is 400,000 miles.

17. INTERIOR DESIGN The rounded top of the window is the top half of an ellipse. Write an equation for the ellipse if the origin is at the midpoint of the bottom edge of the window.



Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

Real-World Link

The Ellipse, also known as President's Park South, has an area of about 16 acres.

18. $\frac{y^2}{10} + \frac{x^2}{5} = 1$

19. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

20. $\frac{(x+8)^2}{144} + \frac{(y-2)^2}{81} = 1$

21. $\frac{(y+11)^2}{144} + \frac{(x-5)^2}{121} = 1$

22. $3x^2 + 9y^2 = 27$

23. $27x^2 + 9y^2 = 81$

24. $7x^2 + 3y^2 - 28x - 12y = -19$

25. $16x^2 + 25y^2 + 32x - 150y = 159$

Write an equation for the ellipse that satisfies each set of conditions.

26. major axis 16 units long and parallel to x-axis, minor axis 9 units long, center at (5, 4)

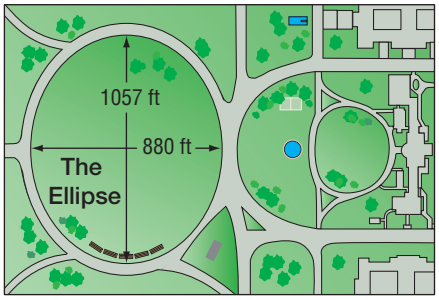
27. endpoints of major axis at (10, 2) and (−8, 2), foci at (6, 2) and (−4, 2)

28. endpoints of minor axis at (0, 5) and (0, −5), foci at (12, 0) and (−12, 0)

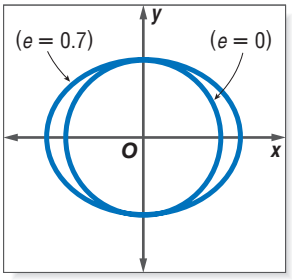
29. Write the equation $10x^2 + 2y^2 = 40$ in standard form.

30. What is the standard form of the equation $x^2 + 6y^2 - 2x + 12y - 23 = 0$?

31. WHITE HOUSE There is an open area south of the White House known as The Ellipse. Write an equation to model The Ellipse. Assume that the origin is at the center of The Ellipse.



32. ASTRONOMY In an ellipse, the ratio $\frac{c}{a}$ is called the eccentricity and is denoted by the letter e. Eccentricity measures the elongation of an ellipse. The closer e is to 0, the more an ellipse looks like a circle. Pluto has the most eccentric orbit in our solar system with $e \approx 0.25$. Find an equation to model the orbit of Pluto, given that the length of the major axis is about 7.34 billion miles. Assume that the major axis is horizontal and that the center of the orbit is the origin.



EXTRA PRACTICE
See pages 912, 935.

MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

33. REASONING Explain why a circle is a special case of an ellipse.

34. OPEN ENDED Write an equation for an ellipse with its center at (2, −5) and a horizontal major axis.

James P. Blair/CORBIS

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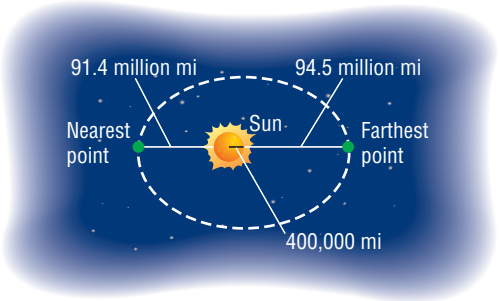
35. CHALLENGE Find an equation for the ellipse with foci at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$ that passes through $(0, 3)$.

36. *Writing in Math* Use the information about the solar system on page 581 and the figure at the right to explain why ellipses are important in the study of the solar system. Explain why an equation that is an accurate model of the path of a planet might be useful.



STANDARDIZED TEST PRACTICE

37. ACT/SAT Winona is making an elliptical target for throwing darts. She wants the target to be 27 inches wide and 15 inches high. Which equation should Winona use to draw the target?

A $\frac{y^2}{13.5} + \frac{x^2}{7.5} = 1$

B $\frac{y^2}{182.25} + \frac{x^2}{56.25} = 1$

C $\frac{y^2}{56.25} + \frac{x^2}{182.25} = 1$

D $\frac{y^2}{7.5} + \frac{x^2}{13.5} = 1$

38. REVIEW What is the standard form of the equation of the conic given below?

$$2x^2 - 4y^2 - 8x - 24y - 16 = 0$$

F $\frac{(x-4)^2}{11} - \frac{(y+3)^2}{3} = 1$

G $\frac{(y-3)^2}{3} - \frac{(x-2)^2}{6} = 1$

H $\frac{(y+3)^2}{4} - \frac{(x+2)^2}{5} = 1$

J $\frac{(x-4)^2}{11} + \frac{(y+3)^2}{3} = 1$

Spiral Review

Write an equation for the circle that satisfies each set of conditions. (Lesson 10-3)

39. center $(3, -2)$, radius 5 units

40. endpoints of a diameter at $(5, -9)$ and $(3, 11)$

41. Write an equation of a parabola with vertex $(3, 1)$ and focus $\left(3, 1\frac{1}{2}\right)$. Then draw the graph. (Lesson 10-2)

MARRIAGE For Exercises 42–44, use the table below that shows the number of married Americans over the last few decades. (Lesson 2-5)

Year	1980	1990	1995	1999	2000	2010
People (millions)	104.6	112.6	116.7	118.9	120.2	?

Source: U.S. Census Bureau

42. Draw a scatter plot in which x is the number of years since 1980.

43. Find a prediction equation.

44. Predict the number of married Americans in 2010.

GET READY for the Next Lesson

PREREQUISITE SKILL Graph the line with the given equation. (Lessons 2-1, 2-2, and 2-3)

45. $y = 2x$

46. $y = -2x$

47. $y = -\frac{1}{2}x$

48. $y = \frac{1}{2}x$

49. $y + 2 = 2(x - 1)$

50. $y + 2 = -2(x - 1)$

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CHAPTER 10

Mid-Chapter Quiz

Lessons 10-1 through 10-4

Find the distance between each pair of points with the given coordinates. (Lesson 10-1)

1. (9, 5), (4, −7)

2. (0, −5), (10, −3)

The coordinates of the endpoints of a segment are given. Find the coordinates of the midpoint of each segment. (Lesson 10-1)

3. (1, 5), (−4, −3)

4. (−3, 8), (−11, −6)

DISTANCE **For Exercises 5 and 6, use the following information.** (Lesson 10-1)

Jessica lives at the corner of 5th Avenue and 12th street. Julie lives at the corner of 15th Avenue and 4th street.

5. How many blocks apart do the two girls live?

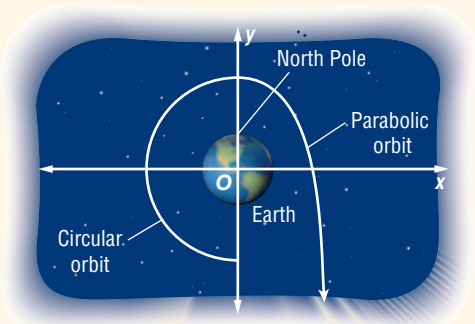
6. If they want to meet for lunch halfway between their houses, where would they meet?

Write each equation in standard form. Identify the vertex, axis of symmetry, and direction of opening of the parabola. (Lesson 10-2)

7. $y = x^2 - 6x + 4$

8. $x = y^2 + 2y - 3$

9. **SPACE SCIENCE** A spacecraft is in a circular orbit 93 miles above Earth. Once it attains the velocity needed to escape Earth's gravity, the spacecraft will follow a parabolic path with the center of Earth as the focus. Suppose the spacecraft reaches escape velocity above the North Pole. Write an equation to model the parabolic path of the spacecraft, assuming that the center of Earth is at the origin and the radius of Earth is 3977 miles. (Lesson 10-2)



Identify the coordinates of the vertex and focus, the equation of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola. (Lesson 10-2)

10. $y^2 = 6x$

11. $y = x^2 + 8x + 20$

12. Find the center and radius of the circle with equation $x^2 + (y - 4)^2 = 49$. Then graph the circle. (Lesson 10-3)

13. **SPRINKLERS** A sprinkler waters a circular section of lawn about 20 feet in diameter. The homeowner decides that placing the sprinkler at (7, 5) will maximize the area of grass being watered. Write an equation to represent the boundary the sprinkler waters. (Lesson 10-3)

14. Write an equation for the circle that has center at (−1, 0) and passes through (2, −6). (Lesson 10-4)

15. **MULTIPLE CHOICE** What is the radius of the circle with equation $x^2 + y^2 + 8x + 8y + 28 = 0$? (Lesson 10-3)

A 2

B 4

C 8

D 28

16. Write an equation of the ellipse with foci at (3, 8) and (3, −6) and endpoints of the major axis at (3, −8) and (3, 10). (Lesson 10-4)

Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with the given equation. Then graph the ellipse. (Lesson 10-4)

17. $\frac{(x-4)^2}{9} + \frac{(y+2)^2}{1} = 1$

18. $16x^2 + 5y^2 + 32x - 10y - 59 = 0$

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10-5 Hyperbolas

Main Ideas

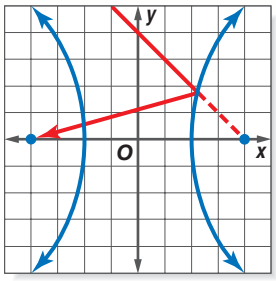
- Write equations of hyperbolas.
- Graph hyperbolas.

New Vocabulary

hyperbola
foci
center
vertex
asymptote
transverse axis
conjugate axis

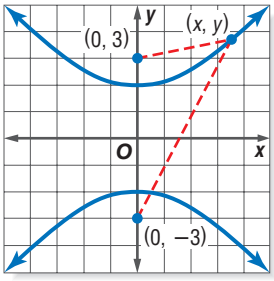
GET READY for the Lesson

A hyperbola is a conic section with the property that rays directed toward one focus are reflected toward the other focus. Notice that, unlike the other conic sections, a hyperbola has two branches.



Equations of Hyperbolas A **hyperbola** is the set of all points in a plane such that the absolute value of the difference of the distances from two fixed points, called the **foci**, is constant.

The hyperbola at the right has foci at (0, 3) and (0, −3). The distances from either of the y-intercepts to the foci are 1 unit and 5 units, so the difference of the distances from any point with coordinates (x, y) on the hyperbola to the foci is 4 or −4 units, depending on the order in which you subtract.



You can use the Distance Formula and the definition of a hyperbola to find an equation of this hyperbola.

The distance between (x, y) and $(0, 3)$ − the distance between (x, y) and $(0, -3)$ $= \pm 4$.

$$\sqrt{x^2 + (y-3)^2} - \sqrt{x^2 + (y+3)^2} = \pm 4$$

$$\sqrt{x^2 + (y-3)^2} = \pm 4 + \sqrt{x^2 + (y+3)^2}$$ Isolate the radicals.

$$x^2 + (y-3)^2 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + (y+3)^2$$

$$x^2 + y^2 - 6y + 9 = 16 \pm 8\sqrt{x^2 + (y+3)^2} + x^2 + y^2 + 6y + 9$$

$$-12y - 16 = \pm 8\sqrt{x^2 + (y+3)^2}$$ Simplify.

$$3y + 4 = \pm 2\sqrt{x^2 + (y+3)^2}$$ Divide each side by −4.

$$9y^2 + 24y + 16 = 4[x^2 + (y+3)^2]$$ Square each side.

$$9y^2 + 24y + 16 = 4x^2 + 4y^2 + 24y + 36$$ Distributive Property

$$5y^2 - 4x^2 = 20$$ Simplify.

$$\frac{y^2}{4} - \frac{x^2}{5} = 1$$ Divide each side by 20.

An equation of this hyperbola is $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

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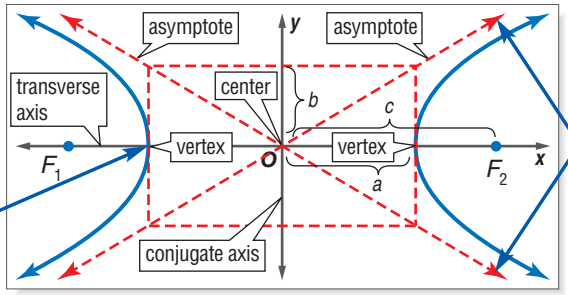
The diagram below shows the parts of a hyperbola.



The point on each branch nearest the center is a **vertex**.

As a hyperbola recedes from its center, the branches approach lines called **asymptotes**.

The distance from the **center** to a vertex of a hyperbola is a units. The distance from the center to a focus is c units. There are two axes of symmetry. The **transverse axis** is a segment of length $2a$ whose endpoints are the vertices of the hyperbola. The **conjugate axis** is a segment of length $2b$ units that is perpendicular to the transverse axis at the center. The values of a, b, and c are related differently for a hyperbola than for an ellipse. For a hyperbola, $c^2 = a^2 + b^2$.

Reading Math

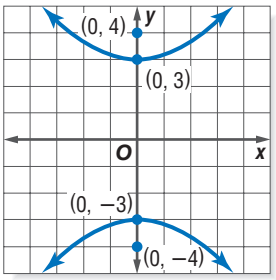
Standard Form In the standard form of a hyperbola, the squared terms are subtracted (−). For an ellipse, they are added (+).

KEY CONCEPT *Equations of Hyperbolas with Centers at the Origin*

Standard Form of Equation	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Vertices	$(a, 0), (-a, 0)$	$(0, a), (0, -a)$
Length of Transverse Axis	$2a$ units	$2a$ units
Length of Conjugate Axis	$2b$ units	$2b$ units
Equations of Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

EXAMPLE 1 Write an Equation for a Graph

Write an equation for the hyperbola shown at the right.



The center is the midpoint of the segment connecting the vertices, or (0, 0).

The value of a is the distance from the center to a vertex, or 3 units. The value of c is the distance from the center to a focus, or 4 units.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$4^2 = 3^2 + b^2$ $c = 4$, $a = 3$

$16 = 9 + b^2$ Evaluate the squares.

$7 = b^2$ Solve for b^2.

Since the transverse axis is vertical, an equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{7} = 1$.

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CHECK Your Progress

1. Write an equation for the hyperbola with vertices at (0, 4) and (0, −4) and foci at (0, 5) and (0, −5).

Real-World EXAMPLE **Write an Equation Given the Foci**

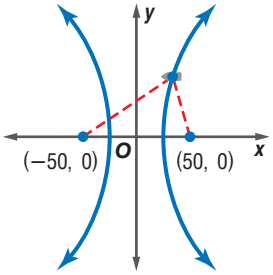
2 NAVIGATION **The LORAN navigational system is based on hyperbolas. Two stations send out signals at the same time. A ship notes the difference in the times at which it receives the signals. The ship is on a hyperbola with the stations at the foci. Suppose a ship determines that the difference of its distances from two stations is 50 nautical miles. Write an equation for a hyperbola on which the ship lies if the stations are at (−50, 0) and (50, 0).**

Real-World Link

LORAN stands for *Long Range Navigation.* The LORAN system is generally accurate to within 0.25 nautical mile.

Source: U.S. Coast Guard

First, draw a figure. By studying either of the x-intercepts, you can see that the difference of the distances from any point on the hyperbola to the stations at the foci is the same as the length of the transverse axis, or $2a$. Therefore, $2a = 50$, or $a = 25$. According to the coordinates of the foci, $c = 50$.



Use the values of a and c to determine the value of b for this hyperbola.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$50^2 = 25^2 + b^2$ $c = 50$, $a = 25$

$2500 = 625 + b^2$ Evaluate the squares.

$1875 = b^2$ Solve for b^2.

Since the transverse axis is horizontal, the equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Substitute the values for a^2 and b^2. An equation of the hyperbola is $\frac{x^2}{625} - \frac{y^2}{1875} = 1$.

CHECK Your Progress

2. Two microphones are set up underwater 3000 feet apart to observe dolphins. Sound travels under water at 5000 feet per second. One microphone picked up the sound of a dolphin 0.25 second before the other microphone picks up the same sound. Find the equation of the hyperbola that describes the possible locations of the dolphin.

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Graph Hyperbolas It is easier to graph a hyperbola if the asymptotes are drawn first. To graph the asymptotes, use the values of a and b to draw a rectangle with dimensions $2a$ and $2b$. The diagonals of the rectangle should intersect at the center of the hyperbola. The asymptotes will contain the diagonals of the rectangle.

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EXAMPLE Graph an Equation in Standard Form

3 **Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then graph the hyperbola.**

The center of this hyperbola is at the origin. According to the equation, $a^2 = 9$ and $b^2 = 4$, so $a = 3$ and $b = 2$. The coordinates of the vertices are $(3, 0)$ and $(-3, 0)$.

$c^2 = a^2 + b^2$ Equation relating a, b, and c for a hyperbola

$c^2 = 3^2 + 2^2$ $a = 3, b = 2$

$c^2 = 13$ Simplify.

$c = \sqrt{13}$ Take the square root of each side.

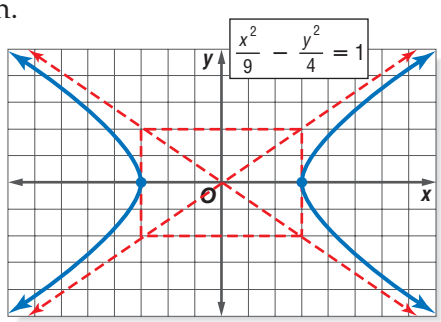
The foci are at $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}x$ or $y = \pm\frac{2}{3}x$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the hyperbola is centered at the origin, it is symmetric about the y-axis. Therefore, the points at $(-8, 4.9)$, $(-7, 4.2)$, $(-6, 3.5)$, $(-5, 2.7)$, $(-4, 1.8)$, and $(-3, 0)$ lie on the graph.

The hyperbola is also symmetric about the x-axis, so the points at $(-8, -4.9)$, $(-7, -4.2)$, $(-6, -3.5)$, $(-5, -2.7)$, $(-4, -1.8)$, $(4, -1.8)$, $(5, -2.7)$, $(6, -3.5)$, $(7, -4.2)$, and $(8, -4.9)$ also lie on the graph.

Draw a 6-unit by 4-unit rectangle. The asymptotes contain the diagonals of the rectangle. Graph the vertices, which, in this case, are the x-intercepts. Use the asymptotes as a guide to draw the hyperbola that passes through the vertices and the other points. The graph does not intersect the asymptotes.



Study Tip

Graphing Calculator

You can graph a hyperbola on a graphing calculator. Similar to an ellipse, first solve the equation for y. Then graph the two equations that result on the same screen.

CHECK Your Progress

3. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $\frac{x^2}{49} - \frac{y^2}{25} = 1$. Then graph the hyperbola.

So far, you have studied hyperbolas that are centered at the origin. A hyperbola may be translated so that its center is at (h, k). This corresponds to replacing x by $x - h$ and y by $y - k$ in both the equation of the hyperbola and the equations of the asymptotes.

KEY CONCEPT *Equations of Hyperbolas with Centers at (h, k)*

Standard Form of Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Direction of Transverse Axis	horizontal	vertical
Equations of Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

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When graphing a hyperbola given an equation that is not in standard form, begin by rewriting the equation in standard form.

EXAMPLE 4 Graph an Equation Not in Standard Form

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $4x^2 - 9y^2 - 32x - 18y + 19 = 0$. Then graph the hyperbola.

Complete the square for each variable to write this equation in standard form.

$$4x^2 - 9y^2 - 32x - 18y + 19 = 0$$ Original equation

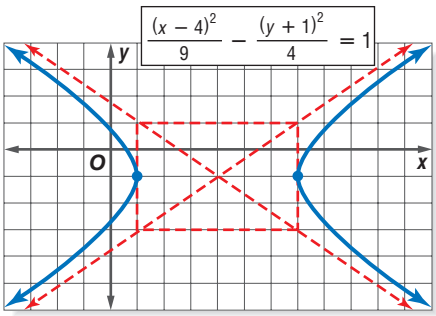
$$4(x^2 - 8x + \blacksquare) - 9(y^2 + 2y + \blacksquare) = -19 + 4(\blacksquare) - 9(\blacksquare)$$ Complete the squares.

$$4(x^2 - 8x + \mathbf{16}) - 9(y^2 + 2y + \mathbf{1}) = -19 + 4(\mathbf{16}) - 9(\mathbf{1})$$

$$4(x - 4)^2 - 9(y + 1)^2 = 36$$ Write the trinomials as perfect squares.

$$\frac{(x - 4)^2}{9} - \frac{(y + 1)^2}{4} = 1$$ Divide each side by 36.

The graph of this hyperbola is the graph from Example 3 translated 4 units to the right and down 1 unit. The vertices are at $(7, -1)$ and $(1, -1)$, and the foci are at $(4 + \sqrt{13}, -1)$ and $(4 - \sqrt{13}, -1)$. The equations of the asymptotes are $y + 1 = \pm\frac{2}{3}(x - 4)$.



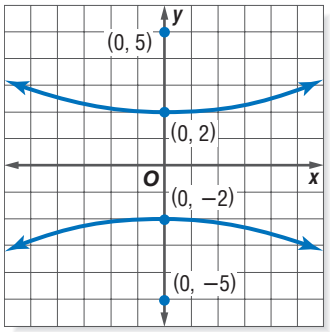
CHECK Your Progress

4. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with equation $9x^2 - 25y^2 - 36x - 50y - 214 = 0$. Then graph the hyperbola.

CHECK Your Understanding

Example 1 (pp. 591–592)

1. Write an equation for the hyperbola shown at right.



Example 2 (p. 592)

2. A hyperbola has foci at $(4, 0)$ and $(-4, 0)$. The value of a is 1. Write an equation for the hyperbola.

3. ASTRONOMY Comets or other objects that pass by Earth or the Sun only once and never return may follow hyperbolic paths. Suppose a comet's path can be modeled by a branch of the hyperbola with equation $\frac{y^2}{225} - \frac{x^2}{400} = 1$. Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola. Then graph the hyperbola.

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Examples 3, 4 (pp. 593, 594)

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

4. $\frac{y^2}{18} - \frac{x^2}{20} = 1$

5. $\frac{(y+6)^2}{20} - \frac{(x-1)^2}{25} = 1$

6. $x^2 - 36y^2 = 36$

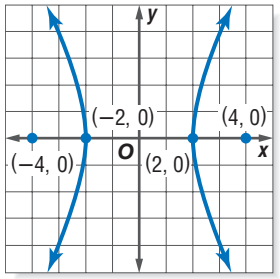
7. $5x^2 - 4y^2 - 40x - 16y - 36 = 0$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–11	1
12–15	2
16–21	3
22–25	4

Write an equation for each hyperbola.

8.



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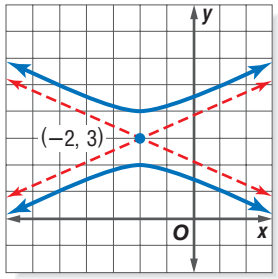

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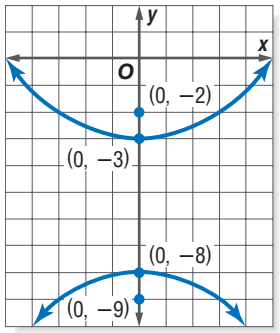
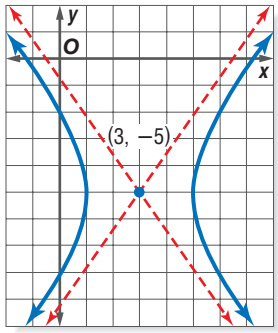


11.



Write an equation for the hyperbola that satisfies each set of conditions.

12. vertices $(-5, 0)$ and $(5, 0)$, conjugate axis of length 12 units

13. vertices $(0, -4)$ and $(0, 4)$, conjugate axis of length 14 units

14. vertices $(9, -3)$ and $(-5, -3)$, foci $\left(2 \pm \sqrt{53}, -3\right)$

15. vertices $(-4, 1)$ and $(-4, 9)$, foci $\left(-4, 5 \pm \sqrt{97}\right)$

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

16. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

17. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

18. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

19. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

20. $\frac{(y-4)^2}{16} - \frac{(x+2)^2}{9} = 1$

21. $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$

22. $x^2 - 2y^2 = 2$

23. $x^2 - y^2 = 4$

24. $y^2 = 36 + 4x^2$

25. $6y^2 = 2x^2 + 12$

26. $\frac{(x+1)^2}{4} - \frac{(y+3)^2}{9} = 1$

27. $\frac{(x+6)^2}{36} - \frac{(y+3)^2}{9} = 1$

28. $y^2 - 3x^2 + 6y + 6x - 18 = 0$

29. $4x^2 - 25y^2 - 8x - 96 = 0$

30. Find an equation for a hyperbola centered at the origin with a horizontal transverse axis of length 8 units and a conjugate axis of length 6 units.

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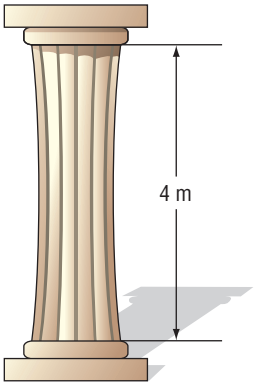
31. What is an equation for the hyperbola centered at the origin with a vertical transverse axis of length 12 units and a conjugate axis of length 4 units?

Real-World Link

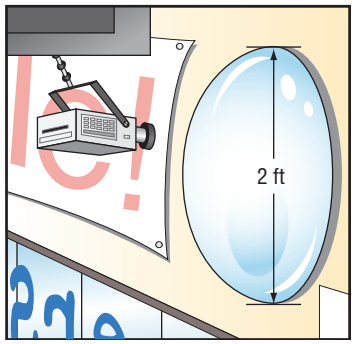
The Parthenon, originally constructed in 447 B.C., has 46 columns around its exterior perimeter.

Source: www.mlahanas.de/Greeks/Arts/Parthenon.htm

32. STRUCTURAL DESIGN An architect's design for a building includes some large pillars with cross sections in the shape of hyperbolas. The curves can be modeled by the equation $\frac{x^2}{0.25} - \frac{y^2}{9} = 1$, where the units are in meters. If the pillars are 4 meters tall, find the width of the top of each pillar and the width of each pillar at the narrowest point in the middle. Round to the nearest centimeter.

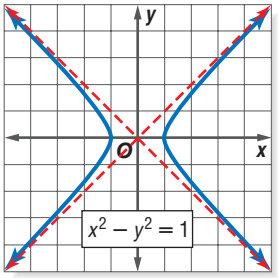


33. PHOTOGRAPHY A curved mirror is placed in a store for a wide-angle view of the room. The equation $\frac{x^2}{1} - \frac{y^2}{3} = 1$ models the curvature of the mirror. A small security camera is placed 3 feet from the vertex of the mirror so that a diameter of 2 feet of the mirror is visible. If the back of the room lies on $x = -18$, what width of the back of the room is visible to the camera?



NONRECTANGULAR HYPERBOLA **For Exercises 34–37, use the following information.**

A hyperbola with asymptotes that are not perpendicular is called a **nonrectangular hyperbola**. Most of the hyperbolas you have studied so far are nonrectangular. A **rectangular hyperbola** is a hyperbola with perpendicular asymptotes. For example, the graph of $x^2 - y^2 = 1$ is a rectangular hyperbola. The graphs of equations of the form $xy = c$, where c is a constant, are rectangular hyperbolas with the coordinate axes as their asymptotes.



34. Plot some points and use them to graph the equation. Be sure to consider negative values for the variables.

35. Find the coordinates of the vertices of the graph of $xy = 2$.

36. Graph $xy = -2$.

37. Describe the transformations that can be applied to the graph of $xy = 2$ to obtain the graph of $xy = -2$.

EXTRA PRACTICE
See pages 913, 935.

Math Online
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H.O.T. Problems

38. OPEN ENDED Find and graph a counterexample to the following statement.

If the equation of a hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *then* $a^2 \geq b^2$.

39. REASONING Describe how the graph of $y^2 - \frac{x^2}{k^2} = 1$ changes as $|k|$ increases.

40. CHALLENGE A hyperbola with a horizontal transverse axis contains the point at (4, 3). The equations of the asymptotes are $y - x = 1$ and $y + x = 5$. Write the equation for the hyperbola.

41. ***Writing in Math*** Explain how hyperbolas and parabolas are different. Include differences in the graphs of hyperbolas and parabolas and differences in the reflective properties of hyperbolas and parabolas.

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STANDARDIZED TEST PRACTICE

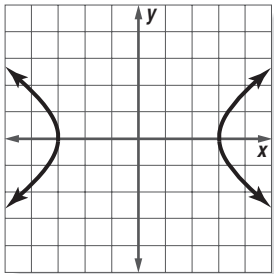
42. ACT/SAT The foci of the graph are at $(\sqrt{13}, 0)$ and $(-\sqrt{13}, 0)$. Which equation does the graph represent?

A $\frac{x^2}{9} - \frac{y^2}{4} = 1$

B $\frac{x^2}{3} - \frac{y^2}{2} = 1$

C $\frac{x^2}{3} - \frac{y^2}{\sqrt{13}} = 1$

D $\frac{x^2}{9} - \frac{y^2}{13} = 1$



43. REVIEW To begin a game, Nate must randomly draw a red, blue, green, or yellow game piece, and a tile from a group of 26 tiles labeled with all the letters of the alphabet. What is the probability that Nate will draw the green game piece and a tile with a letter from his name?

F $\frac{1}{26}$

G $\frac{1}{13}$

H $\frac{3}{52}$

J $\frac{1}{2}$

Spiral Review

Write an equation for the ellipse that satisfies each set of conditions. (Lesson 10-4)

44. endpoints of major axis at (1, 2) and (9, 2), endpoints of minor axis at (5, 1) and (5, 3)

45. major axis 8 units long and parallel to y-axis, minor axis 6 units long, center at $(-3, 1)$

46. foci at (5, 4) and $(-3, 4)$, major axis 10 units long

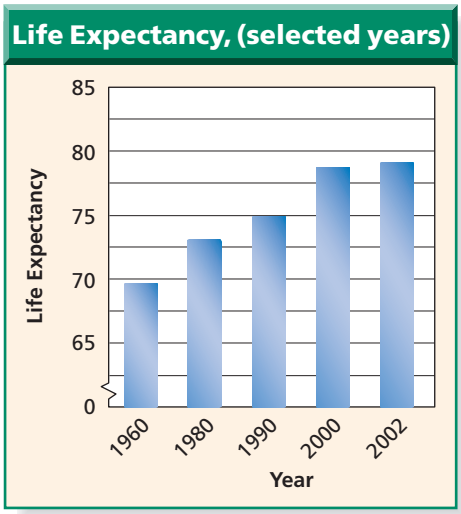
47. Find the center and radius of the circle with equation $x^2 + y^2 - 10x + 2y + 22 = 0$. Then graph the circle. (Lesson 10-3)

Solve each equation by factoring. (Lesson 5-2)

48. $x^2 + 6x + 8 = 0$

49. $2q^2 + 11q = 21$

50. LIFE EXPECTANCY Refer to the graph at the right. What was the average rate of change of life expectancy from 1960 to 2002? (Lesson 2-3)



Source: National Center for Health Statistics

51. Solve $|2x + 1| = 9$. (Lesson 1-4)

52. Simplify $7x + 8y + 9y - 5x$. (Lesson 1-2)

GET READY for the Next Lesson

PREREQUISITE SKILL **Each equation is of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Identify the values of A, B, and C.** (Lesson 6-1)

53. $2x^2 + 3xy - 5y^2 = 0$

54. $-3x^2 + xy + 2y^2 + 4x - 7y = 0$

55. $x^2 - 4x + 4y + 2 = 0$

56. $-xy - 2x - 3y + 6 = 0$

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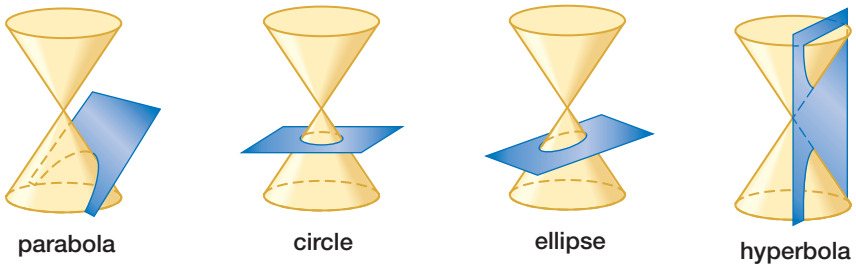
10-6 Conic Sections

Main Ideas

- Write equations of conic sections in standard form.
- Identify conic sections from their equations.

GET READY for the Lesson

Recall that parabolas, circles, ellipses, and hyperbolas are called *conic sections* because they are the cross sections formed when a double cone is sliced by a plane.



parabola circle ellipse hyperbola

Standard Form The equation of any conic section can be written in the form of a general second-degree equation in two variables $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, and C are not all zero. If you are given an equation in this general form, you may be able to complete the square to write the equation in one of the standard forms you have learned.

Reading Math

Ellipses In this lesson, the word *ellipse* means an ellipse that is not a circle.

CONCEPT SUMMARY *Standard Form of Conic Section*

Conic Section	Standard Form of Equation
Parabola	$y = a(x-h)^2 + k$ or $x = a(y-k)^2 + h$
Circle	$(x-h)^2 + (y-k)^2 = r^2$
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$, $a \neq b$
Hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

EXAMPLE 1 Rewrite an Equation of a Conic Section

Write the equation $x^2 + 4y^2 - 6x - 7 = 0$ in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*. Then graph the equation.

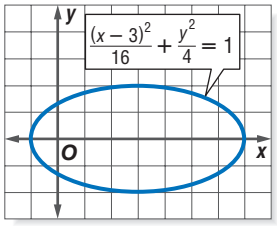
$x^2 + 4y^2 - 6x - 7 = 0$ Original equation

$x^2 - 6x + \blacksquare + 4y^2 = 7 + \blacksquare$ Isolate terms.

$x^2 - 6x + 9 + 4y^2 = 7 + 9$ Complete the square.

$(x-3)^2 + 4y^2 = 16$ $x^2 - 6x + 9 = (x-3)^2$

$\frac{(x-3)^2}{16} + \frac{y^2}{4} = 1$ Divide each side by 16.



The graph of the equation is an ellipse with its center at (3, 0).

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CHECK Your Progress

1. Write the equation $x^2 + y^2 - 4x - 6y - 3 = 0$ in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

Identify Conic Sections Instead of writing the equation in standard form, you can determine what type of conic section an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, represents by looking at A and C.

CONCEPT SUMMARY *Identifying Conic Sections*

Conic Section	Relationship of A and C
Parabola	$A = 0$ or $C = 0$, but not both.
Circle	$A = C$
Ellipse	A and C have the same sign and $A \neq C$.
Hyperbola	A and C have opposite signs.

EXAMPLE 2 **Analyze an Equation of a Conic Section**

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

a. $y^2 - 2x^2 - 4x - 4y - 4 = 0$

$A = -2$ and $C = 1$. Since A and C have opposite signs, the graph is a hyperbola.

b. $4x^2 + 4y^2 + 20x - 12y + 30 = 0$

$A = 4$ and $C = 4$. Since $A = C$, the graph is a circle.

c. $y^2 - 3x + 6y + 12 = 0$

$C = 1$. Since there is no x^2 term, $A = 0$. The graph is a parabola.

CHECK Your Progress

2A. $3x^2 + 3y^2 - 6x + 9y - 15 = 0$

2B. $4x^2 + 3y^2 + 12x - 9y + 14 = 0$

2C. $y^2 = 3x$

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CHECK Your Understanding

Example 1 (pp. 598–599)

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

1. $y = x^2 + 3x + 1$

2. $y^2 - 2x^2 - 16 = 0$

3. $x^2 + y^2 = x + 2$

4. $x^2 + 4y^2 + 2x - 24y + 33 = 0$

Example 2 (p. 599)

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

5. $y^2 - x - 10y + 34 = 0$

6. $3x^2 + 2y^2 + 12x - 28y + 104 = 0$

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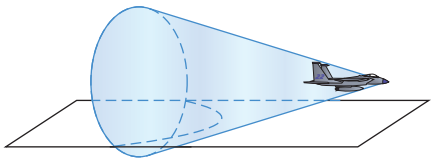
AVIATION **For Exercises 7 and 8, use the following information.**

When an airplane flies faster than the speed of sound, it produces a shock wave in the shape of a cone. Suppose the shock wave generated by a jet intersects the ground in a curve that can be modeled by the equation $x^2 - 14x + 4 = 9y^2 - 36y$.



7. Identify the shape of the curve.

8. Graph the equation.

Exercises

HOMEWORK HELP

For Exercises	See Examples
9–18	1
19–25	2

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

9. $6x^2 + 6y^2 = 162$

10. $4x^2 + 2y^2 = 8$

11. $x^2 = 8y$

12. $4y^2 - x^2 + 4 = 0$

13. $(x - 1)^2 - 9(y - 4)^2 = 36$

14. $y + 4 = (x - 2)^2$

15. $(y - 4)^2 = 9(x - 4)$

16. $x^2 + y^2 + 4x - 6y = -4$

17. $x^2 + y^2 + 6y + 13 = 40$

18. $x^2 - y^2 + 8x = 16$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

19. $x^2 + y^2 - 8x - 6y + 5 = 0$

20. $3x^2 - 2y^2 + 32y - 134 = 0$

21. $y^2 + 18y - 2x = -84$

22. $7x^2 - 28x + 4y^2 + 8y = -4$

Real-World Career

Pilot

While flying the plane, a pilot must also be constantly scanning flight instruments, monitoring the engine, and communicating with the air traffic controller.

MathOnline

For more information, go to algebra2.com.

For Exercises 23–25, match each equation below with the situation that it could represent.

a. $9x^2 + 4y^2 - 36 = 0$

b. $0.004x^2 - x + y - 3 = 0$

c. $x^2 + y^2 - 20x + 30y - 75 = 0$

23. **SPORTS** the flight of a baseball

24. **PHOTOGRAPHY** the oval opening in a picture frame

25. **GEOGRAPHY** the set of all points that are 20 miles from a landmark

AVIATION **For Exercises 26–28, use the following information.**

A military jet performs for an air show. The path of the plane during one trick can be modeled by a conic section with equation $24x^2 + 1000y - 31{,}680x - 45{,}600 = 0$, where distances are represented in feet.

26. Identify the shape of the curved path of the jet. Write the equation in standard form.

27. If the jet begins its path upward or ascent at (0, 0), what is the horizontal distance traveled by the jet from the beginning of the ascent to the end of the descent?

28. What is the maximum height of the jet?

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LIGHT **For Exercises 29 and 30, use the following information.**

A lamp standing near a wall throws an arc of light in the shape of a conic section. Suppose the edge of the light can be represented by the equation $3y^2 - 2y - 4x^2 + 2x - 8 = 0$.

29. Identify the shape of the edge of the light.

30. Graph the equation.

WATER **For Exercises 31 and 32, use the following information.**

If two stones are thrown into a lake at different points, the points of intersection of the resulting ripples will follow a conic section. Suppose the conic section has the equation $x^2 - 2y^2 - 2x - 5 = 0$.

31. Identify the shape of the curve.

32. Graph the equation.

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

33. $x^2 + 2y^2 = 2x + 8$

34. $x^2 - 8y + y^2 + 11 = 0$

35. $9y^2 + 18y = 25x^2 + 216$

36. $3x^2 + 4y^2 + 8y = 8$

37. $x^2 + 4y^2 - 11 = 2(4y - x)$

38. $y + x^2 = -(8x + 23)$

39. $6x^2 - 24x - 5y^2 - 10y - 11 = 0$

40. $25y^2 + 9x^2 - 50y - 54x = 119$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

41. $5x^2 + 6x - 4y = x^2 - y^2 - 2x$

42. $2x^2 + 12x + 18 - y^2 = 3(2 - y^2) + 4y$

43. Identify the shape of the graph of the equation $2x^2 + 3x - 4y + 2 = 0$.

44. What type of conic section is represented by the equation $y^2 - 6y = x^2 - 8$?

EXTRA PRACTICE
See pages 913, 935.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

45. **OPEN ENDED** Write an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A = 2$, that represents a circle.

46. **REASONING** Explain why the graph of the equation $x^2 + y^2 - 4x + 2y + 5 = 0$ is a single point.

CHALLENGE **For Exercises 47 and 48, use the following information.**

The graph of an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ is a special case of a hyperbola.

47. Identify the graph of such an equation.

48. Explain how to obtain such a set of points by slicing a double cone with a plane.

49. **REASONING** Refer to Exercise 32 on page 587. Eccentricity can be studied for conic sections other than ellipses. The expression for the eccentricity of a hyperbola is $\frac{c}{a}$, just as for an ellipse. The eccentricity of a parabola is 1. Find inequalities for the eccentricities of noncircular ellipses and hyperbolas, respectively.

50. ***Writing in Math*** Use the information about conic sections on page 598 to explain how you can use a flashlight to make conic sections. Explain how you could point the flashlight at a ceiling or wall to make a circle and how you could point the flashlight to make a branch of a hyperbola.

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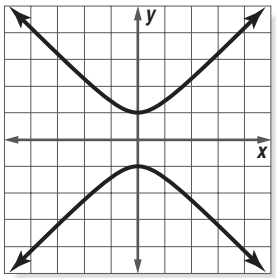
STANDARDIZED TEST PRACTICE

51. ACT/SAT What is the equation of the graph?



A $y = x^2 + 1$ B $y - x = 1$

C $y^2 - x^2 = 1$ D $x^2 + y^2 = 1$

52. REVIEW The graph of $\left(\frac{x}{4}\right)^2 - \left(\frac{y}{5}\right)^2 = 1$ is a hyperbola. Which set of equations represents the asymptotes of the hyperbola's graph?

F $y = \frac{4}{5}x, y = -\frac{4}{5}x$

G $y = \frac{1}{4}x, y = -\frac{1}{4}x$

H $y = \frac{5}{4}x, y = -\frac{5}{4}x$

J $y = \frac{1}{5}x, y = -\frac{1}{5}x$

Spiral Review

Write an equation of the hyperbola that satisfies each set of conditions. (Lesson 10-5)

53. vertices (5, 10) and (5, −2), conjugate axis of length 8 units

54. vertices (6, −6) and (0, −6), foci $\left(3 \pm \sqrt{13}, -6\right)$

55. Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $4x^2 + 9y^2 - 24x + 72y + 144 = 0$. Then graph the ellipse. (Lesson 10-4)

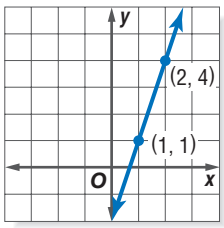
Simplify. Assume that no variable equals 0. (Lesson 6-1)

56. $(x^3)^4$ **57.** $(m^5n^{-3})^2m^2n^7$ **58.** $\frac{x^2y^{-3}}{x^{-5}y}$

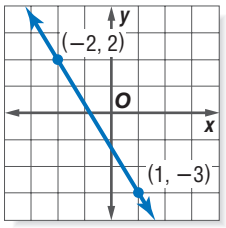
59. HEALTH The prediction equation $y = 205 - 0.5x$ relates a person's maximum heart rate for exercise y and age x. Use the equation to find the maximum heart rate for an 18-year old. (Lesson 2-5)

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

60.



61.



GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each system of equations.** (Lesson 3-2)

62. $y = x + 4$
$2x + y = 10$

63. $4x + y = 14$
$4x - y = 10$

64. $x + 5y = 10$
$3x - 2y = -4$

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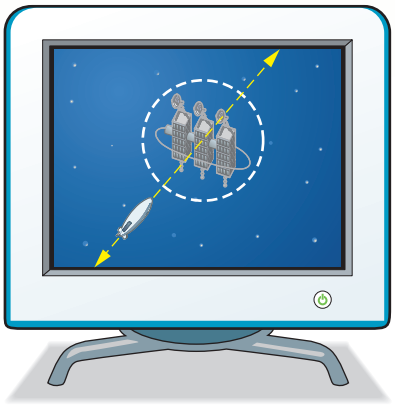
10-7 Solving Quadratic Systems

Main Ideas

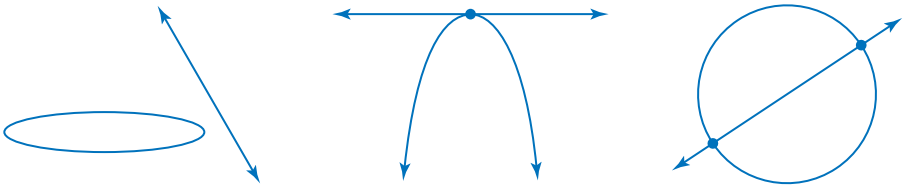
- Solve systems of quadratic equations algebraically and graphically.
- Solve systems of quadratic inequalities graphically.

GET READY for the Lesson

Suppose you are playing a computer game in which an enemy space station is located at the origin in a coordinate system. The space station is surrounded by a circular force field of radius 50 units. If the spaceship you control is flying toward the center along the line with equation $y = 3x$, the point where the ship hits the force field is a solution of a system of equations.



Systems of Quadratic Equations If the graphs of a system of equations are a conic section and a line, the system may have zero, one, or two solutions. Some of the possible situations are shown below.



no solutions | one solution | two solutions

You have solved systems of linear equations graphically and algebraically. You can use similar methods to solve systems involving quadratic equations.

Study Tip

Look Back

To review **solving systems of linear equations**, see Lesson 3-2.

EXAMPLE Linear-Quadratic System

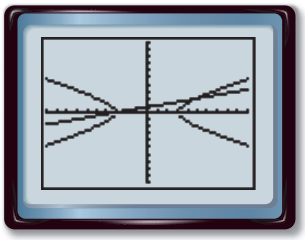
1 **Solve the system of equations.**

$x^2 - 4y^2 = 9$

$4y - x = 3$

You can use a graphing calculator to help visualize the relationships of the graphs of the equations and predict the number of solutions.

Solve each equation for y to obtain $y = \pm\frac{\sqrt{x^2 - 9}}{2}$ and $y = \frac{1}{4}x + \frac{3}{4}$. Enter $y = \frac{\sqrt{x^2 - 9}}{2}$, $y = -\frac{\sqrt{x^2 - 9}}{2}$, and $y = \frac{1}{4}x + \frac{3}{4}$ on the Y= screen. The graph indicates that the hyperbola and line intersect in two points. So the system has two solutions.



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Use substitution to solve the system. First rewrite $4y - x = 3$ as $x = 4y - 3$.

$x^2 - 4y^2 = 9$ First equation in the system

$(4y - 3)^2 - 4y^2 = 9$ Substitute $4y - 3$ for x.

$12y^2 - 24y = 0$ Simplify.

$y^2 - 2y = 0$ Divide each side by 12.

$y(y - 2) = 0$ Factor.

$y = 0$ or $y - 2 = 0$ Zero Product Property

$y = 2$ Solve for y.

Now solve for x.

$x = 4y - 3$	Equation for x in terms of y	$x = 4y - 3$
$= 4(0) - 3$	Substitute the y-values.	$= 4(2) - 3$
$= -3$	Simplify.	$= 5$

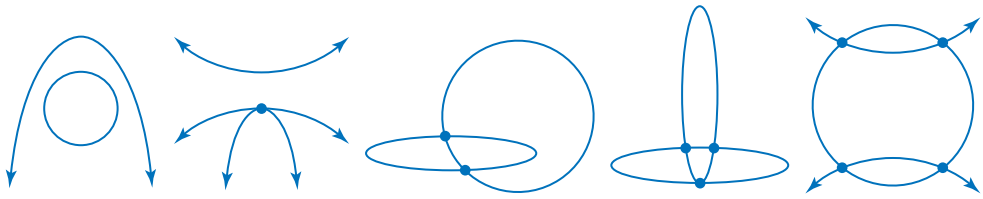
The solutions of the system are $(-3, 0)$ and $(5, 2)$. Based on the graph, these solutions are reasonable.

CHECK Your Progress **Solve each system of equations.**

1A. $y = x - 1$
$x^2 + y^2 = 25$

1B. $x + y = 1$
$y = x^2 - 5$

If the graphs of a system of equations are two conic sections, the system may have zero, one, two, three, or four solutions. Here are possible situations.



no solutions | one solution | two solutions | three solutions | four solutions

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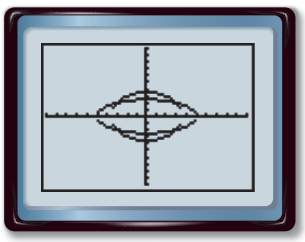
EXAMPLE **Quadratic-Quadratic System**

2 **Solve the system of equations.**

$y^2 = 13 - x^2$

$x^2 + 4y^2 = 25$

A graph of the system indicates that the circle and ellipse intersect in four points. So, this system has four solutions. Use the elimination method to solve.



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$-x^2 - y^2 = -13$ Rewrite the first original equation.

$(+)\ x^2 + 4y^2 = 25$ Second original equation

$3y^2 = 12$ Add.

$y^2 = 4$ Divide each side by 3.

$y = \pm 2$ Take the square root of each side.

Study Tip

Graphing Calculators

If you use ZSquare on the ZOOM menu, the graph of the first equation will look like a circle.

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Substitute 2 and -2 for y in either of the original equations and solve for x.

$x^2 + 4y^2 = 25$	Second original equation	$x^2 + 4y^2 = 25$
$x^2 + 4(2)^2 = 25$	Substitute for y.	$x^2 + 4(-2)^2 = 25$
$x^2 = 9$	Subtract 16 from each side.	$x^2 = 9$
$x = \pm 3$	Take the square root of each side.	$x = \pm 3$

The solutions are $(3, 2)$, $(-3, 2)$, $(-3, -2)$, and $(3, -2)$.

CHECK Your Progress **Solve each system of equations.**

2A. $x^2 + y^2 = 36$
$x^2 + 9y^2 = 36$

2B. $x^2 - y^2 = 8$
$x^2 + y^2 = 120$

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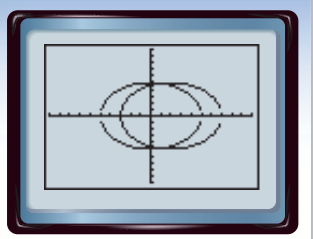
A graphing calculator can be used to approximate solutions of a system of quadratic equations.

GRAPHING CALCULATOR LAB

Quadratic Systems

THINK AND DISCUSS

The calculator screen shows the graphs of two circles.



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1. Write the system of equations represented.
2. Enter the equations into a TI-83/84 Plus and use the intersect feature on the CALC menu to solve the system. Round to the nearest hundredth.
3. Solve the system algebraically.
4. Can you always find the exact solution of a system using a graphing calculator? Explain.

Systems of Quadratic Inequalities Systems of quadratic inequalities are solved by graphing. As with linear inequalities, examine the inequality symbol to determine whether to include the boundary.

Study Tip

Graphing Quadratic Inequalities

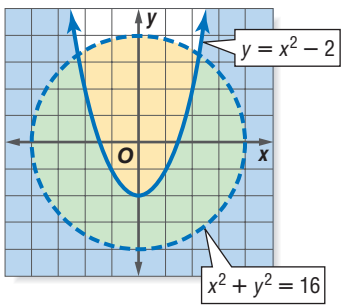
If you are unsure about which region to shade, you can test one or more points, as you did with linear inequalities.

EXAMPLE **System of Quadratic Inequalities**

3 **Solve the system of inequalities by graphing.**

$y \leq x^2 - 2$
$x^2 + y^2 < 16$

The intersection of the graphs, shaded green, represents the solution of the system.



CHECK $(0, -3)$ is in the shaded area. Use this point to check your solution.

$y \leq x^2 - 2$	$x^2 + y^2 < 16$
$-3 \leq (0)^2 - 2$	$0^2 + (-3)^2 < 16$
$-3 \leq -2$ true	$9 < 16$ true

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CHECK Your Progress

3. Solve by graphing. $x^2 + y^2 \leq 49$ and $y \geq x^2 + 1$

CHECK Your Understanding

Example 1 (pp. 603–604)

Find the exact solution(s) of each system of equations.

1. $y = 5$
$y^2 = x^2 + 9$

2. $y - x = 1$
$x^2 + y^2 = 25$

Example 2 (pp. 604–605)

3. $3x = 8y^2$
$8y^2 - 2x^2 = 16$

4. $5x^2 + y^2 = 30$
$9x^2 - y^2 = -16$

5. CELL PHONES A person using a cell phone can be located in respect to three cellular towers. In a coordinate system where a unit represents one mile, the caller is determined to be 50 miles from a tower at the origin, 40 miles from a tower at (0, 30), and 13 miles from a tower at (35, 18). Where is the caller?

Example 3 (pp. 605–606)

Solve each system of inequalities by graphing.

6. $x + y < 4$
$9x^2 - 4y^2 \geq 36$

7. $x^2 + y^2 < 25$
$4x^2 - 9y^2 < 36$

Exercises

HOMEWORK HELP

For Exercises	See Examples
8–13	1
14–19	2
20–25	3

Find the exact solution(s) of each system of equations.

8. $y = x + 2$
$y = x^2$

9. $y = x + 3$
$y = 2x^2$

10. $x^2 + y^2 = 36$
$y = x + 2$

11. $y^2 + x^2 = 9$
$y = 7 - x$

12. $\frac{x^2}{30} + \frac{y^2}{6} = 1$
$x = y$

13. $\frac{x^2}{36} - \frac{y^2}{4} = 1$
$x = y$

14. $4x + y^2 = 20$
$4x^2 + y^2 = 100$

15. $y + x^2 = 3$
$x^2 + 4y^2 = 36$

16. $x^2 + y^2 = 64$
$x^2 + 64y^2 = 64$

17. $y^2 + x^2 = 25$
$y^2 + 9x^2 = 25$

18. $y^2 = x^2 - 25$
$x^2 - y^2 = 7$

19. $y^2 = x^2 - 7$
$x^2 + y^2 = 25$

Solve each system of inequalities by graphing.

20. $x + 2y > 1$
$x^2 + y^2 \leq 25$

21. $x + y \leq 2$
$4x^2 - y^2 \geq 4$

22. $x^2 + y^2 \geq 4$
$4y^2 + 9x^2 \leq 36$

23. $x^2 + y^2 < 36$
$4x^2 + 9y^2 > 36$

24. $y^2 < x$
$x^2 - 4y^2 < 16$

25. $x^2 \leq y$
$y^2 - x^2 \geq 4$

26. Graph each system of equations. Use the graph to solve the system.

a. $4x - 3y = 0$
$x^2 + y^2 = 25$

b. $y = 5 - x^2$
$y = 2x^2 + 2$

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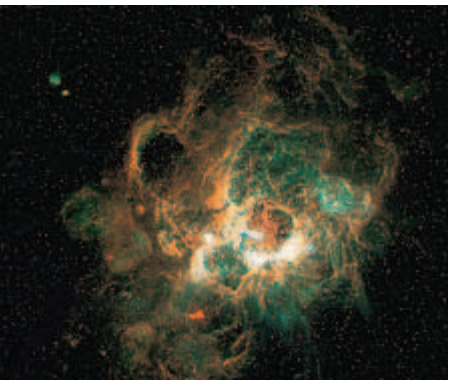

Real-World Link

The astronomical unit (AU) is the mean distance between Earth and the Sun. One AU is about 93 million miles or 150 million kilometers

Source: infoplease.com

ASTRONOMY **For Exercises 27 and 28, use the following information.**

The orbit of Pluto can be modeled by the equation $\frac{x^2}{39.5^2} + \frac{y^2}{38.3^2} = 1$, where the units are astronomical units. Suppose a comet is following a path modeled by the equation $x = y^2 + 20$.

27. Find the point(s) of intersection of the orbits of Pluto and the comet. Round to the nearest tenth.

28. Will the comet necessarily hit Pluto? Explain.

29. Where do the graphs of $y = 2x + 1$ and $2x^2 + y^2 = 11$ intersect?

30. What are the coordinates of the points that lie on the graphs of both $x^2 + y^2 = 25$ and $2x^2 + 3y^2 = 66$?

31. ROCKETS Two rockets are launched at the same time, but from different heights. The height y in feet of one rocket after t seconds is given by $y = -16t^2 + 150t + 5$. The height of the other rocket is given by $y = -16t^2 + 160t$. After how many seconds are the rockets at the same height?

32. ADVERTISING The corporate logo for an automobile manufacturer is shown at the right. Write a system of three equations to model this logo.

SATELLITES **For Exercises 33–35, use the following information.**

Two satellites are placed in orbit about Earth. The equations of the two orbits are $\frac{x^2}{(300)^2} + \frac{y^2}{(900)^2} = 1$ and $\frac{x^2}{(600)^2} + \frac{y^2}{(690)^2} = 1$, where distances are in kilometers and Earth is the center of each curve.

33. Solve each equation for y.

34. Use a graphing calculator to estimate the intersection points of the two orbits.

35. Compare the orbits of the two satellites.

Graphing Calculator

Write a system of equations that satisfies each condition. Use a graphing calculator to verify that you are correct.

36. two parabolas that intersect in two points

37. a hyperbola and a circle that intersect in three points

38. a circle and an ellipse that do not intersect

39. a circle and an ellipse that intersect in four points

40. a hyperbola and an ellipse that intersect in two points

41. two circles that intersect in three points

EXTRA PRACTICE
See pages 913, 935.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

42. REASONING Sketch a parabola and an ellipse that intersect in exactly three points.

43. OPEN ENDED Write a system of quadratic equations for which (2, 6) is a solution.

CHALLENGE **For Exercises 44–48, find all values of k for which the system of equations has the given number of solutions. If no values of k meet the condition, write *none*.**

$$x^2 + y^2 = k^2 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

44. no solutions **45.** one solution **46.** two solutions

47. three solutions **48.** four solutions

(l)Space Telescope Science Institute/NA

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49. Which One Doesn't Belong? Which system of equations is NOT like the others? Explain your reasoning.

$x^2 + y^2 = 16$
$x + y = 3$

$x^2 + y^2 = 25$
$\frac{x^2}{25} - \frac{y^2}{16} = 1$

$x^2 + y^2 = 20$
$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$y - 2x = -5$
$y^2 + x = 9$

50. *Writing in Math* Use the information on page 603 to explain how systems of equations apply to video games. Include a linear-quadratic system of equations that applies to this situation and the coordinates of the point at which the spaceship will hit the force field, assuming that the spaceship moves from the bottom of the screen toward the center.

STANDARDIZED TEST PRACTICE

51. ACT/SAT How many solutions does the system of equations $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$ and $(x - 3)^2 + y^2 = 9$ have?

A 0

B 1

C 2

D 4

52. REVIEW Given: Two angles are supplementary. One angle is 25° more than the measure of the other angle. Conclusion: The measures of the angles are 65° and 90°. This conclusion—

F is contradicted by the first statement given.

G is verified by the first statement given.

H invalidates itself because a 90° angle cannot be supplementary to another.

J verifies itself because 90° is 25° more than 65°.

Spiral Review

Write each equation in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola*. Then graph the equation. (Lesson 10-6)

53. $x^2 + y^2 + 4x + 2y - 6 = 0$

54. $9x^2 + 4y^2 - 24y = 0$

55. Find the coordinates of the vertices and foci and the equations of the asymptotes of the hyperbola with the equation $6y^2 - 2x^2 = 24$. Then graph the hyperbola. (Lesson 10-5)

Cross-Curricular Project

Algebra and Earth Science

Earthquake Extravaganza It is time to complete your project. Use the information and data you have gathered about earthquakes to prepare a research report or Web page. Be sure to include graphs, tables, diagrams, and any calculations you need for the earthquake you chose.

MathOnline **Cross-Curricular Project at at algebra2.com**

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CHAPTER 10

Study Guide and Review

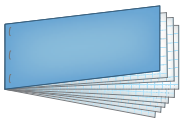
STUDY TO GO

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Midpoint and Distance Formulas (Lesson 10-1)

- $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Circles (Lesson 10-2)

- The equation of a circle with center (h, k) and radius r can be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Parabolas (Lesson 10-3)

Standard Form	$y = a(x - h)^2 + k$	$x = a(y - k)^2 + h$
Axis of Symmetry	$x = h$	$y = k$

Ellipses (Lesson 10-4)

Standard Form of Equation	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical

Hyperbolas (Lesson 10-5)

Standard Form	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$
Transverse Axis	horizontal	vertical

Solving Quadratic Systems (Lesson 10-7)

- Systems of quadratic equations can be solved using substitution and elimination.
- A system of quadratic equations can have zero, one, two, three, or four solutions.

Key Vocabulary

asymptote (p. 591)
center of a circle (p. 574)
center of a hyperbola (p. 591)
center of an ellipse (p. 582)
circle (p. 574)
conic section (p. 567)
conjugate axis (p. 591)
directrix (p. 567)
ellipse (p. 581)
foci of a hyperbola (p. 590)
foci of an ellipse (p. 581)
focus of a parabola (p. 567)
hyperbola (p. 590)
latus rectum (p. 569)
major axis (p. 582)
minor axis (p. 582)
parabola (p. 567)
transverse axis (p. 591)
vertex of a hyperbola (p. 591)

Vocabulary Check

Tell whether each statement is *true* or *false*. If the statement is false, correct it to make it true.

1. An ellipse is the set of all points in a plane such that the sum of the distances from two given points in the plane, called the foci, is constant.
2. The major axis is the longer of the two axes of symmetry of an ellipse.
3. A parabola is the set of all points that are the same distance from a given point called the directrix and a given line called the focus.
4. The radius is the distance from the center of a circle to any point on the circle.
5. The conjugate axis of a hyperbola is a line segment parallel to the transverse axis.
6. A conic section is formed by slicing a double cone by a plane.
7. The set of all points in a plane that are equidistant from a given point in a plane, called the center, forms a circle.

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Lesson-by-Lesson Review

10–1 Midpoint and Distance Formulas (pp. 562–566)

Find the midpoint of the line segment with endpoints at the given coordinates.

8. (1, 2), (4, 6)

9. (−8, 0), (−2, 3)

10. $\left(\frac{3}{5}, -\frac{7}{4}\right), \left(\frac{1}{4}, -\frac{2}{5}\right)$

11. (13, 24), (19, 28)

Find the distance between each pair of points with the given coordinates.

12. (−2, 10), (−2, 13)

13. (8, 5), (−9, 4)

14. (7, −3), (1, 2)

15. $\left(\frac{5}{4}, \frac{1}{2}\right), \left(\frac{1}{4}, \frac{3}{2}\right)$

HIKING **For Exercises 16 and 17, use the following information.**

Marc wants to hike from his camp to a waterfall. The waterfall is 5 miles south and 8 miles east of his campsite.

16. How far away is the waterfall?

17. Marc wants to stop for lunch halfway to the waterfall. If the camp is at the origin, where should he stop?

Example 1 **Find the midpoint of a segment whose endpoints are at (−5, 9) and (11, −1).**

Let $(x_1, y_1) = (-5, 9)$ and $(x_2, y_2) = (11, -1)$.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-5 + 11}{2}, \frac{9 + (-1)}{2}\right)$$

$$= \left(\frac{6}{2}, \frac{8}{2}\right) \text{ or } (3, 4) \quad \text{Simplify.}$$

Example 2 **Find the distance between $P(6, -4)$ and $Q(-3, 8)$. Let $(x_1, y_1) = (6, -4)$ and $(x_2, y_2) = (-3, 8)$.**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{Distance Formula}$$

$$= \sqrt{(-3 - 6)^2 + [8 - (-4)]^2}$$

$$= \sqrt{81 + 144} \quad \text{Subtract.}$$

$$= \sqrt{225} \text{ or } 15 \text{ units} \quad \text{Simplify.}$$

10–2 Parabolas (pp. 567–573)

(continued on the next page)

Identify the coordinates of the vertex and focus, the equation of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

18. $(x - 1)^2 = 12(y - 1)$

19. $y + 6 = 16(x - 3)^2$

20. $x^2 - 8x + 8y + 32 = 0$

21. $x = 16y^2$

22. Write an equation for a parabola with vertex (0, 1) and focus (0, −1). Then graph the parabola.

Example 3 **Graph $4y - x^2 = 14x - 27$.**

Write the equation in the form $y = a(x - h)^2 + k$ by completing the square.

$$4y = x^2 + 14x - 27 \quad \text{Isolate the terms with } x.$$

$$4y = (x^2 + 14x + \blacksquare) - 27 - \blacksquare$$

$$4y = (x^2 + 14x + 49) - 27 - 49$$

$$4y = (x + 7)^2 - 76 \quad x^2 + 14x + 49 = (x + 7)^2$$

$$y = \frac{1}{4}(x + 7)^2 - 19 \quad \text{Divide each side by 4.}$$

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10–2 Parabolas (pp. 567–573)

23. SPORTS When a golf ball is hit, the path it travels is shaped like a parabola. Suppose a golf ball is hit from ground level, reaches a maximum height of 100 feet, and lands 400 feet away. Assuming the ball was hit at the origin, write an equation of the parabola that models the flight of the ball.

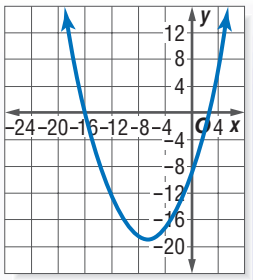
vertex: $(-7, -19)$

axis of symmetry: $x = -7$

direction of opening: upward since $a > 0$

focus: $\left(-7, -19 + \frac{1}{4\left(\frac{1}{4}\right)}\right)$ or $(-7, -18)$

directrix: $y = -19 - \frac{1}{4\left(\frac{1}{4}\right)}$ or $y = -20$



10–3 Circles (pp. 574–579)

Write an equation for the circle that satisfies each set of conditions.

24. center $(2, -3)$, radius 5 units

25. center $(-4, 0)$, radius $\frac{3}{4}$ units

26. endpoints of a diameter at $(9, 4)$ and $(-3, -2)$

27. center at $(-1, 2)$, tangent to x-axis

Find the center and radius of the circle with the given equation. Then graph the circle.

28. $x^2 + y^2 = 169$

29. $(x + 5)^2 + (y - 11)^2 = 49$

30. $x^2 + y^2 - 6x + 16y - 152 = 0$

31. $x^2 + y^2 + 6x - 2y - 15 = 0$

32. WEATHER On average the circular eye of a tornado is about 200 feet in diameter. Suppose a satellite photo showed the center of its eye at the point $(72, 39)$. Write an equation to represent the possible boundary of a tornado's eye.

Example 4 **Graph $x^2 + y^2 + 8x - 24y + 16 = 0$.**

First write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$.

$$x^2 + y^2 + 8x - 24y + 16 = 0$$

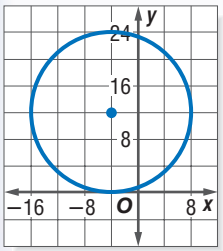
$$x^2 + 8x + \blacksquare + y^2 + -24y + \blacksquare = -16 + \blacksquare + \blacksquare$$

$$x^2 + 8x + 16 + y^2 + -24y + 144 = -16 + 16 + 144$$

$$(x + 4)^2 + (y - 12)^2 = 144$$

The center of the circle is at $(-4, 12)$ and the radius is 12.

Now draw the graph.



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10–4 Ellipses (pp. 581–588)

33. Write an equation for the ellipse with endpoints of the major axis at (4, 1) and (−6, 1) and endpoints of the minor axis at (−1, 3) and (−1, −1).

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

34. $\frac{x^2}{16} + \frac{y^2}{25} = 1$

35. $\frac{(x+2)^2}{16} + \frac{(y-3)^2}{9} = 1$

36. $x^2 + 4y^2 - 2x + 16y + 13 = 0$

37. The Oval Office in the White House is an ellipse. The major axis is 10.9 meters and the minor axis is 8.8 meters. Write an equation to model the Oval Office. Assume that the origin is at the center of the Oval Office.

Example 5 **Graph $x^2 + 3y^2 - 16x + 24y + 31 = 0$.**

First write the equation in standard form by completing the squares.

$$x^2 + 3y^2 - 16x + 24y + 31 = 0$$

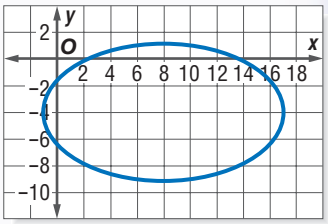
$$x^2 - 16x + \blacksquare + 3(y^2 + 8y + \blacksquare) = -31 + \blacksquare + 3(\blacksquare)$$

$$x^2 - 16x + 64 + 3(y^2 + 8y + 16) = -31 + 64 + 3(16)$$

$$(x-8)^2 + 3(y+4)^2 = 81$$

$$\frac{(x-8)^2}{81} + \frac{(y+4)^2}{27} = 1$$

The center of the ellipse is at (8, −4). The length of the major axis is 18, and the length of the minor axis is $6\sqrt{3}$.



10–5 Hyperbolas (pp. 590–597)

(continued on the next page)

38. Write an equation for a hyperbola that has vertices at (2, 5) and (2, 1) and a conjugate axis of length 6 units.

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

39. $\frac{y^2}{4} - \frac{x^2}{9} = 1$

40. $16x^2 - 25y^2 - 64x - 336 = 0$

41. $\frac{(x-2)^2}{1} - \frac{(y+1)^2}{9} = 1$

42. $9y^2 - 16x^2 = 144$

Example 6 **Graph $9x^2 - 4y^2 + 18x + 32y - 91 = 0$.**

Complete the square for each variable to write this equation in standard form.

$$9x^2 - 4y^2 + 18x + 32y - 91 = 0$$

$$9(x^2 + 2x + \blacksquare) - 4(y^2 - 8y + \blacksquare) = 91 + 9(\blacksquare) - 4(\blacksquare)$$

$$9(x^2 + 2x + 1) - 4(y^2 - 8y + 16) = 91 + 9(1) - 4(16)$$

$$9(x+1)^2 - 4(y-4)^2 = 36$$

$$\frac{(x+1)^2}{4} - \frac{(y-4)^2}{9} = 1$$

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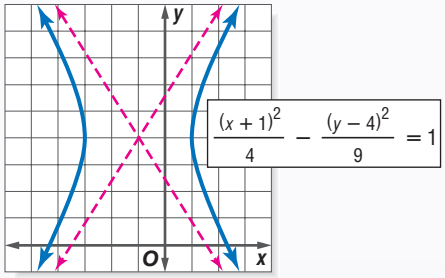
Mixed Problem Solving
For mixed problem-solving practice, see page 935.

10–5 Hyperbolas (pp. 590–597)

43. MIRRORS A hyperbolic mirror is a mirror in the shape of one branch of a hyperbola. Such a mirror reflects light rays directed at one focus toward the other focus. Suppose a hyperbolic mirror is modeled by the upper branch of the hyperbola with equation $\frac{y^2}{9} - \frac{x^2}{16} = 1$. A light source is located at $(-10, 0)$. Where should the light from the source hit the mirror so that the light will be reflected to $(0, -5)$?

The center is at $(-1, 4)$. The vertices are at $(-3, 4)$ and $(1, 4)$ and the foci are at $(-1 \pm \sqrt{13}, 4)$. The equations of the asymptotes are $y - 4 = \pm\frac{3}{2}(x + 1)$.



10–6 Conic Sections (pp. 598–602)

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

44. $-4x^2 + y^2 + 8x - 8 = 0$

45. $x^2 + 4x - y = 0$

46. $x^2 + y^2 - 4x - 6y + 4 = 0$

47. $9x^2 + 4y^2 = 36$

Without writing the equation in standard form, state whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*.

48. $7x^2 + 9y^2 = 63$

49. $5y^2 + 2y + 4x - 13x^2 = 81$

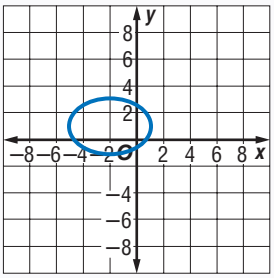
50. $x^2 - 8x + 16 = 6y$

51. $x^2 + 4x + y^2 - 285 = 0$

52. ASTRONOMY A satellite travels in a hyperbolic orbit. It reaches a vertex of its orbit at $(9, 0)$ and then travels along a path that gets closer and closer to the line $y = \frac{2}{9}x$. Write an equation that describes the path of the satellite if the center of its hyperbolic orbit is at $(0, 0)$.

Example 7 **Without writing the equation in standard form, state whether the graph of $4x^2 + 9y^2 + 16x - 18y - 11 = 0$ is a *parabola*, *circle*, *ellipse*, or *hyperbola*.**

In this equation, $A = 4$ and $C = 9$. Since A and C are both positive and $A \neq C$, the graph is an ellipse.



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10–7 Solving Quadratic Systems (pp. 603–608)

Find the exact solution(s) of each system of equations.

53. $x^2 + y^2 - 18x + 24y + 200 = 0$
$4x + 3y = 0$

54. $4x^2 + y^2 = 16$
$x^2 + 2y^2 = 4$

Solve each system of inequalities by graphing.

55. $y < x$
$y > x^2 - 4$

56. $x^2 + y^2 \leq 9$
$x^2 + 4y^2 \leq 16$

57. ARCHITECTURE An architect is building the front entrance of a building in the shape of a parabola with the equation $y = -\frac{1}{10}(x - 10)^2 + 20$. While the entrance is being built the construction team puts in two support beams with equations $y = -x + 10$ and $y = x - 10$. Where do the support beams meet the parabola?

Example 8 **Solve the system of equations.**

$x^2 + y^2 + 2x - 12y + 12 = 0$
$y + x = 0$

Use substitution to solve the system. First, rewrite $y + x = 0$ as $y = -x$.

$$x^2 + y^2 + 2x - 12y + 12 = 0$$
$$x^2 + (-x)^2 + 2x - 12(-x) + 12 = 0$$
$$2x^2 + 14x + 12 = 0$$
$$x^2 + 7x + 6 = 0$$
$$(x + 6)(x + 1) = 0$$

$x + 6 = 0$ or $x + 1 = 0$ Zero Product Property

$x = -6$ $x = -1$ Solve for x.

Now solve for y.

$y = -x$ $y = -x$ Equation for y in terms of x

$= -(-6)$ $= -(-1)$ Substitute.

$= 6$ $= 1$

The solutions of the system are $(-6, 6)$ and $(-1, 1)$.

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CHAPTER 10

Practice Test

Find the midpoint of the line segment with endpoints at the given coordinates.

1. $(7, 1), (-5, 9)$
2. $\left(\frac{3}{8}, -1\right), \left(-\frac{8}{5}, 2\right)$
3. $(-13, 0), (-1, -8)$

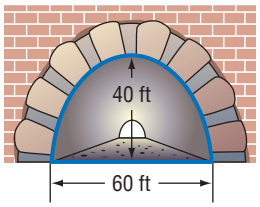
Find the distance between each pair of points with the given coordinates.

4. $(-6, 7), (3, 2)$
5. $\left(\frac{1}{2}, \frac{5}{2}\right), \left(-\frac{3}{4}, -\frac{11}{4}\right)$
6. $(8, -1), (8, -9)$

State whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation.

7. $x^2 + 4y^2 = 25$
8. $x^2 = 36 - y^2$
9. $4x^2 - 26y^2 + 10 = 0$
10. $-(y^2 - 24) = x^2 + 10x$
11. $\frac{1}{3}x^2 - 4 = y$
12. $y = 4x^2 + 1$
13. $(x + 4)^2 = 7(y + 5)$
14. $25x^2 + 49y^2 = 1225$
15. $5x^2 - y^2 = 49$
16. $\frac{y^2}{9} - \frac{x^2}{25} = 1$

17. **TUNNELS** The opening of a tunnel is in the shape of a semielliptical arch. The arch is 60 feet wide and 40 feet high. Find the height of the arch 12 feet from the edge of the tunnel.



18. Solve the system of inequalities by graphing.
$x^2 - y^2 \geq 1$
$x^2 + y^2 \leq 16$

Find the exact solution(s) of each system of equations.

19. $x^2 + y^2 = 100$
$y = 2 - x$
20. $x^2 + 2y^2 = 6$
$x + y = 1$
21. $x^2 - y^2 - 12x + 12y = 36$
$x^2 + y^2 - 12x - 12y + 36 = 0$

FORESTRY **For Exercises 22 and 23, use the following information.**
A forest ranger at an outpost in the Fishlake National Forest in Utah and another ranger at the primary station both heard an explosion. The outpost and the primary station are 6 kilometers apart.

22. If one ranger heard the explosion 6 seconds before the other, write an equation that describes all the possible locations of the explosion. Place the two ranger stations on the x-axis with the midpoint between the stations at the origin. The transverse axis is horizontal. (*Hint:* The speed of sound is about 0.35 kilometer per second.)
23. Draw a sketch of the possible locations of the explosion. Include the ranger stations in the drawing.

24. **MULTIPLE CHOICE** Which is NOT the equation of a parabola?
A $y = 2x^2 + 4x - 9$
B $3x + 2y^2 + y + 1 = 0$
C $x^2 + 2y^2 + 8y = 8$
D $x = \frac{1}{2}(y - 1)^2 + 5$

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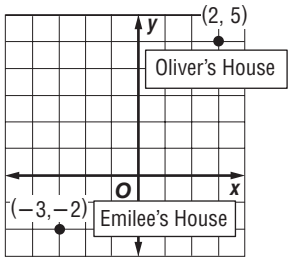
CHAPTER 10

Standardized Test Practice

Cumulative, Chapters 1–10

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A coordinate grid is placed over a map. Emilee's house is located at (−3, −2) and Oliver's house is located at (2, 5). A side of each square represents one block. What is the approximate distance between Emilee's house and Oliver's house?



A 3.2 blocks **C** 12.0 blocks

B 8.6 blocks **D** 17.2 blocks

2. A diameter of a circle has endpoints $A(4, 6)$ and $B(-3, -1)$. Find the approximate length of the radius.

F 2.5 units

G 4.9 units

H 5.1 units

J 9.9 units

TEST-TAKING TIP

Question 2 Always write down your calculations on scrap paper or in the test booklet, even if you think you can do the calculations in your head. Writing down your calculations will help you avoid making simple mistakes.

3. Two parallel lines have equations $y = -2x + 3$ and $y = mx - 4$. What is the value of m in the second linear equation?

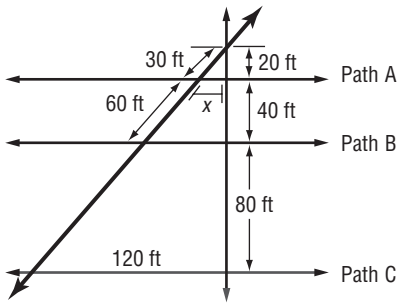
A -2

B $-\frac{1}{2}$

C $\frac{1}{2}$

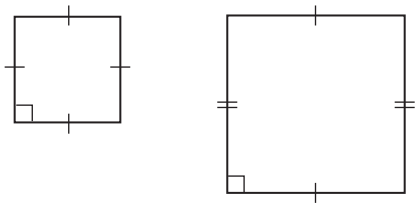
D 2

4. GRIDDABLE Carla received a map of some walking paths through her college campus. Paths A, B, and C are parallel. What is the length x to the nearest tenth of a foot?

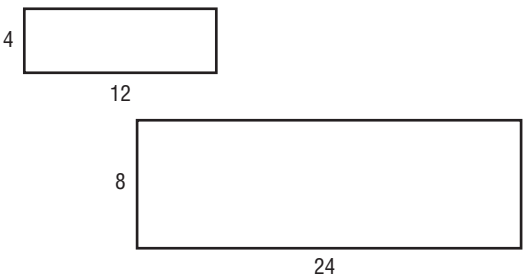


5. Use the information in each diagram to find the pair of similar polygons.

F



G



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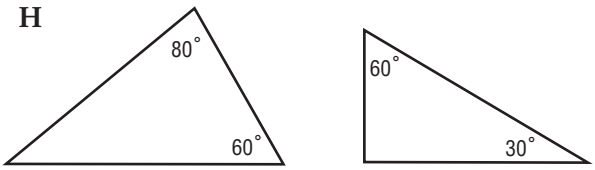
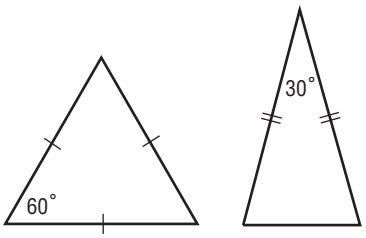


J



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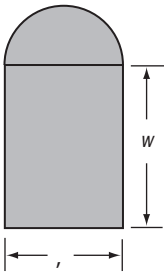
6. Find the equation that can be used to determine the total area of the composite figure below.



A $A = \ell w + \frac{1}{2}\ell w$

B $A = \ell w + \pi\left(\frac{1}{2}\ell\right)^2$

C $A = \ell w + \frac{1}{2}\pi\ell^2$

D $A = \ell w + \pi\left(\frac{1}{2}\ell\right)^2\left(\frac{1}{2}\right)$

7. The table shows one of the dimensions of a square tent and the number of people that can fit in the tent.

Length of Tent (yards)	Number of People
2	7
5	28
6	39
8	67
12	147

Let ℓ represent the length of the tent and n represent the number of people that can fit in the tent. Identify the equation that best represents the relationship between the length of the tent and the number of people that can fit in the tent.

F $\ell = n^2 + 3$ **H** $\ell = 3n + 1$

G $n = \ell^2 + 3$ **J** $n = 3\ell + 1$

8. Margo took her brother to lunch. The bill with tax was $38.69. If the sales tax was 6%, what was her bill before the sales tax?

A $2.32

B $36.37

C $36.50

D $41.01

9. How many faces, edges, and vertices does a pentagonal pyramid have?

F 5 faces, 8 edges, and 5 vertices

G 6 faces, 10 edges, and 6 vertices

H 7 faces, 15 edges, and 10 vertices

J 6 faces, 12 edges, and 8 vertices

Pre-AP

Record your answers on a sheet of paper. Show your work.

10. The endpoints of a diameter of a circle are at $(-1, 0)$ and $(5, -8)$.

a. What are the coordinates of the center of the circle? Explain your method.

b. Find the radius of the circle. Explain your method.

c. Write an equation of the circle.

NEED EXTRA HELP?									
If You Missed Question...	1	2	4	5	6	7	8	9	10
Go to Lesson or Page...	10-1	10-1	1-4	2-4	1-1	2-4	750	754	10-3

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UNIT 4
Discrete Mathematics

Focus

Use multiple representations, technology, applications and modeling, and numerical fluency in discrete problem-solving contexts.

CHAPTER 11
Sequences and Series

BIG Idea Use sequences and series as well as tools and technology to represent, analyze, and solve real-life problems.

CHAPTER 12
Probability and Statistics

BIG Idea Use probability and statistical models to describe everyday situations involving chance.

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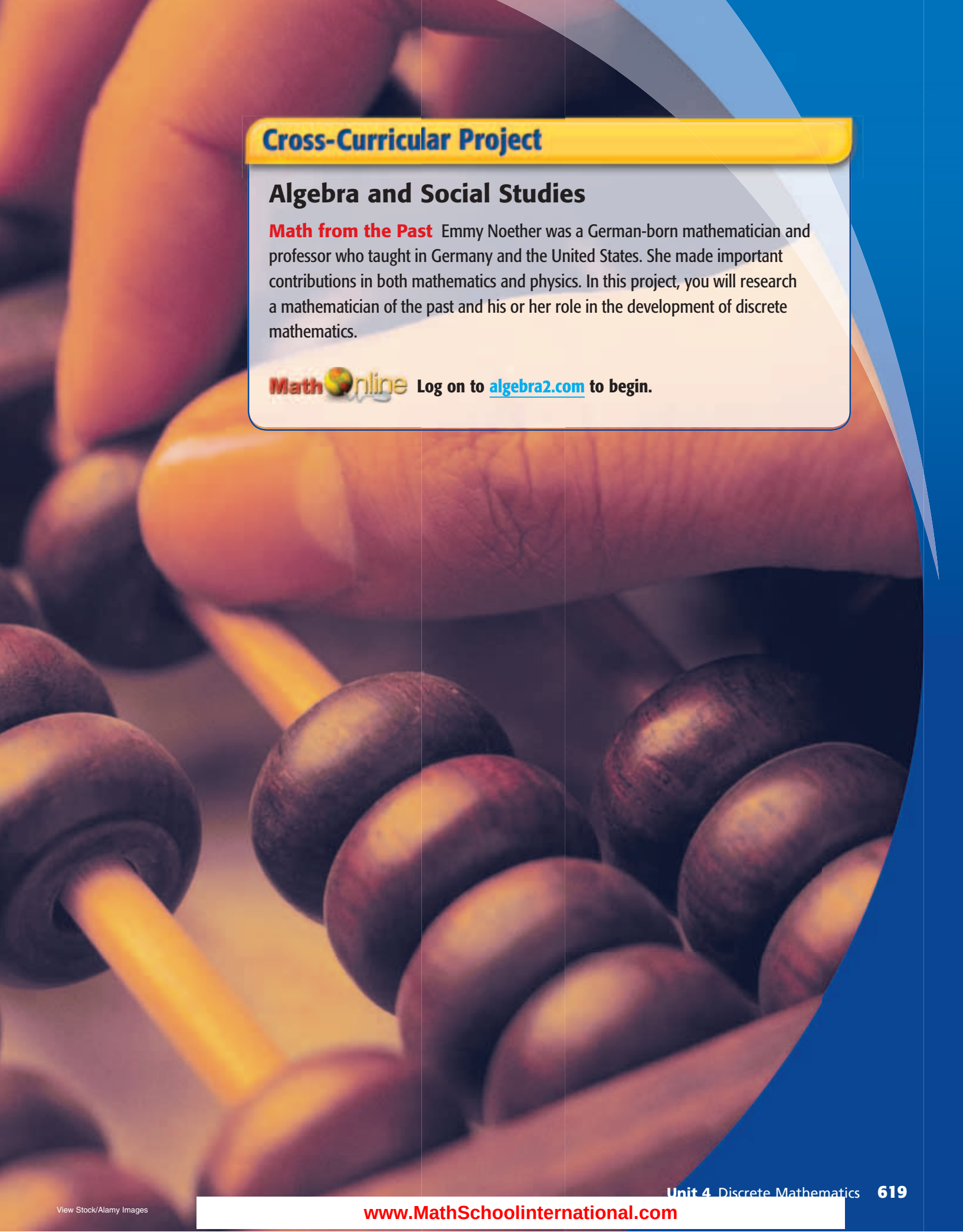

Cross-Curricular Project

Algebra and Social Studies

Math from the Past Emmy Noether was a German-born mathematician and professor who taught in Germany and the United States. She made important contributions in both mathematics and physics. In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

MathOnline **Log on to algebra2.com to begin.**

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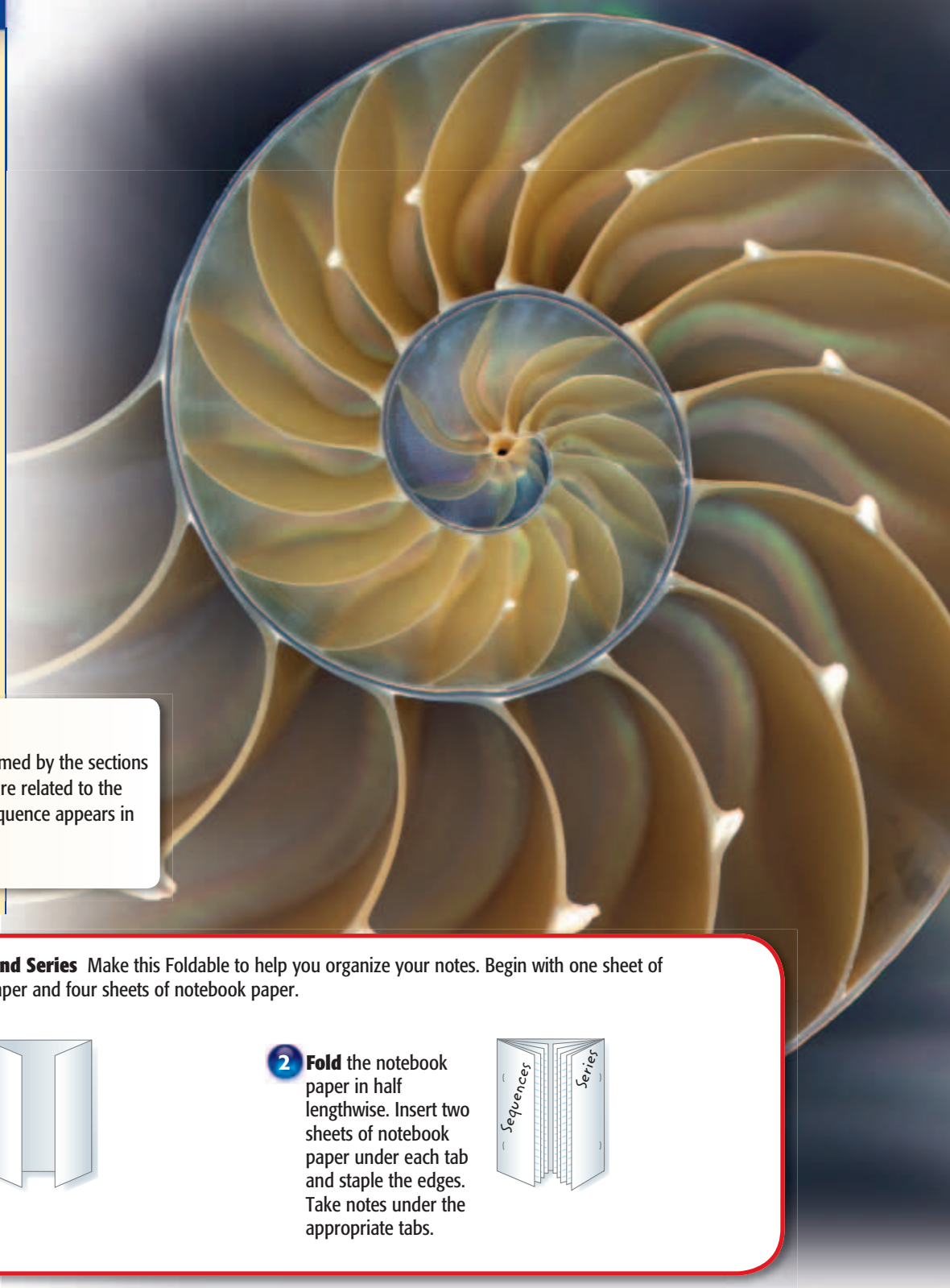
CHAPTER 11

Sequences and Series



BIG Ideas

- Use arithmetic and geometric sequences and series.
- Use special sequences and iterate functions.
- Expand powers by using the Binomial Theorem.
- Prove statements by using mathematical induction.

Key Vocabulary

arithmetic sequence (p. 622)
arithmetic series (p. 629)
geometric sequence (p. 636)
geometric series (p. 643)
inductive hypothesis (p. 670)
mathematical induction (p. 670)
recursive formula (p. 658)

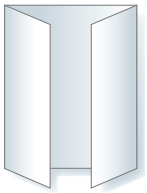
Real-World Link

Chambered Nautilus The spiral formed by the sections of the shell of a chambered nautilus are related to the Fibonacci sequence. The Fibonacci sequence appears in many objects naturally.

FOLDABLES Study Organizer

Sequences and Series Make this Foldable to help you organize your notes. Begin with one sheet of 11″ by 17″ paper and four sheets of notebook paper.

1. **Fold** the short sides of the 11″ by 17″ paper to meet in the middle.



2. **Fold** the notebook paper in half lengthwise. Insert two sheets of notebook paper under each tab and staple the edges. Take notes under the appropriate tabs.

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GET READY for Chapter 11

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Solve each equation. (Lesson 1-3)

1. $-40 = 10 + 5x$
2. $162 = 2x^4$
3. $12 - 3x = 27$
4. $3x^3 + 4 = -20$
5. **FAIR** Jeremy goes to a state fair with \$36. The entrance fee is \$12 and each ride costs \$4. How many rides can Jeremy go on? (Lesson 1-3)

Graph each function. (Lesson 2-1)

6. $\{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$
7. $\{(1, -20), (2, -16), (3, -12), (4, -8), (5, -4)\}$
8. $\left\{(1, 64), (2, 16), (3, 4), (4, 1), \left(5, \frac{1}{4}\right)\right\}$
9. $\left\{(1, 2), (2, 3), \left(3, \frac{7}{2}\right), \left(4, \frac{15}{4}\right), \left(5, \frac{31}{8}\right)\right\}$
10. **HOBBIES** Arthur has a collection of 21 model cars. He decides to buy 2 more model cars every time he goes to the toy store. The function $C(t) = 21 + 2t$ counts the number of model cars $C(t)$ he has after t trips to the toy store. How many model cars will he have after he has been to the toy store 6 times? (Lesson 1-3)

Evaluate each expression for the given value(s) of the variable(s). (Lesson 1-1)

11. $x + (y - 1)z$ if $x = 3$, $y = 8$, and $z = 2$
12. $\frac{x}{2}(y + z)$ if $x = 10$, $y = 3$, and $z = 25$
13. $a \cdot b^{c-1}$ if $a = 2$, $b = \frac{1}{2}$, and $c = 7$
14. $\frac{a(1 - bc)^2}{1 - b}$ if $a = -2$, $b = 3$, and $c = 5$

QUICKReview

EXAMPLE 1

Solve the equation $14 = 2x^3 + 700$.

$-686 = 2x^3$ Subtract 700 from each side.

$-343 = x^3$ Divide each side by 2.

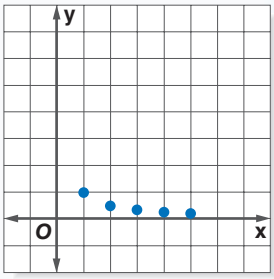
$\sqrt[3]{-343} = \sqrt[3]{x^3}$ Take the cube root of each side.

$-7 = x$ Simplify.

EXAMPLE 2

Graph the function $\left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$.

The domain of a function is the set of all possible x-values. So, the domain of this function is $\{1, 2, 3, 4, 5\}$. The range of a function is the set of all possible y-values. So, the range of this function is $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$.



EXAMPLE 3

Evaluate the expression 2^{m+k+b} if $m = 4$, $k = -5$, and $b = 1$.

$2^{4+(-5)+1}$ Substitute.

$= 2^0$ Simplify.

$= 1$ Zero Exponent Rule

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11-1 Arithmetic Sequences

Main Ideas

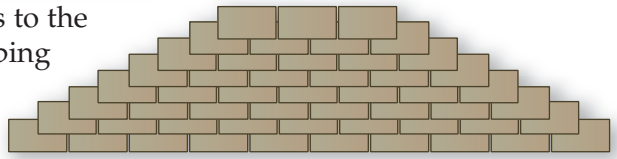
- Use arithmetic sequences.
- Find arithmetic means.

New Vocabulary

sequence
term
arithmetic sequence
common difference
arithmetic means

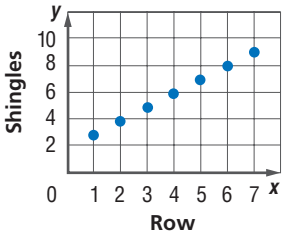
GET READY for the Lesson

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one more shingle is needed in each successive row.



Row	1	2	3	4	5	6	7
Shingles	3	4	5	6	7	8	9

Arithmetic Sequences The numbers 3, 4, 5, 6, …, representing the number of shingles in each row, are an example of a sequence of numbers. A **sequence** is a list of numbers in a particular order. Each number in a sequence is called a **term**. The first term is symbolized by a_1, the second term is symbolized by a_2, and so on. A sequence can also be thought of as a discrete function whose domain is the set of positive integers over some interval.



Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence. An **arithmetic sequence** is a sequence in which each term after the first is found by adding a constant, called the **common difference**, to the previous term.

Study Tip

Sequences
The numbers in a sequence may not be ordered. For example, the numbers 84, 102, 97, 72, 93, 84, 87, 92, … are a sequence that represents the number of games won by the Houston Astros each season beginning with 1997.

EXAMPLE 1 Find the Next Terms

Find the next four terms of the arithmetic sequence 55, 49, 43, … .

Find the common difference d by subtracting two consecutive terms.

$49 - 55 = -6$ and $43 - 49 = -6$ So, $d = -6$.

Now add -6 to the third term of the sequence, and then continue adding -6 until the next four terms are found.

$$43 \xrightarrow{+(-6)} 37 \xrightarrow{+(-6)} 31 \xrightarrow{+(-6)} 25 \xrightarrow{+(-6)} 19$$

The next four terms of the sequence are 37, 31, 25, and 19.

CHECK Your Progress

1. Find the next four terms of the arithmetic sequence $-1.6, -0.7, 0.2, \ldots$.

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It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term a_1 and the common difference d. Consider the sequence in Example 1.

Sequence	numbers	55	49	43	37	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of d and the First Term	numbers	$55 + 0(-6)$	$55 + 1(-6)$	$55 + 2(-6)$	$55 + 3(-6)$	…	$55 + (n - 1)(-6)$
	symbols	$a_1 + 0 \cdot d$	$a_1 + 1 \cdot d$	$a_1 + 2 \cdot d$	$a_1 + 3 \cdot d$	…	$a_1 + (n - 1)d$

The following formula generalizes this pattern for any arithmetic sequence.

KEY CONCEPT ***n*th Term of an Arithmetic Sequence**

The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by the following formula, where n is any positive integer.

$$a_n = a_1 + (n - 1)d$$

You can use the formula to find a term in a sequence given the first term and the common difference or given the first term and some successive terms.

Real-World Link

A hydraulic crane uses fluid to transmit forces from point to point. The brakes of a car use this same principle.

Source: howstuffworks.com

Real-World EXAMPLE **Find a Particular Term**

2 **CONSTRUCTION** **The table at the right shows typical costs for a construction company to rent a crane for one, two, three, or four months. If the sequence continues, how much would it cost to rent the crane for twelve months?**

Months	Cost ($)
1	75,000
2	90,000
3	105,000
4	120,000

Explore Since the difference between any two successive costs is \$15,000, the costs form an arithmetic sequence with common difference 15,000.

Plan You can use the formula for the nth term of an arithmetic sequence with $a_1 = 75{,}000$ and $d = 15{,}000$ to find a_{12}, the cost for twelve months.

Solve

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_{12} = 75{,}000 + (12 - 1)15{,}000$ $n = 12$, $a_1 = 75{,}000$, $d = 15{,}000$

$a_{12} = 240{,}000$ Simplify.

It would cost \$240,000 to rent the crane for twelve months.

Check You can find terms of the sequence by adding 15,000. a_5 through a_{12} are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, \$240,000 is correct.

CHECK Your Progress

2. The construction company has a budget of \$350,000 for crane rental. The job is expected to last 18 months. Will the company be able to afford the crane rental for the entire job? Explain.

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If you are given some of the terms of a sequence, you can use the formula for the nth term of a sequence to write an equation to help you find the nth term.

EXAMPLE Write an Equation for the nth Term

3 **Write an equation for the nth term of the arithmetic sequence 8, 17, 26, 35, ...**

In this sequence, $a_1 = 8$ and $d = 9$. Use the nth term formula to write an equation.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_n = 8 + (n - 1)9$ $a_1 = 8, d = 9$

$a_n = 8 + 9n - 9$ Distributive Property

$a_n = 9n - 1$ Simplify.

An equation is $a_n = 9n - 1$.

Study Tip

Checking Solutions

You can check to see that the equation you wrote to describe a sequence is correct by finding the first few terms of the sequence.

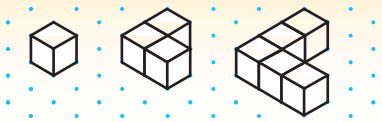
CHECK Your Progress

3. Write an equation for the nth term of the arithmetic sequence -1.5, -3.5, -5.5, ...

ALGEBRA LAB

Arithmetic Sequences

Study the figures below. The length of an edge of each cube is 1 centimeter.



MODEL AND ANALYZE

1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.

2. Find the volumes of the four figures.

3. Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure n.

4. What would the volume of the twelfth figure be?

Arithmetic Means Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two nonsuccessive terms of an arithmetic sequence are called **arithmetic means**. In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

19, 30, 41, 52, 63, 74, 85, 96, …

three arithmetic means between 30 and 74

The formula for the nth term of a sequence can be used to find arithmetic means between given terms of a sequence.

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EXAMPLE 4 Find Arithmetic Means

Find the four arithmetic means between 16 and 91.

Study Tip

Alternate Method

You may prefer this method. The four means will be $16 + d$, $16 + 2d$, $16 + 3d$, and $16 + 4d$. The common difference is $d = 91 - (16 + 4d)$ or $d = 15$.

You can use the nth term formula to find the common difference. In the sequence 16, __?__, __?__, __?__, __?__, 91, …, a_1 is 16 and a_6 is 91.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$a_6 = 16 + (6 - 1)d$ $n = 6, a_1 = 16$

$91 = 16 + 5d$ $a_6 = 91$

$75 = 5d$ Subtract 16 from each side.

$15 = d$ Divide each side by 5.

Now use the value of d to find the four arithmetic means.

$$16 \xrightarrow{+15} 31 \xrightarrow{+15} 46 \xrightarrow{+15} 61 \xrightarrow{+15} 76$$

The arithmetic means are 31, 46, 61, and 76. **CHECK** $76 + 15 = 91$ ✓

CHECK Your Progress

4. Find the three arithmetic means between 15.6 and 60.4.

CHECK Your Understanding

Example 1 (p. 622)

Find the next four terms of each arithmetic sequence.

1. 12, 16, 20, ...

2. 3, 1, −1, ...

Find the first five terms of each arithmetic sequence described.

3. $a_1 = 5, d = 3$

4. $a_1 = 14, d = -2$

5. $a_1 = \frac{1}{2}, d = \frac{1}{4}$

6. $a_1 = 0.5, d = -0.2$

Example 2 (p. 623)

7. Find a_{13} for the arithmetic sequence −17, −12, −7, ...

Find the indicated term of each arithmetic sequence.

8. $a_1 = 3, d = -5, n = 24$

9. $a_1 = -5, d = 7, n = 13$

10. $a_1 = -4, d = \frac{1}{3}, n = 8$

11. $a_1 = 6.6, d = 1.05, n = 32$

12. ENTERTAINMENT A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at $5000 for the first game and increases $500 each time there is no winner. Ellis has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

Example 3 (p. 624)

13. Write an equation for the nth term of the arithmetic sequence −26, −15, −4, 7, ...

14. Complete: 68 is the __?__th term of the arithmetic sequence −2, 3, 8, ...

Example 4 (p. 625)

15. Find the three arithmetic means between 44 and 92.

16. Find the three arithmetic means between 2.5 and 12.5.

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
17–26	1
27–34	2
35–40	3
41–44	4

Find the next four terms of each arithmetic sequence.

17. 9, 16, 23, ...
18. 31, 24, 17, ...
19. $-6, -2, 2, \ldots$
20. $-8, -5, -2, \ldots$

Find the first five terms of each arithmetic sequence described.

21. $a_1 = 2, d = 13$
22. $a_1 = 41, d = 5$
23. $a_1 = 6, d = -4$
24. $a_1 = 12, d = -3$

25. Find a_8 if $a_n = 4 + 3n$.
26. If $a_n = 1 - 5n$, what is a_{10}?

Find the indicated term of each arithmetic sequence.

27. $a_1 = 3, d = 7, n = 14$
28. $a_1 = -4, d = -9, n = 20$
29. $a_1 = 35, d = 3, n = 101$
30. $a_1 = 20, d = 4, n = 81$
31. a_{12} for $-17, -13, -9, \ldots$
32. a_{12} for $8, 3, -2, \ldots$

Real-World Link
Upon its completion in 1370, the Leaning Tower of Pisa leaned about 1.7 meters from vertical. Today, it leans about 5.2 meters from vertical.

Source: Associated Press

33. TOWER OF PISA To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the sixth second?

34. GEOLOGY Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (*Hint*: $a_1 = 0.75$)

Complete the statement for each arithmetic sequence.

35. 170 is the __?__ term of $-4, 2, 8, \ldots$.
36. 124 is the __?__ term of $-2, 5, 12, \ldots$.

Write an equation for the *n*th term of each arithmetic sequence.

37. 7, 16, 25, 34, ...
38. 18, 11, 4, −3, ...
39. $-3, -5, -7, -9, \ldots$
40. $-4, 1, 6, 11, \ldots$

Find the arithmetic means in each sequence.

41. 55, __?__, __?__, __?__, 115
42. 10, __?__, __?__, −8
43. −8, __?__, __?__, __?__, __?__, 7
44. 3, __?__, __?__, __?__, __?__, __?__, 27

Find the next four terms of each arithmetic sequence.

45. $\frac{1}{3}, 1, \frac{5}{3}, \ldots$
46. $\frac{18}{5}, \frac{16}{5}, \frac{14}{5}, \ldots$
47. 6.7, 6.3, 5.9, ...
48. 1.3, 3.8, 6.3, ...

Find the first five terms of each arithmetic sequence described.

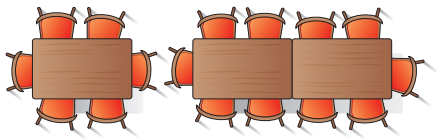
49. $a_1 = \frac{4}{3}, d = -\frac{1}{3}$
50. $a_1 = \frac{5}{8}, d = \frac{3}{8}$

Sylvan H. Witter/Visuals Unlimited
www.MathSchoolinternational.com

51. VACATION DAYS Kono's employer gives him 1.5 vacation days for each month he works. If Kono has 11 days at the end of one year and takes no vacation time during the next year, how many days will he have at the end of that year?

52. DRIVING Olivia was driving her car at a speed of 65 miles per hour. To exit the highway, she began decelerating at a rate of 5 mph per second. How long did it take Olivia to come to a stop?

SEATING For Exercises 53–55, use the following information.
The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.



53. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.

54. Write an equation representing the nth number in this pattern.

55. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.

Find the indicated term of each arithmetic sequence.

56. $a_1 = 5, d = \frac{1}{3}, n = 12$

57. $a_1 = \frac{5}{2}, d = -\frac{3}{2}, n = 11$

58. a_{21} for 121, 118, 115, ...

59. a_{43} for 5, 9, 13, 17, ...

Use the given information to write an equation that represents the nth number in each arithmetic sequence.

60. The 15th term of the sequence is 66. The common difference is 4.

61. The 100th term of the sequence is 100. The common difference is 7.

62. The tenth term of the sequence is 84. The 21st term of the sequence is 161.

63. The 63rd term of the sequence is 237. The 90th term of the sequence is 75.

64. The 18th term of a sequence is 367. The 30th term of the sequence is 499. How many terms of this sequence are less than 1000?

EXTRA PRACTICE
See pages 914, 936.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

65. OPEN ENDED Write a real-life application that can be described by an arithmetic sequence with common difference −5.

66. REASONING Explain why the sequence 4, 5, 7, 10, 14, ... is not arithmetic.

67. CHALLENGE The numbers x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y.

68. ***Writing in Math*** Use the information on pages 622 and 623 to explain the relationship between n and a_n in the formula for the nth term of an arithmetic sequence. If n is the independent variable and a_n is the dependent variable, what kind of equation relates n and a_n? Explain what a_1 and d mean in the context of the graph.

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STANDARDIZED TEST PRACTICE

69. ACT/SAT What is the first term in the arithmetic sequence?

$$____, 8\tfrac{1}{3}, 7, 5\tfrac{2}{3}, 4\tfrac{1}{3}, \ldots$$

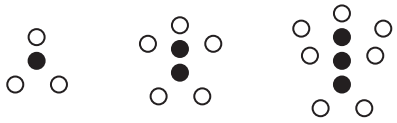
A 3

B $9\frac{2}{3}$

C $10\frac{1}{3}$

D 11

70. REVIEW The figures below show a pattern of filled circles and white circles that can be described by a relationship between 2 variables.



Which rule relates w, the number of white circles, to f, the number of dark circles?

F $w = 3f$

G $f = \frac{1}{2}w - 1$

H $w = 2f + 1$

J $f = \frac{1}{3}w$

Spiral Review

Find the exact solution(s) of each system of equations. (Lesson 10-7)

71. $x^2 + 2y^2 = 33$
$x^2 + y^2 - 19 = 2x$

72. $x^2 + 2y^2 = 33$
$x^2 - y^2 = 9$

Write each equation in standard form. State whether the graph of the equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. Then graph the equation. (Lesson 10-6)

73. $y^2 - 3x + 6y + 12 = 0$

74. $x^2 - 14x + 4 = 9y^2 - 36y$

75. If y varies directly as x and $y = 5$ when $x = 2$, find y when $x = 6$. (Lesson 8-4)

Simplify each expression. (Lesson 8-1)

76. $\dfrac{39a^3b^4}{13a^4b^3}$

77. $\dfrac{k+3}{5k\ell} \cdot \dfrac{10k\ell}{k+3}$

78. $\dfrac{5y - 15z}{42x^2} \div \dfrac{y - 3z}{14x}$

Find all the zeros of each function. (Lesson 6-8)

79. $f(x) = 8x^3 - 36x^2 + 22x + 21$

80. $g(x) = 12x^4 + 4x^3 - 3x^2 - x$

81. SAVINGS Mackenzie has $57 in her bank account. She begins receiving a weekly allowance of $15, of which she deposits 20% in her bank account. Write an equation that represents how much money is in Mackenzie's account after x weeks. (Lesson 2-4)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression for the given values of the variable.** (Lesson 1-1)

82. $3n - 1; n = 1, 2, 3, 4$

83. $6 - j; j = 1, 2, 3, 4$

84. $4m + 7; m = 1, 2, 3, 4, 5$

85. $4 - 2k; k = 3, 4, 5, 6, 7$

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11-2 Arithmetic Series

Main Ideas

- Find sums of arithmetic series.
- Use sigma notation.

New Vocabulary

series
arithmetic series
sigma notation
index of summation

GET READY for the Lesson

Austin, Texas has a strong musical tradition. It is home to many indoor and outdoor music venues where new and established musicians perform regularly. Some of these venues are amphitheaters that generally get wider as the distance from the stage increases.

Suppose a section of an amphitheater can seat 18 people in the first row and each row can seat 4 more people than the previous row.

Arithmetic Series The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is $18 + 22 + 26 + 30$ or 96. A **series** is an indicated sum of the terms of a sequence. Since 18, 22, 26, 30 is an arithmetic sequence, $18 + 22 + 26 + 30$ is an **arithmetic series**.

Study Tip

Indicated Sum

The sum of a series is the result when the terms of the series are added. An *indicated sum* is the expression that illustrates the series, which includes the terms + or −.

S_n represents the sum of the first n terms of a series. For example, S_4 is the sum of the first four terms.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Write S_9 in two different orders and add the two equations.

$$\begin{aligned} S_9 &= 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60 \\ (+)\ S_9 &= 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4 \\ \hline 2S_9 &= 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 \end{aligned}$$

$$2S_9 = 9(64)$$ Note that the sum had 9 terms.

$$S_9 = \frac{9}{2}(64)$$ The sum of the first and last terms of the series is 64.

An arithmetic series S_n has n terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.

Allen Matheson/photohome.com
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KEY CONCEPT — *Sum of an Arithmetic Series*

The sum S_n of the first n terms of an arithmetic series is given by

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \text{ or } S_n = \frac{n}{2}(a_1 + a_n).$$

EXAMPLE Find the Sum of an Arithmetic Series

1 **Find the sum of the first 100 positive integers.**

The series is $1 + 2 + 3 + \ldots + 100$. Since you can see that $a_1 = 1$, $a_{100} = 100$, and $d = 1$, you can use either sum formula for this series.

Method 1		Method 2
$S_n = \frac{n}{2}(a_1 + a_n)$	Sum formula	$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$
$S_{100} = \frac{100}{2}(1 + 100)$	$n = 100, a_1 = 1, a_{100} = 100, d = 1$	$S_{100} = \frac{100}{2}[2(1) + (100 - 1)1]$
$S_{100} = 50(101)$	Simplify.	$S_{100} = 50(101)$
$S_{100} = 5050$	Multiply.	$S_{100} = 5050$

CHECK Your Progress

1. Find the sum of the first 50 positive even integers.

Real-World EXAMPLE Find the First Term

Real-World Link

99.0% of teens ages 12–17 listen to the radio at least once a week. 79.1% listen at least once a day.

Source: Radio Advertising Bureau

2 **RADIO** **A radio station is giving away a total of $124,000 in August. If they increase the amount given away each day by $100, how much should they give away the first day?**

You know the values of n, S_n, and d. Use the sum formula that contains d.

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d] \quad \text{Sum formula}$$

$$S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100] \quad n = 31, d = 100$$

$$124{,}000 = \frac{31}{2}(2a_1 + 3000) \quad S_{31} = 124{,}000$$

$$8000 = 2a_1 + 3000 \quad \text{Multiply each side by } \frac{2}{31}.$$

$$5000 = 2a_1 \quad \text{Subtract 3000 from each side.}$$

$$2500 = a_1 \quad \text{Divide each side by 2.}$$

The radio station should give away $2500 the first day.

CHECK Your Progress

2. **EXERCISE** Aiden did pushups every day in March. He started on March 1st and increased the number of pushups done each day by one. He did a total of 1085 pushups for the month. How many pushups did Aiden do on March 1st?

Online Personal Tutor at algebra2.com

Michelle Bridwell/PhotoEdit

www.MathSchoolinternational.com

Sometimes it is necessary to use both a sum formula and the formula for the nth term to solve a problem.

EXAMPLE 3 Find the First Three Terms

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.

Step 1 Since you know a_1, a_n, and S_n, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find n.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$741 = \frac{n}{2}(9 + 105)$$
$$741 = 57n$$
$$13 = n$$

Step 2 Find d.

$$a_n = a_1 + (n - 1)d$$
$$105 = 9 + (13 - 1)d$$
$$96 = 12d$$
$$8 = d$$

Step 3 Use d to determine a_2 and a_3.

$a_2 = 9 + 8$ or 17 $\qquad$ $a_3 = 17 + 8$ or 25

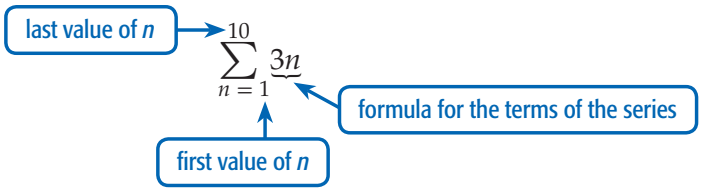
The first three terms are 9, 17, and 25.

CHECK Your Progress

3. Find the first three terms of an arithmetic series in which $a_1 = -16$, $a_n = 33$, and $S_n = 68$.

Sigma Notation Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \cdots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read *the sum of 3n as n goes from 1 to 10.*

Concepts in Motion
Animation
algebra2.com



The variable, in this case n, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of n are 1, 2, 3, and so on, through 10.

There are many ways to represent a given series. If changes are made to the first and last values of the variable and to the formula for the terms of the series, the same terms can be produced. For example, the following expressions produce the same terms.

$$\sum_{r=4}^{9} (r - 3) \qquad \sum_{s=2}^{7} (s - 1) \qquad \sum_{j=0}^{5} (j + 1)$$

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EXAMPLE Evaluate a Sum in Sigma Notation

4 Evaluate $\sum_{j=5}^{8}(3j-4)$.

Method 1

Find the terms by replacing j with 5, 6, 7, and 8. Then add.

$$\begin{aligned}\sum_{j=5}^{8}(3j-4) &= [3(5)-4]+[3(6)-4]+[3(7)-4]+[3(8)-4]\\ &= 11+14+17+20\\ &= 62\end{aligned}$$

Method 2

Since the sum is an arithmetic series, use the formula $S_n = \frac{n}{2}(a_1 + a_n)$.

There are 4 terms, $a_1 = 3(5) - 4$ *or* 11, and $a_4 = 3(8) - 4$ or 20.

$$\begin{aligned}S_4 &= \frac{4}{2}(11+20)\\ &= 62\end{aligned}$$

CHECK Your Progress

4. Evaluate $\sum_{k=2}^{6}(2k+1)$.

You can use the sum and sequence features on a graphing calculator to find the sum of a series.

GRAPHING CALCULATOR LAB

Sums of Series

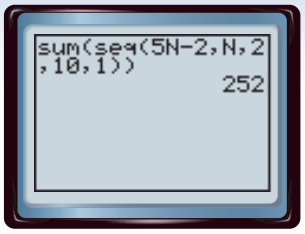
Study Tip

Graphing Calculators

On the TI-83/84 Plus, sum(is located on the LIST MATH menu. The function seq(is located on the LIST OPS menu.

The calculator screen shows the evaluation of $\sum_{N=2}^{10}(5N-2)$. The first four entries for seq(are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.



The last entry is always 1 for the types of series that we are considering.

THINK AND DISCUSS

1. Explain why you can use any letter for the index of summation.

2. Evaluate $\sum_{n=1}^{8}(2n-1)$ and $\sum_{j=5}^{12}(2j-9)$. Make a conjecture as to their relationship and explain why you think it is true.

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CHECK Your Understanding

Example 1 (p. 630)

Find the sum of each arithmetic series.

1. $5 + 11 + 17 + \cdots + 95$

2. $12 + 17 + 22 + \cdots + 102$

3. $38 + 35 + 32 + \cdots + 2$

4. $101 + 90 + 79 + \cdots + 2$

5. TRAINING To train for a race, Rosmaria runs 1.5 hours longer each week than she did the previous week. In the first week, Rosmaria ran 3 hours. How much time will Rosmaria spend running if she trains for 12 weeks?

Examples 1, 2 (p. 630)

Find S_n for each arithmetic series described.

6. $a_1 = 4, a_n = 100, n = 25$

7. $a_1 = 40, n = 20, d = -3$

8. $d = -4, n = 21, a_n = 52$

9. $d = 5, n = 16, a_n = 72$

Example 2 (p. 630)

Find a_1 for each arithmetic series described.

10. $d = 8, n = 19, S_{19} = 1786$

11. $d = -2, n = 12, S_{12} = 96$

Example 3 (p. 631)

Find the first three terms of each arithmetic series described.

12. $a_1 = 11, a_n = 110, S_n = 726$

13. $n = 8, a_n = 36, S_n = 120$

Example 4 (p. 632)

Find the sum of each arithmetic series.

14. $\sum_{n=1}^{7} (2n + 1)$

15. $\sum_{k=3}^{7} (3k + 4)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
16–21, 34–37	1
22–25	1, 2
26–29	2
30–33	3
38–43	4

Find S_n for each arithmetic series described.

16. $a_1 = 7, a_n = 79, n = 8$

17. $a_1 = 58, a_n = -7, n = 26$

18. $a_1 = 7, d = -2, n = 9$

19. $a_1 = 3, d = -4, n = 8$

20. $a_1 = 5, d = \frac{1}{2}, n = 13$

21. $a_1 = 12, d = \frac{1}{3}, n = 13$

22. $d = -3, n = 21, a_n = -64$

23. $d = 7, n = 18, a_n = 72$

24. TOYS Jamila is making a wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?

25. CONSTRUCTION A construction company will be fined for each day it is late completing a bridge. The daily fine will be \$4000 for the first day and will increase by \$1000 each day. Based on its budget, the company can only afford \$60,000 in total fines. What is the maximum number of days it can be late?

Find a_1 for each arithmetic series described.

26. $d = 3.5, n = 20, S_{20} = 1005$

27. $d = -4, n = 42, S_{42} = -3360$

28. $d = 0.5, n = 31, S_{31} = 573.5$

29. $d = -2, n = 18, S_{18} = 18$

EXTRA PRACTICE

See pages 914, 936.

MathOnline

Self-Check Quiz at algebra2.com

Find the first three terms of each arithmetic series described.

30. $a_1 = 17, a_n = 197, S_n = 2247$

31. $a_1 = -13, a_n = 427, S_n = 18{,}423$

32. $n = 31, a_n = 78, S_n = 1023$

33. $n = 19, a_n = 103, S_n = 1102$

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Find the sum of each arithmetic series.

34. $6 + 13 + 20 + 27 + \cdots + 97$

35. $7 + 14 + 21 + 28 + \cdots + 98$

36. $34 + 30 + 26 + \cdots + 2$

37. $16 + 10 + 4 + \cdots + (-50)$

38. $\sum_{n=1}^{6} (2n + 11)$

39. $\sum_{n=1}^{5} (2 - 3n)$

40. $\sum_{k=7}^{11} (42 - 9k)$

41. $\sum_{t=19}^{23} (5t - 3)$

42. $\sum_{n=1}^{300} (7n - 3)$

43. $\sum_{k=1}^{150} (11 + 2k)$

Real-World Link

Six missions of the Apollo Program landed humans on the Moon. Apollo 11 was the first mission to do so.

Source: nssdc.gsfc.nasa.gov

Find S_n for each arithmetic series described.

44. $a_1 = 43, n = 19, a_n = 115$

45. $a_1 = 76, n = 21, a_n = 176$

46. $a_1 = 91, d = -4, a_n = 15$

47. $a_1 = -2, d = \frac{1}{3}, a_n = 9$

48. $d = \frac{1}{5}, n = 10, a_n = \frac{23}{10}$

49. $d = -\frac{1}{4}, n = 20, a_n = -\frac{53}{12}$

50. Find the sum of the first 1000 positive even integers.

51. What is the sum of the multiples of 3 between 3 and 999, inclusive?

52. AEROSPACE On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than it did the previous second. How far would an object fall in the first ten seconds after being dropped?

53. SALARY Mr. Vacarro's salary this year is $41,000. If he gets a raise of $2500 each year, how much will Mr. Vacarro earn in ten years?

Graphing Calculator

Use a graphing calculator to find the sum of each arithmetic series.

54. $\sum_{n=21}^{75} (2n + 5)$

55. $\sum_{n=10}^{50} (3n - 1)$

56. $\sum_{n=20}^{60} (4n + 3)$

57. $\sum_{n=17}^{90} (1.5n + 13)$

58. $\sum_{n=22}^{64} (-n + 70)$

59. $\sum_{n=26}^{50} (-2n + 100)$

H.O.T. Problems

60. OPEN ENDED Write an arithmetic series for which $S_5 = 10$.

CHALLENGE State whether each statement is *true* or *false*. Explain your reasoning.

61. Doubling each term in an arithmetic series will double the sum.

62. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum.

63. ***Writing in Math*** Use the information on page 629 to explain how arithmetic series apply to amphitheaters. Explain what the sequence and the series that can be formed from the given numbers represent, and show two ways to find the seating capacity of the amphitheater if it has ten rows of seats.

NASA

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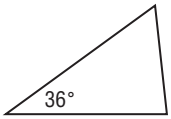
STANDARDIZED TEST PRACTICE

64. ACT/SAT The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is 36°, what is the measure of the largest angle?



A 75° **B** 84° **C** 90° **D** 97°

65. REVIEW How many 5-inch cubes can be placed completely inside a box that is 10 inches long, 15 inches wide, and 5 inches tall?

F 5 **H** 20

G 6 **J** 15

Spiral Review

Find the indicated term of each arithmetic sequence. (Lesson 11-1)

66. $a_1 = 46, d = 5, n = 14$ **67.** $a_1 = 12, d = -7, n = 22$

Solve each system of inequalities by graphing. (Lesson 10-7)

68. $9x^2 + y^2 < 81$
$x^2 + y^2 \geq 16$

69. $(y - 3)^2 \geq x + 2$
$x^2 \leq y + 4$

Write an equivalent logarithmic equation. (Lesson 9-2)

70. $5^x = 45$ **71.** $7^3 = x$ **72.** $b^y = x$

73. PAINTING Two employees of a painting company paint houses together. One painter can paint a house alone in 3 days, and the other painter can paint the same size house alone in 4 days. How long will it take them to paint one house if they work together? (Lesson 8-6)

Simplify. (Lesson 7-5)

74. $5\sqrt{3} - 4\sqrt{3}$ **75.** $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$ **76.** $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$

Solve each equation by completing the square. (Lesson 5-5)

77. $x^2 + 9x + 20.25 = 0$ **78.** $9x^2 + 96x + 256 = 0$ **79.** $x^2 - 3x - 20 = 0$

Use a graphing calculator to find the value of each determinant. (Lesson 4-5)

80. $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$ **81.** $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$ **82.** $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

83. $a + 4b = 6$
$3a + 2b = -2$

84. $10x - y = 13$
$3x - 4y = 15$

85. $3c - 7d = -1$
$2c - 6d = -6$

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate the expression $a \cdot b^{n-1}$ for the given values of a, b, and n.** (Lesson 1-1)

86. $a = 1, b = 2, n = 5$ **87.** $a = 2, b = -3, n = 4$ **88.** $a = 18, b = \frac{1}{3}, n = 6$

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11-3 Geometric Sequences

Main Ideas

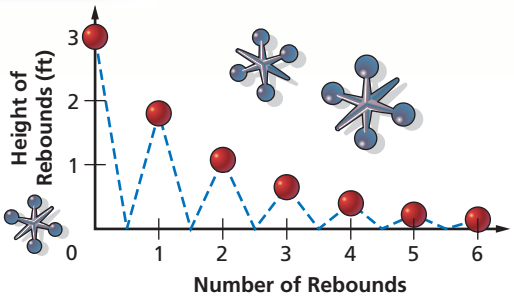
- Use geometric sequences.
- Find geometric means.

New Vocabulary

geometric sequence
common ratio
geometric means

GET READY for the Lesson

When you drop a ball, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball's rebounds form a sequence.



Geometric Sequences The height of the first rebound of the ball is 3(0.6) or 1.8 feet. The height of the second rebound is 1.8(0.6) or 1.08 feet. The height of the third rebound is 1.08(0.6) or 0.648 feet. The sequence of heights is an example of a **geometric sequence**. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant r called the **common ratio**.

As with an arithmetic sequence, you can label the terms of a geometric sequence as a_1, a_2, a_3, and so on, $a_1 \neq 0$. The nth term is a_n and the previous term is a_{n-1}. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

STANDARDIZED TEST EXAMPLE — Find the Next Term

1 What is the missing term in the geometric sequence: 8, 20, 50, 125, ____ ?

A 75 **B** 200 **C** 250 **D** 312.5

Test-Taking Tip

Since the terms of this sequence are increasing, the missing term must be greater than 125. You can immediately eliminate 75 as a possible answer.

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the common ratio is 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: 125(2.5) = 312.5. The answer is D.

CHECK Your Progress

1. What is the missing term in the geometric sequence: −120, 60, −30, 15, _____ ?

F −7.5 **G** 0 **H** 7.5 **J** 10

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You have seen that each term of a geometric sequence after the first term can be expressed in terms of r and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of r and the first term a_1. Study the patterns in the table for the sequence 2, 6, 18, 54, … .

Sequence	**numbers**	2	6	18	54	…	
	symbols	a_1	a_2	a_3	a_4	…	a_n
Expressed in Terms of *r* and the Previous Term	**numbers**	2	2(3)	6(3)	18(3)	…	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	…	$a_{n-1} \cdot r$
Expressed in Terms of *r* and the First Term	**numbers**	2	2(3)	2(9)	2(27)	…	
		$2(3^0)$	$2(3^1)$	$2(3^2)$	$2(3^3)$	…	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	…	$a_1 \cdot r^{n-1}$

The three entries in the last column all describe the nth term of a geometric sequence. This leads to the following formula.

COncepts in MOtion
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KEY CONCEPT ***n*th Term of a Geometric Sequence**

The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by the following formula, where n is any positive integer.

$$a_n = a_1 \cdot r^{n-1}$$

Study Tip

Finding a Term

For small values of r and n, it may be easier to multiply by r successively to find a given term than to use the formula.

EXAMPLE **Find a Term Given the First Term and the Ratio**

2 **Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_8 = (-3) \cdot (-2)^{8-1}$ $n = 8, a_1 = -3, r = -2$

$a_8 = (-3) \cdot (-128)$ $(-2)^7 = -128$

$a_8 = 384$ Multiply.

CHECK Your Progress

2. Find the sixth term of a geometric sequence for which $a_1 = -\frac{1}{9}$ and $r = 3$.

EXAMPLE **Write an Equation for the *n*th Term**

3 **Write an equation for the nth term of the geometric sequence 3, 12, 48, 192, … .**

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_n = 3 \cdot 4^{n-1}$ $a_1 = 3, r = 4$

CHECK Your Progress

3. Write an equation for the nth term of the geometric sequence 18, −3, $\frac{1}{2}$, $-\frac{1}{12}$, … .

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You can also use the formula for the nth term if you know the common ratio and one term of a geometric sequence, but not the first term.

EXAMPLE 4 Find a Term Given One Term and the Ratio

Find the tenth term of a geometric sequence for which $a_4 = 108$ and $r = 3$.

Step 1 Find the value of a_1.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_4 = a_1 \cdot 3^{4-1}$	$n = 4, r = 3$
$108 = 27a_1$	$a_4 = 108$
$4 = a_1$	Divide each side by 27.

Step 2 Find a_{10}.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_{10} = 4 \cdot 3^{10-1}$	$n = 10, a_1 = 4, r = 3$
$a_{10} = 78{,}732$	Use a calculator.

The tenth term is 78,732.

CHECK Your Progress

4. Find the eighth term of a geometric sequence for which $a_3 = 16$ and $r = 4$.

Geometric Means In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called *arithmetic means*. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called **geometric means**. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a sequence.

EXAMPLE 5 Find Geometric Means

Study Tip

Alternate Method

You may prefer this method. The three means will be $2.25r$, $2.25r^2$, and $2.25r^3$. Then the common ratio is $r = \frac{576}{2.25r^3}$ or $r^4 = \frac{576}{2.25}$. Thus, $r = 4$.

Find three geometric means between 2.25 and 576.

Use the nth term formula to find the value of r. In the sequence 2.25, $\frac{?}{}$, $\frac{?}{}$, $\frac{?}{}$, 576, a_1 is 2.25 and a_5 is 576.

$a_n = a_1 \cdot r^{n-1}$	Formula for nth term
$a_5 = 2.25 \cdot r^{5-1}$	$n = 5, a_1 = 2.25$
$576 = 2.25r^4$	$a_5 = 576$
$256 = r^4$	Divide each side by 2.25.
$\pm 4 = r$	Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find three geometric means.

$r = 4$	$r = -4$
$a_2 = 2.25(4)$ or 9	$a_2 = 2.25(-4)$ or -9
$a_3 = 9(4)$ or 36	$a_3 = -9(-4)$ or 36
$a_4 = 36(4)$ or 144	$a_4 = 36(-4)$ or -144

The geometric means are 9, 36, and 144, or -9, 36, and -144.

CHECK Your Progress

5. Find two geometric means between 4 and 13.5.

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CHECK Your Understanding

Example 1 (p. 636)

1. Find the next two terms of the geometric sequence 20, 30, 45, … .
2. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.

3. **STANDARDIZED TEST PRACTICE** What is the missing term in the geometric sequence: $-\frac{1}{4}, \frac{1}{2}, -1, 2, ___$?

   A -4 **B** $-3\frac{1}{2}$ **C** $3\frac{1}{2}$ **D** 4

Example 2 (p. 637)

4. Find a_9 for the geometric sequence 60, 30, 15, … .
5. Find a_8 for the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$.

Find the indicated term of each geometric sequence.

6. $a_1 = 7, r = 2, n = 4$
7. $a_1 = 3, r = \frac{1}{3}, n = 5$

Example 3 (p. 637)

8. Write an equation for the nth term of the geometric sequence 4, 8, 16, … .
9. Write an equation for the nth term of the geometric sequence 15, 5, $\frac{5}{3}$, … .

Example 4 (p. 638)

Find the indicated term of each geometric sequence.

10. $a_3 = 24, r = \frac{1}{2}, n = 7$
11. $a_3 = 32, r = -0.5, n = 6$

Example 5 (p. 638)

12. Find two geometric means between 1 and 27.
13. Find two geometric means between 2 and 54.

Exercises

HOMEWORK HELP

For Exercises	See Examples
14–19	1
20–27	2
28, 29	3
30–33	4
34–37	5

Find the next two terms of each geometric sequence.

14. 405, 135, 45, …
15. 81, 108, 144, …
16. 16, 24, 36, …
17. 162, 108, 72, …

Find the first five terms of each geometric sequence described.

18. $a_1 = 2, r = -3$
19. $a_1 = 1, r = 4$
20. Find a_7 if $a_1 = 12$ and $r = \frac{1}{2}$.
21. Find a_6 if $a_1 = \frac{1}{3}$ and $r = 6$.

22. **INTEREST** An investment pays interest so that each year the value of the investment increases by 10%. How much is an initial investment of $1000 worth after 5 years?

23. **SALARIES** Geraldo's current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

Find the indicated term of each geometric sequence.

24. $a_1 = \frac{1}{3}, r = 3, n = 8$
25. $a_1 = \frac{1}{64}, r = 4, n = 9$
26. a_9 for $a_1 = \frac{1}{5}, 1, 5, \ldots$
27. a_7 for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, \ldots$
28. $a_4 = 16, r = 0.5, n = 8$
29. $a_6 = 3, r = 2, n = 12$

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Real-World Link

The largest ever ice construction was an ice palace built for a carnival in St. Paul, Minnesota, in 1992. It contained 10.8 million pounds of ice.

Source: *The Guinness Book of Records*

Write an equation for the nth term of each geometric sequence.

30. 36, 12, 4, …

31. 64, 16, 4, …

32. −2, 10, −50, …

33. 4, −12, 36, …

Find the geometric means in each sequence.

34. 9, ?, ?, ?, 144

35. 4, ?, ?, ?, 324

36. 32, ?, ?, ?, ?, 1

37. 3, ?, ?, ?, ?, 96

Find the next two terms of each geometric sequence.

38. $\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \ldots$

39. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$

40. 1.25, −1.5, 1.8, …

41. 1.4, −3.5, 8.75, …

Find the first five terms of each geometric sequence described.

42. $a_1 = 243, r = \frac{1}{3}$

43. $a_1 = 576, r = -\frac{1}{2}$

44. ART A one-ton ice sculpture is melting so that it loses one-eighth of its weight per hour. How much of the sculpture will be left after five hours? Write your answer in pounds.

MEDICINE For Exercises 45 and 46, use the following information.

Iodine-131 is a radioactive element used to study the thyroid gland.

45. RESEARCH Use the Internet or other resource to find the *half-life* of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

46. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

EXTRA PRACTICE
See pages 914, 936.
MathOnline
Self-Check Quiz at algebra2.com

Find the indicated term of each geometric sequence.

47. $a_1 = 16{,}807, r = \frac{3}{7}, n = 6$

48. $a_1 = 4096, r = \frac{1}{4}, n = 8$

49. a_8 for 4, −12, 36, …

50. a_6 for 540, 90, 15, …

51. $a_4 = 50, r = 2, n = 8$

52. $a_4 = 1, r = 3, n = 10$

H.O.T. Problems

53. OPEN ENDED Write a geometric sequence with a common ratio of $\frac{2}{3}$.

54. FIND THE ERROR Marika and Lori are finding the seventh term of the geometric sequence 9, 3, 1, … . Who is correct? Explain your reasoning.

Marika

$r = \frac{3}{9}$ or $\frac{1}{3}$

$a_7 = 9\left(\frac{1}{3}\right)^{7-1}$

$= \frac{1}{81}$

Lori

$r = \frac{9}{3}$ or 3

$a_7 = 9 \cdot 3^{7-1}$

$= 6561$

55. Which One Doesn't Belong? Identify the sequence that does not belong with the other three. Explain your reasoning.

1, 4, 16, …

3, 9, 27, …

9, 16, 25, …

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Bill Horseman/Stock Boston

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CHALLENGE **Determine whether each statement is *true* or *false*. If true, explain. If false, provide a counterexample.**

56. Every sequence is either arithmetic or geometric.

57. There is no sequence that is both arithmetic and geometric.

58. ***Writing in Math*** Use the information on pages 636 and 637 to explain the relationship between n and a_n in the formula for the nth term of a geometric sequence. If n is the independent variable and a_n is the dependent variable, what kind of equation relates n and a_n? Explain what r represents in the context of the relationship.

STANDARDIZED TEST PRACTICE

59. **ACT/SAT** The first four terms of a geometric sequence are shown in the table. What is the tenth term in the sequence?

a_1	144
a_2	72
a_3	36
a_4	18

A 0

B $\frac{9}{64}$

C $\frac{9}{32}$

D $\frac{9}{16}$

60. **REVIEW** The table shows the cost of jelly beans depending on the amount purchased. Which conclusion can be made based on the table?

Cost of Jelly Beans	
Number of Pounds	Cost
5	\$14.95
20	\$57.80
50	\$139.50
100	\$269.00

F The cost of 10 pounds of jelly beans would be more than \$30.

G The cost of 200 pounds of jelly beans would be less than \$540.

H The cost of jelly beans is always more than \$2.70 per pound.

J The cost of jelly beans is always less than \$2.97 per pound.

Spiral Review

Find S_n for each arithmetic series described. (Lesson 11-2)

61. $a_1 = 11, a_n = 44, n = 23$

62. $a_1 = -5, d = 3, n = 14$

Find the arithmetic means in each sequence. (Lesson 11-1)

63. $15, __?__, __?__, 27$

64. $-8, __?__, __?__, __?__, -24$

65. **GEOMETRY** Find the perimeter of a triangle with vertices at $(2, 4)$, $(-1, 3)$ and $(1, -3)$. (Lesson 10-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.** (Lesson 1-1)

66. $\frac{1 - 2^7}{1 - 2}$

67. $\frac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}}$

68. $\frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)}$

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EXTEND 11-3

Graphing Calculator Lab

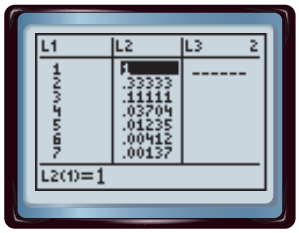
Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as n increases, a_n approaches 0. The value that the terms of a sequence approach, in this case 0, is called the **limit** of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

ACTIVITY 1 Find the limit of the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \ldots$.

Step 1 **Enter the sequence.**

- The formula for this sequence is $a_n = \left(\frac{1}{3}\right)^{n-1}$.
- Position the cursor on L1 in the STAT EDIT Edit … screen and enter the formula seq(N,N,1,10,1). This generates the values 1, 2, …, 10 of the index N.
- Position the cursor on L2 and enter the formula seq((1/3)^(N-1),N,1,10,1). This generates the first ten terms of the sequence.

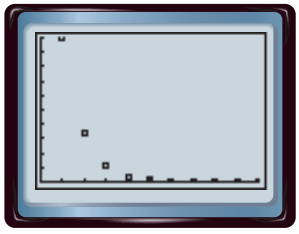


KEYSTROKES: *Review sequences in the Graphing Calculator Lab on page 632.*

Notice that as n increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for $n \geq 8$ the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

Step 2 **Graph the sequence.**

- Use a STAT PLOT to graph the sequence. Use L1 as the Xlist and L2 as the Ylist.



KEYSTROKES: *Review STAT PLOTs on page 92.*

The graph also shows that, as n increases, the terms approach 0. In fact, for $n \geq 6$, the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

EXERCISES

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. $a_n = \left(\frac{1}{2}\right)^n$
2. $a_n = \left(-\frac{1}{2}\right)^n$
3. $a_n = 4^n$
4. $a_n = \frac{1}{n^2}$
5. $a_n = \frac{2^n}{2^n + 1}$
6. $a_n = \frac{n^2}{n + 1}$

Other Calculator Keystrokes at algebra2.com

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11-4 Geometric Series

Main Ideas

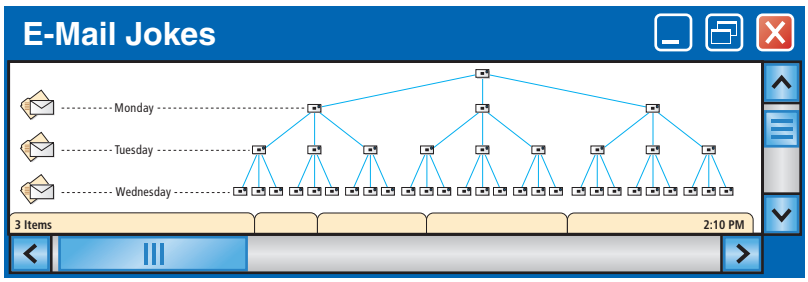
- Find sums of geometric series.
- Find specific terms of geometric series.

New Vocabulary

geometric series

GET READY for the Lesson

Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.



Geometric Series Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. The indicated sum of the numbers in the sequence, $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, is called a **geometric series**.

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above. Multiply each term in the series by the common ratio and subtract the result from the original series.

$$\begin{aligned} S_8 &= 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \\ (-)\ 3S_8 &= \quad 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561 \\ \hline (1-3)S_8 &= 1 + 0 + 0 + \ 0 + \ 0 + \ 0 + \ 0 + \ 0 - 6561 \end{aligned}$$

first term in series

$$S_8 = \frac{1 - 6561}{1 - 3} \text{ or } 3280$$

last term in series multiplied by common ratio; in this case, a_9

common ratio

The expression for S_8 can be written as $S_8 = \frac{a_1 - a_1 r^8}{1 - r}$. A rational expression like this can be used to find the sum of any geometric series.

Study Tip

Terms of Geometric Sequences

Remember that a_9 can also be written as $a_1 r^8$.

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KEY CONCEPT — **Sum of a Geometric Series**

The sum S_n of the first n terms of a geometric series is given by

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} \text{ or } S_n = \frac{a_1(1 - r^n)}{1 - r}, \text{ where } r \neq 1.$$

Real-World Link

The development of vaccines for many diseases has helped to prevent infection. Vaccinations are commonly given to children.

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \cdots + 4$ is such a series. In general, the sum of n terms of a geometric series with $r = 1$ is $n \cdot a_1$.

Real-World EXAMPLE Find the Sum of the First *n* Terms

1 **HEALTH** **Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic, and each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?**

This is a geometric series with $a_1 = 5$, $r = 4$, and $n = 10$.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_{15} = \frac{5(1 - 4^{10})}{1 - 4}$ $n = 10, a_1 = 5, r = 4$

$S_{15} = 1{,}747{,}625$ Use a calculator.

After ten weeks, 1,747,625 people have been affected by the illness.

CHECK Your Progress

1. **GAMES** Maria arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Maria use?

You can use sigma notation to represent geometric series.

EXAMPLE Evaluate a Sum Written in Sigma Notation

2 Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1 Since the sum is a geometric series, you can use the formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$

$S_6 = \frac{5(1 - 2^6)}{1 - 2}$ $n = 6, a_1 = 5, r = 2$

$S_6 = \frac{5(-63)}{-1}$ $2^6 = 64$

$S_6 = 315$ Simplify.

David Kelly Crow/PhotoEdit

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Method 2 Find the terms by replacing n with 1, 2, 3, 4, 5, and 6. Then add.

$$\sum_{n=1}^{6} 5 \cdot 2^{n-1}$$

$= 5(2^{1-1}) + 5(2^{2-1}) + 5(2^{3-1}) + 5(2^{4-1}) + 5(2^{5-1}) + 5(2^{6-1})$ Write out the sum.

$= 5(1) + 5(2) + 5(4) + 5(8) + 5(16) + 5(32)$ Simplify.

$= 5 + 10 + 20 + 40 + 80 + 160$ Multiply.

$= 315$ Add.

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{5} -4 \cdot 3^{n-1}$.

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How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? You can use the formula for the nth term of a geometric sequence or series, $a_n = a_1 \cdot r^{n-1}$, to find an expression involving r^n.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$a_n \cdot r = a_1 \cdot r^{n-1} \cdot r$ Multiply each side by r.

$a_n \cdot r = a_1 \cdot r^n$ $r^{n-1} \cdot r^1 = r^{n-1+1}$ or r^n

Now substitute $a_n \cdot r$ for $a_1 \cdot r^n$ in the formula for the sum of geometric series. The result is $S_n = \frac{a_1 - a_n r}{1 - r}$.

EXAMPLE **Use the Alternate Formula for a Sum**

3 **Find the sum of a geometric series for which $a_1 = 15{,}625$, $a_n = -5$, and $r = -\frac{1}{5}$.**

Since you do not know the value of n, use the formula derived above.

$S_n = \frac{a_1 - a_n r}{1 - r}$ Alternate sum formula

$= \frac{15{,}625 - (-5)\left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}$ $a_1 = 15{,}625$; $a_n = -5$; $r = -\frac{1}{5}$

$= \frac{15{,}624}{\frac{6}{5}}$ or 13,020 Simplify.

CHECK Your Progress

3. Find the sum of a geometric series for which $a_1 = 1000$, $a_n = 125$, and $r = \frac{1}{2}$.

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Specific Terms You can use the formula for the sum of a geometric series to help find a particular term of the series.

EXAMPLE 4 Find the First Term of a Series

Find a_1 in a geometric series for which $S_8 = 39{,}360$ and $r = 3$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$ Sum formula

$$39{,}360 = \frac{a_1(1 - 3^8)}{1 - 3}$$ $S_8 = 39{,}360;\ r = 3;\ n = 8$

$$39{,}360 = \frac{-6560a_1}{-2}$$ Subtract.

$$39{,}360 = 3280a_1$$ Divide.

$$12 = a_1$$ Divide each side by 3280.

CHECK Your Progress

4. Find a_1 in a geometric series for which $S_7 = 258$ and $r = -2$.

CHECK Your Understanding

Example 1 (p. 644)

Find S_n for each geometric series described.

1. $a_1 = 5, r = 2, n = 14$

2. $a_1 = 243, r = -\frac{2}{3}, n = 5$

Find the sum of each geometric series.

3. $54 + 36 + 24 + 16 + \cdots$ to 6 terms

4. $3 - 6 + 12 - \cdots$ to 7 terms

5. WEATHER Heavy rain caused a river to rise. The river rose three inches the first day, and each day it rose twice as much as the previous day. How much did the river rise in five days?

Example 2 (pp. 644–645)

Find the sum of each geometric series.

6. $\sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1}$

7. $\sum_{n=1}^{7} 81\left(\frac{1}{3}\right)^{n-1}$

8. $\sum_{n=1}^{12} \frac{1}{6}(-2)^n$

9. $\sum_{n=1}^{8} \frac{1}{3} \cdot 5^{n-1}$

10. $\sum_{n=1}^{6} 100\left(\frac{1}{2}\right)^{n-1}$

11. $\sum_{n=1}^{9} \frac{1}{27}(-3)^{n-1}$

Example 3 (p. 645)

Find S_n for each geometric series described.

12. $a_1 = 12, a_5 = 972, r = -3$

13. $a_1 = 3, a_n = 46{,}875, r = -5$

14. $a_1 = 5, a_n = 81{,}920, r = 4$

15. $a_1 = -8, a_6 = -256, r = 2$

Example 4 (p. 646)

Find the indicated term for each geometric series described.

16. $S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1$

17. $S_n = 33, a_n = 48, r = -2; a_1$

18. $S_n = 443, r = \frac{1}{3}, n = 6; a_1$

19. $S_n = -242, a_n = -162, r = 3; a_1$

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
20–25, 28–31	1
26, 27	3
32, 33	2
34–37	4

Find S_n for each geometric series described.

20. $a_1 = 2, a_6 = 486, r = 3$

21. $a_1 = 3, a_8 = 384, r = 2$

22. $a_1 = 4, r = -3, n = 5$

23. $a_1 = 5, r = 3, n = 12$

24. $a_1 = 2401, r = -\frac{1}{7}, n = 5$

25. $a_1 = 625, r = \frac{3}{5}, n = 5$

26. $a_1 = 1296, a_n = 1, r = -\frac{1}{6}$

27. $a_1 = 343, a_n = -1, r = -\frac{1}{7}$

28. GENEALOGY In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

29. LEGENDS There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have $1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

Day	Payment
1	1¢
2	2¢
3	4¢
4	8¢
⋮	⋮
30	?
Total	?

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Real-World Link

Some of the best-known legends involving a king are the Arthurian legends. According to the legends, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.

Find the sum of each geometric series.

30. $4096 - 512 + 64 - \cdots$ to 5 terms

31. $7 + 21 + 63 + \cdots$ to 10 terms

32. $\sum_{n=1}^{9} 5 \cdot 2^{n-1}$

33. $\sum_{n=1}^{6} 2(-3)^{n-1}$

Find the indicated term for each geometric series described.

34. $S_n = 165, a_n = 48, r = -\frac{2}{3}; a_1$

35. $S_n = 688, a_n = 16, r = -\frac{1}{2}; a_1$

36. $S_n = -364, r = -3, n = 6; a_1$

37. $S_n = 1530, r = 2, n = 8; a_1$

Find S_n for each geometric series described.

38. $a_1 = 162, r = \frac{1}{3}, n = 6$

39. $a_1 = 80, r = -\frac{1}{2}, n = 7$

40. $a_1 = 625, r = 0.4, n = 8$

41. $a_1 = 4, r = 0.5, n = 8$

42. $a_2 = -36, a_5 = 972, n = 7$

43. $a_3 = -36, a_6 = -972, n = 10$

44. $a_1 = 4, a_n = 236{,}196, r = 3$

45. $a_1 = 125, a_n = \frac{1}{125}, r = \frac{1}{5}$

Find the sum of each geometric series.

46. $\frac{1}{16} + \frac{1}{4} + 1 + \cdots$ to 7 terms

47. $\frac{1}{9} - \frac{1}{3} + 1 - \cdots$ to 6 terms

48. $\sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^{n-1}$

49. $\sum_{n=1}^{20} 3 \cdot 2^{n-1}$

50. $\sum_{n=1}^{16} 4 \cdot 3^{n-1}$

51. $\sum_{n=1}^{7} 144\left(-\frac{1}{2}\right)^{n-1}$

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EXTRA PRACTICE
See pages 915, 936.
Math Online
Self-Check Quiz at algebra2.com

Find the indicated term for each geometric series described.

52. $S_n = 315, r = 0.5, n = 6; a_2$

53. $S_n = 249.92, r = 0.2, n = 5, a_3$

54. WATER TREATMENT A certain water filtration system can remove 80% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system three times, what percent of the original contaminants will be removed from the water sample?

Graphing Calculator

Use a graphing calculator to find the sum of each geometric series.

55. $\sum_{n=1}^{20} 3(-2)^{n-1}$

56. $\sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^{n-1}$

57. $\sum_{n=1}^{10} 5(0.2)^{n-1}$

58. $\sum_{n=1}^{13} 6\left(\frac{1}{3}\right)^{n-1}$

H.O.T. Problems

59. OPEN ENDED Write a geometric series for which $r = \frac{1}{2}$ and $n = 4$.

60. REASONING Explain how to write the series $2 + 12 + 72 + 432 + 2592$ using sigma notation.

61. CHALLENGE If a_1 and r are integers, explain why the value of $\frac{a_1 - a_1 r^n}{1 - r}$ must also be an integer.

62. REASONING Explain, using geometric series, why the polynomial $1 + x + x^2 + x^3$ can be written as $\frac{x^4 - 1}{x - 1}$, assuming $x \neq 1$.

63. *Writing in Math* Use the information on page 643 to explain how e-mailing a joke is related to a geometric series. Include an explanation of how the situation could be changed to make it better to use a formula than to add terms.

STANDARDIZED TEST PRACTICE

64. ACT/SAT The first term of a geometric series is -1, and the common ratio is -3. How many terms are in the series if its sum is 182?

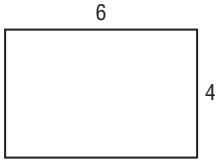
A 6

B 7

C 8

D 9

65. REVIEW Which set of dimensions corresponds to a rectangle similar to the one shown below?



F 3 units by 1 unit

G 12 units by 9 units

H 13 units by 8 units

J 18 units by 12 units

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Spiral Review

Find the geometric means in each sequence. (Lesson 11-3)

66. $\frac{1}{24}, \underline{?}, \underline{?}, \underline{?}, 54$

67. $-2, \underline{?}, \underline{?}, \underline{?}, \underline{?}, -\frac{243}{16}$

Find the sum of each arithmetic series. (Lesson 11-2)

68. $50 + 44 + 38 + \cdots + 8$

69. $\sum_{n=1}^{12} (2n + 3)$

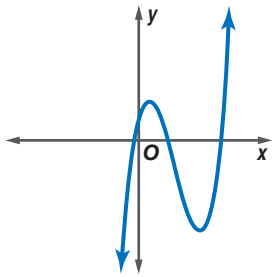
Solve each equation. Check your solutions. (Lesson 8-6)

70. $\frac{1}{y+1} - \frac{3}{y-3} = 2$

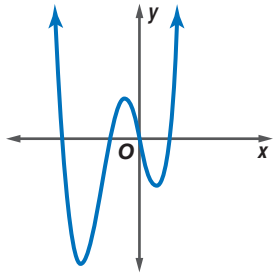
71. $\frac{6}{a-7} = \frac{a-49}{a^2-7a} + \frac{1}{a}$

Determine whether each graph represents an odd-degree polynomial function or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-4)

72.



73.



Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 5-3)

74. $3d^2 + 2d - 8$

75. $42pq - 35p + 18q - 15$

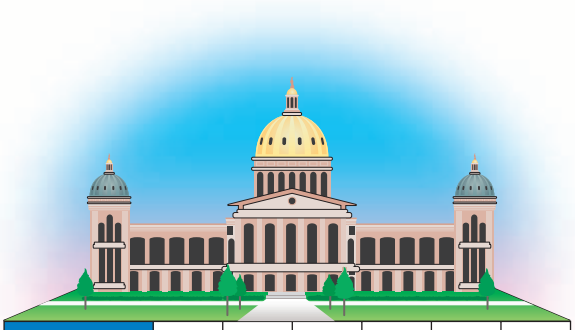
76. $13xyz + 3x^2z + 4k$

VOTING **For Exercises 77–79, use the table that shows the percent of the Iowa population of voting age that voted in each presidential election from 1984–2004.** (Lesson 2-5)

77. Draw a scatter plot in which x is the number of elections since the 1984 election.

78. Find a linear prediction equation.

79. Predict the percent of the Iowa voting age population that will vote in the 2012 election.



Year	1984	1988	1992	1996	2000	2004
Percent	76.0	75.7	72.4	80.2	73.2	75.0

Source: sos.state.ia.us

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $\frac{a}{1-b}$ for the given values of a and b.** (Lesson 1-1)

80. $a = 1, b = \frac{1}{2}$

81. $a = 3, b = -\frac{1}{2}$

82. $a = \frac{1}{3}, b = -\frac{1}{3}$

83. $a = \frac{1}{2}, b = \frac{1}{4}$

84. $a = -1, b = 0.5$

85. $a = 0.9, b = -0.5$

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11-5 Infinite Geometric Series

Main Ideas

- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

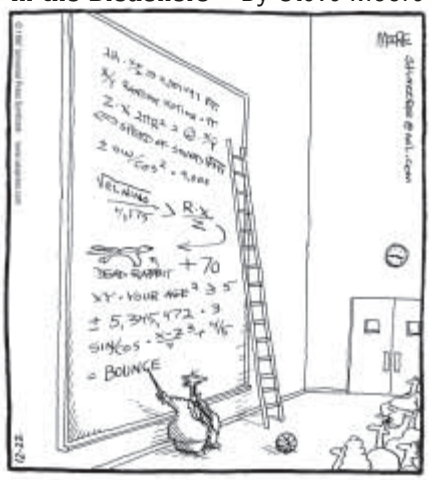
New Vocabulary

infinite geometric series
partial sum
convergent series

GET READY for the Lesson

Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball on page 636. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series.**

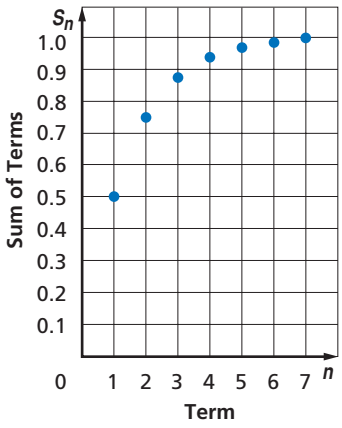
In the Bleachers By Steve Moore



"And that, ladies and gentlemen, is the way the ball bounces."

Infinite Geometric Series Consider the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$. You have already learned how to find the sum S_n of the first n terms of a geometric series. For an infinite series, S_n is called a **partial sum** of the series. The table and graph show some values of S_n.

n	S_n
1	$\frac{1}{2}$ or 0.5
2	$\frac{3}{4}$ or 0.75
3	$\frac{7}{8}$ or 0.875
4	$\frac{15}{16}$ or 0.9375
5	$\frac{31}{32}$ or 0.96875
6	$\frac{63}{64}$ or 0.984375
7	$\frac{127}{128}$ or 0.9921875



Study Tip

Absolute Value
Recall that $|r| < 1$ *means* $-1 < r < 1$.

Notice that as n increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which $|r| < 1$.

Let's look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.

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$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$ Sum of first n terms of a finite geometric series

$$= \frac{a_1}{1 - r} - \frac{a_1 r^n}{1 - r}$$ Write the fraction as a difference of fractions.

If $-1 < r < 1$, the value of r^n will approach 0 as n increases. Therefore, the partial sums of an infinite geometric series will approach $\frac{a_1}{1 - r} - \frac{a_1(0)}{1 - r}$ or $\frac{a_1}{1 - r}$. An infinite series that has a sum is called a **convergent series.**

Study Tip

Formula for Sum if $-1 < r < 1$

To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with $r = \frac{1}{2}$ and $a_1 = 100$.

Term Number	Term	Partial Sum
1	100	100
2	50	150
3	25	175
⋮	⋮	⋮
10		

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

KEY CONCEPT *Sum of an Infinite Geometric Series*

The sum S of an infinite geometric series with $-1 < r < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

An infinite geometric series for which $|r| \geq 1$ does not have a sum. Consider the series $1 + 3 + 9 + 27 + 81 + \ldots$. In this series, $a_1 = 1$ and $r = 3$. The table shows some of the partial sums of this series. As n increases, S_n rapidly increases and has no limit.

n	S_n
5	121
10	29,524
15	7,174,453
20	1,743,392,200

EXAMPLE Sum of an Infinite Geometric Series

1 **Find the sum of each infinite geometric series, if it exists.**

a. $\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots$

Step 1 Find the value of r to determine if the sum exists.

$a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{8}$, so $r = \frac{\frac{3}{8}}{\frac{1}{2}}$ or $\frac{3}{4}$.

Since $\left|\frac{3}{4}\right| < 1$, the sum exists.

Step 2 Use the formula for the sum of an infinite geometric series.

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{\frac{1}{2}}{1 - \frac{3}{4}}$ $a_1 = \frac{1}{2}, r = \frac{3}{4}$

$= \frac{\frac{1}{2}}{\frac{1}{4}}$ or 2 Simplify.

b. $1 - 2 + 4 - 8 + \ldots$

$a_1 = 1$ and $a_2 = -2$, so $r = \frac{-2}{1}$ or -2. Since $|-2| \geq 1$, the sum does not exist.

CHECK Your Progress

1A. $3 + 9 + 27 + 51 + \ldots$ **1B.** $-3 + \frac{1}{3} - \frac{1}{27} + \ldots$

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You can use sigma notation to represent infinite series. An *infinity symbol* ∞ is placed above the Σ to indicate that a series is infinite.

EXAMPLE Infinite Series in Sigma Notation

2 Evaluate $\sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1}$.

$S = \frac{a_1}{1-r}$ Sum formula

$= \frac{24}{1-\left(-\frac{1}{5}\right)}$ $a_1 = 24, r = -\frac{1}{5}$

$= \frac{24}{\frac{6}{5}}$ or 20 Simplify.

CHECK Your Progress

2. Evaluate $\sum_{n=1}^{\infty} 11\left(\frac{1}{3}\right)^{n-1}$.

Repeating Decimals The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction.

EXAMPLE Write a Repeating Decimal as a Fraction

3 Write $0.\overline{39}$ as a fraction.

Method 1

$0.\overline{39} = 0.393939\ldots$

$= 0.39 + 0.0039 + 0.000039 + \ldots$

$= \frac{39}{100} + \frac{39}{10{,}000} + \frac{39}{1{,}000{,}000} + \ldots$

$S = \frac{a_1}{1-r}$ Sum formula

$= \frac{\frac{39}{100}}{1-\frac{1}{100}}$ $a_1 = \frac{39}{100}, r = \frac{1}{100}$

$= \frac{\frac{39}{100}}{\frac{99}{100}}$ Subtract.

$= \frac{39}{99}$ or $\frac{13}{33}$ Simplify.

Method 2

$S = 0.\overline{39}$ Label the given decimal.

$S = 0.393939\ldots$ Repeating decimal

$100S = 39.393939\ldots$ Multiply each side by 100.

$99S = 39$ Subtract the second equation from the third.

$S = \frac{39}{99}$ or $\frac{13}{33}$ Divide each side by 99.

Study Tip

Bar Notation

Remember that decimals with bar notation such as $0.\overline{2}$ and $0.\overline{47}$ represent $0.222222\ldots$ and $0.474747\ldots$, respectively.

CHECK Your Progress

3. Write $0.\overline{47}$ as a fraction.

Extra Examples at algebra2.com

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CHECK Your Understanding

Example 1 (p. 651)

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 36, r = \frac{2}{3}$

2. $a_1 = 18, r = -1.5$

3. $16 + 24 + 36 + \cdots$

4. $\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \cdots$

5. CLOCKS Altovese's grandfather clock is broken. When she sets the pendulum in motion by holding it against the side of the clock and letting it go, it swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings before it stops?

Example 2 (p. 652)

Find the sum of each infinite geometric series, if it exists.

6. $\sum_{n=1}^{\infty} 6\,(-0.4)^{n-1}$

7. $\sum_{n=1}^{\infty} 40\left(\frac{3}{5}\right)^{n-1}$

8. $\sum_{n=1}^{\infty} 35\left(-\frac{3}{4}\right)^{n-1}$

9. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{8}\right)^{n-1}$

Example 3 (p. 652)

Write each repeating decimal as a fraction.

10. $0.\overline{5}$

11. $0.\overline{73}$

12. $0.\overline{175}$

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–22, 32–34	1
23–27	2
28–31	3

Find the sum of each infinite geometric series, if it exists.

13. $a_1 = 4, r = \frac{5}{7}$

14. $a_1 = 14, r = \frac{7}{3}$

15. $a_1 = 12, r = -0.6$

16. $a_1 = 18, r = 0.6$

17. $16 + 12 + 9 + \cdots$

18. $-8 - 4 - 2 - \cdots$

19. $12 - 18 + 24 - \cdots$

20. $18 - 12 + 8 - \cdots$

21. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$

22. $\frac{5}{3} + \frac{25}{3} + \frac{125}{3} + \cdots$

23. $\sum_{n=1}^{\infty} 48\left(\frac{2}{3}\right)^{n-1}$

24. $\sum_{n=1}^{\infty} \left(\frac{3}{8}\right)\left(\frac{3}{4}\right)^{n-1}$

25. $\sum_{n=1}^{\infty} \frac{1}{2}(3)^{n-1}$

26. $\sum_{n=1}^{\infty} 10{,}000\left(\frac{1}{101}\right)^{n-1}$

27. $\sum_{n=1}^{\infty} \frac{1}{100}\left(\frac{101}{99}\right)^{n-1}$

Write each repeating decimal as a fraction.

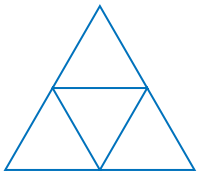
28. $0.\overline{7}$

29. $0.\overline{1}$

30. $0.\overline{36}$

31. $0.\overline{82}$

GEOMETRY For Exercises 32 and 33, refer to equilateral triangle *ABC*, which has a perimeter of 39 centimeters. If the midpoints of the sides are connected, a smaller equilateral triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely.



32. Write an infinite geometric series to represent the sum of the perimeters of all of the triangles.

33. Find the sum of the perimeters of all of the triangles.

EXTRA PRACTICE
See pages 915, 936.

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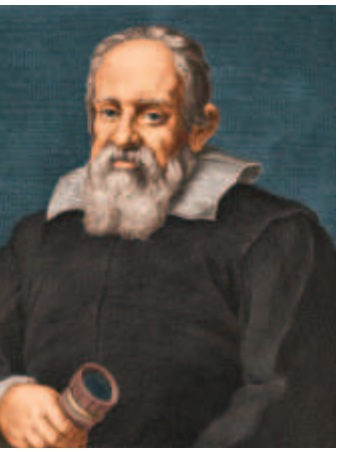

Real-World Link

Galileo Galilei performed experiments with wooden ramps and metal balls to study the physics of acceleration.

Source: galileoandeinstein.physics.virginia.edu

34. PHYSICS In a physics experiment, a steel ball on a flat track is accelerated and then allowed to roll freely. After the first minute, the ball has rolled 120 feet. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?

Find the sum of each infinite geometric series, if it exists.

35. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$

36. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$

37. $3 + 1.8 + 1.08 + \ldots$

38. $1 - 0.5 + 0.25 - \ldots$

39. $\sum_{n=1}^{\infty} 3(0.5)^{n-1}$

40. $\sum_{n=1}^{\infty} (1.5)(0.25)^{n-1}$

Write each repeating decimal as a fraction

41. $0.\overline{246}$

42. $0.\overline{427}$

43. $0.4\overline{5}$

44. $0.2\overline{31}$

45. SCIENCE MUSEUM An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulled the object down and let it go. The object traveled a distance of 1.2 feet upward before heading back the other way. Each time the object changed direction, it moved only 80% as far as it did in the previous direction. Find the total distance the object traveled.

46. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

47. The sum of an infinite geometric series is 125, and the value of r is 0.4. Find the first three terms of the series.

48. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

49. The first term of an infinite geometric series is -8, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

H.O.T. Problems

50. OPEN ENDED Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ using sigma notation in two different ways.

51. REASONING Explain why $0.999999\ldots = 1$.

52. FIND THE ERROR Conrado and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \cdots$. Conrado says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

Conrado

$$S = \frac{-\frac{1}{3}}{1 - \left(-\frac{4}{3}\right)}$$

$$= -\frac{1}{7}$$

53. CHALLENGE Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum S of a general infinite geometric series, multiply each side of the equation by r, and subtract equations.

54. *Writing in Math* Use the information on page 650 to explain how an infinite geometric series applies to a bouncing ball. Explain how to find the total distance traveled, both up and down, by the bouncing ball described on page 636.

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STANDARDIZED TEST PRACTICE

55. ACT/SAT What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

A 3

B 4

C 9

D 12

56. REVIEW What is the sum of the infinite geometric series $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$?

F $\frac{2}{3}$

G 1

H $1\frac{1}{3}$

J $1\frac{2}{3}$

Spiral Review

Find S_n for each geometric series described. (Lesson 11-4)

57. $a_1 = 1, a_6 = -243, r = -3$

58. $a_1 = 72, r = \frac{1}{3}, n = 7$

59. PHYSICS A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes? (Lesson 11-3)

Solve each equation or inequality. Check your solution. (Lesson 9-1)

60. $6^x = 216$

61. $2^{2x} = \frac{1}{8}$

62. $3^{x-2} \geq 27$

Simplify each expression. (Lesson 8-2)

63. $\frac{-2}{ab} + \frac{5}{a^2}$

64. $\frac{1}{x-3} - \frac{2}{x+1}$

65. $\frac{1}{x^2+6x+8} + \frac{3}{x+4}$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers. (Lesson 5-3)

66. 6, −6

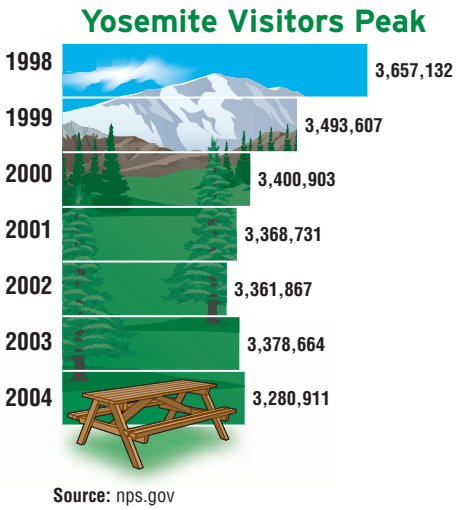
67. −2, −7

68. 6, 4

RECREATION **For Exercises 69 and 70, refer to the graph at the right.** (Lesson 2-3)

69. Find the average rate of change of the number of visitors to Yosemite National Park from 1998 to 2004.

70. Interpret your answer to Exercise 69.



Source: nps.gov

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each function value.** (Lesson 2-1)

71. $f(x) = 2x, f(1)$

72. $g(x) = 3x - 3, g(2)$

73. $h(x) = -2x + 2, h(0)$

74. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$

75. $g(x) = x^2, g(2)$

76. $h(x) = 2x^2 - 4, h(0)$

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CHAPTER 11

Mid-Chapter Quiz

Lessons 11-1 through 11-5

Find the indicated term of each arithmetic sequence. (Lesson 11-1)

1. $a_1 = 7, d = 3, n = 14$
2. $a_1 = 2, d = \frac{1}{2}, n = 8$

For Exercises 3 and 4, refer to the following information. (Lesson 11-1)

READING Amber makes a New Year's resolution to read 50 books by the end of the year.

3. By the end of February, Amber has read 9 books. If she reads 3 books each month for the rest of the year, will she meet her goal? Explain.
4. If Amber has read 10 books by the end of April, how many will she have to read on average each month in order to meet her goal?

5. **MULTIPLE CHOICE** The figures below show a pattern of filled squares and white squares that can be described by a relationship between 2 variables.

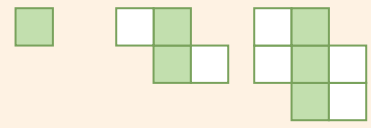


Figure 1 Figure 2 Figure 3

Which rule relates f, the number of filled squares, to w, the number of white squares? (Lesson 11-1)

A $w = f - 1$ C $f = \frac{1}{2}w - 1$

B $w = 2f - 2$ D $f = w - 1$

Find the sum of each arithmetic series described. (Lesson 11-2)

6. $a_1 = 5, a_n = 29, n = 11$
7. $6 + 12 + 18 + \cdots + 96$

8. **BANKING** Veronica has a savings account with $1500 dollars in it. At the end of each month, the balance in her account has increased by 0.25%. How much money will Veronica have in her savings account at the end of one year? (Lesson 11-3)

9. **GAMES** In order to help members of a group get to know each other, sometimes the group plays a game. The first person states his or her name and an interesting fact about himself or herself. The next person must repeat the first person's name and fact and then say his or her own. Each person must repeat the information for all those who preceded him or her. If there are 20 people in a group, what is the total number of times the names and facts will be stated? (Lesson 11-2)

10. Find a_7 for the geometric sequence 729, −243, 81, … . (Lesson 11-3)

Find the sum of each geometric series, if it exists. (Lessons 11-4 and 11-5)

11. $a_1 = 5, r = 3, n = 12$
12. $5 + 1 + \frac{1}{5} + \cdots$
13. $\sum_{n=1}^{6} 2(-3)^{n-1}$
14. $\sum_{n=1}^{\infty} 8\left(\frac{2}{3}\right)^{n-1}$
15. $\sum_{n=1}^{\infty} -13\left(\frac{1}{3}\right)^{n-1}$
16. $\sum_{n=1}^{\infty} \frac{1}{100}\left(\frac{10}{9}\right)^{n-1}$

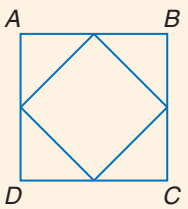
Write each repeating decimal as a fraction. (Lesson 11-5)

17. $0.\overline{17}$
18. $0.\overline{256}$
19. $1.\overline{27}$
20. $3.1\overline{5}$

GEOMETRY **For Exercises 21 and 22, refer to square $ABCD$, which has a perimeter of 120 inches.** (Lesson 11-5)

If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.

21. Write an infinite geometric series to represent the sum of the perimeters of all of the squares.
22. Find the sum of the perimeters of all of the squares.



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EXPLORE 11-6

Spreadsheet Lab

Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the *principal*, or original amount of the loan. This process is called *amortization*. You can use a spreadsheet to analyze the payments, interest, and balance on a loan.

EXAMPLE

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be $\frac{9\%}{12}$ or 0.75%. You can find the balance after a payment by multiplying the balance after the previous payment by 1 + 0.0075 or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

Loans

	A	B
1	Interest rate	=495*(1+A2)−A5
2	0.0075	=B1*(1+A2)−A5
3		=B2*(1+A2)−A5
4	Monthly Payment	=B3*(1+A2)−A5
5	43.29	=B4*(1+A2)−A5
6		=B5*(1+A2)−A5

Sheet 1 Sheet 2 Sheet 3

The spreadsheet shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.

Exercises

1. Let b_n be the balance left on Marisela's loan after n months. Write an equation relating b_n and b_{n+1}.
2. What percent of Marisela's loan remains to be paid after half a year?
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan?
5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?

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11-6 Recursion and Special Sequences

Main Ideas

- Recognize and use special sequences.
- Iterate functions.

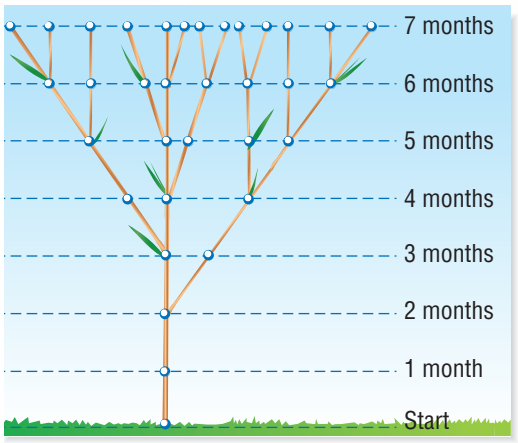
New Vocabulary

Fibonacci sequence
recursive formula
iteration

GET READY for the Lesson

A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Month	1	2	3	4	5
Shoots	1	1	2	3	5



Special Sequences Notice that each term in the sequence is the sum of the two previous terms. For example, $8 = 3 + 5$ and $13 = 5 + 8$. This sequence is called the **Fibonacci sequence**, and it is found in many places in nature.

first term	a_1		1
second term	a_2		1
third term	a_3	$a_1 + a_2$	$1 + 1 = 2$
fourth term	a_4	$a_2 + a_3$	$1 + 2 = 3$
⋮	⋮	⋮	
nth term	a_n	$a_{n-2} + a_{n-1}$	

The formula $a_n = a_{n-2} + a_{n-1}$ is an example of a **recursive formula**. This means that each term is formulated from one or more previous terms.

EXAMPLE 1 Use a Recursive Formula

Find the first five terms of the sequence in which $a_1 = 4$ and $a_{n+1} = 3a_n - 2$, $n \geq 1$.

$a_{n+1} = 3a_n - 2$ Recursive formula

$a_{1+1} = 3a_1 - 2$ $n = 1$

$a_2 = 3(4) - 2$ or 10 $a_1 = 4$

$a_{2+1} = 3a_2 - 2$ $n = 2$

$a_3 = 3(10) - 2$ or 28 $a_2 = 10$

$a_{3+1} = 3a_3 - 2$ $n = 3$

$a_4 = 3(28) - 2$ or 82 $a_3 = 28$

$a_{4+1} = 3a_4 - 2$ $n = 4$

$a_5 = 3(82) - 2$ or 244 $a_4 = 82$

The first five terms of the sequence are 4, 10, 28, 82, and 244.

CHECK Your Progress

1. Find the first five terms of the sequence in which $a_1 = -1$ and $a_{n+1} = 2a_n + 4$, $n \geq 1$.

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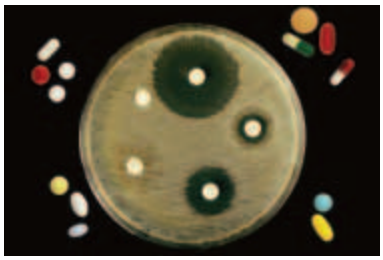
Real-World EXAMPLE 2 Find and Use a Recursive Formula

MEDICAL RESEARCH **A pharmaceutical company is experimenting with a new drug. An experiment begins with 1.0×10^9 bacteria. A dose of the drug that is administered every four hours can kill 4.0×10^8 bacteria. Between doses of the drug, the number of bacteria increases by 50%.**



Real-World Link

In 1928, Alexander Fleming found that penicillin mold could destroy certain types of bacteria. Production increases allowed the price of penicillin to fall from about $20 per dose in 1943 to $0.55 per dose in 1946.

Source: inventors.about.com

a. Write a recursive formula for the number of bacteria alive before each application of the drug.

Let b_n represent the number of bacteria alive just before the nth application of the drug. 4.0×10^8 of these will be killed by the drug, leaving $b_n - 4.0 \times 10^8$. The number b_{n+1} of bacteria before the next application will have increased by 50%. So $b_{n+1} = 1.5(b_n - 4.0 \times 10^8)$, or $1.5b_n - 6.0 \times 10^8$.

b. Find the number of bacteria alive before the fifth application.

Before the first application of the drug, there were 1.0×10^9 bacteria alive, so $b_1 = 1.0 \times 10^9$.

$b_{n+1} = 1.5b_n - 6.0 \times 10^8$ Recursive formula

$b_{1+1} = 1.5b_1 - 6.0 \times 10^8$ $n = 1$

$b_2 = 1.5(1.0 \times 10^9) - 6.0 \times 10^8$

$= 9.0 \times 10^8$

$b_{2+1} = 1.5b_2 - 6.0 \times 10^8$ $n = 2$

$b_3 = 1.5(9.0 \times 10^8) - 6.0 \times 10^8$

$= 7.5 \times 10^8$

$b_{3+1} = 1.5b_3 - 6.0 \times 10^8$ $n = 3$

$b_4 = 1.5(7.5 \times 10^8) - 6.0 \times 10^8$

$= 5.25 \times 10^8$

$b_{4+1} = 1.5b_4 - 6.0 \times 10^8$ $n = 4$

$b_5 = 1.5(5.25 \times 10^8) - 6.0 \times 10^8$

$= 1.875 \times 10^8$

Before the fifth dose, there would be 1.875×10^8 bacteria alive.

CHECK Your Progress

A stronger dose of the drug can kill 6.0×10^8 bacteria.

2A. Write a recursive formula for the number of bacteria alive before each dose of the drug.

2B. How many of the stronger doses of the drug will kill all the bacteria?

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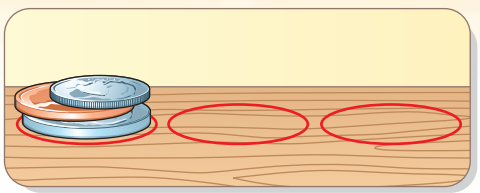
ALGEBRA LAB

Special Sequences

The object of the *Towers of Hanoi* game is to move a stack of n coins from one position to another in the fewest number a_n of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.



(continued on the next page)

Dr. Dennis Drenner/Getty Images

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MODEL AND ANALYZE

1. Draw three circles on a sheet of paper, as shown. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?
2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)
3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

MAKE A CONJECTURE

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number a_n of moves required to move a stack of n different sized coins.

Study Tip

Look Back

To review the **composition of functions**, see Lesson 7-5.

Iteration **Iteration** is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is $f \circ f(x)$ or $f(f(x))$. If you compose a function with itself two times, the result is $f \circ f \circ f(x)$ or $f(f(f(x)))$, and so on.

You can use iteration to recursively generate a sequence. Start with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1)$ or $f(f(x_0))$, $x_3 = f(x_2)$ or $f(f(f(x_0)))$, and so on.

EXAMPLE **Iterate a Function**

3 **Find the first three iterates x_1, x_2, and x_3 of the function $f(x) = 2x + 3$ for an initial value of $x_0 = 1$.**

$x_1 = f(x_0)$	Iterate the function.
$= f(1)$	$x_0 = 1$
$= 2(1) + 3$ or 5	Simplify.
$x_2 = f(x_1)$	Iterate the function.
$= f(5)$	$x_1 = 5$
$= 2(5) + 3$ or 13	Simplify.

$x_3 = f(x_2)$	Iterate the function.
$= f(13)$	$x_2 = 13$
$= 2(13) + 3$ or 29	Simplify.

The first three iterates are 5, 13, and 29.

CHECK Your Progress

3. Find the first four iterates, x_1, x_2, x_3, x_4, of the function $f(x) = x^2 - 2x - 1$ for an initial value of $x_0 = -1$.

CHECK Your Understanding

Example 1 (p. 658) **Find the first five terms of each sequence.**

1. $a_1 = 12, a_{n+1} = a_n - 3$
2. $a_1 = -3, a_{n+1} = a_n + n$
3. $a_1 = 0, a_{n+1} = -2a_n - 4$
4. $a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n$

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Example 2 (p. 659)

BANKING **For Exercises 5 and 6, use the following information.**
Rita has deposited $1000 in a bank account. At the end of each year, the bank posts 3% interest to her account, but then takes out a $10 annual fee.

5. Let b_0 be the amount Rita deposited. Write a recursive equation for the balance b_n in her account at the end of n years.

6. Find the balance in the account after four years.

Example 3 (p. 660)

Find the first three iterates of each function for the given initial value.

7. $f(x) = 3x - 4, x_0 = 3$ **8.** $f(x) = -2x + 5, x_0 = 2$ **9.** $f(x) = x^2 + 2, x_0 = -1$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–17	1
18–21	3
22–27	2

Find the first five terms of each sequence.

10. $a_1 = -6, a_{n+1} = a_n + 3$ **11.** $a_1 = 13, a_{n+1} = a_n + 5$

12. $a_1 = 2, a_{n+1} = a_n - n$ **13.** $a_1 = 6, a_{n+1} = a_n + n + 3$

14. $a_1 = 9, a_{n+1} = 2a_n - 4$ **15.** $a_1 = 4, a_{n+1} = 3a_n - 6$

16. If $a_0 = 7$ and $a_{n+1} = a_n + 12$ for $n \geq 0$, find the value of a_5.

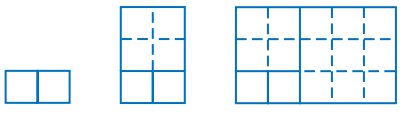
17. If $a_0 = 1$ and $a_{n+1} = -2.1$ for $n \geq 0$, then what is the value of a_4?

Find the first three iterates of each function for the given initial value.

18. $f(x) = 9x - 2, x_0 = 2$ **19.** $f(x) = 4x - 3, x_0 = 2$

20. $f(x) = 3x + 5, x_0 = -4$ **21.** $f(x) = 5x + 1, x_0 = -1$

GEOMETRY **For Exercises 22–24, use the following information.**
Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process.



22. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with two original squares.

23. Write a recursive formula for the sequence of lengths added.

24. Identify the sequence in Exercise 23.

GEOMETRY **For Exercises 25–27, study the triangular numbers shown below.**

25. Write a sequence of the first five triangular numbers.

26. Write a recursive formula for the nth triangular number t_n.

27. What is the 200th triangular number?

28. LOANS Miguel's monthly car payment is $234.85. The recursive formula $b_n = 1.005b_{n-1} - 234.85$ describes the balance left on the loan after n payments. Find the balance of the $10,000 loan after each of the first eight payments.

Real-World Career
Loan Officer
Loan officers help customers through the loan application process. Their work may require frequent travel.

Math Online
For more information, go to algebra2.com.

David Young-Wolff/PhotoEdit
www.MathSchoolinternational.com

EXTRA PRACTICE
See pages 915, 936.

Math Online
Self-Check Quiz at algebra2.com

29. ECONOMICS If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function $c(x) = 1.02x$. Find the cost of a \$70 MP3 player in four years if the rate of inflation remains constant.

Find the first three iterates of each function for the given initial value.

30. $f(x) = 2x^2 - 5, x_0 = -1$

31. $f(x) = 3x^2 - 4, x_0 = 1$

32. $f(x) = 2x^2 + 2x + 1, x_0 = \frac{1}{2}$

33. $f(x) = 3x^2 - 3x + 2, x_0 = \frac{1}{3}$

H.O.T. Problems

34. OPEN ENDED Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.

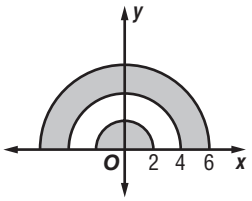
35. REASONING Is the statement $x_n \neq x_{n-1}$ *sometimes*, *always*, or *never* true if $x_n = f(x_{n-1})$? Explain.

36. CHALLENGE Are there a function $f(x)$ and an initial value x_0 such that the first three iterates, in order, are 4, 4, and 7? Explain.

37. *Writing in Math* Use the information on page 658 to explain how the Fibonacci sequence is illustrated in nature. Include the 13th term in the sequence, with an explanation of what it tells you about the plant described.

STANDARDIZED TEST PRACTICE

38. ACT/SAT The figure is made of three concentric semicircles. What is the total area of the shaded regions?



A 4π units^2

B 10π units^2

C 12π units^2

D 20π units^2

39. REVIEW If x is a real number, for what values of x is the equation $\frac{4x - 16}{4} = x - 4$ true?

F all values of x

G some values of x

H no values of x

J impossible to determine

Spiral Review

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

40. $9 + 6 + 4 + \cdots$

41. $\frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$

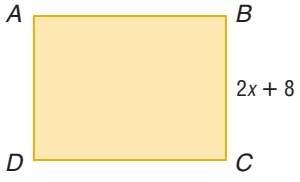
42. $4 - \frac{8}{3} + \frac{16}{9} + \cdots$

Find the sum of each geometric series. (Lesson 11-4)

43. $2 - 10 + 50 - \cdots$ to 6 terms

44. $3 + 1 + \frac{1}{3} + \cdots$ to 7 terms

45. GEOMETRY The area of rectangle $ABCD$ is $6x^2 + 38x + 56$ square units. Its width is $2x + 8$ units. What is the length of the rectangle? (Lesson 6-3)



GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.**

46. $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

47. $\frac{4 \cdot 3}{2 \cdot 1}$

48. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1}$

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EXTEND 11-6

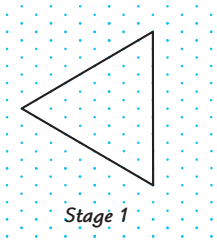
Algebra Lab
Fractals

Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

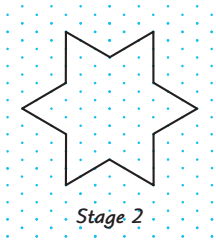
You can use isometric dot paper to draw stages of the construction of a fractal called the *von Koch snowflake*.

ACTIVITY

Stage 1 Draw an equilateral triangle with sides of length 9 units on the dot paper.



Stage 2 Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.



Imagine continuing this process infinitely. The von Koch snowflake is the shape that these stages approach.

Model and Analyze the Results

1. Copy and complete the table. Draw stage 3, if necessary.

Stage	1	2	3	4
Number of Segments	3	8		
Length of each Segment	9	3		
Perimeter	27	36		

2. Write recursive formulas for the number s_n of segments in Stage n, the length ℓ_n of each segment in Stage n, and the perimeter P_n of Stage n.

3. Write nonrecursive formulas for s_n, ℓ_n, and P_n.

4. What is the perimeter of the von Koch snowflake? Explain.

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$.

6. Find the sum of the series in Exercise 5. Explain your steps.

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain.

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11-7 The Binomial Theorem

Main Ideas

- Use Pascal's triangle to expand powers of binomials.
- Use the Binomial Theorem to expand powers of binomials.

New Vocabulary

Pascal's triangle
Binomial Theorem
factorial

GET READY for the Lesson

According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG BGBG BGGB GBBG GBGB GGBB

Pascal's Triangle You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Expand a few powers of the binomial $b + g$.

$$(b+g)^0 = 1b^0g^0$$
$$(b+g)^1 = 1b^1g^0 + 1b^0g^1$$
$$(b+g)^2 = 1b^2g^0 + 2b^1g^1 + 1b^0g^2$$
$$(b+g)^3 = 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3$$
$$(b+g)^4 = 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4$$

The coefficient 4 of the b^1g^3 term in the expansion of $(b + g)^4$ gives the number of sequences of births that result in one boy and three girls.

Here are some patterns in any binomial expansion of the form $(a + b)^n$.

1. There are $n + 1$ terms.
2. The exponent n of $(a + b)^n$ is the exponent of a in the first term and the exponent of b in the last term.
3. In successive terms, the exponent of a decreases by one, and the exponent of b increases by one.
4. The sum of the exponents in each term is n.
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal's triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.

$(a+b)^0$ 1
$(a+b)^1$ 1 1
$(a+b)^2$ 1 2 1
$(a+b)^3$ 1 3 3 1
$(a+b)^4$ 1 4 6 4 1
$(a+b)^5$ 1 5 10 10 5 1

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Real-World Link
Although he did not discover it, Pascal's triangle is named for the French mathematician Blaise Pascal (1623–1662).

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EXAMPLE 1 Use Pascal's Triangle

Expand $(x + y)^7$.

Write two more rows of Pascal's triangle. Then use the patterns of a binomial expansion and the coefficients to write the expansion.

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$(x + y)^7 = 1x^7y^0 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7x^1y^6 + 1x^0y^7$$
$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

CHECK Your Progress

1. Expand $(c + d)^8$.

The Binomial Theorem Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

$(a + b)^0$				1			
$(a + b)^1$			1		$\frac{1}{1}$		
$(a + b)^2$		1		$\frac{2}{1}$		$\frac{2 \cdot 1}{1 \cdot 2}$	
$(a + b)^3$	1		$\frac{3}{1}$		$\frac{3 \cdot 2}{1 \cdot 2}$		$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}$
$(a + b)^4$	1	$\frac{4}{1}$		$\frac{4 \cdot 3}{1 \cdot 2}$		$\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}$		$\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$

Eliminate common factors that are shown in color.

Study Tip

Terms

The expansion of a binomial to the nth power has $n + 1$ terms. For example, $(a - b)^6$ has 7 terms.

This pattern is summarized in the **Binomial Theorem**.

KEY CONCEPT *Binomial Theorem*

If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$.

EXAMPLE 2 Use the Binomial Theorem

Study Tip

Coefficients

Notice that in terms having the same coefficients, the exponents are reversed, as in $15a^4b^2$ and $15a^2b^4$.

Expand $(a - b)^6$.

Use the sequence $1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

$$(a - b)^6 = 1a^6(-b)^0 + \frac{6}{1}a^5(-b)^1 + \frac{6 \cdot 5}{1 \cdot 2}a^4(-b)^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3(-b)^3 + \ldots$$
$$+ 1a^0(-b)^6$$
$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

CHECK Your Progress

2. Expand $(w + z)^5$.

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Study Tip

Graphing Calculators

On a TI-83/84 Plus, the factorial symbol, !, is located on the MATH PRB menu.

The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written 4! and is read *4 factorial*. In general, if n is a positive integer, then $n! = n(n-1)(n-2)(n-3) \ldots 2 \cdot 1$. *By definition, $0! = 1$.*

EXAMPLE Factorials

3 **Evaluate $\frac{8!}{3!5!}$.**

$$\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot \overset{1}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}}{3 \cdot 2 \cdot 1 \cdot \underset{1}{\cancel{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}}$$ Note that $8! = 8 \cdot 7 \cdot 6 \cdot 5!$, so $\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!}$ or $\frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$.

$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} \text{ or } 56$$

CHECK Your Progress

3. Evaluate $\frac{12!}{8!4!}$.

Study Tip

Missing Steps

If you don't understand a step like $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6!}{3!3!}$, work it out on a piece of scrap paper.

$$\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{1 \cdot 2 \cdot 3 \cdot 3!}$$

$$= \frac{6!}{3!3!}$$

The Binomial Theorem can be written in factorial notation and in sigma notation.

KEY CONCEPT *Binomial Theorem, Factorial Form*

$$(a+b)^n = \frac{n!}{n!0!}a^n b^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0 b^n$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}b^k$$

EXAMPLE Use a Factorial Form of the Binomial Theorem

4 **Expand $(2x+y)^5$.**

$$(2x+y)^5 = \sum_{k=0}^{5} \frac{5!}{(5-k)!k!}(2x)^{5-k}y^k$$ Binomial Theorem, factorial form

$$= \frac{5!}{5!0!}(2x)^5 y^0 + \frac{5!}{4!1!}(2x)^4 y^1 + \frac{5!}{3!2!}(2x)^3 y^2 + \frac{5!}{2!3!}(2x)^2 y^3 + \frac{5!}{1!4!}(2x)^1 y^4 + \frac{5!}{0!5!}(2x)^0 y^5$$ Let $k = 0, 1, 2, 3, 4$, and 5.

$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}(2x)^4 y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}(2x)^3 y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1}(2x)^2 y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1}(2x)y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}y^5$$

$$= 32x^5 + 80x^4 y + 80x^3 y^2 + 40x^2 y^3 + 10xy^4 + y^5$$ Simplify.

CHECK Your Progress

4. Expand $(q-3r)^4$.

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Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of k is always one less than the number of the term you are finding.

EXAMPLE 5 Find a Particular Term

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10 - k)!k!} p^{10-k}q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10 - k)!k!} p^{10-k}q^k = \frac{10!}{(10 - 4)!4!} p^{10-4}q^4 \qquad k = 4$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} p^6q^4 \qquad \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6!4!} \text{ or } \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 210p^6q^4 \qquad \text{Simplify.}$$

CHECK Your Progress

5. Find the eighth term in the expansion of $(x - y)^{12}$.

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CHECK Your Understanding

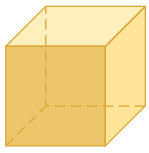
Examples 1, 2, 4 (pp. 665, 666)

Expand each power.

1. $(p + q)^5$ **2.** $(t + 2)^6$ **3.** $(x - 3y)^4$

Examples 2, 4 (pp. 665, 666)

4. GEOMETRY Write an expanded expression for the volume of the cube at the right.



Example 3 (p. 666)

Evaluate each expression.

5. 8! **6.** 10!

7. $\frac{13!}{9!}$ **8.** $\frac{12!}{2!10!}$

Example 5 (p. 667)

Find the indicated term of each expansion.

9. fourth term of $(a + b)^8$ **10.** fifth term of $(2a + 3b)^{10}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–16	1, 2, 4
17–20	3
21–26	5

Expand each power.

11. $(a - b)^3$ **12.** $(m + n)^4$ **13.** $(r + s)^8$

14. $(m - a)^5$ **15.** $(x + 3)^5$ **16.** $(a - 2)^4$

Evaluate each expression.

17. 9! **18.** 13! **19.** $\frac{9!}{7!}$ **20.** $\frac{7!}{4!}$

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Find the indicated term of each expansion.

21. sixth term of $(x - y)^9$

22. seventh term of $(x + y)^{12}$

23. fourth term of $(x + 2)^7$

24. fifth term of $(a - 3)^8$

25. SCHOOL Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

Cross-Curricular Project

Math Online Pascal's triangle displays many patterns. Visit algebra2.com to continue work on your project.

26. INTRAMURALS Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Expand each power.

27. $(2b - x)^4$

28. $(2a + b)^6$

29. $(3x - 2y)^5$

30. $(3x + 2y)^4$

31. $\left(\frac{a}{2} + 2\right)^5$

32. $\left(3 + \frac{m}{3}\right)^5$

Evaluate each expression.

33. $\frac{12!}{8!4!}$

34. $\frac{14!}{5!9!}$

Find the indicated term of each expansion.

35. fifth term of $(2a + 3b)^{10}$

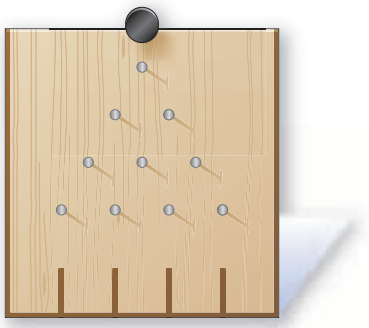
36. fourth term of $(2x + 3y)^9$

37. fourth term of $\left(x + \frac{1}{3}\right)^7$

38. sixth term of $\left(x - \frac{1}{2}\right)^{10}$

39. GENETICS The color of a particular flower may be either red, white, or pink. If the flower has two red alleles R, the flower is red. If the flower has two white alleles w, the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?

40. GAMES The diagram shows the board for a game in which disks are dropped down a chute. A pattern of nails and dividers causes the disks to take various paths to the sections at the bottom. How many paths through the board lead to each bottom section?



EXTRA PRACTICE

See pages 916, 936.

Math Online Self-Check Quiz at algebra2.com

H.O.T. Problems

41. OPEN ENDED Write a power of a binomial for which the first term of the expansion is $625x^4$.

42. CHALLENGE Explain why $\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}$ without finding the value of any of the expressions.

43. *Writing in Math* Use the information on page 664 to explain how the power of a binomial describes the number of boys and girls in a family. Include the expansion of $(b + g)^5$ and an explanation of what it tells you about sequences of births of boys and girls in families with five children.

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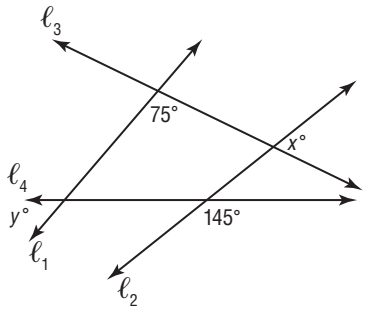
STANDARDIZED TEST PRACTICE

44. ACT/SAT If four lines intersect as shown, what is the value of $x + y$?



A 70 B 115 C 140 D 220

45. REVIEW $(2x - 2)^4 =$

F $16x^4 + 64x^3 - 96x^2 - 64x + 16$

G $16x^4 - 32x^3 - 192x^2 - 64x + 16$

H $16x^4 - 64x^3 + 96x^2 - 64x + 16$

J $16x^4 + 32x^3 - 192x^2 - 64x + 16$

Spiral Review

Find the first five terms of each sequence. (Lesson 11-6)

46. $a_1 = 7, a_{n+1} = a_n - 2$

47. $a_1 = 3, a_{n+1} = 2a_n - 1$

48. MINIATURE GOLF A wooden pole swings back and forth over the cup on a miniature golf hole. One player pulls the pole to the side and lets it go. Then it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pole swings before coming to rest? (Lesson 11-5)

Without writing the equation in standard form, state whether the graph of each equation is a *parabola*, *circle*, *ellipse*, or *hyperbola*. (Lesson 10-6)

49. $x^2 - 6x - y^2 - 3 = 0$

50. $4y - x + y^2 = 1$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

51. $\log_2 5$

52. $\log_3 10$

53. $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 8-3)

54. $f(x) = \dfrac{1}{x^2 + 5x + 6}$

55. $f(x) = \dfrac{x + 2}{x^2 + 3x - 4}$

56. $f(x) = \dfrac{x^2 + 4x + 3}{x + 3}$

GET READY for the Next Lesson

PREREQUISITE SKILL **State whether each statement is *true* or *false* when $n = 1$. Explain.** (Lesson 1-1)

57. $1 = \dfrac{n(n+1)}{2}$

58. $1 = \dfrac{(n+1)(2n+1)}{2}$

59. $1 = \dfrac{n^2(n+1)^2}{4}$

60. $3^n - 1$ is even.

61. $7^n - 3^n$ is divisible by 4.

62. $2^n - 1$ is prime.

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11-8 Proof and Mathematical Induction

Main Ideas

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

New Vocabulary

mathematical induction
inductive hypothesis

GET READY for the Lesson

Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?

Mathematical Induction

Mathematical induction is used to prove statements about positive integers. This proof uses three steps.

KEY CONCEPT *Mathematical Induction*

Step 1 Show that the statement is true for some positive integer n.

Step 2 Assume that the statement is true for some positive integer k, where $k \geq n$. This assumption is called the **inductive hypothesis.**

Step 3 Show that the statement is true for the next positive integer $k + 1$. If so, we can assume that the statement is true for any positive integer.

EXAMPLE **Summation Formula**

Study Tip

Step 1
In many cases, it will be helpful to let $n = 1$.

1 Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Step 1 When $n = 1$, the left side of the given equation is 1^2 or 1. The right side is $\dfrac{1(1+1)[2(1)+1]}{6}$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume $1^2 + 2^2 + 3^2 + \cdots + k^2 = \dfrac{k(k+1)(2k+1)}{6}$ for a positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$$

$= \dfrac{k(k+1)(2k+1)}{6} + (k+1)^2$ Add $(k+1)^2$ to each side.

$= \dfrac{k(k+1)(2k+1) + 6(k+1)^2}{6}$ Add.

$= \dfrac{(k+1)[k(2k+1) + 6(k+1)]}{6}$ Factor.

$= \dfrac{(k+1)[2k^2 + 7k + 6]}{6}$ Simplify.

$= \dfrac{(k+1)(k+2)(2k+3)}{6}$ Factor.

$= \dfrac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$

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The last expression is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus, the equation is true for $n = k + 1$. This proves the conjecture.

CHECK Your Progress

1. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

EXAMPLE 2 Divisibility

Prove that $7^n - 1$ is divisible by 6 for all positive integers n.

Step 1 When $n = 1$, $7^n - 1 = 7^1 - 1$ or 6. Since 6 is divisible by 6, the statement is true for $n = 1$.

Step 2 Assume that $7^k - 1$ is divisible by 6 for some positive integer k. This means that there is a whole number r such that $7^k - 1 = 6r$.

Step 3 Show that the statement is true for $n = k + 1$.

$7^k - 1$	$= 6r$	Inductive hypothesis
7^k	$= 6r + 1$	Add 1 to each side.
$7(7^k)$	$= 7(6r + 1)$	Multiply each side by 7.
7^{k+1}	$= 42r + 7$	Simplify.
$7^{k+1} - 1$	$= 42r + 6$	Subtract 1 from each side.
$7^{k+1} - 1$	$= 6(7r + 1)$	Factor.

Since r is a whole number, $7r + 1$ is a whole number. Therefore, $7^{k+1} - 1$ is divisible by 6. Thus, the statement is true for $n = k + 1$.

This proves that $7^n - 1$ is divisible by 6 for all positive integers n.

CHECK Your Progress

2. Prove that $10^n - 1$ is divisible by 9 for all positive integers n.

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Review Vocabulary

Counterexample
a specific case that shows that a statement is false (Lesson 1-2)

Counterexamples Of course, not every equation that you can write is true. You can show that an equation is not always true by finding a *counterexample*.

EXAMPLE 3 Counterexample

Find a counterexample for $1^4 + 2^4 + 3^4 + \cdots + n^4 = 1 + (4n - 4)^2$.

n	Left Side of Formula	Right Side of Formula	
1	1^4 or 1	$1 + [4(1) - 4]^2 = 1 + 0^2$ or 1	true
2	$1^4 + 2^4 = 1 + 16$ or 17	$1 + [4(2) - 4]^2 = 1 + 4^2$ or 17	true
3	$1^4 + 2^4 + 3^4 = 1 + 16 + 81$ or 98	$1 + [4(3) - 4]^2 = 1 + 64$ or 65	false

The value $n = 3$ is a counterexample for the equation.

CHECK Your Progress

3. Find a counterexample for the statement that $2n^2 + 11$ is prime for all positive integers n.

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CHECK Your Understanding

Example 1 (pp. 670–671)

Prove that each statement is true for all positive integers.

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

2. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

3. PARTIES Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after n guests have arrived, a total of $\dfrac{n(n-1)}{2}$ handshakes have taken place.

Example 2 (p. 671)

Prove that each statement is true for all positive integers.

4. $4^n - 1$ is divisible by 3.

5. $5^n + 3$ is divisible by 4.

Example 3 (p. 672)

Find a counterexample for each statement.

6. $1 + 2 + 3 + \cdots + n = n^2$

7. $2^n + 3^n$ is divisible by 4.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–11	1
12, 13	2
14, 15	1, 2
16–21	3

Prove that each statement is true for all positive integers.

8. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

9. $2 + 5 + 8 + \cdots + (3n - 1) = \dfrac{n(3n+1)}{2}$

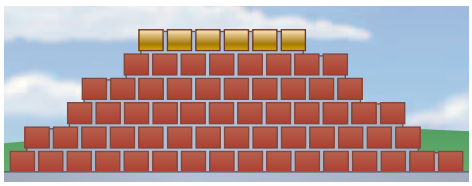
10. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

11. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

12. $8^n - 1$ is divisible by 7.

13. $9^n - 1$ is divisible by 8.

14. ARCHITECTURE A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top n rows is $n^2 + 5n$.



15. NATURE The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers are Fibonacci numbers, as are the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If f_n stands for the nth Fibonacci number, prove that $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

Find a counterexample for each statement.

16. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(3n-1)}{2}$

17. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 12n^3 - 23n^2 + 12n$

18. $3^n + 1$ is divisible by 4.

19. $2^n + 2n^2$ is divisible by 4.

20. $n^2 - n + 11$ is prime.

21. $n^2 + n + 41$ is prime.

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Prove that each statement is true for all positive integers.

22. $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$

23. $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

24. $12^n + 10$ is divisible by 11.

25. $13^n + 11$ is divisible by 12.

26. ARITHMETIC SERIES Use mathematical induction to prove the formula $a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] = \frac{n}{2}[2a_1 + (n-1)d]$ for the sum of an arithmetic series.

27. GEOMETRIC SERIES Use mathematical induction to prove the formula $a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} = \frac{a_1(1 - r^n)}{1 - r}$ for the sum of a finite geometric series.

EXTRA PRACTICE
See pages 916, 936.
MathOnline
Self-Check Quiz at algebra2.com

28. PUZZLES Show that a 2^n by 2^n checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.

H.O.T. Problems

29. OPEN ENDED Write an expression of the form $b^n - 1$ that is divisible by 2 for all positive integers n.

30. CHALLENGE Refer to Example 2. Explain how to use the Binomial Theorem to show that $7^n - 1$ is divisible by 6 for all positive integers n.

31. *Writing in Math* Use the information on page 670 to explain how the concept of a ladder can help you prove statements about numbers.

STANDARDIZED TEST PRACTICE

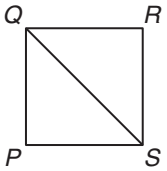
32. ACT/SAT $PQRS$ is a square. What is the ratio of the length of diagonal $\overline{QS}$ to the length of side $\overline{RS}$?

A 2

B $\sqrt{2}$

C 1

D $\frac{\sqrt{2}}{2}$



33. REVIEW The lengths of the bases of an isosceles trapezoid are 15 centimeters and 29 centimeters. If the perimeter of this trapezoid is 94 centimeters, what is the area?

F 500 cm^2

G 515 cm^2

H 528 cm^2

J 550 cm^2

Spiral Review

Expand each power. (Lesson 11-7)

34. $(x + y)^6$

35. $(a - b)^7$

36. $(2x + y)^8$

Find the first three iterates of each function for the given initial value. (Lesson 11-6)

37. $f(x) = 3x - 2, x_0 = 2$

38. $f(x) = 4x^2 - 2, x_0 = 1$

39. BIOLOGY Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? (Lesson 9-2)

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CHAPTER 11

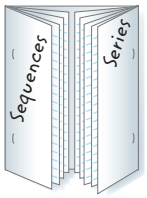
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Arithmetic Sequences and Series (Lessons 11-1 and 11-2)

- The nth term a_n of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$, where n is any positive integer.
- The sum S_n of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$.

Geometric Sequences and Series (Lessons 11-3 to 11-5)

- The nth term a_n of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$, where n is any positive integer.
- The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.
- The sum S of an infinite geometric series with $-1 < r < 1$ is given by $S = \frac{a_1}{1 - r}$.

Recursion and Special Sequences (Lesson 11-6)

- In a recursive formula, each term is formulated from one or more previous terms.

The Binomial Theorem (Lesson 11-7)

- The Binomial Theorem:

$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{(n - k)!k!} a^{n-k} b^k$$

Mathematical Induction (Lesson 11-8)

- Mathematical induction is a method of proof used to prove statements about the positive integers.

Key Vocabulary

arithmetic means (p. 624)
arithmetic sequence (p. 622)
arithmetic series (p. 629)
Binomial Theorem (p. 665)
common difference (p. 622)
common ratio (p. 636)
convergent series (p. 651)
factorial (p. 666)
Fibonacci sequence (p. 658)
geometric means (p. 638)
geometric sequence (p. 636)
geometric series (p. 643)
index of summation (p. 631)
inductive hypothesis (p. 670)
infinite geometric series (p. 650)
iteration (p. 660)
mathematical induction (p. 670)
partial sum (p. 650)
Pascal's triangle (p. 664)
recursive formula (p. 658)
sequence (p. 622)
series (p. 629)
sigma notation (p. 631)
term (p. 622)

Vocabulary Check

Choose the term from the list above that best completes each statement.

1. A(n) _____ of an infinite series is the sum of a certain number of terms.
2. If a sequence has a common ratio, then it is a(n) _____ .
3. Using _____ , the series $2 + 5 + 8 + 11 + 14$ can be written as $\sum_{n=1}^{5} (3n - 1)$.
4. Eleven and 17 are two _____ between 5 and 23 in the sequence 5, 11, 17, 23.
5. Using the _____ , $(a - 2)^4$ can be expanded to $a^4 - 8a^3 + 24a^2 - 32a + 16$.
6. The _____ of the sequence 3, 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$ is $\frac{2}{3}$.
7. The _____ $11 + 16.5 + 22 + 27.5 + 33$ has a sum of 110.
8. A(n) _____ is expressed as $n! = n(n - 1)(n - 2) \ldots 2 \cdot 1$.

Vocabulary Review at algebra2.com

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Lesson-by-Lesson Review

11–1 Arithmetic Sequences (pp. 622-628)

Find the indicated term of each arithmetic sequence.

9. $a_1 = 6, d = 8, n = 5$

10. $a_1 = -5, d = 7, n = 22$

11. $a_1 = 5, d = -2, n = 9$

12. $a_1 = -2, d = -3, n = 15$

Find the arithmetic means in each sequence.

13. $-7, ___, ___, ___, 9$

14. $12, ___, ___, 4$

15. $9, ___, ___, ___, ___, -6$

16. $56, ___, ___, ___, 28$

17. GLACIERS The fastest glacier is recorded to have moved 12 kilometers every three months. If the glacier moved at a constant speed, how many kilometers did it move in one year?

Example 1 **Find the 12th term of an arithmetic sequence if $a_1 = -17$ and $d = 4$.**

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$a_{12} = -17 + (12 - 1)4$ $n = 12, a_1 = -17, d = 4$

$a_{12} = 27$ Simplify.

Example 2 **Find the two arithmetic means between 4 and 25.**

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$a_4 = 4 + (4 - 1)d$ $n = 4, a_1 = 4$

$25 = 4 + 3d$ $a_4 = 25$

$7 = d$ Simplify.

The arithmetic means are $4 + 7$ or 11 and $11 + 7$ or 18.

11–2 Arithmetic Series (pp. 629-635)

Find S_n for each arithmetic series.

18. $a_1 = 12, a_n = 117, n = 36$

19. $4 + 10 + 16 + \ldots + 106$

20. $10 + 4 + (-2) + \ldots + (-50)$

21. Evaluate $\sum_{n=2}^{13} (3n + 1)$.

22. PATTERNS On the first night of a celebration, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and all the candles from the previous nights is continued for seven nights. Find the total number of candle lightings.

Example 3 **Find S_n for the arithmetic series with $a_1 = 34$, $a_n = 2$, and $n = 9$.**

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_9 = \frac{9}{2}(34 + 2)$ $n = 9, a_1 = 34, a_n = 2$

$= 162$ Simplify.

Example 4 **Evaluate $\sum_{n=5}^{11} (2n - 3)$.**

Use the formula $S_n = \frac{n}{2}(a_1 + a_n)$. There are 7 terms, $a_1 = 2(5) - 3$ or 7, and $a_7 = 2(11) - 3$ or 19.

$S_7 = \frac{7}{2}(19 + 7)$

$= 91$

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CHAPTER 11

Study Guide and Review

11–3 Geometric Sequences (pp. 636-641)

Find the indicated term of each geometric sequence.

23. $a_1 = 2, r = 2, n = 5$

24. $a_1 = 7, r = 2, n = 4$

25. $a_1 = 243, r = -\frac{1}{3}, n = 5$

26. a_6 for $\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

Find the geometric means in each sequence.

27. 3, ____ , ____ , 24

28. 7.5, ____ , ____ , ____ , 120

29. SAVINGS Kathy has a savings account with a current balance of $5000. What would Kathy's account balance be after five years if she receives 3% interest annually?

Example 5 Find the fifth term of a geometric sequence for which $a_1 = 7$ and $r = 3$.

$a_n = a_1 \cdot r^{n-1}$ Formula for the nth term

$a_5 = 7 \cdot 3^{5-1}$ $n = 5, a_1 = 7, r = 3$.

$a_5 = 567$ The fifth term is 567.

Example 6 Find two geometric means between 1 and 8.

$a_n = a_1 \cdot r^{n-1}$ Formula for the nth term

$a_4 = 1 \cdot r^{4-1}$ $n = 4$ and $a_1 = 1$

$8 = r^3$ $a_4 = 8$

$2 = r$ Simplify.

The geometric means are 1(2) or 2 and 2(2) or 4.

11–4 Geometric Series (pp. 643-649)

Find S_n for each geometric series.

30. $a_1 = 12, r = 3, n = 5$

31. $4 - 2 + 1 - \ldots$ to 6 terms

32. $256 + 192 + 144 + \ldots$ to 7 terms

33. Evaluate $\sum_{n=1}^{5} \left(-\frac{1}{2}\right)^{n-1}$.

34. TELEPHONES Joe started a phone tree to give information about a party to his friends. Joe starts by calling 3 people. Then each of those 3 people calls 3 people, and each person who receives a call then calls 3 more people. How many people have been called after 4 layers of the phone tree? (*Hint:* Joe is considered the first layer.)

Example 7 Find the sum of a geometric series for which $a_1 = 7$, $r = 3$, and $n = 14$.

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ Sum formula

$S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3}$ $n = 14, a_1 = 7, r = 3$

$S_{14} = 16{,}740{,}388$ Use a calculator.

Example 8 Evaluate $\sum_{n=1}^{5} \left(\frac{3}{4}\right)^{n-1}$.

$S_5 = \frac{1\left[1 - \left(\frac{3}{4}\right)^5\right]}{1 - \frac{3}{4}}$ $n = 5, a_1 = 1, r = \frac{3}{4}$

$= \frac{\frac{781}{1024}}{\frac{1}{4}}$ $\frac{3^5}{4} = \frac{243}{1024}$

$= \frac{781}{256}$

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Mixed Problem Solving
For mixed problem-solving practice, see page 936.

11–5 Infinite Geometric Series (pp. 650-655)

Find the sum of each infinite geometric series, if it exists.

35. $a_1 = 6, r = \frac{11}{12}$

36. $\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \cdots$

37. $\sum_{n=1}^{\infty} -2\left(-\frac{5}{8}\right)^{n-1}$

38. GEOMETRY If the midpoints of the sides of $\triangle ABC$ are connected, a smaller triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely. Find the sum of the perimeters of all of the triangles if the perimeter of $\triangle ABC$ is 30 centimeters.

Example 9 **Find the sum of the infinite geometric series for which $a_1 = 18$ and $r = -\frac{2}{7}$.**

$S = \frac{a_1}{1 - r}$ Sum formula

$= \frac{18}{1 - \left(-\frac{2}{7}\right)}$ $a_1 = 18, r = -\frac{2}{7}$

$= \frac{18}{\frac{9}{7}}$ or 14 Simplify.

11–6 Recursion and Special Sequences (pp. 658-662)

Find the first five terms of each sequence.

39. $a_1 = -2, a_{n+1} = a_n + 5$

40. $a_1 = 3, a_{n+1} = 4a_n - 10$

Find the first three iterates of each function for the given initial value.

41. $f(x) = -2x + 3, x_0 = 1$

42. $f(x) = 7x - 4, x_0 = 2$

43. SAVINGS Toni has a savings account with a \$15,000 balance. She has a 4% interest rate that is compounded monthly. Every month Toni makes a \$1000 withdrawal from the account to cover her expenses. The recursive formula $b_n = 1.04b_{n-1} - 1000$ describes the balance in Toni's savings account after n months. Find the balance of Toni's account after the first four months. Round your answer to the nearest dollar.

Example 10 **Find the first five terms of the sequence in which $a_1 = 2$, $a_{n+1} = 2a_n - 1$.**

$a_{n+1} = 2a_n - 1$ Recursive formula

$a_{1+1} = 2a_1 - 1$ $n = 1$

$a_2 = 2(2) - 1$ or 3 $a_1 = 2$

$a_{2+1} = 2a_2 - 1$ $n = 2$

$a_3 = 2(3) - 1$ or 5 $a_2 = 3$

$a_{3+1} = 2a_3 - 1$ $n = 3$

$a_4 = 2(5) - 1$ or 9 $a_3 = 5$

$a_{4+1} = 2a_4 - 1$ $n = 4$

$a_5 = 2(9) - 1$ or 17 $a_4 = 9$

The first five terms of the sequence are 2, 3, 5, 9, and 17.

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CHAPTER 11

Study Guide and Review

11–7 The Binomial Theorem (pp. 664-669)

Expand each power.

44. $(x - 2)^4$

45. $(3r + s)^5$

Find each indicated term of each expansion.

46. fourth term of $(x + 2y)^6$

47. second term of $(4x - 5)^{10}$

48. SCHOOL Mr. Brown is giving a four-question multiple-choice quiz. Each question can be answered A, B, C, or D. How many ways could a student answer the questions using each answer A, B, C, or D once?

Example 11 Expand $(a - 2b)^4$.

$$(a - 2b)^4$$

$$= \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k}(-2b)^k$$

$$= \frac{4!}{4!0!} a^4(-2b)^0 + \frac{4!}{3!1!} a^3(-2b)^1 + \frac{4!}{2!2!} a^2(-2b)^2 + \frac{4!}{1!3!} a^1(-2b)^3 + \frac{4!}{0!4!} a^0(-2b)^4$$

$$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$$

11–8 Proof and Mathematical Induction (pp. 670-674)

Prove that each statement is true for all positive integers.

49. $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

50. $6^n - 1$ is divisible by 5.

51. $3^n - 1$ is divisible by 2.

52. $1 + 4 + 7 + \ldots (3n - 2) = \frac{n(3n-1)}{2}$

Find a counterexample for each statement.

53. $n^2 - n + 13$ is prime.

54. $13^n + 11$ is divisible by 24.

55. $9^{n+1} - 1$ is divisible by 16.

56. $n^2 + n + 1$ is prime.

Example 12 Prove that $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for positive integers n.

Step 1 When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1}{4}(5^1 - 1)$ or 1. Thus, the equation is true for $n = 1$.

Step 2 Assume that $1 + 5 + 25 + \cdots + 5^{k-1} = \frac{1}{4}(5^k - 1)$ for some positive integer k.

Step 3 Show that the given equation is true for $n = k + 1$.

$1 + 5 + 25 + \ldots + 5^{k-1} + 5^{(k+1)-1}$

$= \frac{1}{4}(5^k - 1) + 5^{(k+1)-1}$ Add to each side.

$= \frac{1}{4}(5^k - 1) + 5^k$ Simplify the exponent.

$= \frac{5^k - 1 + 4 \cdot 5^k}{4}$ Common denominator

$= \frac{5 \cdot 5^k - 1}{4}$ Distributive Property

$= \frac{1}{4}(5^{k+1} - 1)$ $5^k = 5^{k+1}$

Thus, the equation is true for $n = k + 1$. The conjecture is proved.

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CHAPTER 11

Practice Test

1. Find the next four terms of the arithmetic sequence 42, 37, 32, … .
2. Find the 27th term of an arithmetic sequence for which $a_1 = 2$ and $d = 6$.

3. **MULTIPLE CHOICE** What is the tenth term in the arithmetic sequence that begins 10, 5.6, 1.2, −3.2, … ?

   A −39.6

   B −29.6

   C 29.6

   D 39.6

4. Find the three arithmetic means between −4 and 16.
5. Find the sum of the arithmetic series for which $a_1 = 7$, $n = 31$, and $a_n = 127$.
6. Find the next two terms of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.
7. Find the sixth term of the geometric sequence for which $a_1 = 5$ and $r = -2$.

8. **MULTIPLE CHOICE** Find the next term in the geometric sequence $8, 6, \frac{9}{2}, \frac{27}{8}, \ldots$.

   F $\frac{11}{8}$

   G $\frac{27}{16}$

   H $\frac{9}{4}$

   J $\frac{81}{32}$

9. Find the two geometric means between 7 and 189.
10. Find the sum of the geometric series for which $a_1 = 125$, $r = \frac{2}{5}$, and $n = 4$.

Find the sum of each series, if it exists.

11. $\sum_{k=3}^{15} (14 - 2k)$
12. $\sum_{n=1}^{\infty} \frac{1}{3}(-2)^{n-1}$
13. $91 + 85 + 79 + \cdots + (-29)$
14. $12 + (-6) + 3 + \left(-\frac{3}{2}\right) + \cdots$

Find the first five terms of each sequence.

15. $a_1 = 1, a_{n+1} = a_n + 3$
16. $a_1 = -3, a_{n+1} = a_n + n^2$

17. Find the first three iterates of $f(x) = x^2 - 3x$ for an initial value of $x_0 = 1$.
18. Expand $(2s - 3t)^5$.
19. What is the coefficient of the fifth term of $(r + 2q)^7$?
20. Find the third term of the expansion of $(x + y)^{10}$.

Prove that each statement is true for all positive integers.

21. $1 + 7 + 49 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$
22. $14^n - 1$ is divisible by 13.

23. Find a counterexample for the following statement.
    The units digit of $7^n - 3$ is never 8.

24. **DESIGN** The pattern in a red and white brick wall starts with 20 red bricks on the bottom row. Each row contains 3 fewer red bricks than the row below. If the top row has no red bricks, how many rows are there and how many red bricks were used?

25. **RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?

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CHAPTER 11

Standardized Test Practice

Cumulative, Chapters 1–11

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. How many 3-inch cubes can be placed completely inside a box that is 15 inches long, 12 inches wide, and 18 inches tall?

A 5

B 20

C 120

D 360

2. Using the table below, which expression can be used to determine the nth term of the sequence?

n	y
1	6
2	10
3	14
4	18

F $y = 6n$

G $y = n + 5$

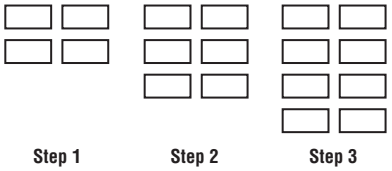
H $y = 2n + 1$

J $y = 2(2n + 1)$

TEST-TAKING TIP

Question 2 Sometimes sketching the graph of a function can help you to see the relationship between n and y and answer the question.

3. GRIDDABLE The pattern of squares below continues infinitely, with more squares being added at each step. How many squares are in the tenth step?



Step 1 Step 2 Step 3

4. The pattern of dots shown below continues infinitely, with more dots being added at each step.

First Step Second Step Third Step

Which expression can be used to determine the number of dots in the nth step?

A $2n$

B $n(n + 2)$

C $n(n + 1)$

D $2(n + 1)$

5. The figures below show a pattern of dark tiles and white tiles that can be described by a relationship between two variables.

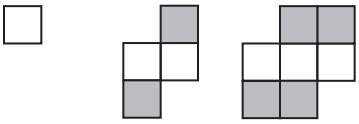


Figure 1 Figure 2 Figure 3

Which rule relates d, the number of dark tiles, to w, the number of white tiles?

F $d = 2w$

G $w = d - 1$

H $d = 2w - 2$

J $w = \frac{1}{2}d + 1$

6. Leland is renting an apartment. He looked at a 3-bedroom apartment for \$950 per month near the downtown area, and a 3-bedroom apartment for \$725 per month on the edge of town. About what percent of the cost of the downtown apartment is Leland saving by renting the apartment on the edge of town?

A 2%

B 24%

C 31%

D 231%

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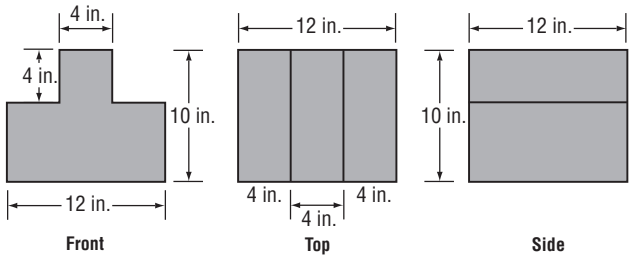
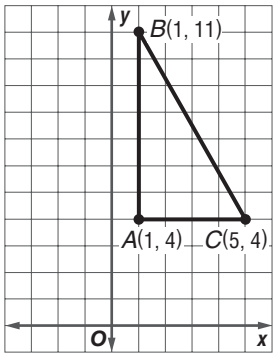
7. What is the volume of a 3-dimensional object with the dimensions shown in the 3 views below?



F 864 in^3
G 1056 in^3
H 1248 in^3
J 1440 in^3

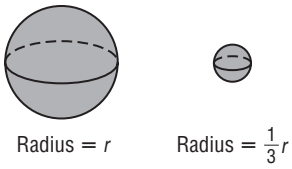
8. $\triangle ABC$ is graphed on the coordinate grid below.



Which set of coordinates represents the vertices of a triangle congruent to $\triangle ABC$?

A (−1, 7), (−1, 15), (3, 8)
B (2, 7), (2, 14), (3, 7)
C (4, 7), (4, 14), (7, 7)
D (−1, 7), (−1, 14), (3, 7)

9. The radius of the larger sphere shown below was multiplied by a factor of $\frac{1}{3}$ to produce the smaller sphere.



How does the volume of the smaller sphere compare to the volume of the larger sphere?

F The volume of the smaller sphere is $\frac{1}{9}$ as large.
G The volume of the smaller sphere is $\frac{1}{\pi^3}$ as large.
H The volume of the smaller sphere is $\frac{1}{27}$ as large.
J The volume of the smaller sphere is $\frac{1}{3}$ as large.

10. GRIDDABLE Marla is putting a binding around a square quilt. The length of the binding was 32 feet. Find the approximate length, in feet, of the diagonal of the square quilt. Round to one decimal place.

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Kyla's annual salary is $50,000. Each year she gets a 6% raise.

a. To the nearest dollar, what will her salary be in four years?

b. To the nearest dollar, what will her salary be in 10 years?

NEED EXTRA HELP?											
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11
Go to Lesson or Page...	11-7	11-1	11-1	11-1	11-1	750	1-1	754	10-3	10-3	11-3

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CHAPTER 12

Probability and Statistics

BIG Ideas

- Solve problems involving independent events, dependent events, permutations, and combinations.
- Find probability and odds.
- Find statistical measures.
- Use the normal, binomial, and exponential distributions.
- Determine whether a sample is unbiased.

Key Vocabulary

event (p. 684)
probability (p. 697)
sample space (p. 684)

Real-World Link

Approval Polls Polls are often conducted to determine how satisfied the public is with the job being performed by elected officials, such as the President and state governors. Results of these polls may determine on which issues an official focuses his or her efforts.

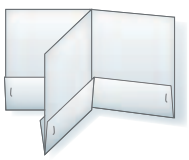
FOLDABLES Study Organizer

Probability and Statistics Make this Foldable to help you organize your notes. Begin with one sheet of 11" by 17" paper.

1. **Fold** 2" tabs on each of the short sides.

2. **Fold** in half in both directions. Open and cut as shown.

3. **Refold** along the width. Staple each pocket.



4. **Label** pockets as *The Counting Principle, Permutations and Combinations, Probability,* and *Statistics.* Place index cards for notes in each pocket.

Ted S. Warren/Associated Press

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GET READY for Chapter 12

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Find each probability if a die is rolled once. (Prerequisite Skill)

1. $P(2)$
2. P(numbers greater than 1)
3. $P(5)$
4. P(even number)
5. P(odd number)
6. P(numbers less than 5)

STAMP COLLECTING **Lynette collects stamps from different countries. She has 12 from Mexico, 5 from Canada, 3 from France, 8 from Great Britain, 1 from Russia and 3 from Germany. Find the probability of each of the following if she accidentally loses one stamp.** (Prerequisite Skill)

7. the stamp is from Canada
8. the stamp is not from Germany or Russia

Expand each binomial. (Lesson 11-7)

9. $(a + b)^3$
10. $(c + d)^4$
11. $(m - n)^5$
12. $(x + y)^6$
13. **COINS** A coin is flipped five times. Each time the coin is flipped the outcome is either a head h or a tail t. The terms of the binomial expansion of $(h + t)^5$ can be used to find the probabilities of each combination of heads and tails. Expand the binomial. (Lesson 11-7)

QUICKReview

EXAMPLE 1

Find the probability of rolling a 1 or a 6 if a die is rolled once.

$$P(1 \text{ or } 6) = \frac{\text{number of desired outcomes}}{\text{number of possible outcomes}}$$

There are 2 desired outcomes since 1 or 6 are both desired. There are 6 possible outcomes since there are 6 sides on a die.

$$P(1 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$$

The probability of a 1 or a 6 being rolled is $\frac{1}{3}$, or about 33%.

EXAMPLE 2

Expand $(g - h)^7$.

Remember Pascal's Triangle when expanding a binomial to a large power.

$$a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$$

Notice the signs in the expansion alternate because the binomial is the difference of two terms. The sum of the exponents of the variables in each term of the expansion is always 7, which is the power the binomial is being raised to. Substitute g for a and h for b.

$$g^7 - 7g^6h + 21g^5h^2 - 35g^4h^3 + 35g^3h^4 - 21g^2h^5 + 7gh^6 - h^7$$

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12-1 The Counting Principle

Main Ideas

- Solve problems involving independent events.
- Solve problems involving dependent events.

New Vocabulary

outcome
sample space
event
independent events
Fundamental Counting Principle
dependent events

GET READY for the Lesson

The number of possible license plates for a state is too great to count by listing all of the possibilities. It is much more efficient to count the number of possibilities by using the Fundamental Counting Principle.

Independent Events An **outcome** is the result of a single trial. For example, the trial of flipping a coin once has two outcomes: head or tail. The set of all possible outcomes is called the **sample space**. An **event** consists of one or more outcomes of a trial. The choices of letters and digits to be put on a license plate are called **independent events** because each letter or digit chosen does *not* affect the choices for the others.

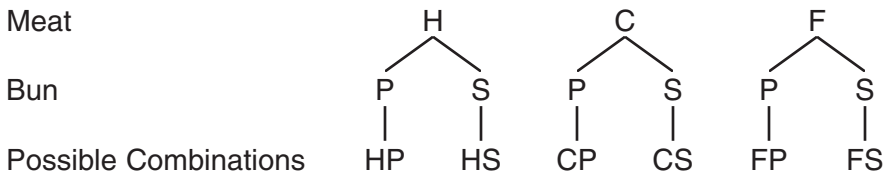
EXAMPLE Independent Events

1 **FOOD** **A sandwich cart offers customers a choice of hamburger, chicken, or fish on either a plain or a sesame seed bun. How many different combinations of meat and a bun are possible?**

First, note that the choice of the type of meat does not affect the choice of the type of bun, so these events are independent.

Method 1 Tree Diagram

H represents hamburger, C, chicken, F, fish, P, plain, and S, sesame seed.



Method 2 Make a Table

Make a table in which each row represents a type of meat and each column represents a type of bun.

There are six possible outcomes.

		Bun	
		Plain	Sesame
Meal	Hamburger	HP	HS
	Chicken	CP	CS
	Fish	FP	FS

CHECK Your Progress

1. A cafeteria offers drink choices of water, coffee, juice, and milk and salad choices of pasta, fruit, and potato. How many different combinations of drink and salad are possible?

Reading Math

There are both *infinite* and *finite* sample spaces. Finite sample spaces have a countable number of possible outcomes, such as rolling a die. Infinite sample spaces have an uncountable number of possible outcomes, such as the probability of a point on a line.

D.F. Harris
www.MathSchoolinternational.com

Notice that in Example 1, there are 3 ways to choose the type of meat, 2 ways to choose the type of bun, and 3 • 2 or 6 total ways to choose a combination of the two. This illustrates the **Fundamental Counting Principle**.

KEY CONCEPT — *Fundamental Counting Principle*

Words If event M can occur in m ways and is followed by event N that can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

Example If event M can occur in 2 ways and event N can occur in 3 ways, then M followed by N can occur in $2 \cdot 3$ or 6 ways.

This rule can be extended to any number of events.

STANDARDIZED TEST EXAMPLE — Fundamental Counting Principle

2 Kim won a contest on a radio station. The prize was a restaurant gift certificate and tickets to a sporting event. She can select one of three different restaurants and tickets to a football, baseball, basketball, or hockey game. How many different ways can she select a restaurant followed by a sporting event?

A 7 **B** 12 **C** 15 **D** 16

Test-Taking Tip

Remember that you can check your answer by making a tree diagram or a table showing the outcomes.

Read the Test Item

Her choice of a restaurant does not affect her choice of a sporting event, so these events are independent.

Solve the Test Item

There are 3 ways she can choose a restaurant and there are 4 ways she can choose the sporting event. By the Fundamental Counting Principle, there are 3 • 4 or 12 total ways she can choose her two prizes. The answer is B.

CHECK Your Progress

2. Dane is renting a tuxedo for prom. Once he has chosen his jacket, he must choose from three types of pants and six colors of vests. How many different ways can he select his attire for the prom?

F 9 **G** 10 **H** 18 **J** 36

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EXAMPLE — More than Two Independent Events

3 **COMMUNICATION** **Many answering machines allow owners to call home and get their messages by entering a 3-digit code. How many codes are possible?**

Reading Math

Independent and *dependent* have the same meaning in mathematics as they do in ordinary language.

The choice of any digit does not affect the other two digits, so the choices of the digits are independent events.

There are 10 possible first digits in the code, 10 possible second digits, and 10 possible third digits. So, there are 10 • 10 • 10 or 1000 possible different code numbers.

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CHECK Your Progress

3. If a garage door opener has a 10-digit keypad and the code to open the door is a 4-digit code, how many codes are possible?

Dependent Events Some situations involve dependent events. With **dependent events**, the outcome of one event *does* affect the outcome of another event. The Fundamental Counting Principle applies to dependent events as well as independent events.

EXAMPLE Dependent Events

4 **SCHOOL** **Charlita wants to take 6 different classes next year. Assuming that each class is offered each period, how many different schedules could she have?**

When Charlita schedules a given class for a given period, she cannot schedule that class for any other period. Therefore, the choices of which class to schedule each period are dependent events.

There are 6 classes Charlita can take during first period. That leaves 5 classes she can take second period. After she chooses which classes to take the first two periods, there are 4 remaining choices for third period, and so on.

Study Tip

Look Back

To review **factorials**, see Lesson 11-7.

Period	1st	2nd	3rd	4th	5th	6th
Number of Choices	6	5	4	3	2	1

There are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 720 schedules that Charlita could have.

Note that $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$.

CHECK Your Progress

4. Each player in a board game uses one of six different pieces. If four players play the game, how many different ways could the players choose their game pieces?

CONCEPT SUMMARY

Independent Events	**Words**	If the outcome of an event does not affect the outcome of another event, the two events are independent.
	Example	Tossing a coin and rolling a die are independent events.
Dependent Events	**Words**	If the outcome of an event does affect the outcome of another event, the two events are dependent.
	Example	Taking a piece of candy from a jar and then taking a second piece without replacing the first are dependent events because taking the first piece affects what is available to be taken next.

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CHECK Your Understanding

Examples 1–4 (pp. 684–686)

State whether the events are *independent* or *dependent*.

1. choosing the color and size of a pair of shoes
2. choosing the winner and runner-up at a dog show

Examples 1, 2 (pp. 684, 685)

3. An ice cream shop offers a choice of two types of cones and 15 flavors of ice cream. How many different 1-scoop ice cream cones can a customer order?

Example 2 (p. 685)

4. **STANDARDIZED TEST PRACTICE** A bookshelf holds 4 different biographies and 5 different mystery novels. How many ways can one book of each type be selected?

A 1 **B** 9 **C** 10 **D** 20

Example 3 (p. 685)

5. Lance's math quiz has eight true-false questions. How many different choices for giving answers to the eight questions are possible?
6. Pizza House offers three different crusts, four sizes, and eight toppings. How many different ways can a customer order a pizza?

Example 4 (p. 686)

7. For a college application, Macawi must select one of five topics on which to write a short essay. She must also select a different topic from the list for a longer essay. How many ways can she choose the topics for the two essays?

Exercises

HOMEWORK HELP

For Exercises	See Examples
8–11	1, 4
12–26	1–4

State whether the events are *independent* or *dependent*.

8. choosing a president, vice-president, secretary, and treasurer for Student Council, assuming that a person can hold only one office
9. selecting a fiction book and a nonfiction book at the library
10. Each of six people guess the total number of points scored in a basketball game. Each person writes down his or her guess without telling what it is.
11. The letters A through Z are written on pieces of paper and placed in a jar. Four of them are selected one after the other without replacing any of them.
12. Tim wants to buy one of three different books he sees in a book store. Each is available in print and on CD. How many book and format choices does he have?
13. A video store has 8 new releases this week. Each is available on videotape and on DVD. How many ways can a customer choose a new release and a format to rent?
14. Carlos has homework in math, chemistry, and English. How many ways can he choose the order in which to do his homework?

Cross-Curricular Project

Math online You can use the Fundamental Counting Principle to list possible outcomes in games. Visit algebra2.com to continue work on your project.

15. The menu for a banquet has a choice of 2 types of salad, 5 main courses, and 3 desserts. How many ways can a salad, a main course, and a dessert be selected to form a meal?
16. A baseball glove manufacturer makes gloves in 4 different sizes, 3 different types by position, 2 different materials, and 2 different levels of quality. How many different gloves are possible?
17. Each question on a five-question multiple-choice quiz has answer choices labeled A, B, C, and D. How many different ways can a student answer the five questions?

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18. PASSWORDS Abby is registering at a Web site. She must select a password containing six numerals to be able to use the site. How many passwords are allowed if no digit may be used more than once?

ENTERTAINMENT **For Exercises 19 and 20, refer to the comic strip. Assume that the books are all different.**

Real-World Link

Before 1995, area codes had the following format.

(XYZ)

X = 2, 3, . . ., or 9

Y = 0 or 1

Z = 0, 1, 2, . . ., or 9

Source: www.nanpa.com

19. How many ways can you arrange the science books?

20. Since the science books are to be together, they can be treated like one book and arranged with the music books. Use your answer to Exercise 19 and the Counting Principle to find the answer to the problem in the comic.

AREA CODES **For Exercises 21 and 22, refer to the information about telephone area codes at the left.**

21. How many area codes were possible before 1995?

22. In 1995, the restriction on the middle digit was removed, allowing any digit in that position. How many total codes were possible after this change was made?

23. How many ways can six different books be arranged on a shelf if one of the books is a dictionary and it must be on an end?

24. In how many orders can eight actors be listed in the opening credits of a movie if the leading actor must be listed first or last?

25. HOME SECURITY How many different 5-digit codes are possible using the keypad shown at the right if the first digit cannot be 0 and no digit may be used more than once?

26. RESEARCH Use the Internet or other resource to find the configuration of letters and numbers on license plates in your state. Then find the number of possible plates.

EXTRA PRACTICE
See pages 916, 937.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

27. OPEN ENDED Describe a situation in which you can use the Fundamental Counting Principle to show that there are 18 total possibilities.

28. REASONING Explain how choosing to buy a car or a pickup truck and then selecting the color of the vehicle could be dependent events.

29. CHALLENGE The members of the Math Club need to elect a president and a vice president. They determine that there are a total of 272 ways that they can fill the positions with two different members. How many people are in the Math Club?

(t)Mitch Kezar/Getty Images, (b)age fotc

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30. ***Writing in Math*** Use the information on page 684 to explain how you can count the maximum number of license plates a state can issue. Explain how to use the Fundamental Counting Principle to find the number of different license plates in a state such as Oklahoma, which has 3 letters followed by 3 numbers. Also explain how a state can increase the number of possible plates without increasing the length of the plate number.

STANDARDIZED TEST PRACTICE

31. ACT/SAT How many numbers between 100 and 999, inclusive, have 7 in the tens place?

A 90

B 100

C 110

D 120

32. REVIEW A coin is tossed four times. How many possible sequences of heads or tails are possible?

F 4

G 8

H 16

J 32

Spiral Review

33. Prove that $4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all positive integers n. (Lesson 11-8)

Find the indicated term of each expansion. (Lesson 11-7)

34. third term of $(x + y)^8$

35. fifth term of $(2a - b)^7$

36. CARTOGRAPHY Edison is located at (9, 3) in the coordinate system on a road map. Kettering is located at (12, 5) on the same map. Each side of a square on the map represents 10 miles. To the nearest mile, what is the distance between Edison and Kettering? (Lesson 10-1)

Solve each equation by factoring. (Lesson 5-3)

37. $x^2 - 16 = 0$

38. $x^2 - 3x - 10 = 0$

39. $3x^2 + 8x - 3 = 0$

Solve each matrix equation. (Lesson 4-1)

40. $[x \quad y] = [y \quad 4]$

41. $\begin{bmatrix} 3y \\ 2x \end{bmatrix} = \begin{bmatrix} x + 8 \\ y - x \end{bmatrix}$

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate each expression.** (Lesson 11-7)

42. $\frac{5!}{2!}$

43. $\frac{6!}{4!}$

44. $\frac{7!}{3!}$

45. $\frac{6!}{1!}$

46. $\frac{4!}{2!2!}$

47. $\frac{6!}{2!4!}$

48. $\frac{8!}{3!5!}$

49. $\frac{5!}{5!0!}$

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12-2 Permutations and Combinations

Main Ideas

- Solve problems involving permutations.
- Solve problems involving combinations.

New Vocabulary

permutation
linear permutation
combination

GET READY for the Lesson

When the manager of a softball team fills out her team's lineup card before the game, the order in which she fills in the names is important because it determines the order in which the players will bat.

Suppose she has 7 possible players in mind for the top 4 spots in the lineup. You know from the Fundamental Counting Principle that there are $7 \cdot 6 \cdot 5 \cdot 4$ or 840 ways that she could assign players to the top 4 spots.

Permutations When a group of objects or people are arranged in a certain order, the arrangement is called a **permutation**. In a permutation, the *order* of the objects is very important. The arrangement of objects or people in a line is called a **linear permutation**.

Notice that $7 \cdot 6 \cdot 5 \cdot 4$ is the product of the first 4 factors of 7!. You can rewrite this product in terms of 7!.

$$7 \cdot 6 \cdot 5 \cdot 4 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$ Multiply by $\frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$ or 1.

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} \text{ or } \frac{7!}{3!}$$ $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and $3! = 3 \cdot 2 \cdot 1$

Notice that 3! is the same as $(7 - 4)!$.

The number of ways to arrange 7 people or objects taken 4 at a time is written $P(7, 4)$. The expression for the softball lineup above is a case of the following formula.

KEY CONCEPT *Permutations*

The number of permutations of n distinct objects taken r at a time is given by

$$P(n, r) = \frac{n!}{(n - r)!}.$$

Reading Math

Permutations The expression $P(n, r)$ reads *the number of permutations of n objects taken r at a time.* It is sometimes written as ${}_nP_r$.

EXAMPLE 1 **Permutation**

FIGURE SKATING **There are 10 finalists in a figure skating competition. How many ways can gold, silver, and bronze medals be awarded?**

Since each winner will receive a different medal, order is important. You must find the number of permutations of 10 things taken 3 at a time.

Mark C. Burnett/Photo Researchers
www.MathSchoolinternational.com

Study Tip

Alternate Method

Notice that in Example 1, all of the factors of $(n - r)!$ are also factors of $n!$. You can also evaluate the expression in the following way.

$$\frac{10!}{(10-3)!}$$
$$= \frac{10!}{7!}$$
$$= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$$
$$= 10 \cdot 9 \cdot 8 \text{ or } 720$$

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{Permutation formula}$$

$$P(10, 3) = \frac{10!}{(10-3)!} \quad n = 10, r = 3$$

$$= \frac{10!}{7!} \quad \text{Simplify.}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} \text{ or } 720 \quad \text{Divide by common factors.}$$

The gold, silver, and bronze medals can be awarded in 720 ways.

CHECK Your Progress

1. A newspaper has nine reporters available to cover four different stories. How many ways can the reporters be assigned to cover the stories?

Suppose you want to rearrange the letters of the word *geometry* to see if you can make a different word. If the two *e*s were not identical, the eight letters in the word could be arranged in $P(8, 8)$ ways. To account for the identical *e*s, divide $P(8, 8)$ by the number of arrangements of *e*. The two *e*s can be arranged in $P(2, 2)$ ways.

$$\frac{P(8, 8)}{P(2, 2)} = \frac{8!}{2!} \quad \text{Divide.}$$

$$= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} \text{ or } 20{,}160 \quad \text{Simplify.}$$

Thus, there are 20,160 ways to arrange the letters in *geometry*.

When some letters or objects are alike, use the rule below to find the number of permutations.

KEY CONCEPT — ***Permutations with Repetitions***

The number of permutations of n objects of which p are alike and q are alike is $\frac{n!}{p!q!}$.

This rule can be extended to any number of objects that are repeated.

EXAMPLE **Permutation with Repetition**

2 **How many different ways can the letters of the word *MISSISSIPPI* be arranged?**

The letter *I* occurs 4 times, *S* occurs 4 times, and *P* occurs twice.

You need to find the number of permutations of 11 letters of which 4 of one letter, 4 of another letter, and 2 of another letter are the same.

$$\frac{11!}{4!4!2!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!4!2!} \text{ or } 34{,}650$$

There are 34,650 ways to arrange the letters.

CHECK Your Progress

2. How many different ways can the letters of the word *DECIDED* be arranged?

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Study Tip

Permutations and Combinations

- If order in an arrangement *is* important, the arrangement is a *permutation*
- If order is *not* important, the arrangement is a *combination*.

Combinations An arrangement or selection of objects in which order is *not* important is called a **combination**. The number of combinations of n objects taken r at a time is written $C(n, r)$. *It is sometimes written* ${}_nC_r$

You know that there are $P(n, r)$ ways to select r objects from a group of n if the order is important. There are $r!$ ways to order the r objects that are selected, so there are $r!$ permutations that are all the same combination. Therefore,

$$C(n, r) = \frac{P(n, r)}{r!} \text{ or } \frac{n!}{(n-r)!r!}.$$

KEY CONCEPT — ***Combinations***

The number of combinations of n distinct objects taken r at a time is given by

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

EXAMPLE Combination

3 **A group of seven students working on a project needs to choose two students to present the group's report. How many ways can they choose the two students?**

Since the order they choose the students is not important, you must find the number of combinations of 7 students taken 2 at a time.

$C(n, r) = \frac{n!}{(n-r)!r!}$ Combination formula

$C(7, 2) = \frac{7!}{(7-2)!2!}$ $n = 7$ and $r = 2$

$= \frac{7!}{5!2!}$ or 21 Simplify.

There are 21 possible ways to choose the two students.

CHECK Your Progress

3. A family with septuplets assigns different chores to the children each week. How many ways can three children be chosen to help with the laundry?

In more complicated situations, you may need to multiply combinations and/or permutations.

EXAMPLE Multiple Events

Study Tip

Deck of Cards

In this text, a *standard deck of cards* always means a deck of 52 playing cards. There are 4 suits—clubs (black), diamonds (red), hearts (red), and spades (black)—with 13 cards in each suit.

4 **Five cards are drawn from a standard deck of cards. How many hands consist of three clubs and two diamonds?**

By the Fundamental Counting Principle, you can multiply the number of ways to select three clubs and the number of ways to select two diamonds.

Only the cards in the hand matter, not the order in which they were drawn, so use combinations.

$C(13, 3)$ Three of 13 clubs are to be drawn.

$C(13, 2)$ Two of 13 diamonds are to be drawn.

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$$C(13, 3) \cdot C(13, 2) = \frac{13!}{(13-3)!13!} \cdot \frac{13!}{(13-2)!13!} \quad \text{Combination formula}$$

$$= \frac{13!}{10!3!} \cdot \frac{13!}{11!2!} \quad \text{Subtract.}$$

$$= 286 \cdot 78 \text{ or } 22{,}308 \quad \text{Simplify.}$$

There are 22,308 hands consisting of 3 clubs and 2 diamonds.

CHECK Your Progress

4. How many five-card hands consist of five cards of the same suit?

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CHECK Your Understanding

Examples 1–4 (pp. 690–693)

Evaluate each expression.

1. $P(5, 3)$ **2.** $P(6, 3)$

3. $C(4, 2)$ **4.** $C(6, 1)$

Examples 1–3 (pp. 690–692)

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

5. seven shoppers in line at a checkout counter

6. an arrangement of the letters in the word *intercept*

7. an arrangement of 4 blue tiles, 2 red tiles, and 3 black tiles in a row

8. choosing 2 different pizza toppings from a list of 6

Example 3 (p. 692)

9. SCHEDULING The Helping Hand Moving Company owns nine trucks. On one Saturday, the company has six customers who need help moving. In how many ways can a group of six trucks be selected from the company's fleet?

Example 4 (pp. 692–693)

10. Six cards are drawn from a standard deck of cards. How many hands will contain three hearts and three spades?

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–14, 21, 22	1
23, 24	2
15–18, 25–28	3
19, 20, 29–31	4

Evaluate each expression.

11. $P(8, 2)$ **12.** $P(9, 1)$

13. $P(7, 5)$ **14.** $P(12, 6)$

15. $C(5, 2)$ **16.** $C(8, 4)$

17. $C(12, 7)$ **18.** $C(10, 4)$

19. $C(12, 4) \cdot C(8, 3)$ **20.** $C(9, 3) \cdot C(6, 2)$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

21. the winner and first, second, and third runners-up in a contest with 10 finalists

22. placing an algebra book, a geometry book, a chemistry book, an English book, and a health book on a shelf

23. an arrangement of the letters in the word *algebra*

24. an arrangement of the letters in the word *parallel*

25. selecting two of eight employees to attend a business seminar

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Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

26. selecting nine books to check out of the library from a reading list of twelve

27. choosing two CDs to buy from ten that are on sale

28. selecting three of fifteen flavors of ice cream at the grocery store

29. How many ways can a hand of five cards consisting of four cards from one suit and one card from another suit be drawn from a standard deck of cards?

30. A student council committee must be composed of two juniors and two sophomores. How many different committees can be chosen from seven juniors and five sophomores?

31. How many ways can a hand of five cards consisting of three cards from one suit and two cards from another suit be drawn from a standard deck of cards?

Real-World Link

The Hawaiian language consists of only twelve letters, the vowels a, e, i, o, and u and the consonants h, k, l m, n, p, and w.

Source: andhawaii.com

32. MOVIES The manager of a four-screen movie theater is deciding which of 12 available movies to show. The screens are in rooms with different seating capacities. How many ways can she show four different movies on the screens?

33. LANGUAGES How many different arrangements of the letters of the Hawaiian word *aloha* are possible?

34. GOVERNMENT How many ways can five members of the 100-member United States Senate be chosen to serve on a committee?

35. LOTTERIES In a multi-state lottery, the player must guess which five of forty-nine white balls numbered from 1 to 49 will be drawn. The order in which the balls are drawn does not matter. The player must also guess which one of forty-two red balls numbered from 1 to 42 will be drawn. How many ways can the player fill out a lottery ticket?

EXTRA PRACTICE
See pages 917, 937.
MathOnline
Self-Check Quiz at algebra2.com

36. CARD GAMES *Hachi-hachi* is a Japanese game that uses a deck of *Hanafuda* cards which is made up of 12 suits, with each suit having four cards. How many 7-card hands can be formed so that 3 are from one suit and 4 are from another?

H.O.T. Problems

37. OPEN ENDED Describe a situation in which the number of outcomes is given by $P(6, 3)$.

38. REASONING Prove that $C(n, n - r) = C(n, r)$.

39. REASONING Determine whether the statement $C(n, r) = P(n, r)$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

40. CHALLENGE Show that $C(n - 1, r) + C(n - 1, r - 1) = C(n, r)$.

41. *Writing in Math* Use the information on page 690 to explain how permutations and combinations apply to softball. Explain how to find the number of 9-person lineups that are possible and how many ways there are to choose 9 players if 16 players show up for a game.

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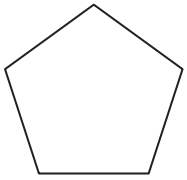
STANDARDIZED TEST PRACTICE

42. ACT/SAT How many diagonals can be drawn in the pentagon?



A 5 C 15

B 10 D 20

43. REVIEW How many ways can eight runners in an Olympic race finish in first, second, and third places?

F 8 H 56

G 24 J 336

Spiral Review

44. Darius can do his homework in pencil or pen, using lined or unlined paper, and on one or both sides of each page. How many ways can he do his homework? (Lesson 12-1)

45. A customer in an ice cream shop can order a sundae with a choice of 10 flavors of ice cream, a choice of 4 flavors of sauce, and with or without a cherry on top. How many different sundaes are possible? (Lesson 12-1)

Find a counterexample for each statement. (Lesson 11-8)

46. $1 + 2 + 3 + \cdots + n = 2n - 1$

47. $5^n + 1$ is divisible by 6.

Solve each equation or inequality. (Lesson 9-5)

48. $3e^x + 1 = 2$

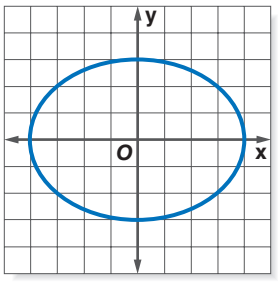
49. $e^{2x} > 5$

50. $\ln (x - 1) = 3$

51. CONSTRUCTION A painter works on a job for 10 days and is then joined by an associate. Together they finish the job in 6 more days. The associate could have done the job in 30 days. How long would it have taken the painter to do the job alone? (Lesson 8-6)

Write an equation for each ellipse. (Lesson 10-4)

52.



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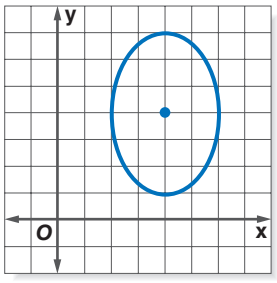

GET READY for the Next Lesson

PREREQUISITE SKILL Evaluate the expression $\frac{x}{x+y}$ for the given values of x and y. (Lesson 1-1)

54. $x = 3, y = 2$

55. $x = 4, y = 4$

56. $x = 2, y = 8$

57. $x = 5, y = 10$

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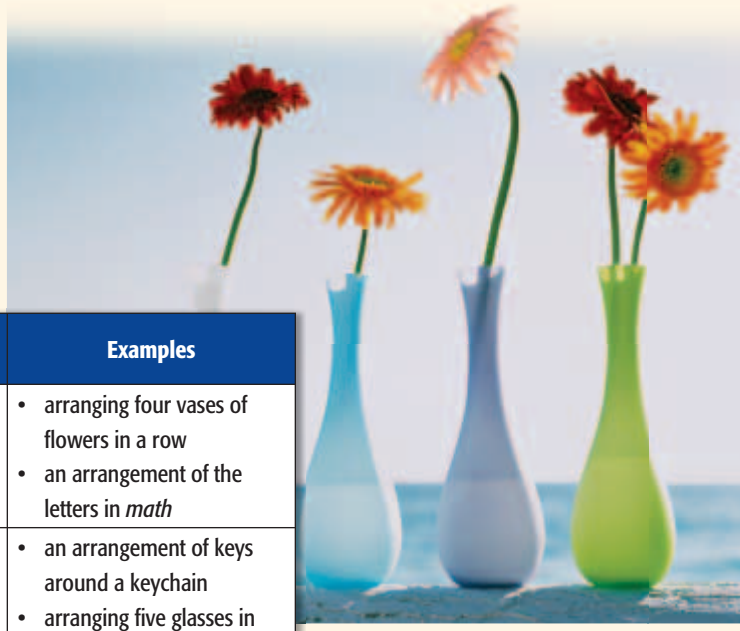
READING MATH

Permutations and Combinations

When solving probability problems, it is helpful to be able to determine whether situations involve *permutations* or *combinations*. Often words in a problem give clues as to which type of arrangement is involved.



Type of Arrangement	Description	Clue Words	Examples
Linear Permutation	The order of objects or people in a line is important.	• arranging *x* • an arrangement of first, second, third	• arranging four vases of flowers in a row • an arrangement of the letters in *math*
Circular Permutation	The order of objects or people in a circle is important.	• an arrangement around • arranging in a circle	• an arrangement of keys around a keychain • arranging five glasses in a circle on a tray
Combination	The order of objects or people is not important.	• selecting *x* of *y* • choosing *x* from *y* • forming *x* from *y*	• selecting 3 of 8 flavors • choosing 2 people from a group of 7

Reading to Learn

Determine whether each situation involves a *permutation* or a *combination*. If it is a permutation, identify it as *linear* or *circular*.

1. choosing six students from a class of 25

2. an arrangement of the letters in *drive*

3. selecting two of nine different side dishes

4. choosing three classes from a list of twelve to schedule for first, second, and third periods

5. arranging eighteen students in a circle for a class discussion

6. arranging seven swimmers in the lanes of a swimming pool

7. selecting five volunteers from a group of ten

8. an arrangement of six small photographs around a central photograph

9. forming a team of twelve athletes from a group of 35 who try out

10. OPEN ENDED Write a combination problem that involves the numbers 4 and 16.

11. Discuss how the definitions of the words *permanent* and *combine* could help you to remember the difference between permutations and combinations.

12. Describe a real-world situation that involves a permutation and a real-world situation that involves a combination. Explain your reasoning.

Michal Venera/Getty Images

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12-3 Probability

Main Ideas

- Use combinations and permutations to find probability.
- Create and use graphs of probability distributions.

New Vocabulary

probability
success
failure
random
random variable
probability distribution
uniform distribution
relative-frequency histogram

GET READY for the Lesson

The risk of getting struck by lightning in any given year is 1 in 750,000. The chances of surviving a lightning strike are 3 in 4. These risks and chances are a way of describing the probability of an event. The **probability** of an event is a ratio that measures the chances of the event occurring.

Probability and Odds Mathematicians often use tossing of coins and rolling of dice to illustrate probability. When you toss a coin, there are only two possible outcomes—heads or tails. A desired outcome is called a **success**. Any other outcome is called a **failure**.

Reading Math

Notation When P is followed by an event in parentheses, P stands for *probability*. When there are two numbers in parentheses, P stands for *permutations*.

KEY CONCEPT Probability of Success and Failure

If an event can succeed in s ways and fail in f ways, then the probabilities of success, $P(S)$, and of failure, $P(F)$, are as follows.

$$P(S) = \frac{s}{s+f} \qquad P(F) = \frac{f}{s+f}$$

The probability of an event occurring is always between 0 and 1, inclusive. The closer the probability of an event is to 1, the more likely the event is to occur. The closer the probability of an event is to 0, the less likely the event is to occur. When all outcomes have an equally likely chance of occurring, we say that the outcomes occur at **random**.

EXAMPLE 1 Probability with Combinations

Monifa has a collection of 32 CDs—18 R&B and 14 rap. As she is leaving for a trip, she randomly chooses 6 CDs to take with her. What is the probability that she selects 3 R&B and 3 rap?

Step 1 Determine how many 6-CD selections meet the conditions.
$C(18, 3)$ Select 3 R&B CDs. Their order does not matter.
$C(14, 3)$ Select 3 rap CDs.

Step 2 Use the Fundamental Counting Principle to find s, the number of successes.

$$C(18, 3) \cdot C(14, 3) = \frac{18!}{15!3!} \cdot \frac{14!}{11!3!} \text{ or } 297{,}024$$

(continued on the next page)

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Step 3 Find the total number, $s + f$, of possible 6-CD selections.

$C(32, 6) = \frac{32!}{26!6!}$ or 906,192 $s + f = 906{,}192$

Step 4 Determine the probability.

$P(3 \text{ R\&B CDs and } 3 \text{ rap CDs}) = \frac{s}{s+f}$ Probability formula

$= \frac{297{,}024}{906{,}192}$ Substitute.

≈ 0.32777 Use a calculator.

The probability of selecting 3 R&B CDs and 3 rap CDs is about 0.32777 or 33%.

CHECK Your Progress

1. A board game is played with tiles with letters on one side. There are 56 tiles with consonants and 42 tiles with vowels. Each player must choose seven of the tiles at the beginning of the game. What is the probability that a player selects four consonants and three vowels?

EXAMPLE 2 Probability with Permutations

Ramon has five books on the floor, one for each of his classes: Algebra 2, chemistry, English, Spanish, and history. Ramon is going to put the books on a shelf. If he picks the books up at random and places them in a row on the same shelf, what is the probability that his English, Spanish, and Algebra 2 books will be the leftmost books on the shelf, but not necessarily in that order?

Step 1 Determine how many book arrangements meet the conditions.

$P(3, 3)$ Place the 3 leftmost books.

$P(2, 2)$ Place the other 2 books.

Step 2 Use the Fundamental Counting Principle to find the number of successes.

$P(3, 3) \cdot P(2, 2) = 3! \cdot 2!$ or 12

Step 3 Find the total number, $s + f$, of possible 5-book arrangements.

$P(5, 5) = 5!$ or 120 $s + f = 120$

Step 4 Determine the probability.

$P(\text{English, Spanish, Algebra 2 followed by other books})$

$= \frac{s}{s+f}$ Probability formula

$= \frac{12}{120}$ Substitute.

$= 0.1$ Use a calculator.

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The probability of placing English, Spanish, and Algebra 2 before the other four books is 0.1 or 10%.

CHECK Your Progress

2. What is the probability that English will be the last book on the shelf?

Personal Tutor at algebra2.com

Probability Distributions Many experiments, such as rolling a die, have numerical outcomes. A **random variable** is a variable whose value is the numerical outcome of a random event. For example, when rolling a die we can let the random variable D represent the number showing on the die. Then D can equal 1, 2, 3, 4, 5, or 6. A **probability distribution** for a particular random variable is a function that maps the sample space to the probabilities of the outcomes in the sample space. The table below illustrates the probability distribution for rolling a die. *A distribution like this one where all of the probabilities are the same is called a* ***uniform distribution****.*

Reading Math

Random Variables The notation $P(X = n)$ is used with random variables. $P(D = 4) = \frac{1}{6}$ is read *the probability that D equals 4 is one sixth.*

D = Roll	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$P(D = 4) = \frac{1}{6}$

To help visualize a probability distribution, you can use a table of probabilities or a graph, called a **relative-frequency histogram**.

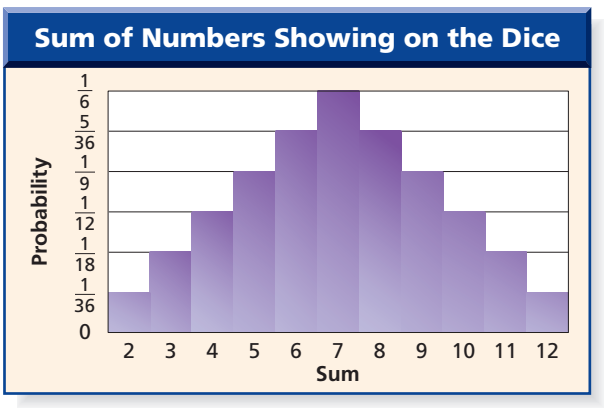
EXAMPLE Probability Distribution

3 **Suppose two dice are rolled. The table and the relative-frequency histogram show the distribution of the sum of the numbers rolled.**

Reading Math

Discrete Random Variables A discrete random variable is a variable that can have a countable number of values. The variable is said to be *random* if the sum of the probabilities is 1.

S = Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$



a. Use the graph to determine which outcome is most likely. What is its probability?

The most likely outcome is a sum of 7, and its probability is $\frac{1}{6}$.

(continued on the next page)

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b. Use the table to find $P(S = 9)$. What other sum has the same probability?

According to the table, the probability of a sum of 9 is $\frac{1}{9}$. The other outcome with a probability of $\frac{1}{9}$ is 5.

CHECK Your Progress

3A. Which outcome(s) is least likely? What is its probability?

3B. Use the table to find $P(S = 3)$. What other sum has the same probability?

CHECK Your Understanding

Example 1 (pp. 697–698)

Suppose you select 2 letters at random from the word *compute*. Find each probability.

1. P(2 vowels) **2.** P(2 consonants) **3.** P(1 vowel, 1 consonant)

Example 2 (pp. 698–699)

ORGANIZATION An administrative assistant has 4 blue file folders, 3 red folders, and 3 yellow folders on her desk. Each folder contains different information, so two folders of the same color should be viewed as being different. She puts the file folders randomly in a box to be taken to a meeting. Find each probability.

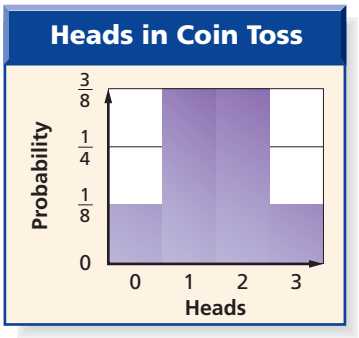
4. P(4 blue, 3 red, 3 yellow, in that order)

5. P(first 2 blue, last 2 blue)

Example 3 (pp. 699–700)

The table and the relative-frequency histogram show the distribution of the number of heads when 3 coins are tossed. Find each probability.

H = Heads	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



6. $P(H = 0)$

7. $P(H = 2)$

Exercises

HOMEWORK HELP

For Exercises	See Examples
8–15	1
16–21	2
22–27	3

Bob is moving and all of his sports cards are mixed up in a box. Twelve cards are baseball, eight are football, and five are basketball. If he reaches in the box and selects them at random, find each probability.

8. P(3 football) **9.** P(3 baseball)

10. P(1 basketball, 2 football) **11.** P(2 basketball, 1 baseball)

12. P(1 football, 2 baseball) **13.** P(1 basketball, 1 football, 1 baseball)

14. P(2 baseball, 2 basketball) **15.** P(2 football, 1 hockey)

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DVDS **Janice has 8 DVD cases on a shelf, one for each season of her favorite TV show. Her brother accidentally knocks them off the shelf onto the floor. When her brother puts them back on the shelf, he does not pay attention to the season numbers and puts the cases back on the shelf randomly. Find each probability.**

16. P(season 5 in the correct position)

17. P(seasons 1 and 8 in the correct positions)

18. P(seasons 1 through 4 in the correct positions)

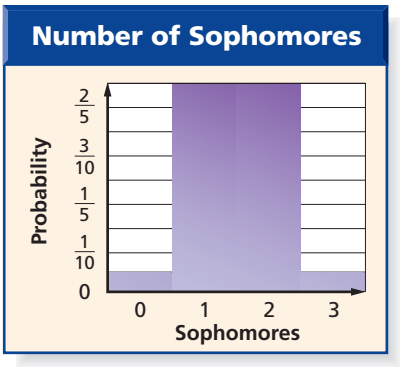
19. P(all even-numbered seasons followed by all odd-numbered seasons)

20. P(all even-numbered seasons in the correct position)

21. P(seasons 5 through 8 in any order followed by seasons 1 through 4 in any order)

Three students are selected at random from a group of 3 sophomores and 3 juniors. The table and relative-frequency histogram show the distribution of the number of sophomores chosen. Find each probability.

Sophomores	0	1	2	3
Probability	$\frac{1}{20}$	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{1}{20}$



22. P(0 sophomores)

23. P(1 sophomore)

24. P(2 sophomores)

25. P(3 sophomores)

26. P(2 juniors)

27. P(1 junior)

Real-World Career

Physician

In addition to the MCAT, most medical schools require applicants to have had one year each of biology, physics, and English, and two years of chemistry in college.

Math Online

For more information, go to algebra2.com.

28. **LOTTERIES** The state of Texas has a lottery in which 5 numbers out of 37 are drawn at random. What is the probability of a given ticket matching all 5 numbers?

ENTRANCE TESTS **For Exercises 29–31, use the table that shows the college majors of the students who took the Medical College Admission Test (MCAT) recently.**

If a student taking the test were randomly selected, find each probability. Express as decimals rounded to the nearest thousandth.

Major	Students
biological sciences	15,819
humanities	963
math or statistics	179
physical sciences	2770
social sciences	2482
specialized health sciences	1431
other	1761

29. P(math or statistics)

30. P(biological sciences)

31. P(physical sciences)

32. **CARD GAMES** The game of euchre (YOO ker) is played using only the 9s, 10s, jacks, queens, kings, and aces from a standard deck of cards. Find the probability of being dealt a 5-card hand containing all four suits.

EXTRA PRACTICE

See pages 917, 937.

Math Online

Self-Check Quiz at algebra2.com

33. **WRITING** Josh types the five entries in the bibliography of his term paper in random order, forgetting that they should be in alphabetical order by author. What is the probability that he actually typed them in alphabetical order?

H.O.T. Problems

34. **OPEN ENDED** Describe an event that has a probability of 0 and an event that has a probability of 1.

Food & Drug Administration/Science Ph

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CHALLENGE **Theoretical probability** is determined using mathematical methods and assumptions about the fairness of coins, dice, and so on. **Experimental probability** is determined by performing experiments and observing the outcomes.

Determine whether each probability is *theoretical* or *experimental*. Then find the probability.

35. Two dice are rolled. What is the probability that the sum will be 12?

36. A baseball player has 126 hits in 410 at-bats this season. What is the probability that he gets a hit in his next at-bat?

37. A hand of 2 cards is dealt from a standard deck of cards. What is the probability that both cards are clubs?

38. ***Writing in Math*** Use the information on page 697 to explain what probability tells you about life's risks. Include a description of the meaning of *success* and *failure* in the case of being struck by lightning and surviving.

STANDARDIZED TEST PRACTICE

39. **ACT/SAT** What is the value of $\frac{6!}{2!}$?

A 3

B 60

C 360

D 720

40. **REVIEW** A jar contains 4 red marbles, 3 green marbles, and 2 blue marbles. If a marble is drawn at random, what is the probability that it is *not* green?

F $\frac{2}{9}$ **G** $\frac{1}{3}$ **H** $\frac{4}{9}$ **J** $\frac{2}{3}$

Spiral Review

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. (Lesson 12-2)

41. arranging 5 different books on a shelf

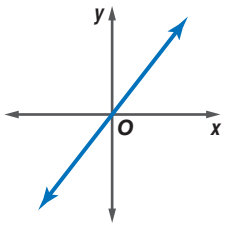
42. arranging the letters of the word *arrange*

43. picking 3 apples from the last 7 remaining at the grocery store

44. How many ways can 4 different gifts be placed into 4 different gift bags if each bag gets exactly 1 gift? (Lesson 12-1)

Identify the type of function represented by each graph. (Lesson 8-5)

45.



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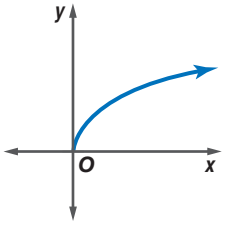

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each product if $a = \frac{3}{5}$, $b = \frac{2}{7}$, $c = \frac{3}{4}$, and $d = \frac{1}{3}$.**

47. ab **48.** bc **49.** cd **50.** bd **51.** ac

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12-4 Multiplying Probabilities

Main Ideas

- Find the probability of two independent events.
- Find the probability of two dependent events.

New Vocabulary

area diagram

GET READY for the Lesson

Yao Ming, of the Houston Rockets, has one of the best field-goal percentages in the National Basketball Association. The table shows the field-goal percentages for three years of his career. For any year, you can determine the probability that Yao will make two field goals in a row based on the probability of his making one field goal.

Season	FG%
2002–03	49.8
2003–04	52.2
2004–05	55.2

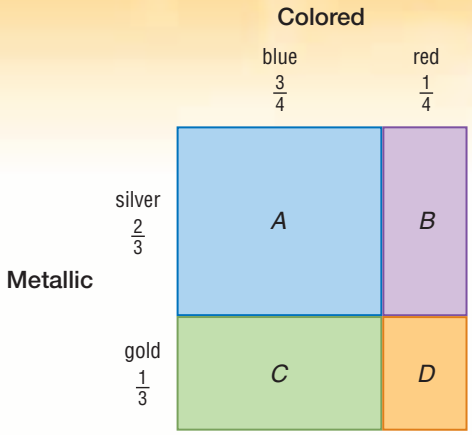
Source: nba.com

Probability of Independent Events In a situation with two events like shooting a field goal and then shooting another, you can find the probability of both events occurring if you know the probability of each event occurring. You can use an **area diagram** to model the probability of the two events occurring.

Algebra Lab

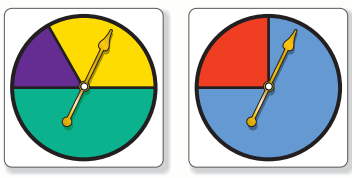
Area Diagrams

Suppose there are 1 red and 3 blue paper clips in one drawer and 1 gold and 2 silver paper clips in another drawer. The area diagram represents the probabilities of choosing one colored paper clip and one metallic paper clip if one of each is chosen at random. For example, rectangle A represents drawing 1 silver clip and 1 blue clip.



MODEL AND ANALYZE

1. Find the areas of rectangles *A*, *B*, *C*, and *D*. Explain what each represents.
2. Find the probability of choosing a red paper clip and a silver paper clip.
3. What are the length and width of the whole square? What is the area? Why does the area need to have this value?
4. Make an area diagram that represents the probability of each outcome if you spin each spinner once. Label the diagram and describe what the area of each rectangle represents.



In Exercise 4 of the lab, spinning one spinner has no effect on the second spinner. These events are independent.

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KEY CONCEPT *Probability of Two Independent Events*

If two events, A and B, are independent, then the probability of both events occurring is $P(A \text{ and } B) = P(A) \cdot P(B)$.

This formula can be applied to any number of independent events.

EXAMPLE 1 Two Independent Events

At a picnic, Julio reaches into an ice-filled cooler containing 8 regular soft drinks and 5 diet soft drinks. He removes a can, then decides he is not really thirsty, and puts it back. What is the probability that Julio and the next person to reach into the cooler both randomly select a regular soft drink?

Explore These events are independent since Julio replaced the can that he removed. The outcome of the second person's selection is not affected by Julio's selection.

Plan Since there are 13 cans, the probability of each person's getting a regular soft drink is $\frac{8}{13}$.

Solve

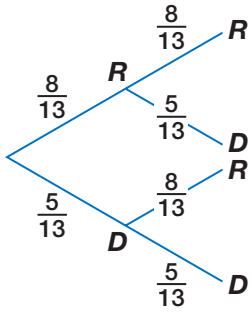
$$P(\text{both regular}) = P(\text{regular}) \cdot P(\text{regular}) \quad \text{Probability of independent events}$$

$$= \frac{8}{13} \cdot \frac{8}{13} \text{ or } \frac{64}{169} \quad \text{Substitute and multiply.}$$

The probability that both people select a regular soft drink is $\frac{64}{169}$ or about 38%.

Check You can verify this result by making a tree diagram that includes probabilities. Let R stand for regular and D stand for diet.

$$P(R, R) = \frac{8}{13} \cdot \frac{8}{13}$$



Study Tip

Alternative Method

You could use the Fundamental Counting Principle to find the number of successes and the number of total outcomes.
both regular = 8 • 8 or 64
total outcomes = 13 • 13 or 169
So, $P(\text{both reg.}) = \frac{64}{169}$

CHECK Your Progress

1. At a promotional event, a radio station lets visitors spin a prize wheel. The wheel has 10 sectors of the same size for posters, 6 for T-shirts, and 2 for concert tickets. What is the probability that two consecutive visitors will win posters?

EXAMPLE 2 Three Independent Events

In a board game, three dice are rolled to determine the number of moves for the players. What is the probability that the first die shows a 6, the second die shows a 6, and the third die does not?

Let A be the event that the first die shows a 6. → $P(A) = \frac{1}{6}$

Let B be the event that the second die shows a 6. → $P(B) = \frac{1}{6}$

Let C be the event that the third die does *not* show a 6. → $P(C) = \frac{5}{6}$

Study Tip

The **complement** of a set is the set of all objects that do *not* belong to the given set. For a six-sided die, showing a 6 is the complement of showing 1, 2, 3, 4, or 5.

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$P(A, B, \text{and } C) = P(A) \cdot P(B) \cdot P(C)$ Probability of independent events

$= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$ or $\frac{5}{216}$ Substitute and multiply.

The probability that the first and second dice show a 6 and the third die does not is $\frac{5}{216}$.

CHECK Your Progress

2. In a state lottery game, each of three cages contains 10 balls. The balls are each labeled with one of the digits 0–9. What is the probability that the first two balls drawn will be even and that the third will be prime?

Study Tip

Conditional Probability

The event of getting a regular soft drink the second time *given* that Julio got a regular soft drink the first time is called a *conditional probability*.

Probability of Dependent Events In Example 1, what is the probability that both people select a regular soft drink if Julio does not put his back in the cooler? In this case, the two events are dependent because the outcome of the first event affects the outcome of the second event.

First selection	Second selection	
$P(\text{regular}) = \frac{8}{13}$	$P(\text{regular}) = \frac{7}{12}$	Notice that when Julio removes his can, there is not only one fewer regular soft drink but also one fewer drink in the cooler.

$P(\text{both regular}) = P(\text{regular}) \cdot P(\text{regular following regular})$

$= \frac{8}{13} \cdot \frac{7}{12}$ or $\frac{14}{39}$ Substitute and multiply.

The probability that both people select a regular soft drink is $\frac{14}{39}$ or about 36%.

KEY CONCEPT *Probability of Two Dependent Events*

If two events, A and B, are dependent, then the probability of both events occurring is $P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$.

This formula can be extended to any number of dependent events.

EXAMPLE **Two Dependent Events**

3 **The host of a game show is drawing chips from a bag to determine the prizes for which contestants will play. Of the 10 chips in the bag, 6 show *television*, 3 show *vacation*, and 1 shows *car*. If the host draws the chips at random and does not replace them, find the probability that he draws a vacation, then a car.**

Because the first chip is not replaced, the events are dependent. Let T represent a television, V a vacation, and C a car.

$P(V \text{ and } C) = P(V) \cdot P(C \text{ following } V)$ Dependent events

$= \frac{3}{10} \cdot \frac{1}{9}$ or $\frac{1}{30}$ After the first chip is drawn, there are 9 left.

The probability of a vacation and then a car is $\frac{1}{30}$ or about 3%.

CHECK Your Progress

3. Use the information above. What is the probability that the host draws two televisions?

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EXAMPLE Three Dependent Events

4 **Three cards are drawn from a standard deck of cards without replacement. Find the probability of drawing a diamond, a club, and another diamond in that order.**

Since the cards are not replaced, the events are dependent. Let D represent a diamond and C a club.

$P(D, C, D) = P(D) \cdot P(C \text{ following } D) \cdot P(D \text{ following } D \text{ and } C)$

$$= \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{12}{50} \text{ or } \frac{13}{850}$$

If the first two cards are a diamond and a club, then 12 of the remaining cards are diamonds.

The probability is $\frac{13}{850}$ or about 1.5%.

CHECK Your Progress

4. Find the probability of drawing three cards of the same suit.

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CHECK Your Understanding

Example 1 (p. 704)

A die is rolled twice. Find each probability.

1. $P(5, \text{ then } 1)$

2. $P(\text{two even numbers})$

Examples 1, 3 (pp. 704, 705)

There are 8 action, 3 comedy, and 5 children's DVDs on a shelf. Suppose two DVDs are selected at random from the shelf. Find each probability.

3. $P(2 \text{ action DVDs})$, if replacement occurs

4. $P(2 \text{ action DVDs})$, if no replacement occurs

5. $P(\text{a comedy DVD, then a children's DVD})$, if no replacement occurs

Examples 2, 4 (pp. 704, 706)

Three cards are drawn from a standard deck of cards. Find each probability.

6. $P(3 \text{ hearts})$, if replacement occurs

7. $P(3 \text{ hearts})$, if no replacement occurs

Determine whether the events are *independent* or *dependent*. Then find the probability.

8. A black die and a white die are rolled. What is the probability that a 3 shows on the black die and a 5 shows on the white die?

9. Yana has 4 black socks, 6 blue socks, and 8 white socks in his drawer. If he selects three socks at random with no replacement, what is the probability that he will first select a blue sock, then a black sock, and then another blue sock?

Example 3 (p. 705)

Two cards are drawn from a standard deck of cards. Find each probability if no replacement occurs.

10. $P(\text{two hearts})$

11. $P(\text{ace, then king})$

Exercises

A die is rolled twice. Find each probability.

12. $P(2, \text{ then } 3)$

13. $P(\text{no 6s})$

14. $P(\text{two 4s})$

15. $P(1, \text{ then any number})$

16. $P(\text{two of the same number})$

17. $P(\text{two different numbers})$

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HOMEWORK HELP	
For Exercises	See Examples
12–20	1
21–29	3
30–35	1–4

The tiles *E, T, F, U, N, X,* and *P* of a word game are placed face down in the lid of the game. If two tiles are chosen at random, find each probability.

18. *P*(*E*, then *N*), if replacement occurs

19. *P*(2 consonants), if replacement occurs

20. *P*(*T*, then *D*), if replacement occurs

21. *P*(*X*, then *P*), if no replacement occurs

22. *P*(2 consonants), if no replacement occurs

23. *P*(selecting the same letter twice), if no replacement occurs

Anita scores well enough at a carnival game that she gets to randomly draw two prizes out of a prize bag. There are 6 purple T-shirts, 8 yellow T-shirts, and 5 T-shirts with a picture of a celebrity on them in the bag. Find each probability.

24. *P*(choosing 2 purple)

25. *P*(choosing 2 celebrity)

26. *P*(choosing a yellow, then a purple)

27. *P*(choosing a celebrity, then a yellow)

28. **ELECTIONS** Tami, Sonia, Malik, and Roger are the four candidates for Student Council president. If their names are placed in random order on the ballot, what is the probability that Malik's name will be first on the ballot followed by Sonia's name second?

29. **CHORES** The five children of the Blanchard family get weekly chores assigned to them at random. Their parents put pieces of paper with the names of the five children in a hat and draw them out. The order of the names pulled determines the order in which the children will be responsible for sorting laundry for the next five weeks. What is the probability that Jim will be responsible for the first week and Emily will be responsible for the fifth week?

Real-World Link

Three hamsters domesticated in 1930 are the ancestors of most of the hamsters sold as pets or used for research.

Source: www.ahc.umn.edu

Determine whether the events are *independent* or *dependent*. Then find the probability.

30. There are 3 miniature chocolate bars and 5 peanut butter cups in a candy dish. Judie chooses 2 of them at random. What is the probability that she chose 2 miniature chocolate bars?

31. A cage contains 3 white and 6 brown hamsters. Maggie randomly selects one, puts it back, and then randomly selects another. What is the probability that both selections were white?

32. A bag contains 7 red, 4 blue, and 6 yellow marbles. If 3 marbles are selected in succession, what is the probability of selecting blue, then yellow, then red, if replacement occurs each time?

33. Jen's purse contains three $1 bills, four $5 bills, and two $10 bills. If she selects three bills in succession, find the probability of selecting a $10 bill, then a $5 bill, and then a $1 bill if the bills are not replaced.

34. What is the probability of getting heads each time if a coin is tossed 5 times?

35. When Ramon plays basketball, he makes an average of two out of every three foul shots he takes. What is the probability that he will make the next three foul shots in a row?

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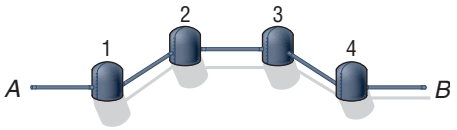
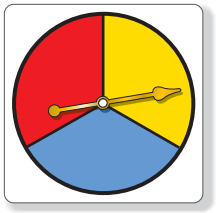
36. UTILITIES A city water system includes a sequence of 4 pumps as shown below. Water enters the system at point A, is pumped through the system by pumps at locations 1, 2, 3, and 4, and exits the system at point B.



If the probability of failure for any one pump is $\frac{1}{100}$, what is the probability that water will flow all the way through the system from A to B?

37. FISHING Suppose a sport fisher has a 35% chance of catching a fish that he can keep each time he goes to a spot. What is the probability that he catches a fish the first 4 times he visits the spot but on the fifth visit he does not?

For Exercises 38–41, suppose you spin the spinner twice.



38. Sketch a tree diagram showing all of the possibilities. Use it to find the probability of spinning red and then blue.

39. Sketch an area diagram of the outcomes. Shade the region on your area diagram corresponding to getting the same color twice.

40. What is the probability that you get the same color on both spins?

41. If you got the same color twice, what is the probability that the color was red?

Real-World Link

There are 2,598,960 different 5-card hands that can be drawn from a 52-card deck. Of these, only 5108 are hands in which all 5 cards are of the same suit.

Find each probability if 13 cards are drawn from a standard deck of cards and no replacement occurs.

42. P(all hearts)

43. P(all red cards)

44. P(all one suit)

45. P(no kings)

For Exercises 46–48, use the following information.
A bag contains 10 marbles. In this problem, a *cycle* means that you draw a marble, record its color, and put it back.

46. You go through the cycle 10 times. If you do not record any black marbles, can you conclude that there are no black marbles in the bag?

47. Can you conclude that there are none if you repeat the cycle 50 times?

48. How many times do you have to repeat the cycle to be certain that there are no black marbles in the bag? Explain your reasoning.

EXTRA PRACTICE
See pages 917, 937.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

49. OPEN ENDED Describe two real-life events that are dependent.

50. FIND THE ERROR Mario and Tamara are calculating the probability of getting a 4 and then a 2 if they roll a die twice. Who is correct? Explain your reasoning.

Mario

$P(4, \text{then } 2) = \frac{1}{6} \cdot \frac{1}{6}$

$= \frac{1}{36}$

Tamara

$P(4, \text{then } 2) = \frac{1}{6} \cdot \frac{1}{5}$

$= \frac{1}{30}$

Ian McKinnell/Getty Images
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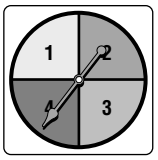
51. CHALLENGE If one bulb in a string of holiday lights fails to work, the whole string will not light. If each bulb in a set has a 99.5% chance of working, what is the maximum number of lights that can be strung together with at least a 90% chance of the whole string lighting?

52. *Writing in Math* Use the information on page 703 to explain how probability applies to basketball. Explain how a value such as one of those in the table could be used to find the chances of Yao Ming making 0, 1, or 2 of 2 successive field goals, assuming the 2 field goals are independent, and a possible reason why 2 field goals might not be independent.

STANDARDIZED TEST PRACTICE

53. ACT/SAT The spinner is spun four times. What is the probability that the spinner lands on 2 each time?



A $\frac{1}{2}$

B $\frac{1}{4}$

C $\frac{1}{16}$

D $\frac{1}{256}$

54. REVIEW A coin is tossed and a die is rolled. What is the probability of a head and a 3?

F $\frac{1}{4}$

G $\frac{1}{8}$

H $\frac{1}{12}$

J $\frac{1}{24}$

Spiral Review

A gumball machine contains 7 red, 8 orange, 9 purple, 7 white, and 5 yellow gumballs. Tyson buys 3 gumballs. Find each probability, assuming that the machine dispenses the gumballs at random. (Lesson 12-3)

55. P(3 red)

56. P(2 white, 1 purple)

57. PHOTOGRAPHY A photographer is taking a picture of a bride and groom together with 6 attendants. How many ways can he arrange the 8 people in a row if the bride and groom stand in the middle? (Lesson 12-2)

Solve each equation. Check your solutions. (Lesson 9-3)

58. $\log_5 5 + \log_5 x = \log_5 30$

59. $\log_{16} c - 2 \log_{16} 3 = \log_{16} 4$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 6-7)

60. $x^3 - x^2 - 10x + 6; x + 3$

61. $x^3 - 7x^2 + 12x; x - 3$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each sum if $a = \frac{1}{2}$, $b = \frac{1}{6}$, $c = \frac{2}{3}$, and $d = \frac{3}{4}$.**

62. $a + b$

63. $b + c$

64. $a + d$

65. $b + d$

66. $c + a$

67. $c + d$

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12-5 Adding Probabilities

Main Ideas

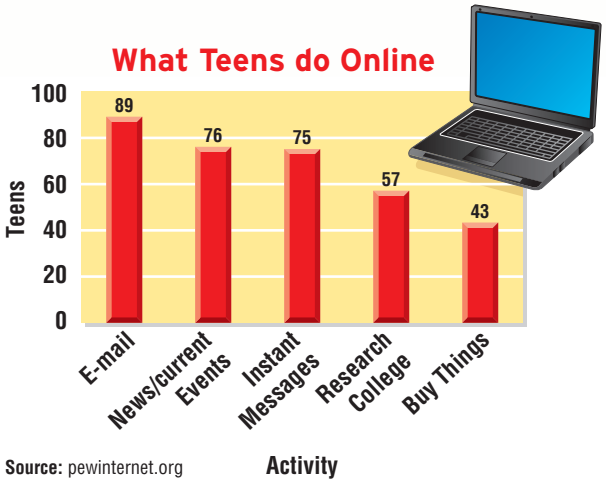
- Find the probability of mutually exclusive events.
- Find the probability of inclusive events.

New Vocabulary

simple event
compound event
mutually exclusive events
inclusive events

GET READY for the Lesson

The graph shows the results of a survey about what teens do online. Determining the probability that a randomly selected teen sends/reads e-mail or buys things online requires adding probabilities.



Mutually Exclusive Events When you roll a die, an event such as rolling a 1 is called a **simple event** because it cannot be broken down into smaller events. An event that consists of two or more simple events is called a **compound event**. For example, the event of rolling an odd number or a number greater than 5 is a compound event because it consists of the simple events rolling a 1, rolling a 3, rolling a 5, or rolling a 6.

When there are two events, it is important to understand how they are related before finding the probability of one or the other event occurring. Suppose you draw a card from a standard deck of cards. What is the probability of drawing a 2 or an ace? Since a card cannot be both a 2 *and* an ace, these are called **mutually exclusive events**. That is, the two events cannot occur at the same time. The probability of drawing a 2 or an ace is found by adding their individual probabilities.

$$P(2 \text{ or ace}) = P(2) + P(\text{ace}) \quad \text{Add probabilities.}$$

$$= \frac{4}{52} + \frac{4}{52} \quad \text{There are 4 twos and 4 aces in a deck.}$$

$$= \frac{8}{52} \text{ or } \frac{2}{13} \quad \text{Simplify.}$$

The probability of drawing a 2 or an ace is $\frac{2}{13}$.

Study Tip

Formula
This formula can be extended to any number of mutually exclusive events.

KEY CONCEPT *Probability of Mutually Exclusive Events*

Words If two events, A and B, are mutually exclusive, then the probability that A or B occurs is the sum of their probabilities.

Symbols $P(A \text{ or } B) = P(A) + P(B)$

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EXAMPLE 1 Two Mutually Exclusive Events

Keisha has a stack of 8 baseball cards, 5 basketball cards, and 6 soccer cards. If she selects a card at random from the stack, what is the probability that it is a baseball or a soccer card?

These are mutually exclusive events, since the card cannot be both a baseball card *and* a soccer card. Note that there is a total of 19 cards.

$P(\text{baseball or soccer}) = P(\text{baseball}) + P(\text{soccer})$ Mutually exclusive events

$= \frac{8}{19} + \frac{6}{19}$ or $\frac{14}{19}$ Substitute and add.

The probability that Keisha selects a baseball or a soccer card is $\frac{14}{19}$.

CHECK Your Progress

1. One teacher must be chosen to supervise a senior class fund-raiser. There are 12 math teachers, 9 language arts teachers, 8 social studies teachers, and 10 science teachers. If the teacher is chosen at random, what is the probability that the teacher is either a language arts teacher or a social studies teacher?

To extend the formula to more than two events, add the probabilities for all of the events.

EXAMPLE 2 Three Mutually Exclusive Events

There are 7 girls and 6 boys on the junior class homecoming committee. A subcommittee of 4 people is being chosen at random to decide the theme for the class float. What is the probability that the subcommittee will have at least 2 girls?

At least 2 girls means that the subcommittee may have 2, 3, or 4 girls. It is not possible to select a group of 2 girls, a group of 3 girls, and a group of 4 girls all in the same 4-member subcommittee, so the events are mutually exclusive. Add the probabilities of each type of committee.

$P(\text{at least 2 girls}) = P(2 \text{ girls}) + P(3 \text{ girls}) + P(4 \text{ girls})$

$= \frac{C(7, 2) \cdot C(6, 2)}{C(13, 4)} + \frac{C(7, 3) \cdot C(6, 1)}{C(13, 4)} + \frac{C(7, 4) \cdot C(6, 0)}{C(13, 4)}$ (2 girls, 2 boys; 3 girls, 1 boy; 4 girls, 0 boys)

$= \frac{315}{715} + \frac{210}{715} + \frac{35}{715}$ or $\frac{112}{143}$ Simplify.

The probability of at least 2 girls on the subcommittee is $\frac{112}{143}$ or about 0.78.

Study Tip

Choosing a Committee

$C(13, 4)$ refers to choosing 4 subcommittee members from 13 committee members. Since order does not matter, the number of combinations is found.

CHECK Your Progress

2. The Cougar basketball team can send 5 players to a basketball clinic. Six guards and 5 forwards would like to attend the clinic. If the players are selected at random, what is the probability that at least 3 of the players selected to attend the clinic will be forwards?

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Inclusive Events What is the probability of drawing a king or a spade from a standard deck of cards? Since it is possible to draw a card that is both a king and a spade, these events are not mutually exclusive. These are called **inclusive events**.

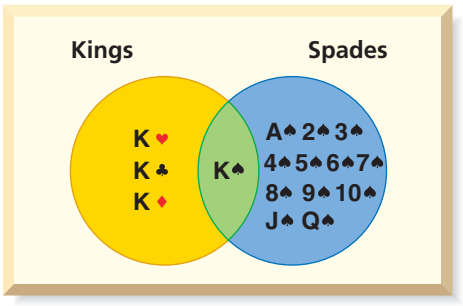
P(king)	P(spade)	P(spade, king)
$\frac{4}{52}$	$\frac{13}{52}$	$\frac{1}{52}$
1 king in each suit	spades	king of spades

Study Tip

Common Misconception

In mathematics, unlike everyday language, the expression *A* or *B* allows the possibility of both *A* and *B* occurring.

In the first two fractions above, the probability of drawing the king of spades is counted twice, once for a king and once for a spade. To find the correct probability, you must subtract P(king of spades) from the sum of the first two probabilities.



$$P(\text{king or spade}) = P(\text{king}) + P(\text{spade}) - P(\text{king of spades})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \text{ or } \frac{4}{13}$$

The probability of drawing a king or a spade is $\frac{4}{13}$.

KEY CONCEPT *Probability of Inclusive Events*

Words If two events, *A* and *B*, are inclusive, then the probability that *A* or *B* occurs is the sum of their probabilities decreased by the probability of both occurring.

Symbols $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

EXAMPLE **Inclusive Events**

3 **EDUCATION** **Suppose that of 1400 students, 550 take Spanish, 700 take biology, and 400 take both Spanish and biology. What is the probability that a student selected at random takes Spanish or biology?**

Since some students take both Spanish and biology, the events are inclusive.

$$P(\text{Spanish}) = \frac{550}{1400} \qquad P(\text{biology}) = \frac{700}{1400} \qquad P(\text{Spanish and biology}) = \frac{400}{1400}$$

$$P(\text{Spanish or biology}) = P(\text{Spanish}) + P(\text{biology}) - P(\text{Spanish and biology})$$

$$= \frac{550}{1400} + \frac{700}{1400} - \frac{400}{1400} \text{ or } \frac{17}{28} \quad \text{Substitute and simplify.}$$

The probability that a student selected at random takes Spanish or biology is $\frac{17}{28}$.

CHECK Your Progress

3. Sixty plastic discs, each with one of the numbers from 1 to 60, are in a bag. LaTanya will win a game if she can pull out any disc with a number divisible by 2 or 3. What is the probability that LaTanya will win?

Concepts in Motion
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CHECK Your Understanding

Examples 1–3 (pp. 711–712)

A die is rolled. Find each probability.

1. P(1 or 6)
2. P(at least 5)
3. P(less than 3)
4. P(even or prime)
5. P(multiple of 3 or 4)
6. P(multiple of 2 or 3)

Examples 2, 3 (pp. 711–712)

A card is drawn from a standard deck of cards. Determine whether the events are *mutually exclusive* or *inclusive*. Then find the probability.

7. P(6 or king)
8. P(queen or spade)

Example 2 (p. 711)

9. SCHOOL There are 8 girls and 8 boys on the Student Senate. Three of the students are seniors. What is the probability that a person selected from the Student Senate is not a senior?

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–19	1, 2
20–23	1–3
24–29	3

Jesse has eight friends who have volunteered to help him with a school fundraiser. Five are boys and 3 are girls. If he randomly selects 3 friends to help him, find each probability.

10. P(2 boys or 2 girls)
11. P(all boys or all girls)
12. P(at least 2 girls)
13. P(at least 1 boy)

Six girls and eight boys walk into a video store at the same time. There are six salespeople available to help them. Find the probability that the salespeople will first help the given numbers of girls and boys.

14. P(4 girls, 2 boys or 4 boys, 2 girls)
15. P(5 girls, 1 boy or 5 boys, 1 girl)
16. P(all girls or all boys)
17. P(at least 4 boys)
18. P(at least 5 girls or at least 5 boys)
19. P(at least 3 girls)

For Exercises 20–23, determine whether the events are *mutually exclusive* or *inclusive*. Then find the probability.

20. There are 4 algebra books, 3 literature books, and 2 biology books on a shelf. If a book is randomly selected, what is the probability of selecting a literature book or an algebra book?

21. A die is rolled. What is the probability of rolling a 5 or a number greater than 3?

22. In the Math Club, 7 of the 20 girls are seniors, and 4 of the 14 boys are seniors. What is the probability of randomly selecting a boy or a senior to represent the Math Club at a statewide math contest?

23. A card is drawn from a standard deck of cards. What is the probability of drawing an ace or a face card? (*Hint:* A face card is a jack, queen, or king.)

24. One tile with each letter of the alphabet is placed in a bag, and one is drawn at random. What is the probability of selecting a vowel or a letter from the word *function?*

25. Each of the numbers from 1 to 30 is written on a card and placed in a bag. If one card is drawn at random, what is the probability that the number is a multiple of 2 or a multiple of 3?

Two cards are drawn from a standard deck of cards. Find each probability.

26. P(both queens or both red)
27. P(both jacks or both face cards)
28. P(both face cards or both black)
29. P(both either black or an ace)

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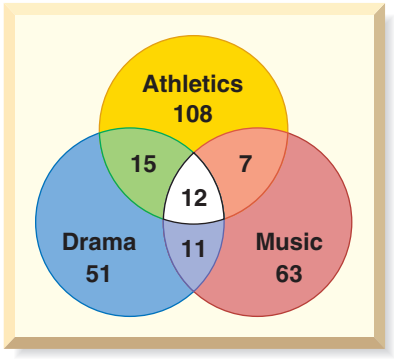
GAMES **For Exercises 30–35, use the following information.**
A certain game has two stacks of 30 tiles with pictures on them. In the first stack of tiles, there are 10 dogs, 4 cats, 5 balls, and 11 horses. In the second stack of tiles, there are 3 flowers, 8 fish, 12 balls, 2 cats, and 5 horses. The top tile in each stack is chosen. Find each probability.

30. P(each is a ball)
31. P(neither is a horse)
32. P(exactly one is a ball)
33. P(exactly one is a fish)
34. P(both are a fish)
35. P(one is a dog and one is a flower)

BASEBALL **For Exercises 36–38, use the following information.**
Albert and Paul are on the school baseball team. Albert has a batting average of .4, and Paul has a batting average of .3. That means that Albert gets a hit 40% of his at bats and Paul gets a hit 30% of his times at bat. What is the probability that—

36. both Albert and Paul are able to get hits their first time at bat?
37. neither Albert nor Paul is able to get a hit their first time at bat?
38. at least one of the two friends is able to get a hit their first time at bat?

SCHOOL **For Exercises 39–41, use the Venn diagram that shows the number of participants in extracurricular activities for a junior class of 324 students.**
Determine each probability if a student is selected at random from the class.



39. P(drama or music)
40. P(drama or athletics)
41. P(athletics and drama, or music and athletics)

EXTRA PRACTICE
See pages 918, 937.
MathOnline
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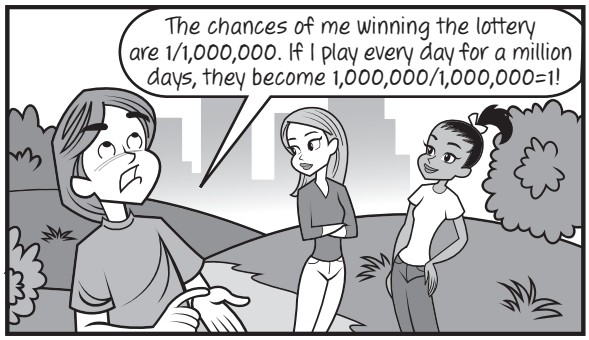
H.O.T. Problems

42. **REASONING** What is wrong with the conclusion in the comic?

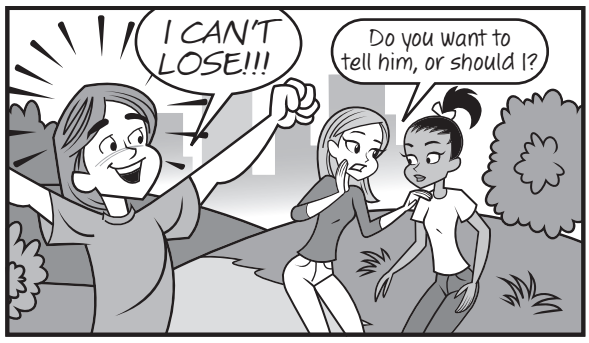


43. **OPEN ENDED** Describe two mutually exclusive events and two inclusive events.

44. **CHALLENGE** A textbook gives the following probability equation for events A and B that are mutually exclusive or inclusive.

$P(A \text{ and } B) = P(A) + P(B) - P(A \text{ or } B)$

Is this correct? Explain.



45. ***Writing in Math*** Use the information on page 710 to explain how probability applies to what teens do online. Include an explanation of whether the events listed in the graphic are mutually exclusive or inclusive.

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STANDARDIZED TEST PRACTICE

46. ACT/SAT In a jar of red and white gumballs, the ratio of white gumballs to red gumballs is 5:4. If the jar contains a total of 180 gumballs, how many of them are red?

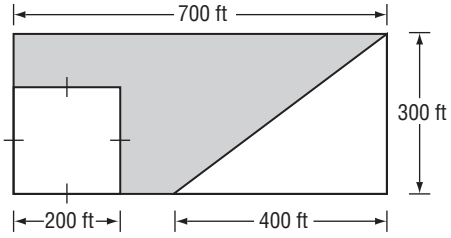
A 45

B 64

C 80

D 100

47. REVIEW What is the area of the shaded part of the rectangle below?



F 90,000 ft^2

G 110,000 ft^2

H 130,000 ft^2

J 150,000 ft^2

Spiral Review

A die is rolled three times. Find each probability. (Lesson 12-4)

48. P(1, then 2, then 3)

49. P(no 4s)

50. P(three 1s)

51. P(three even numbers)

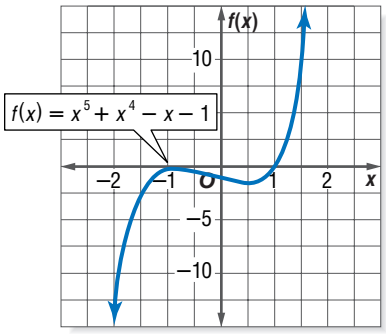
52. BOOKS Dan has twelve books on his shelf that he has not read yet. There are seven novels and five biographies. He wants to take four books with him on vacation. What is the probability that he randomly selects two novels and two biographies? (Lesson 12-3)

Find the sum of each series. (Lessons 11-2 and 11-4)

53. $2 + 4 + 8 + \cdots + 128$

54. $\sum_{n=1}^{3} (5n - 2)$

55. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all factors of the polynomial. (Lesson 6-7)



SPEED SKATING **For Exercises 56 and 57, use the following information.**
In 2001, Catriona LeMay Doan set a world record for women's speed skating by skating approximately 13.43 meters per second in the 500-meter race. (Lesson 2-6)

56. Suppose she could maintain that speed. Write an equation that represents how far she could travel in t seconds.

57. What type of function does the equation in Exercise 56 represent?

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the mean, median, mode, and range for each set of data. Round to the nearest hundredth, if necessary.** (Pages 759 and 760)

58. 298, 256, 399, 388, 276

59. 3, 75, 58, 7, 34

60. 4.8, 5.7, 2.1, 2.1, 4.8, 2.1

61. 80, 50, 65, 55, 70, 65, 75, 50

62. 61, 89, 93, 102, 45, 89

63. 13.3, 15.4, 12.5, 10.7

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CHAPTER 12

Mid-Chapter Quiz

Lessons 12-1 through 12-5

1. **RESTAURANT** At Burger Hut, you can order your hamburger with or without cheese, onions, or pickles, and rare, medium, or well-done. How many different ways can you order your hamburger? (Lesson 12-1)
2. **AUTOMOBILES** For a particular model of car, a dealer offers 3 sizes of engines, 2 types of stereos, 18 body colors, and 7 upholstery colors. How many different possibilities are available for that model? (Lesson 12-1)
3. **CODES** How many codes consisting of a letter followed by 3 digits can be made if no digit can be used more than once? (Lesson 12-1)
4. **ROUTES** There are 4 different routes a student can bike from his house to school. In how many ways can he make a round trip if he uses a different route coming than going? (Lesson 12-1)

Evaluate each expression. (Lesson 12-2)

5. $P(12, 3)$
6. $C(8, 3)$

Determine whether each situation involves a permutation or a combination. Then find the number of possibilities. (Lesson 12-2)

7. 8 cars in a row parked next to a curb
8. a hand of 6 cards from a standard deck of cards

9. **MULTIPLE CHOICE** A box contains 10 silver, 9 green, 8 blue, 11 pink, and 12 yellow paper clips. If a paperclip is drawn at random, what is the probability that it is *not* yellow? (Lesson 12-3)

   A $\frac{1}{5}$
   B $\frac{6}{25}$
   C $\frac{19}{25}$
   D $\frac{3}{5}$

Two cards are drawn from a standard deck of cards. Find each probability. (Lesson 12-3)

10. P(2 aces)
11. P(1 heart, 1 club)
12. P(1 queen, 1 king)

A bag contains colored marbles as shown in the table below. Two marbles are drawn at random from the bag. Find each probability. (Lesson 12-4)

Color	Number
red	5
green	3
blue	2

13. P(red, then green) if replacement occurs
14. P(red, then green) if no replacement occurs
15. P(2 red) if no replacement occurs
16. P(2 red) if replacement occurs

A twelve-sided die has sides numbered 1 through 12. The die is rolled once. Find each probability. (Lesson 12-5)

17. P(4 or 5)
18. P(even or a multiple of 3)
19. P(odd or a multiple 4)

20. **MULTIPLE CHOICE** In a box of chocolate and yellow cupcakes, the ratio of chocolate cupcakes to yellow cupcakes is 3:2. If the box contains 20 cupcakes, how many of them are chocolate? (Lesson 12-5)

   F 9
   G 10
   H 11
   J 12

21. **MULTIPLE CHOICE** A company received job applications from 2000 people. Six hundred of the applicants had the desired education, 1200 had the desired work experience, and 400 had both the desired education and work experience. What is the probability that an applicant selected at random will have the desired education or work experience?

   A $\frac{3}{10}$
   B $\frac{1}{2}$
   C $\frac{7}{10}$
   D $\frac{9}{10}$

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12-6 Statistical Measures

Main Ideas

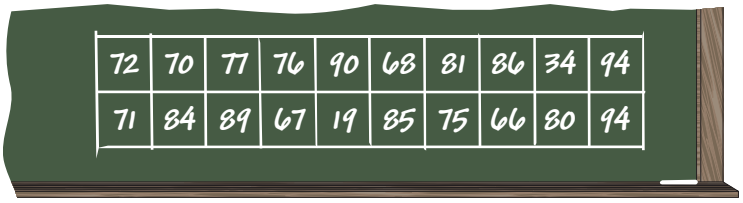
- Use measures of central tendency to represent a set of data.
- Find measures of variation for a set of data.

New Vocabulary

univariate data
measure of central tendency
measure of variation
dispersion
variance
standard deviation

GET READY for the Lesson

On Mr. Dent's most recent Algebra 2 test, his students earned the following scores.



When his students ask how they did on the test, which measure of central tendency should Mr. Dent use to describe the scores?

Measures of Central Tendency Data with one variable, such as the test scores, are called **univariate data**. Sometimes it is convenient to have one number that describes a set of data. This number is called a **measure of central tendency**, because it represents the center or middle of the data. The most commonly used measures of central tendency are the *mean*, *median*, and *mode*.

When deciding which measure of central tendency to use to represent a set of data, look closely at the data itself.

Study Tip

Look Back

To review **outliers**, see Lesson 2-5.

CONCEPT SUMMARY *Measures of Tendency*

Use	When . . .
mean	the data are spread out, and you want an average of the values
median	the data contain outliers
mode	the data are tightly clustered around one or two values

EXAMPLE 1 Choose a Measure of Central Tendency

SWEEPSTAKES **A sweepstakes offers a first prize of $10,000, two second prizes of $100, and one hundred third prizes of $10. Which measure of central tendency best represents the available prizes?**

Since 100 of the 103 prizes are $10, the mode ($10) best represents the available prizes. Notice that in this case the median is the same as the mode.

CHECK Your Progress

1. Which measure of central tendency would the organizers of the sweepstakes be most likely to use in their advertising?

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Measures of Variation

Measures of variation or **dispersion** measure how spread out or scattered a set of data is. The simplest measure of variation to calculate is the *range*, the difference between the greatest and the least values in a set of data. Variance and standard deviation are measures of variation that indicate how much the data values differ from the mean.

Reading Math

Symbols The symbol σ is the lower case Greek letter *sigma*. $\bar{x}$ is read *x bar*.

To find the **variance** σ^2 of a set of data, follow these steps.

1. Find the mean, $\bar{x}$.
2. Find the difference between each value in the set of data and the mean.
3. Square each difference.
4. Find the mean of the squares.

The **standard deviation** σ is the square root of the variance.

KEY CONCEPT — *Standard Deviation*

If a set of data consists of the n values $x_1, x_2, \ldots, x_n$ and has mean $\bar{x}$, then the standard deviation σ is given by the following formula.

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

EXAMPLE 2 Standard Deviation

STATES **The table shows the populations in millions of 11 eastern states as of the 2000 Census. Find the variance and standard deviation of the data to the nearest tenth.**

State	Population	State	Population	State	Population
NY	19.0	MD	5.3	RI	1.0
PA	12.3	CT	3.4	DE	0.8
NJ	8.4	ME	1.3	VT	0.6
MA	6.3	NH	1.2	–	–

Source: U.S. Census Bureau

Step 1 Find the mean. Add the data and divide by the number of items.

$$\bar{x} = \frac{19.0 + 12.3 + 8.4 + 6.3 + 5.3 + 3.4 + 1.3 + 1.2 + 1.0 + 0.8 + 0.6}{11}$$

$\approx 5.4\overline{18}$ The mean is about 5.4 million people.

Step 2 Find the variance.

$$\sigma^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}$$ Variance formula

$$= \frac{(19.0 - 5.4)^2 + (12.3 - 5.4)^2 + \cdots + (8.0 - 5.4)^2 + (0.6 - 5.4)^2}{11}$$

$$= \frac{344.4}{11}$$ Simplify.

$\approx 31.3\overline{09}$ The variance is about 31.3.

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Step 3 Find the standard deviation.

$\sigma^2 \approx 31.3$ Take the square root of each side.

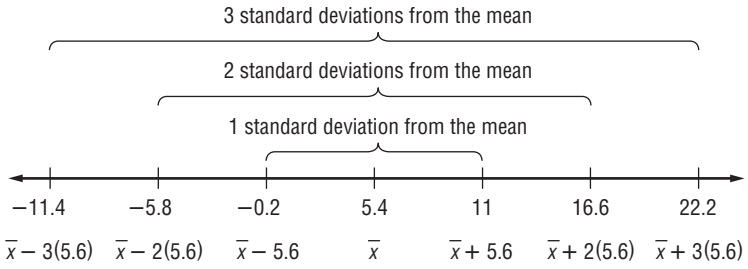
$\sigma \approx 5.594640292$ The standard deviation is about 5.6 million people.

CHECK Your Progress

2. The leading number of home runs in Major League Baseball for the 1994–2004 seasons were 43, 50, 52, 56, 70, 65, 50, 73, 57, 47, and 48. Find the variance and standard deviation of the data to the nearest tenth.

Online Personal Tutor at algebra2.com

Most of the members of a set of data are within 1 standard deviation of the mean. The data in Example 2 can be broken down as shown below.



Looking at the original data, you can see that most of the states' populations were between 2.4 million and 20.2 million. That is, the majority of members of the data set were within 1 standard deviation of the mean.

You can use a TI-83/84 Plus graphing calculator to find statistics for the data in Example 2.

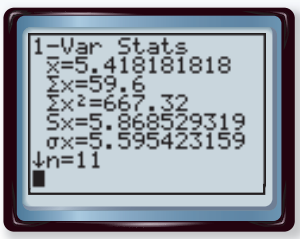
GRAPHING CALCULATOR LAB

One-Variable Statistics

The TI-83/84 Plus can compute a set of one-variable statistics from a list of data. These statistics include the mean, variance, and standard deviation. Enter the data into L1.

KEYSTROKES: STAT ENTER 19.0 ENTER 12.3 ENTER . . .

Then use STAT ▶ 1 ENTER to show the statistics. The mean $\bar{x}$ is about 5.4, the sum of the values $\sum x$ is 59.6, the standard deviation σx is about 5.6, and there are $n = 11$ data items. If you scroll down, you will see the least value (minX = .6), the three quartiles (1, 3.4, and 8.4), and the greatest value (maxX = 19).



THINK AND DISCUSS

1. Find the variance of the data set.

2. Enter the data set in list L1 but without the outlier 19.0. What are the new mean, median, and standard deviation?

3. Did the mean or median change less when the outlier was deleted?

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CHECK Your Understanding

Example 1 (pp. 717–718)

EDUCATION For Exercises 1 and 2, use the following information.
The table below shows the amounts of money spent on education per student in a recent year in two regions of the United States.

Pacific States		Southwest Central States	
State	Expenditures per Student ($)	State	Expenditures per Student ($)
Alaska	9564	Texas	6771
California	7405	Arkansas	6276
Washington	7039	Louisiana	6567
Oregon	7642	Oklahoma	6229

Source: *The World Almanac*

1. Find the mean for each region.
2. For which region is the mean more representative of the data? Explain.

Example 2 (pp. 718–719)

Find the variance and standard deviation of each set of data to the nearest tenth.

3. {48, 36, 40, 29, 45, 51, 38, 47, 39, 37}
4. {321, 322, 323, 324, 325, 326, 327, 328, 329, 330}
5. {43, 56, 78, 81, 47, 42, 34, 22, 78, 98, 38, 46, 54, 67, 58, 92, 55}

Exercises

HOMEWORK HELP	
For Exercises	See Examples
6–13, 24–30	2
14–23	1

Find the variance and standard deviation of each set of data to the nearest tenth.

6. {400, 300, 325, 275, 425, 375, 350}
7. {5, 4, 5, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 9}
8. {2.4, 5.6, 1.9, 7.1, 4.3, 2.7, 4.6, 1.8, 2.4}
9. {4.3, 6.4, 2.9, 3.1, 8.7, 2.8, 3.6, 1.9, 7.2}
10. {234, 345, 123, 368, 279, 876, 456, 235, 333, 444}
11. {13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 67, 56, 34, 99, 44, 55}

12.

Stem	Leaf
4	4 5 6 7 7
5	3 5 6 7 8 9
6	7 7 8 9 9 9

4 | 5 = 45

13.

Stem	Leaf
5	7 7 7 8 9
6	3 4 5 5 6 7
7	2 3 4 5 6

6 | 3 = 63

BASKETBALL For Exercises 14 and 15, use the following information.
The table below shows the rebounding totals for the members of the 2005 Charlotte Sting.

162	145	179	37	44	53	70	65	47	35	71	5	5

Source: WNBA

14. Find the mean, median, and mode of the data to the nearest tenth.
15. Which measure of central tendency best represents the data? Explain your answer.

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ADVERTISING **For Exercises 16–18, use the following information.**
An electronics store placed an ad in the newspaper showing five flat-screen TVs for sale. The ad says, "Our flat-screen TVs average $695." The prices of the flat-screen TVs are $1200, $999, $1499, $895, $695, $1100, $1300, and $695.

16. Find the mean, median, and mode of the prices.

17. Which measure is the store using in its ad? Why did they choose it?

18. As a consumer, which measure would you want to see advertised? Explain.

EDUCATION **For Exercises 19 and 20, use the following information.**
The Millersburg school board is negotiating a pay raise with the teacher's union. Three of the administrators have salaries of $90,000 each. However, a majority of the teachers have salaries of about $45,000 per year.

19. You are a member of the school board and would like to show that the current salaries are reasonable. Would you quote the mean, median, or mode as the "average" salary to justify your claim? Explain.

20. You are the head of the teacher's union and maintain that a pay raise is in order. Which of the mean, median, or mode would you quote to justify your claim? Explain your reasoning.

Real-World Link
While the Mall of America does not have the most gross leasable area, it is the largest fully enclosed retail and entertainment complex in the United States.
Source: Mall of America

SHOPPING MALLS **For Exercises 21–23, use the following information.**
The table lists the areas of some large shopping malls in the United States.

Mall	Gross Leasable Area (ft^2)
1 Del Amo Fashion Center, Torrance, CA	3,000,000
2 South Coast Plaza/Crystal Court, Costa Mesa, CA	2,918,236
3 Mall of America, Bloomington, MN	2,472,500
4 Lakewood Center Mall, Lakewood, CA	2,390,000
5 Roosevelt Field Mall, Garden City, NY	2,300,000
6 Gurnee Mills, Gurnee, IL	2,200,000
7 The Galleria, Houston, TX	2,100,000
8 Randall Park Mall, North Randall, OH	2,097,416
9 Oakbrook Shopping Center, Oak Brook, IL	2,006,688
10 Sawgrass Mills, Sunrise, FL	2,000,000
10 The Woodlands Mall, The Woodlands, TX	2,000,000
10 Woodfield, Schaumburg, IL	2,000,000

Source: Blackburn Marketing Service

21. Find the mean, median, and mode of the gross leasable areas.

22. You are a realtor who is trying to lease mall space in different areas of the country to a large retailer. Which measure would you talk about if the customer felt that the malls were too large for his store? Explain.

23. Which measure would you talk about if the customer had a large inventory? Explain.

SCHOOL **For Exercises 24–26, use the frequency table at the right that shows the scores on a multiple-choice test.**

Score	Frequency
90	3
85	2
80	3
75	7
70	6
65	4

24. Find the variance and standard deviation of the scores.

25. What percent of the scores are within one standard deviation of the mean?

26. What percent of the scores are within two standard deviations of the mean?

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FOOTBALL **For Exercises 27–30, use the weights in pounds of the starting offensive linemen of the football teams from three high schools.**

Jackson	Washington	King
170, 165, 140, 188, 195	144, 177, 215, 225, 197	166, 175, 196, 206, 219

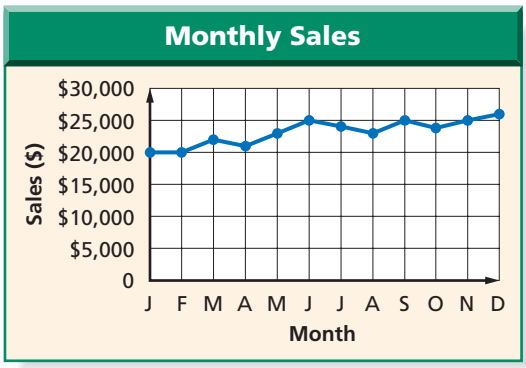
27. Find the standard deviation of the weights for Jackson High.

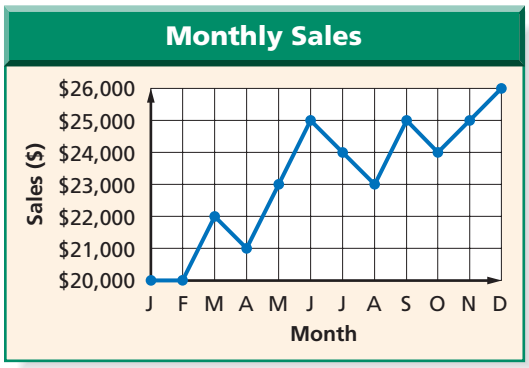
28. Find the standard deviation of the weights for Washington High.

29. Find the standard deviation of the weights for King High.

30. Which team had the most variation in weights? How do you think this variation will impact their play?

For Exercises 31–33, consider the two graphs below.





31. Explain why the graphs made from the same data look different.

32. Describe a situation where the first graph might be used.

33. Describe a situation where the second graph might be used.

EXTRA PRACTICE
See pages 918, 937.
MathOnline
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H.O.T. Problems

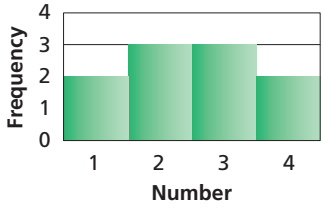
34. **OPEN ENDED** Give a sample set of data with a variance and standard deviation of 0.

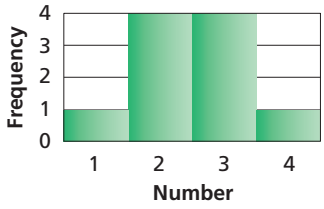
35. **REASONING** Find a counterexample for the following statement.
The standard deviation of a set of data is always less than the variance.

CHALLENGE **For Exercises 36 and 37, consider the two sets of data.**
A = {1, 2, 2, 2, 2, 3, 3, 3, 3, 4}, B = {1, 1, 2, 2, 2, 3, 3, 3, 4, 4}

36. Find the mean, median, variance, and standard deviation of each set of data.

37. Explain how you can tell which histogram below goes with each data set without counting the frequencies in the sets.





38. **Which One Doesn't Belong?** Identify the term that does not belong with the other three. Explain your reasoning.

mode	variance	mean	median

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39. ***Writing in Math*** Use the information on page 717 to explain what statistics a teacher should tell the class after a test. Include the mean, median, and mode of the given data set and which measure of central tendency you think best represents the test scores and why. How will the measures of central tendency be affected if Mr. Dent adds 5 points to each score?

STANDARDIZED TEST PRACTICE

40. ACT/SAT What is the mean of the numbers represented by $x + 1$, $3x - 2$, and $2x - 5$?

A $2x - 2$

B $\frac{6x - 7}{3}$

C $\frac{x + 1}{3}$

D $x + 4$

41. REVIEW A school has two backup generators having probabilities of 0.9 and 0.95, respectively, of operating in case of power outage. Find the probability that at least one backup generator operates during a power outage.

F 0.855

G 0.89

H 0.95

J 0.995

Spiral Review

Determine whether the events are *mutually exclusive* or *inclusive*. Then find the probability. (Lesson 12-5)

42. A card is drawn from a standard deck of cards. What is the probability that it is a 5 or a spade?

43. A jar of change contains 5 quarters, 8 dimes, 10 nickels, and 19 pennies. If a coin is pulled from the jar at random, what is the probability that it is a nickel or a dime?

Two cards are drawn from a standard deck of cards. Find each probability. (Lesson 12-4)

44. P(ace, then king) if replacement occurs

45. P(ace, then king) if no replacement occurs

46. P(heart, then club) if no replacement occurs

47. P(heart, then club) if replacement occurs

48. BUSINESS The Energy Booster Company keeps their stock of Health Aid liquid in a tank that is a rectangular prism. Its sides measure $x - 1$ centimeters, $x + 3$ centimeters, and $x - 2$ centimeters. Suppose they would like to bottle their Health Aid in $x - 3$ containers of the same size. How much liquid in cubic centimeters will remain unbottled? (Lesson 6-6)

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each percent.**

49. 68% of 200

50. 68% of 500

51. 95% of 400

52. 95% of 500

53. 99% of 400

54. 99% of 500

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12-7 The Normal Distribution

Main Ideas

- Determine whether a set of data appears to be normally distributed or skewed.
- Solve problems involving normally distributed data.

New Vocabulary

discrete probability distribution
continuous probability distribution
normal distribution
skewed distribution

GET READY for the Lesson

The frequency table below lists the heights of the 2005 New England Patriots. However, it does not show how these heights compare to the average height of a professional football player. To make that comparison, you can determine how the heights are distributed.

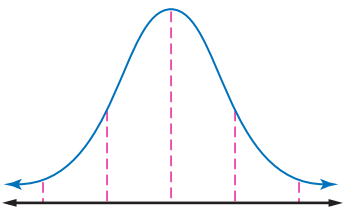
Height (in.)	70	71	72	73	74	75	76	77	78	79	80
Frequency	13	3	5	7	10	9	14	2	4	0	1

Source: www.nfl.com

Normal and Skewed Distributions The probability distributions you have studied thus far are **discrete probability distributions** because they have only a finite number of possible values. A discrete probability distribution can be represented by a histogram. For a **continuous probability distribution**, the outcome can be any value in an interval of real numbers. Continuous probability distributions are represented by curves instead of histograms.

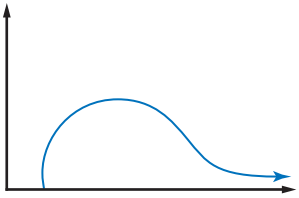
The curve at the right represents a continuous probability distribution. Notice that the curve is symmetric. Such a curve is often called a *bell curve*. Many distributions with symmetric curves or histograms are **normal distributions**.

Normal Distribution

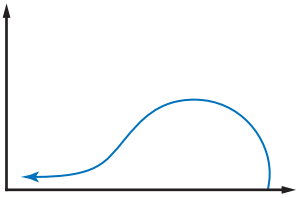


A curve or histogram that is not symmetric represents a **skewed distribution**. For example, the distribution for a curve that is high at the left and has a tail to the right is said to be *positively skewed*. Similarly, the distribution for a curve that is high at the right and has a tail to the left is said to be *negatively skewed*.

Positively Skewed



Negatively Skewed



Study Tip

Skewed Distributions

In a positively skewed distribution, the long tail is in the positive direction. These are sometimes said to be *skewed to the right*. In a negatively skewed distribution, the long tail is in the negative direction. These are sometimes said to be *skewed to the left*.

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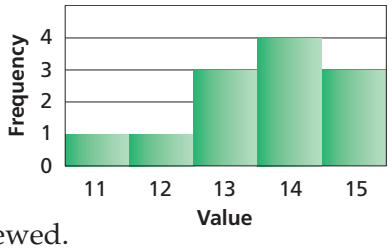
EXAMPLE 1 Classify a Data Distribution

Determine whether the data {14, 15, 11, 13, 13, 14, 15, 14, 12, 13, 14, 15} appear to be *positively skewed*, *negatively skewed*, or *normally distributed*.

Make a frequency table for the data. Then use the table to make a histogram.

Value	11	12	13	14	15
Frequency	1	1	3	4	3



Since the histogram is high at the right and has a tail to the left, the data are negatively skewed.

Concepts in Motion
Animation
algebra2.com

CHECK Your Progress

1. Determine whether the data {25, 27, 20, 22, 28, 20, 24, 22, 20, 21, 21, 26} appear to be *positively skewed*, *negatively skewed*, or *normally distributed*.

Use Normal Distributions Standardized test scores, the lengths of newborn babies, the useful life and size of manufactured items, and production levels can all be represented by normal distributions. In all of these cases, the number of data values must be large for the distribution to be approximately normal.

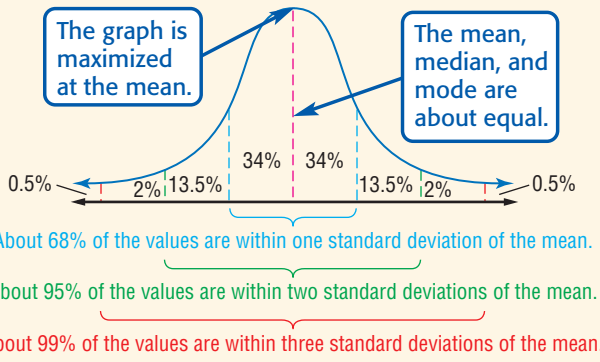
Study Tip

Normal Distributions

If you randomly select an item from data that are normally distributed, the probability that the one you pick will be within one standard deviation of the mean is 0.68. If you do this 1000 times, about 680 of those picked will be within one standard deviation of the mean.

KEY CONCEPT *Normal Distribution*

Normal distributions have these properties.

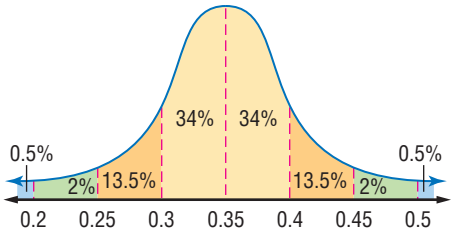


EXAMPLE 2 Normal Distribution

PHYSIOLOGY The reaction times for a hand-eye coordination test administered to 1800 teenagers are normally distributed with a mean of 0.35 second and a standard deviation of 0.05 second.

a. About how many teens had reaction times between 0.25 and 0.45 second?

Draw a normal curve. Label the mean and the mean plus or minus multiples of the standard deviation.



(continued on the next page)

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Reading Math

Normally Distributed Random Variable

A *normally distributed random variable* is a variable whose values are arbitrary but whose statistical distribution is normal.

The values 0.25 and 0.45 are 2 standard deviations *below and above* the mean, respectively. Therefore, about 95% of the data are between 0.25 and 0.45. Since $1800 \times 95\% = 1710$, we know that about 1710 of the teenagers had reaction times between 0.25 and 0.45 second.

b. What is the probability that a teenager selected at random had a reaction time greater than 0.4 second?

The value 0.4 is one standard deviation above the mean. You know that about 100% − 68% or 32% of the data are more than one standard deviation away from the mean. By the symmetry of the normal curve, half of 32%, or 16%, of the data are more than one standard deviation above the mean.

The probability that a teenager selected at random had a reaction time greater than 0.4 second is about 16% or 0.16.

CHECK Your Progress

In a recent year, the mean and standard deviation for scores on the ACT were 21.0 and 4.7. Assume that the scores were normally distributed.

2A. If 1,000,000 people took the test, about how many of them scored between 16.3 and 25.7?

2B. What is the probability that a test taker scored higher than 30.4?

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CHECK Your Understanding

Example 1 (p. 725)

1. The table at the right shows recent composite ACT scores. Determine whether the data appear to be *positively skewed*, *negatively skewed*, or *normally distributed*.

Score	Percent of Students
33–36	1
28–32	9
24–27	19
20–23	29
16–19	27
13–15	12

Source: ACT.org

Example 2 (pp. 725–726)

For Exercises 2–4, use the following information.

Mr. Bash gave a quiz in his social studies class. The scores were normally distributed with a mean of 21 and a standard deviation of 2.

2. What percent would you expect to score between 19 and 23?

3. What percent would you expect to score between 23 and 25?

4. What is the probability that a student chosen at random scored between 17 and 25?

QUALITY CONTROL **For Exercises 5–8, use the following information.**

The useful life of a certain car battery is normally distributed with a mean of 100,000 miles and a standard deviation of 10,000 miles. The company makes 20,000 batteries a month.

5. About how many batteries will last between 90,000 and 110,000 miles?

6. About how many batteries will last more than 120,000 miles?

7. About how many batteries will last less than 90,000 miles?

8. What is the probability that if you buy a car battery at random, it will last between 80,000 and 110,000 miles?

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Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–11	1
12–23	2

Determine whether the data in each table appear to be *positively skewed*, *negatively skewed*, or *normally distributed*.

9.

U.S. Population	
Age	Percent
0–19	28.7
20–39	29.3
40–59	25.5
60–79	13.3
80–99	3.2
100+	0.0

Source: U.S. Census Bureau

10.

Record High U.S. Temperatures	
Temperature (°F)	Number of States
100–104	3
105–109	8
110–114	16
115–119	13
120–124	7
125–129	2
130–134	1

Source: *The World Almanac*

11. SCHOOL The frequency table at the right shows the grade-point averages (GPAs) of the juniors at Stanhope High School. Do the data appear to be *positively skewed*, *negatively skewed*, or *normally distributed*? Explain.

GPA	Frequency
0.0–0.4	4
0.5–0.9	4
1.0–1.4	2
1.5–1.9	32
2.0–2.4	96
2.5–2.9	91
3.0–3.4	110
3.5–4.0	75

HEALTH For Exercises 12 and 13, use the following information.
The cholesterol level for adult males of a specific racial group is normally distributed with a mean of 4.8 and a standard deviation of 0.6.

12. About what percent of the males have cholesterol below 4.2?

13. About how many of the 900 men in a study have cholesterol between 4.2 and 6.0?

Real-World Link
Doctors recommend that people maintain a total blood cholesterol of 200 mg/dL or less.

Source: americanheart.org

VENDING For Exercises 14–16, use the following information.
A vending machine usually dispenses about 8 ounces of coffee. Lately, the amount varies and is normally distributed with a standard deviation of 0.3 ounce.

14. What percent of the time will you get more than 8 ounces of coffee?

15. What percent of the time will you get less than 8 ounces of coffee?

16. What percent of the time will you get between 7.4 and 8.6 ounces of coffee?

MANUFACTURING For Exercises 17–19, use the following information.
The sizes of CDs made by a company are normally distributed with a standard deviation of 1 millimeter. The CDs are supposed to be 120 millimeters in diameter, and they are made for drives 122 millimeters wide.

17. What percent of the CDs would you expect to be greater than 120 millimeters?

18. If the company manufactures 1000 CDs per hour, how many of the CDs made in one hour would you expect to be between 119 and 122 millimeters?

19. About how many CDs per hour will be too large to fit in the drives?

FOOD For Exercises 20–23, use the following information.
The shelf life of a particular snack chip is normally distributed with a mean of 180 days and a standard deviation of 30 days.

20. About what percent of the products last between 150 and 210 days?

21. About what percent of the products last between 180 and 210 days?

22. About what percent of the products last less than 90 days?

23. About what percent of the products last more than 210 days?

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EXTRA PRACTICE
See pages 918, 937.

Math Online
Self-Check Quiz at algebra2.com

RAINFALL For Exercises 24–26, use the table at the right.

24. Find the mean.

25. Find the standard deviation.

26. If the data are normally distributed, what percent of the time will the annual precipitation in these cities be between 16.97 and 7.69 inches?

Average Annual Precipitation

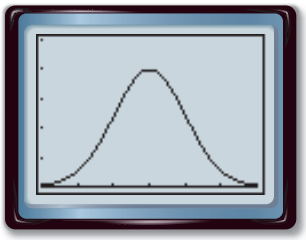
City	Precipitation (in.)
Albuquerque	9
Boise	12
Phoenix	8
Reno	7
Salt Lake City	17
San Francisco	20

Source: noaa.gov

H.O.T. Problems

27. OPEN ENDED Sketch a positively skewed graph. Describe a situation in which you would expect data to be distributed this way.

28. CHALLENGE The graphing calculator screen shows the graph of a normal distribution for a large set of test scores whose mean is 500 and whose standard deviation is 100. If every test score in the data set were increased by 25 points, describe how the mean, standard deviation, and graph of the data would change.



[200, 800] scl: 100 by [0, 0.005] scl: 0.001

29. *Writing in Math* Use the information on page 724 to explain how the heights of professional athletes are distributed. Include a histogram of the given data, and an explanation of whether you think the data is normally distributed.

STANDARDIZED TEST PRACTICE

30. ACT/SAT If $x + y = 5$ and $xy = 6$, what is the value of $x^2 + y^2$?

A 13

B 17

C 25

D 37

31. REVIEW Jessica wants to create several different 7-character passwords. She wants to use arrangements of the first three letters of her name, *followed* by arrangements of 4 digits in 1987, the year of her birth. How many different passwords can she create?

F 672 **G** 288 **H** 576 **J** 144

Spiral Review

Find the variance and standard deviation of each set of data to the nearest tenth. (Lesson 12-6)

32. {7, 16, 9, 4, 12, 3, 9, 4}

33. {12, 14, 28, 19, 11, 7, 10}

A card is drawn from a standard deck of cards. Find each probability. (Lesson 12-5)

34. P(jack or queen)

35. P(ace or heart)

36. P(2 or face card)

GET READY for the Next Lesson

PREREQUISITE SKILL **Use a calculator to evaluate each expression to four decimal places.** (Lesson 9-5)

37. e^{-4}

38. e^{3}

39. $e^{\frac{1}{2}}$

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12-8 Exponential and Binomial Distribution

Main Ideas

- Use exponential distributions to find exponential probabilities.
- Use binomial distributions to find binomial probabilities.

New Vocabulary

exponential distribution
exponential probability
binomial distribution
binomial probability

GET READY for the Lesson

The average length of time that a student at East High School spends talking on the phone per day is 1 hour. What is the probability that a randomly chosen student talks on the phone for more than 2 hours?

Exponential Distributions You can use exponential distributions to predict the probabilities of events based on time. They are most commonly used to measure *reliability*, which is the amount of time that a product lasts. Exponential distributions apply to situations where the time spent on an event, or the amount of time that an event lasts, is important.

Exponential distributions are represented by the following functions.

KEY CONCEPT *Exponential Distribution Functions*

The formula $f(x) = e^{-mx}$ gives the probability $f(x)$ that something lasts longer or costs more than the given value x, where m is the multiplicative inverse of the mean amount of time.

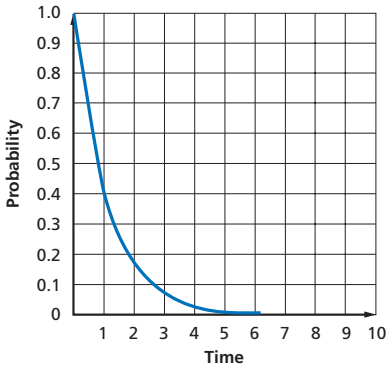
The formula $f(x) = 1 - e^{-mx}$ gives the probability $f(x)$ that something does not last as long or costs less than the given value x, where m is the multiplicative inverse of the mean amount of time.

Study Tip

Look Back

To review inverses, see Lesson 1–2.

Exponential distributions are represented by a curve similar to the one shown. The x-axis usually represents length of time, or money. The y-axis represents probability, so the range will be from 0 to 1.



EXAMPLE **Exponential Distribution**

1 **Refer to the application above. What is the probability that a randomly chosen student talks on the phone for more than 2 hours?**

First, find the m, the inverse of the mean. Because the mean is 1, the multiplicative inverse is 1.

$f(x) = e^{-mx}$ Exponential Distribution Function

$f(2) = e^{-1(2)}$ Replace x with 2 and m with 1.

$= e^{-2}$ Simplify.

≈ 0.135 or 13.5% Use a calculator.

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There is a 13.5% chance that a randomly selected East High School student talks on the phone for more than 2 hours a day. This appears to be a reasonable solution because few students spend either a short amount of time or a long amount of time on the phone.

CHECK Your Progress

1. If computers last an average of 3 years, what is the probability that a randomly selected computer will last more than 4 years?

EXAMPLE 2 Exponential Distribution

If athletic shoes last an average of 1.5 years, what is the probability that a randomly selected pair of athletic shoes will last less than 6 months?

The question asks for the probability that a pair of shoes lasts *less* than 6 months, so we will use the second exponential distribution function. The mean is 1.5 or $\frac{3}{2}$, so the multiplicative inverse m is $\frac{2}{3}$.

$f(x) = 1 - e^{-mx}$ Exponential Distribution Function

$f\left(\frac{1}{2}\right) = 1 - e^{-\frac{2}{3}\left(\frac{1}{2}\right)}$ Replace x with $\frac{1}{2}$ (6 mo $= \frac{1}{2}$ yr) and m with $\frac{2}{3}$.

$= 1 - e^{-\frac{1}{3}}$ Simplify.

≈ 0.2835 or 28.35% Use a calculator.

There is a 28.35% chance that a randomly selected pair of athletic shoes will last less than 6 months.

CHECK Your Progress

2. If the average lifespan of a dog is 12 years, what is the probability that a randomly selected dog will live less than 2 years?

Binomial Distributions In a binomial distribution, all of the trials are independent and have only two possible outcomes, success or failure. The probability of success is the same in every trial. The outcome of one trial does not affect the probabilities of any future trials. The random variable is the number of successes in a given number of trials.

KEY CONCEPT *Binomial Distribution Functions*

The probability of x successes in n independent trials is

$$P(x) = C(n, x)\, p^x\, q^{n-x},$$

where p is the probability of success of an individual trial and q is the probability of failure on that same individual trial ($p + q = 1$).

The expectation for a binomial distribution is

$$E(X) = np,$$

where n is the total number of trials and p is the probability of success.

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EXAMPLE 3 Binomial Probability

A chocolate company makes boxes of assorted chocolates, 40% of which are dark chocolate on average. The production line mixes the chocolates randomly and packages 10 per box.

a. What is the probability that at least 3 chocolates in a given box are dark chocolates?

A success is a dark chocolate, so $p = 0.4$ and $q = 1 - 0.4$ or 0.6. You could add the probabilities of having exactly 3, 4, 5, 6, 7, 8, 9, or 10 dark chocolates, but it is easier to calculate the probability of the box having exactly 0, 1, or 2 chocolates and then subtracting that sum from 1.

$P(\geq 3 \text{ dark chocolates})$

$= 1 - P(< 3)$

$= 1 - P(0) - P(1) - P(2)$ Mutually exclusive events subtracted from 1

$= 1 - C(10, 0)(0.4)^0(0.6)^{10} - C(10, 1)(0.4)^1(0.6)^9 - C(10, 2)(0.4)^2(0.6)^8$

$= 0.8327$ or 83.27% Simplify.

The probability of at least three chocolates being dark chocolates is 0.8327 or 83.27%.

b. What is the expected number of dark chocolates in a box?

$E(X) = np$ Expectation for a binomial distribution

$= 10(0.4)$ $n = 10$ and $p = 0.4$

$= 4$ Multiply.

The expected number of dark chocolates in a box is 4.

CHECK Your Progress

3. If 20% of the chocolates are white chocolates, what is the probability that at least one chocolate in a given box is a white chocolate?

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CHECK Your Understanding

Examples 1, 2 (pp. 729–730)

For Exercises 1 and 2, use the following information.
The average amount of time high school students spend on homework is 2 hours per day.

1. What is the probability that a randomly selected student spends more than 3 hours per day on homework?

2. What is the probability that a randomly selected student spends less than 1 hour per day on homework?

Examples 3 (p. 731)

For Exercises 3 and 4, use the following information.
Mary's cat is having kittens. The probability of a kitten being male is 0.5.

3. If Mary's cat has 4 kittens, what is the probability that at least 3 will be male?

4. What is the expected number of males in a litter of 6?

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Exercises

HOMEWORK HELP

For Exercises	See Examples
5-6, 9–14, 18–20	1–2
7–8, 15–17	3

For Exercises 5 and 6, use the following information.

The average life span of a certain type of car tire is 4 years.

5. What is the probability that a randomly selected set of 4 tires will last more than 9 years?
6. What is the probability that a randomly selected set of tires will last fewer than 2.5 years?

GARDENING **For Exercises 7 and 8, use the following information.**

Dan is planting 24 irises in his front yard. The flowers he bought were a combination of two varieties, blue and white. The flowers are not blooming yet, but Dan knows that the probability of having a blue flower is 75%.

7. What is the probability that at least 20 of the flowers will be blue?
8. What is the expected number of white irises in Dan's garden?

Real-World Link

There are hundreds of species and cultivations of iris in all colors of the rainbow. Iris vary from tiny woodland ground covers, to 4-feet-tall flowers that flourish in the sun, to species that thrive in swampy soil. There is an iris that will do well in virtually every garden.

Source: hgic.clemson.edu

For Exercises 9–14, use the following information.

An exponential distribution has a mean of 0.5. Find each probability.

9. $x > 1.5$
10. $x > 3$
11. $x > \frac{1}{4}$
12. $x < 1$
13. $x < \frac{1}{3}$
14. $x < 2.5$

For Exercises 15–17, use the following information.

A binomial distribution has a 60% rate of success. There are 18 trials.

15. What is the probability that there will be at least 12 successes?
16. What is the probability that there will be 12 failures?
17. What is the expected number of successes?

RELIABILITY **For Exercises 18–20, use the following information.**

A light bulb has an average life of 8 months.

18. What is the probability that a randomly chosen bulb will last more than 13 months?
19. What is the probability that a randomly chosen bulb will last less than 6 months?
20. There is an 80% chance that a randomly chosen light bulb will last more than how long?

JURY DUTY **For Exercises 21–23, use the following information.**

A jury of twelve people is being selected for trial. The probability that a juror will be male is 0.5. The probability that a juror will vote to convict is 0.75.

21. What is the probability that more than 3 jurors will be men?
22. What is the probability that fewer than 6 jurors will vote to convict?
23. What is the expected number of votes for conviction?

EXTRA PRACTICE

See pages 919, 937.

MathOnline

Self-Check Quiz at algebra2.com

H.O.T. Problems

24. **OPEN ENDED** Sketch the graph of an exponential distribution function. Describe a situation in which you would expect data to be distributed in this way.

Steve Chenn/CORBIS

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25. REASONING An exponential distribution function has a mean of 2. A fellow student says that the probability that $x > 2$ is 0.5. Determine whether this is *sometimes*, *always*, or *never* true. Explain your reasoning.

26. CHALLENGE The average amount of money spent per day by students in Mrs. Ross's class for lunch is \$2. In this class, 90% of students spend less than what amount per day?

27. *Writing in Math* Your school has received a grant, and the administration is considering adding a new science wing to the building. You have been asked to poll a sample of your classmates to find out if they support using the funding for the science wing project. Describe how you could use binomial distribution to predict the number of people in the school who would support the science wing project.

STANDARDIZED TEST PRACTICE

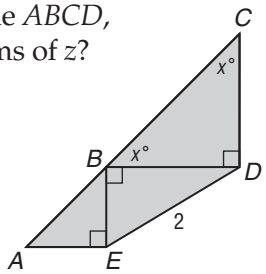
28. ACT/SAT In rectangle $ABCD$, what is $x + y$ in terms of z?

A $90 + z$

B $190 - z$

C $180 + z$

D $270 - z$



29. REVIEW Your gym teacher is randomly distributing 15 red dodge balls and 10 yellow dodge balls. What is the probability that the first ball that she hands out will be yellow and the second will be red?

F $\frac{1}{24}$

G $\frac{1}{4}$

H $\frac{2}{5}$

J $\frac{23}{25}$

Spiral Review

A set of 260 data values is normally distributed with a mean of 50 and a standard deviation of 5.5. (Lesson 12-7)

30. What percent of the data lies between 39 and 61?

31. What is the probability that a data value selected at random is greater than 39?

A die is rolled, Find each probability. (Lesson 12-5)

32. P(even)

33. P (1 or 6)

34. P(prime number)

Simplify each expression. (Lesson 6-2)

35. $(x - 7)(x + 9)$

36. $(4b^2 + 7)^2$

37. $(3q - 6) - (q + 13) + (-2q + 11)$

GET READY for the Next Lesson

PREREQUISITE SKILL Find the indicated term of each expression. (Lesson 11-7)

38. third term of $(a + b)^7$

39. fourth term of $(c + d)^8$

40. fifth term of $(x + y)^9$

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EXTEND 12-8

Algebra Lab
Simulations

A **simulation** uses a probability experiment to mimic a real-life situation. You can use a simulation to solve the following problem about **expected value**.

A brand of cereal is offering one of six different prizes in every box. If the prizes are equally and randomly distributed within the cereal boxes, how many boxes, on average, would you have to buy in order to get a complete set?

ACTIVITY 1

Work in pairs or small groups to complete Steps 1 through 4.

Step 1 Use the six numbers on a die to represent the six different prizes.

Step 2 Roll the die and record which prize was in the first box of cereal. Use a tally sheet like the one shown.

Step 3 Continue to roll the die and record the prize number until you have a complete set of prizes. Stop as soon as you have a complete set. This is the end of one trial in your simulation. Record the number of boxes required for this trial.

Step 4 Repeat steps 1, 2, and 3 until your group has carried out 25 trials. Use a new tally sheet for each trial.

Simulation Tally Sheet	
Prize Number	**Boxes Purchased**
1	
2	
3	
4	
5	
6	
Total Needed	

Analyze the Results

1. Create two different statistical graphs of the data collected for 25 trials.
2. Determine the mean, median, maximum, minimum, and standard deviation of the total number of boxes needed in the 25 trials.
3. Combine the small-group results and determine the mean, median, maximum, minimum, and standard deviation of the number of boxes required for all the trials conducted by the class.
4. If you carry out 25 additional trials, will your results be the same as in the first 25 trials? Explain.
5. Should the small-group results or the class results give a better idea of the average number of boxes required to get a complete set of prizes? Explain.
6. If there were 8 prizes instead of 6, would you need to buy more boxes of cereal or fewer boxes of cereal on average?
7. **DESIGN A SIMULATION** What if one of the 6 prizes was more common than the other 5? Suppose one prize appears in 25% of all the boxes and the other 5 prizes are equally and randomly distributed among the remaining 75% of the boxes? Design and carry out a new simulation to predict the average number of boxes you would need to buy to get a complete set. Include some measures of central tendency and dispersion with your data.

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12-9 Binomial Experiments

Main Ideas

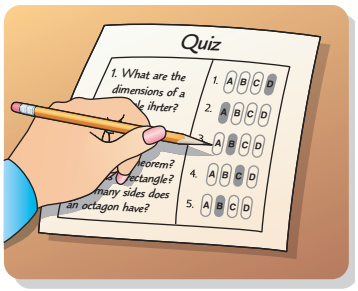
- Use binomial experiments to find probabilities.
- Find probabilities for binomial experiments.

New Vocabulary

binomial experiment

GET READY for the Lesson

What is the probability of getting exactly 4 questions correct on a 5-question multiple-choice quiz if you guess at every question?



Binomial Expansions You can use the Binomial Theorem to find probabilities in certain situations where there are two possible outcomes. The 5 possible ways of getting 4 questions right r and 1 question wrong w are shown at the right. This chart shows the combination of 5 things (answer choices) taken 4 at a time (right answers) or $C(5, 4)$.

w	r	r	r	r
r	w	r	r	r
r	r	w	r	r
r	r	r	w	r
r	r	r	r	w

The terms of the binomial expansion of $(r + w)^5$ can be used to find the probabilities of each combination of right and wrong.

$$(r + w)^5 = r^5 + 5r^4w + 10r^3w^2 + 10r^2w^3 + 5rw^4 + w^5$$

Study Tip

Look Back

To review the **Binomial Theorem**, see Lesson 11-7.

Coefficient	Term	Meaning
$C(5, 5) = 1$	r^5	1 way to get all 5 questions right
$C(5, 4) = 5$	$5r^4w$	5 ways to get 4 questions right and 1 question wrong
$C(5, 3) = 10$	$10r^3w^2$	10 ways to get 3 questions right and 2 questions wrong
$C(5, 2) = 10$	$10r^2w^3$	10 ways to get 2 questions right and 3 questions wrong
$C(5, 1) = 5$	$5rw^4$	5 ways to get 1 question right and 4 questions wrong
$C(5, 0) = 1$	w^5	1 way to get all 5 questions wrong

The probability of getting a question right that you guessed on is $\frac{1}{4}$. So, the probability of getting the question wrong is $\frac{3}{4}$. To find the probability of getting 4 questions right and 1 question wrong, substitute $\frac{1}{4}$ for r and $\frac{3}{4}$ for w in the term $5r^4w$.

$$P(4 \text{ right, } 1 \text{ wrong}) = 5r^4w$$

$$= 5\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) \qquad r = \frac{1}{4}, w = \frac{3}{4}$$

$$= \frac{15}{1024} \qquad \text{Multiply.}$$

The probability of getting exactly 4 questions correct is $\frac{15}{1024}$ or about 1.5%.

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EXAMPLE Binomial Theorem

1 **If a family has 4 children, what is the probability that they have 3 boys and 1 girl?**

There are two possible outcomes for the gender of each of their children: boy or girl. The probability of a boy b is $\frac{1}{2}$, and the probability of a girl g is $\frac{1}{2}$.

$(b + g)^4 = b^4 + 4b^3g + 6b^2g^2 + 4bg^3 + g^4$

The term $4b^3g$ represents 3 boys and 1 girl.

$P(3 \text{ boys, } 1 \text{ girl}) = 4b^3g$

$= 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$ $b = \frac{1}{2}, g = \frac{1}{2}$

$= \frac{1}{4}$ The probability is $\frac{1}{4}$ or 25%.

CHECK Your Progress

1. If a coin is flipped six times, what is the probability that the coin lands heads up four times and tails up two times?

Binomial Experiments Problems like Example 1 that can be solved using binomial expansion are called **binomial experiments**.

KEY CONCEPT *Binomial Experiments*

A binomial experiment exists if and only if all of these conditions occur.

- There are exactly two possible outcomes for each trial.
- There is a fixed number of trials.
- The trials are independent.
- The probabilities for each trial are the same.

A binomial experiment is sometimes called a *Bernoulli experiment*.

Real-World Link

As of 2005, the National Hockey League record for most goals in a game by one player is seven. A player has scored five or more goals in a game 53 times in league history.

Source: NHL

EXAMPLE Binomial Experiment

2 **SPORTS** **Suppose that when hockey star Martin St. Louis takes a shot, he has a $\frac{1}{7}$ probability of scoring a goal. He takes 6 shots in a game.**

a. What is the probability that he will score exactly 2 goals?

The probability that he scores on a given shot is $\frac{1}{7}$, and the probability that he does not is $\frac{6}{7}$. There are $C(6, 2)$ ways to choose the 2 shots that score.

$P(2 \text{ goals}) = C(6, 2)\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^4$ If he scores on 2 shots, he fails to score on 4 shots.

$= \frac{6 \cdot 5}{2}\left(\frac{1}{7}\right)^2\left(\frac{6}{7}\right)^4$ $C(6, 2) = \frac{6!}{4!2!}$

$= \frac{19{,}440}{117{,}649}$ Simplify.

The probability of exactly 2 goals is $\frac{19{,}440}{117{,}649}$ or about 17%.

Chris O'Meara/AP/Wide World Photos

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b. What is the probability that he will score at least 2 goals?

Instead of adding the probabilities of getting exactly 2, 3, 4, 5, and 6 goals, it is easier to subtract the probabilities of getting exactly 0 or 1 goal from 1.

$$P(\text{at least 2 goals}) = 1 - P(0 \text{ goals}) - P(1 \text{ goal})$$

$$= 1 - C(6, 0)\left(\frac{1}{7}\right)^0\left(\frac{6}{7}\right)^6 - C(6, 1)\left(\frac{1}{7}\right)^1\left(\frac{6}{7}\right)^5$$

$$= 1 - \frac{46{,}656}{117{,}649} - \frac{46{,}656}{117{,}649} \quad \text{Simplify.}$$

$$= \frac{24{,}337}{117{,}649} \quad \text{Subtract.}$$

The probability that Martin will score at least 2 goals is $\frac{24{,}337}{117{,}649}$ or about 21%.

CHECK Your Progress

A basketball player has a free-throw percentage of 75% before the last game of the season. The player takes 5 free throws in the final game.

2A. What is the probability that he will make exactly two free throws?

2B. What is the probability that he will make at least two free throws?

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CHECK Your Understanding

Examples 1, 2 (pp. 730–731)

Find each probability if a coin is tossed 3 times.

1. P(exactly 2 heads) **2.** P(0 heads) **3.** P(at least 1 head)

Four cards are drawn from a standard deck of cards. Each card is replaced before the next one is drawn. Find each probability.

4. P(4 jacks) **5.** P(exactly 3 jacks) **6.** P(at most 1 jack)

SPORTS Lauren Wible of Bucknell University was the 2005 NCAA Division I women's softball batting leader with a batting average of .524. This means that the probability of her getting a hit in a given at-bat was 0.524.

7. Find the probability of her getting 4 hits in 4 at-bats.

8. Find the probability of her getting exactly 2 hits in 4 at-bats.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–30	1, 2

Find each probability if a coin is tossed 5 times.

9. P(5 tails) **10.** P(0 tails)

11. P(exactly 2 tails) **12.** P(exactly 1 tail)

13. P(at least 4 tails) **14.** P(at most 2 tails)

Find each probability if a die is rolled 4 times.

15. P(exactly one 3) **16.** P(exactly three 3s)

17. P(at most two 3s) **18.** P(at least three 3s)

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As a maintenance manager, Jackie Thomas is responsible for managing the maintenance of an office building. When entering a room after hours, the probability that she selects the correct key on the first try is $\frac{1}{5}$. If she enters 6 rooms in an evening, find each probability.

19. P(never the correct key)
20. P(always the correct key)
21. P(correct exactly 4 times)
22. P(correct exactly 2 times)
23. P(no more than 2 times correct)
24. P(at least 4 times correct)

Prisana guesses at all 10 true/false questions on her history test. Find each probability.

25. P(exactly 6 correct)
26. P(exactly 4 correct)
27. P(at most half correct)
28. P(at least half correct)

Real-World Link

The word *Internet* was virtually unknown until the mid-1980s. By 1997, 19 million Americans were using the Internet. That number tripled in 1998 and passed 100 million in 1999.

Source: UCLA

29. CARS According to a recent survey, about 1 in 3 new cars is leased rather than bought. What is the probability that 3 of 7 randomly selected new cars are leased?

30. INTERNET In a recent year, it was estimated that 55% of U.S. adult Internet users had access to high-speed Internet connections at home or on the job. What is the probability that exactly 2 out of 5 randomly selected U.S. adults had access to high-speed Internet connections?

If a thumbtack is dropped, the probability of it landing point-up is 0.3. If 10 tacks are dropped, find each probability.

31. P(at least 8 points up)
32. P(at most 3 points up)

33. COINS A fair coin is tossed 6 times. Find the probability of each outcome.

Graphing Calculator

BINOMIAL DISTRIBUTION **For Exercises 34 and 35, use the following information.** You can use a TI-83/84 Plus graphing calculator to investigate the graph of a binomial distribution.

Step 1 Enter the number of trials in L1. Start with 10 trials.

KEYSTROKES: STAT 1 ▲ 2nd [LIST] ▶ 5 X,T,θ,*n* , X,T,θ,*n* , 0 , 10) ENTER

Step 2 Calculate the probability of success for each trial in L2.

KEYSTROKES: ▶ ▲ 2nd [DISTR] 0 10 , .5 , 2nd [L1]) ENTER

Step 3 Graph the histogram.

KEYSTROKES: 2nd [STAT PLOT]

Use the arrow and ENTER keys to choose ON, the histogram, L1 as the Xlist, and L2 as the frequency. Use the window [0, 10] scl: 1 by [0, 0.5] scl: 0.1.

34. Replace the 10 in the keystrokes for steps 1 and 2 to graph the binomial distribution for several values of n less than or equal to 47. You may have to adjust your viewing window to see all of the histogram. Make sure Xscl is 1.

35. What type of distribution does the binomial distribution start to resemble as n increases?

EXTRA PRACTICE
See pages 919, 937.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. OPEN ENDED Describe a situation for which the P(2 or more) can be found by using a binomial expansion.

Steve Chenn/CORBIS
www.MathSchoolinternational.com

37. REASONING Explain why each experiment is not a binomial experiment.

a. rolling a die and recording whether a 1, 2, 3, 4, 5, or 6 comes up

b. tossing a coin repeatedly until it comes up heads

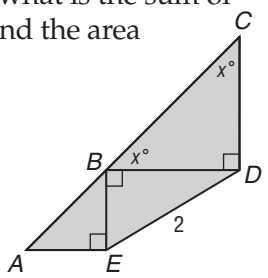
c. removing marbles from a bag and recording whether each one is black or white, if no replacement occurs

38. CHALLENGE Find the probability of exactly m successes in n trials of a binomial experiment where the probability of success in a given trial is p.

39. *Writing in Math* Use the information on page 735 to explain how you can determine whether guessing is worth it. Explain how to find the probability of getting any number of questions right on a 5-question multiple-choice quiz when guessing and the probability of each score.

STANDARDIZED TEST PRACTICE

40. ACT/SAT If $DE = 2$, what is the sum of the area of $\triangle ABE$ and the area of $\triangle BCD$?



A 2 **C** 4

B 3 **D** 5

41. REVIEW An examination consists of 10 questions. A student must answer only one of the first two questions and only six of the remaining ones. How many choices of questions does the student have?

F 112 **H** 44

G 56 **J** 30

Spiral Review

A set of 400 test scores is normally distributed with a mean of 75 and a standard deviation of 8. (Lesson 12-7)

42. What percent of the test scores lie between 67 and 83?

43. How many of the test scores are greater than 91?

44. What is the probability that a randomly-selected score is less than 67?

45. A salesperson had sales of \$11,000, \$15,000, \$11,000, \$16,000, \$12,000, and \$12,000 in the last six months. Which measure of central tendency would he be likely to use to represent these data when he talks with his supervisor? Explain. (Lesson 12-6)

Graph each inequality. (Lesson 2-7)

46. $x \geq -3$ **47.** $x + y \leq 4$ **48.** $y > |5x|$

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $2\sqrt{\frac{p(1-p)}{n}}$ for the given values of p and n. Round to the nearest thousandth if necessary.** (Lesson 5-2)

49. $p = 0.5, n = 100$ **50.** $p = 0.5, n = 400$ **51.** $p = 0.25, n = 500$

52. $p = 0.75, n = 1000$ **53.** $p = 0.3, n = 500$ **54.** $p = 0.6, n = 1000$

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EXPLORE 12-10

Algebra Lab

Testing Hypotheses

A **hypothesis** is a statement to be tested. Testing a hypothesis to determine whether it is supported by the data involves five steps.

Step 1 State the hypothesis. The statement should include a *null hypothesis*, which is the hypothesis to be tested, and an *alternative hypothesis*.

Step 2 Design the experiment.

Step 3 Conduct the experiment and collect the data.

Step 4 Evaluate the data. Decide whether to reject the null hypothesis.

Step 5 Summarize the results.

ACTIVITY **Test the following hypothesis.**

People react to sound and touch at the same rate.

You can measure reaction time by having someone drop a ruler and then having someone else catch it between their fingers. The distance the ruler falls will depend on their reaction time. Half of the class will investigate the time it takes to react when someone is told the ruler has dropped. The other half will measure the time it takes to react when the catcher is alerted by touch.

Step 1 The null hypothesis H_0 and alternative hypothesis H_1 are as follows. These statements often use $=$, $\neq$, $<$, $>$, $\geq$, and $\leq$.

- H_0: reaction time to sound $=$ reaction time to touch
- H_1: reaction time to sound $\neq$ reaction time to touch

Step 2 You will need to decide the height from which the ruler is dropped, the position of the person catching the ruler, the number of practice runs, and whether to use one try or the average of several tries.

Step 3 Conduct the experiment in each group and record the results.

Step 4 Organize the results so that they can be compared.

Step 5 Based on the results of your experiment, do you think the hypothesis is true? Explain.

Analyze the Results

State the null and alternative hypotheses for each conjecture.

1. A teacher feels that playing classical music during a math test will cause the test scores to change (either up or down). In the past, the average test score was 73.
2. An engineer thinks that the mean number of defects can be decreased by using robots on an assembly line. Currently, there are 18 defects for every 1000 items.
3. **MAKE A CONJECTURE** Design an experiment to test the following hypothesis. *Pulse rates increase 20% after moderate exercise.*

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12-10 Sampling and Error

Main Ideas

- Determine whether a sample is unbiased.
- Find margins of sampling error.

New Vocabulary

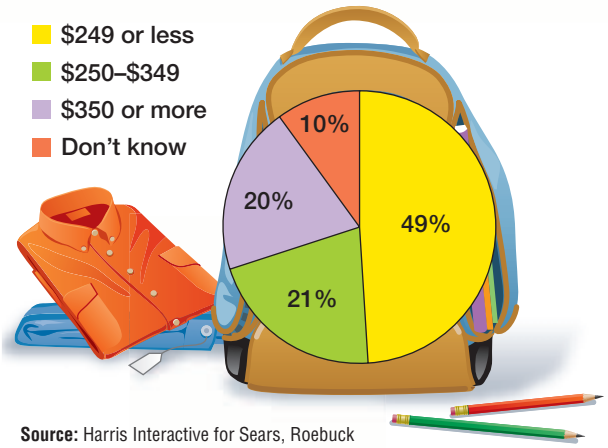
unbiased sample
margin of sampling error

GET READY for the Lesson

A survey was conducted asking mothers how much they spend per student on back-to-school clothing. The results of the survey are shown.

When polling organizations want to find how the public feels about an issue, they survey a small portion of the population.

Back-to-School Clothes Spending



Source: Harris Interactive for Sears, Roebuck

Bias To be sure that survey results are representative of the population, polling organizations need to make sure that they poll a random or **unbiased sample** of the population.

EXAMPLE Biased and Unbiased Samples

1 **State whether each method would produce a random sample. Explain.**

a. asking every tenth person coming out of a gym how many times a week they exercise to determine how often city residents exercise

This would not result in a random sample because the people surveyed probably exercise more often than the average person.

b. surveying people going into an Italian restaurant to find out people's favorite type of food

This would probably not result in a random sample because the people surveyed would probably be more likely than others to prefer Italian food.

CHECK Your Progress

1. asking every player at a golf course what sport they prefer to watch on TV

Study Tip

Random Sample

A sample of size n is random when every possible sample of size n has an equal chance of being selected.

Margin of Error The **margin of sampling error (ME)** gives a limit on the difference between how a sample responds and how the total population would respond.

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KEY CONCEPT — *Margin of Sampling Error*

If the percent of people in a sample responding in a certain way is p and the size of the sample is n, then 95% of the time, the percent of the population responding in that same way will be between $p - ME$ and $p + ME$, where

$$ME = 2\sqrt{\frac{p(1-p)}{n}}.$$

That is, the probability is 0.95 that $p \pm ME$ will contain the true population results.

EXAMPLE 2 Find a Margin of Error

2 **In a survey of 1000 randomly selected adults, 37% answered "yes" to a particular question. What is the margin of error?**

$ME = 2\sqrt{\frac{p(1-p)}{n}}$ Formula for margin of sampling error

$= 2\sqrt{\frac{0.37(1-0.37)}{1000}}$ $p = 37\%$ or 0.37, $n = 1000$

≈ 0.030535 Use a calculator.

The margin of error is about 3%. This means that there is a 95% chance that the percent of people in the whole population who would answer "yes" is between 37 − 3 or 34% and 37 + 3 or 40%.

CHECK Your Progress

2. In a survey of 625 randomly selected teens, 78% said that they purchase music. What is the margin of error in this survey?

EXAMPLE 3 Analyze a Margin of Error

3 **HEALTH In a recent Gallup Poll, 25% of the people surveyed said they had smoked cigarettes in the past week. The margin of error was 3%. How many people were surveyed?**

$ME = 2\sqrt{\frac{p(1-p)}{n}}$ Formula for margin of sampling error

$0.03 = 2\sqrt{\frac{0.25(1-0.25)}{n}}$ $ME = 0.03$, $p = 0.25$

$0.015 = \sqrt{\frac{0.25(0.75)}{n}}$ Divide each side by 2.

$0.000225 = \frac{0.25(0.75)}{n}$ Square each side.

$n = \frac{0.25(0.75)}{0.000225}$ Multiply by n and divide by 0.000225.

$n \approx 833.33$ About 833 people were surveyed.

Real-World Link

The percent of smokers in the United States population declined from 38.7% in 1985 to 23.3% in 2000. New therapies, like the nicotine patch, are helping more people to quit.

Source: U.S. Department of Health and Human Services

CHECK Your Progress

3. In a recent survey, 15% of the people surveyed said they had missed a class or a meeting because they overslept. The margin of error was 4%. How many people were surveyed?

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Aaron Haupt

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CHECK Your Understanding

Example 1 (p. 741)

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer.

1. the government sending a tax survey to everyone whose social security number ends in a particular digit
2. surveying college students in the honors program to determine the average time students at the college study each day

Example 2 (p. 742)

For Exercises 3–5, find the margin of sampling error to the nearest percent.

3. $p = 72\%, n = 100$
4. $p = 31\%, n = 500$
5. In a survey of 350 randomly selected homeowners, 54% stated that they are planning a major home improvement project in the next six months.

Example 3 (p. 742)

MEDIA **For Exercises 6 and 7, use the following information.**
A survey found that 57% of consumers said they will not have any debt from holiday spending. Suppose the survey had a margin of error of 3%.

6. What does the 3% indicate about the results?
7. How many people were surveyed?

Exercises

HOMEWORK HELP

For Exercises	See Examples
8–11	1
12–21	2, 3

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer.

8. pointing with your pencil at a class list with your eyes closed as a way to find a sample of students in your class
9. putting the names of all seniors in a hat, then drawing names from the hat to select a sample of seniors
10. asking every twentieth person on a list of registered voters to determine which political candidate is favored
11. finding the heights of all the boys on the varsity basketball team to determine the average height of all the boys in your school

For Exercises 12–21, find the margin of sampling error to the nearest percent.

12. $p = 81\%, n = 100$
13. $p = 16\%, n = 400$
14. $p = 54\%, n = 500$
15. $p = 48\%, n = 1000$
16. $p = 33\%, n = 1000$
17. $p = 67\%, n = 1500$
18. A poll asked people to name the most serious problem facing the country. Forty-six percent of the 800 randomly selected people said crime.
19. In a recent survey, 431 full-time employees were asked if the Internet has made them more or less productive at work. 27% said it made them more productive.
20. Three hundred sixty-seven of 425 high school students said pizza was their favorite food in the school cafeteria.
21. Nine hundred thirty-four of 2150 subscribers to a particular newspaper said their favorite sport was football.
22. **SHOPPING** According to a recent poll, 33% of shoppers planned to spend $1000 or more during a holiday season. The margin of error was 3%. How many people were surveyed?
23. **ELECTION PREDICTION** One hundred people were asked whether they would vote for Candidate A or Candidate B in an upcoming election. How many said "Candidate A" if the margin of error was 9.6%?

EXTRA PRACTICE
See pages 919, 937.
MathOnline
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24. ECONOMICS In a recent poll, 83% of the 1020 people surveyed said they supported raising the minimum wage. What was the margin of error?

25. PHYSICIANS In a recent poll, 61% of the 1010 people surveyed said they considered being a physician to be a very prestigious occupation. What was the margin of error?

H.O.T. Problems

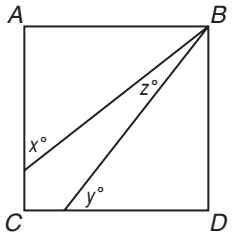
26. OPEN ENDED Give examples of a biased sample and an unbiased sample. Explain your reasoning.

27. REASONING Explain what happens to the margin of sampling error when the size of the sample n increases. Why does this happen?

28. *Writing in Math* Use the information on page 742 to explain how surveys are used in marketing. Find the margin of error for those who spend $249 or less if 807 mothers were surveyed. Explain what this margin of error means.

STANDARDIZED TEST PRACTICE

29. ACT/SAT In rectangle $ABCD$, what is $x + y$ in terms of z?



A $90 + z$

B $190 - z$

C $180 + z$

D $270 - z$

30. REVIEW If $xy^{-2} + y^{-1} = y^{-2}$, then the value of x *cannot* equal which of the following?

F -1

G 0

H 1

J 2

Spiral Review

A student guesses at all 5 questions on a true-false quiz. Find each probability. (Lesson 12-8)

31. P(all 5 correct)

32. P(exactly 4 correct)

33. P(at least 3 correct)

A set of 250 data values is normally distributed with a mean of 50 and a standard deviation of 5.5. (Lesson 12-7)

34. What percent of the data lies between 39 and 61?

35. What is the probability that a data value selected at random is greater than 39?

Cross-Curricular Project

Algebra and Social Studies

Math from the Past It is time to complete your project. Use the information and data you have gathered about the history of mathematics to prepare a presentation or web page. Be sure to include transparencies and a sample mathematics problem or idea in the presentation.

MathOnline **Cross-Curricular Project at algebra2.com**

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CHAPTER 12

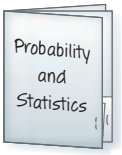
Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

The Counting Principle, Permutations, and Combinations (Lessons 12-1 and 12-2)

- Fundamental Counting Principle: If event M can occur in m ways and is followed by event N that can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.
- Permutation: order of objects is important.
- Combination: order of objects is not important.

Probability (Lessons 12-3 and 12-4)

- Two independent events: $P(A \text{ and } B) = P(A) \cdot P(B)$
- Two dependent events: $P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$
- Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$
- Inclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Statistical Measures (Lesson 12-5)

- To represent a set of data, use the mean if the data are spread out, the median when the data has outliers, or the mode when the data are tightly clustered around one or two values.
- Standard deviation for n values: $\bar{x}$ is the mean,

$$\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

The Normal Distribution (Lesson 12-6)

- The graph is maximized at the mean and the data are symmetric about the mean.

Binomial Experiments, Sampling, and Error (Lessons 12-7 and 12-8)

- A binomial experiment exists if and only if there are exactly two possible outcomes, a fixed number of independent trials, and the possibilities for each trial are the same.

Key Vocabulary

binomial distribution (p. 730)
binomial experiment (p. 730)
binomial probability (p 731)
combination (p. 692)
compound event (p. 710)
dependent events (p. 686)
event (p. 684)
exponential distribution (p 729)
exponential probability (p. 729)
inclusive events (p. 712)
independent events (p. 684)
measure of variation (p. 718)
mutually exclusive events (p. 710)
normal distribution (p. 724)
outcome (p. 684)
permutation (p. 690)
probability (p. 697)
probability distribution (p. 699)
random (p. 697)
random variable (p. 699)
relative-frequency histogram (p. 699)
sample space (p. 684)
simple event (p. 710)
standard deviation (p. 718)
unbiased sample (p. 741)
uniform distribution (p. 699)
univariate data (p. 717)
variance (p. 718)

Vocabulary Check

Choose the term that best matches each statement or phrase. Choose from the list above.

1. the ratio of the number of ways an event can succeed to the number of possible outcomes
2. an arrangement of objects in which order does not matter
3. two or more events in which the outcome of one event affects the outcome of another event
4. a function that is used to predict the probabilities of an event based on time
5. two events in which the outcome can never be the same
6. an arrangement of objects in which order matters
7. the set of all possible outcomes
8. an event that consists of two or more simple events

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Lesson-by-Lesson Review

12–1 The Counting Principle (pp. 684–689)

9. PASSWORDS The letters a, c, e, g, i, and k are used to form 6-letter passwords. How many passwords can be formed if the letters can be used more than once in any given password?

Example 1 **How many different license plates are possible with two letters followed by three digits?**

There are 26 possibilities for each letter. There are 10 possibilities, the digits 0–9, for each number. Thus, the number of possible license plates is as follows.

$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^2 \cdot 10^3$ or 676,000

12–2 Permutations and Combinations (pp. 690–695)

10. A committee of 3 is selected from Jillian, Miles, Mark, and Nikia. How many committees contain 2 boys and 1 girl?

11. Five cards are drawn from a standard deck of cards. How many different hands consist of four queens and one king?

12. A box of pencils contains 4 red, 2 white, and 3 blue pencils. How many different ways can 2 red, 1 white, and 1 blue pencil be selected?

Example 2 **A basket contains 3 apples, 6 oranges, 7 pears, and 9 peaches. How many ways can 1 apple, 2 oranges, 6 pears, and 2 peaches be selected?**

This involves the product of four combinations, one for each type of fruit.

$$C(3, 1) \cdot C(6, 2) \cdot C(7, 6) \cdot C(9, 2)$$
$$= \frac{3!}{(3-1)!1!} \cdot \frac{6!}{(6-2)!2!} \cdot \frac{7!}{(7-6)!6!} \cdot \frac{9!}{(9-2)!2!}$$
$$= 3 \cdot 15 \cdot 7 \cdot 36 \text{ or } 11{,}340 \text{ ways}$$

12–3 Probability (pp. 697–702)

13. A bag contains 4 blue marbles and 3 green marbles. One marble is drawn from the bag at random. What is the probability that the marble drawn is blue?

14. COINS The table shows the distribution of the number of heads occurring when four coins are tossed. Find $P(H = 3)$.

H = Heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Example 3 **A bag of golf tees contains 23 red, 19 blue, 16 yellow, 21 green, 11 orange, 19 white, and 17 black tees. What is the probability that if you choose a tee from the bag at random, you will choose a green tee?**

There are 21 ways to choose a green tee and $23 + 19 + 16 + 11 + 19 + 17$ or 105 ways not to choose a green tee. So, s is 21 and f is 105.

$$P(\text{green tee}) = \frac{s}{s+f}$$
$$= \frac{21}{21 + 105} \text{ or } \frac{1}{6}$$

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12–4 Multiplying Probabilities (pp. 703–709)

Determine whether the events are *independent* or *dependent*. Then find the probability.

15. Two dice are rolled. What is the probability that each die shows a 4?

16. Two cards are drawn from a standard deck of cards without replacement. Find the probability of drawing a heart and a club in that order.

17. Luz has 2 red, 2 white, and 3 blue marbles in a cup. If she draws two marbles at random and does not replace the first one, find the probability of a white marble and then a blue marble.

Example 4 **There are 3 dimes, 2 quarters, and 5 nickels in Langston's pocket. If he reaches in and selects three coins at random without replacing any of them, what is the probability that he will choose a dime d, then a quarter q, and then a nickel n?**

Because the outcomes of the first and second choices affect the later choices, these are dependent events.

$$P(d, \text{ then } q, \text{ then } n) = \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{5}{8} \text{ or } \frac{1}{24}$$

The probability is $\frac{1}{24}$ or about 4.2%.

12–5 Adding Probabilities (pp. 710–715)

Determine whether the events are *mutually exclusive* or *inclusive*. Then find the probability.

18. A die is rolled. What is the probability of rolling a 6 or a number less than 4?

19. A die is rolled. What is the probability of rolling a 6 or a number greater than 4?

20. A card is drawn from a standard deck of cards. What is the probability of drawing a king or a red card?

21. There are 5 English, 2 math, and 3 chemistry books on a shelf. If a book is randomly selected, what is the probability of selecting a math book or a chemistry book?

Example 5 **Trish has four \$1 bills and six \$5 bills. She takes three bills from her wallet at random. What is the probability that Trish will select at least two \$1 bills?**

$P(\text{at least two \$1})$

$$= P(\text{two \$1, \$5}) + P(\text{three \$1, no \$5})$$

$$= \frac{C(4, 2) \cdot C(6, 1)}{C(10, 3)} + \frac{C(4, 3) \cdot C(6, 0)}{C(10, 3)}$$

$$= \frac{\frac{4! \cdot 6!}{(4-2)!2!(6-1)!1!}}{C(10, 3)} + \frac{\frac{4! \cdot 6!}{(4-3)!3!(6-0)!0!}}{C(10, 3)}$$

$$= \frac{36}{120} + \frac{4}{120} \text{ or } \frac{1}{3}$$

The probability is $\frac{1}{3}$ or about 33%.

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CHAPTER 12

Study Guide and Review

12–6 Statistical Measures (pp. 717–723)

FOOD **For Exercises 22 and 23, use the frequency table that shows the number of dried apricots per box.**

Apricot Count	Frequency
19	1
20	3
21	5
22	4

22. Find the mean, median, mode, and standard deviation of the apricots to the nearest tenth.

23. For how many boxes is the number of apricots within one standard deviation of the mean?

Example 6 **Find the variance and standard deviation for {100, 156, 158, 159, 162, 165, 170, 190}.**

Step 1 Find the mean.

$$\frac{100 + 156 + 158 + 159 + 162 + 165 + 170 + 190}{8}$$

$$= \frac{1260}{8} \text{ or } 157.5$$

Step 2 Find the standard deviation.

$$\sigma^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}$$

$$\sigma^2 = \frac{(100 - 157.5)^2 + \ldots + (190 - 157.5)^2}{8}$$

$\sigma^2 = \frac{4600}{8}$ Simplify.

$\sigma^2 = 575$ Divide.

$\sigma \approx 23.98$ Take the square root of each side.

12–7 The Normal Distribution (pp. 724–728)

UTILITIES **For Exercises 24–27, use the following information.**

The utility bills in a city of 5000 households are normally distributed with a mean of $180 and a standard deviation of $16.

24. About how many utility bills were between $164 and $196?

25. About how many bills exceeded $212?

26. About how many bills were under $164?

27. What is the probability that a random bill is between $164 and $180?

BASEBALL **For Exercises 28 and 29, use the following information.**

The average age of a major league baseball player is normally distributed with a mean of 28 and a standard deviation of 4 years.

28. About what percent of major league baseball players are younger than 24?

29. If a team has 35 players, about how many are between the ages of 24 and 32?

Example 7 **Mr. Byrum gave an exam to his 30 Algebra 2 students at the end of the first semester. The scores were normally distributed with a mean score of 78 and a standard deviation of 6.**

a. What percent of the class would you expect to have scored between 72 and 84?
Since 72 and 84 are 1 standard deviation to the left and right of the mean, respectively, 34% + 34% or 68% of the students scored within this range.

b. What percent of the class would you expect to have scored between 90 and 96?
90 to 96 on the test includes 2% of the students.

c. Approximately how many students scored between 84 and 90?
84 to 90 on the test includes 13.5% of the students; $0.135 \times 30 = 4$ students.

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Mixed Problem Solving
For mixed problem-solving practice, see page 937.

12–8 Exponential and Binomial Distribution (pp. 729-733)

30. The average person has a pair of automobile windshield wiper blades for 6 months. What is the probability that a randomly selected automobile has a pair of windshield wiper blades older than one year?

LAWS For Exercises 31 and 32, use the following information.

A polling company wants to estimate how many people are in favor of a new environmental law. The polling company polls 20 people. The probability that a person is in favor of the law is 0.5.

31. What is the probability that exactly 12 people are in favor of the new law?

32. What is the expected number of people in favor of the law?

Example 8 **According to a recent survey, the average teenager spends one hour a day on an outdoor activity. What is the probability that a randomly selected teenager spends more than 1.5 hours per day outside?**

Use the first exponential distribution function. The mean is 1, and the inverse of the mean is 1.

$f(x) = e^{-mx}$ Exponential Distribution Function

$= e^{-1(1.5)}$ Replace m with 1 and x with 1.5.

$= e^{-1.5}$ Simplify.

≈ 0.2231 or 22.31% Use a calculator.

There is a 22.31% chance that a randomly selected teenager spends more than 1.5 hours a day outside.

12–9 Binomial Experiments (pp. 735–739)

Find each probability if a number cube is rolled twelve times.

33. P(twelve 3s) **34.** P(exactly one 3)

35. WORLD CULTURES The Cayuga Indians played a game of chance called *Dish*, in which they used 6 flattened peach stones blackened on one side. They placed the peach stones in a wooden bowl and tossed them. The winner was the first person to get a prearranged number of points. Assume that each face (black or neutral) of each stone has an equal chance of showing up. Find the probability of each possible outcome.

Example 9 **To practice for a jigsaw puzzle competition, Laura and Julian completed four jigsaw puzzles. The probability that Laura places the last piece is $\frac{3}{5}$, and the probability that Julian places the last piece is $\frac{2}{5}$. What is the probability that Laura will place the last piece of at least two puzzles?**

$P = L^4 + 4L^3J + 6L^2J^2$

$= \left(\frac{3}{5}\right)^4 + 4\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right) + 6\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2$

$= \frac{81}{625} + \frac{216}{625} + \frac{216}{625}$ or 0.8208

The probability is about 82%.

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12–10 Sampling and Error (pp. 741–744)

36. ELECTION According to a poll of 300 people, 39% said that they favor Mrs. Smith in an upcoming election. What is the margin of sampling error?

37. FREEDOMS In a poll asking people to name their most valued freedom, 51% of the randomly selected people said it was the freedom of speech. Find the margin of sampling error if 625 people were randomly selected.

38. SPORTS According to a recent survey of mothers with children who play sports, 63% of them would prefer that their children not play football. Suppose the margin of error is 4.5%. How many mothers were surveyed?

Example 10 **In a survey taken at a local high school, 75% of the student body stated that they thought school lunches should be free. This survey had a margin of error of 2%. How many people were surveyed?**

$$ME = 2\sqrt{\frac{p(1-p)}{n}}$$ Margin of sampling error

$$0.02 = 2\sqrt{\frac{0.75(1-0.75)}{n}}$$ $ME = 0.02, p = 0.75$

$$0.01 = \sqrt{\frac{0.75(1-0.75)}{n}}$$ Divide each side by 2.

$$0.0001 = \frac{0.75(0.25)}{n}$$ Square each side.

$$n = \frac{0.75(0.25)}{0.0001}$$ Multiply and divide.

$$= 1875$$ Simplify.

There were about 1875 people in the survey.

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CHAPTER 12

Practice Test

Evaluate each expression.

1. $P(7, 3)$
2. $C(7, 3)$
3. $P(13, 5)$
4. $C(13, 5)$

5. How many ways can 9 bowling balls be arranged on the upper rack of a bowling ball shelf?

6. How many different outfits can be made if you choose 1 each from 11 skirts, 9 blouses, 3 belts, and 7 pairs of shoes?

7. How many ways can the letters of the word *probability* be arranged?

8. How many different soccer teams consisting of 11 players can be formed from 18 players?

9. Eleven points are equally spaced on a circle. How many ways can five of these points be chosen as the vertices of a pentagon?

10. A number is drawn at random from a hat that contains all the numbers from 1 to 100. What is the probability that the number is less than 16?

11. Two cards are drawn in succession from a standard deck of cards without replacement. What is the probability that both cards are greater than 2 and less than 9?

12. A shipment of 10 television sets contains 3 defective sets. How many ways can a hospital purchase 4 of these sets and receive at least 2 of the defective sets?

13. In a row of 10 parking spaces in a parking lot, how many ways can 4 cars park?

14. While shooting arrows, William Tell can hit an apple 9 out of 10 times. What is the probability that he will hit it exactly 4 out of 7 times?

15. Ten people are going on a camping trip in three cars that hold 5, 2, and 4 passengers, respectively. How many ways is it possible to transport the people to their campsite?

16. The number of colored golf balls in a box is shown in the table below.

Color	Number of Golf Balls
white	5
red	3

Three golf balls are drawn from the box in succession, each being replaced in the box before the next draw is made. What is the probability that all 3 golf balls are the same color?

For Exercises 17–19, use the following information.

In a ten-question multiple-choice test with four choices for each question, a student who was not prepared guesses on each item. Find each probability.

17. 6 questions correct
18. at least 8 questions correct
19. fewer than 8 questions correct

20. **MULTIPLE CHOICE** The average amount of money that a student spends for lunch is \$4. What is the probability that a randomly selected student spends less than \$3 on lunch?

A 0.36 **C** 0.49

B 0.47 **D** 0.52

21. **MULTIPLE CHOICE** A mail-order computer company offers a choice of 4 amounts of memory, 2 sizes of hard drives, and 2 sizes of monitors. How many different systems are available to a customer?

F 8

G 16

H 32

J 64

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CHAPTER 12

Standardized Test Practice

Cumulative, Chapters 1–12

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Ms. Rudberg has a list of the yearly salaries of the staff members in her department. Which measure of data describes the middle income value of the salaries?

A mean

B median

C mode

D range

2. A survey of 90 physical trainers found that 15 said they went for a run at least 5 times per week. Of that group, 5 said they also swim during the week and at least 25% run and swim every week. Which conclusion is valid based on the information given?

F The report is accurate because 15 out of 90 is 25%.

G The report is accurate because 5 out of 15 is 33%, which is at least 25%.

H The report is inaccurate because 5 out of 90 is only 3.3%.

J The report is inaccurate because she does not know if the swimming is really exercising.

3. GRIDDABLE Anna is training to run a 10-kilometer race. The table below lists the times she received in different races. The times are listed in minutes. What was her mean time in minutes for a 10-kilometer race?

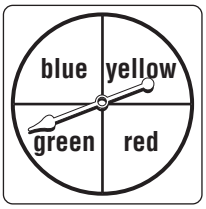
7.25	8.10
7.40	6.75
7.20	7.35
7.10	7.25
8.00	7.45

4. Mariah has 6 books on her bookshelf. Two are literature books, one is a science book, two are math books, and one is a dictionary. What is the probability that she randomly chooses a science book and the dictionary?

A $\frac{1}{3}$ **C** $\frac{1}{12}$

B $\frac{1}{4}$ **D** $\frac{1}{36}$

5. Peter is playing a game where he spins the spinner pictured below and then rolls a die. What is the probability that the spinner lands on yellow and he rolls an even number on the die?



F $\frac{3}{4}$ **H** $\frac{1}{8}$

G $\frac{5}{12}$ **J** $\frac{1}{24}$

6. Lynette has a 4-inch by 6-inch picture of her brother. She gets an enlargement made so that the new print has dimensions that are 4 times the dimensions of her original picture. How does the area of the enlargement compare to the area of the original picture?

TEST-TAKING TIP

Question 1 To prepare for a standardized test, make flash cards of key mathematical terms, such as *mean* and *median*. Use the glossary in this text to determine the important terms and their correct definitions.

Standardized Test Practice at algebra2.com

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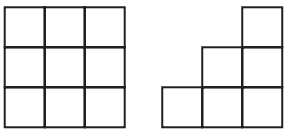
7. Lauren works 8-hour shifts at a book store. She makes $7 an hour and receives a 20% commission on her sales. How much does she need to sell in one shift to earn exactly $80 before taxes are deducted?

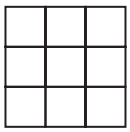
F $30
G $86
H $120
J $400

8. A rectangular solid has a volume of 35 cubic inches. If the length, width and height are all changed to 3 times their original size, what will be the volume of the new rectangular solid?

A 38 in^3
B 105 in^3
C 315 in^3
D 945 in^3

9. The top, side and front views of an object built with cubes are shown below.





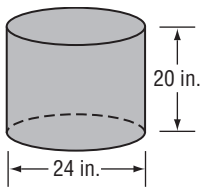
How many cubes are needed to construct this object?

F 9
G 11
H 18
J 21

10. Petra has made a game for her daughter's birthday party. The playing board is a circle divided evenly into 8 sectors. If the circle has a radius of 18 inches, what is the approximate area of one of the sectors?

A 4 in^2
B 32 in^2
C 127 in^2
D 254 in^2

11. GRIDDABLE Kara has a cylindrical container that she needs to fill with dirt so she can plant some flowers.



What is the volume of the cylinder in cubic inches rounded to the nearest cubic inch?

Pre-AP

Record your answers on a sheet of paper. Show your work.

12. When working at Taco King, Naomi wears a uniform that consists of a shirt, a pair of pants, and a tie. She has 6 uniform shirts, 3 uniform pants, and 4 uniform ties.

a. How many different combinations of shirt, pants, and tie can she make?

b. How many different combinations of shirts and pants can she make?

c. Two of her shirts are red, 3 are blue, and 1 is white. If she wears a different shirt for six days in a row and chooses the shirts at random, what is the probability that she wears a red shirt the first two days?

NEED EXTRA HELP?												
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12
Go to Lesson...	12–6	12–4	12–6	12–3	12–4	1–1	2–4	1–1	2–7	10–3	6–8	12–1

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Unit 5
Trigonometry

Focus

Trigonometry is used in navigation, physics, and construction, among other fields. In this unit, you will learn about trigonometric functions, graphs, and identities.

CHAPTER 13
Trigonometric Funtions

BIG Idea Understand and apply trigonometry to various problems.

BIG Idea Understand and apply the laws of sines and cosines.

CHAPTER 14
Trigonometric Graphs and Identities

BIG Idea Comprehend and manipulate the trigonometric functions, graphs and identities.

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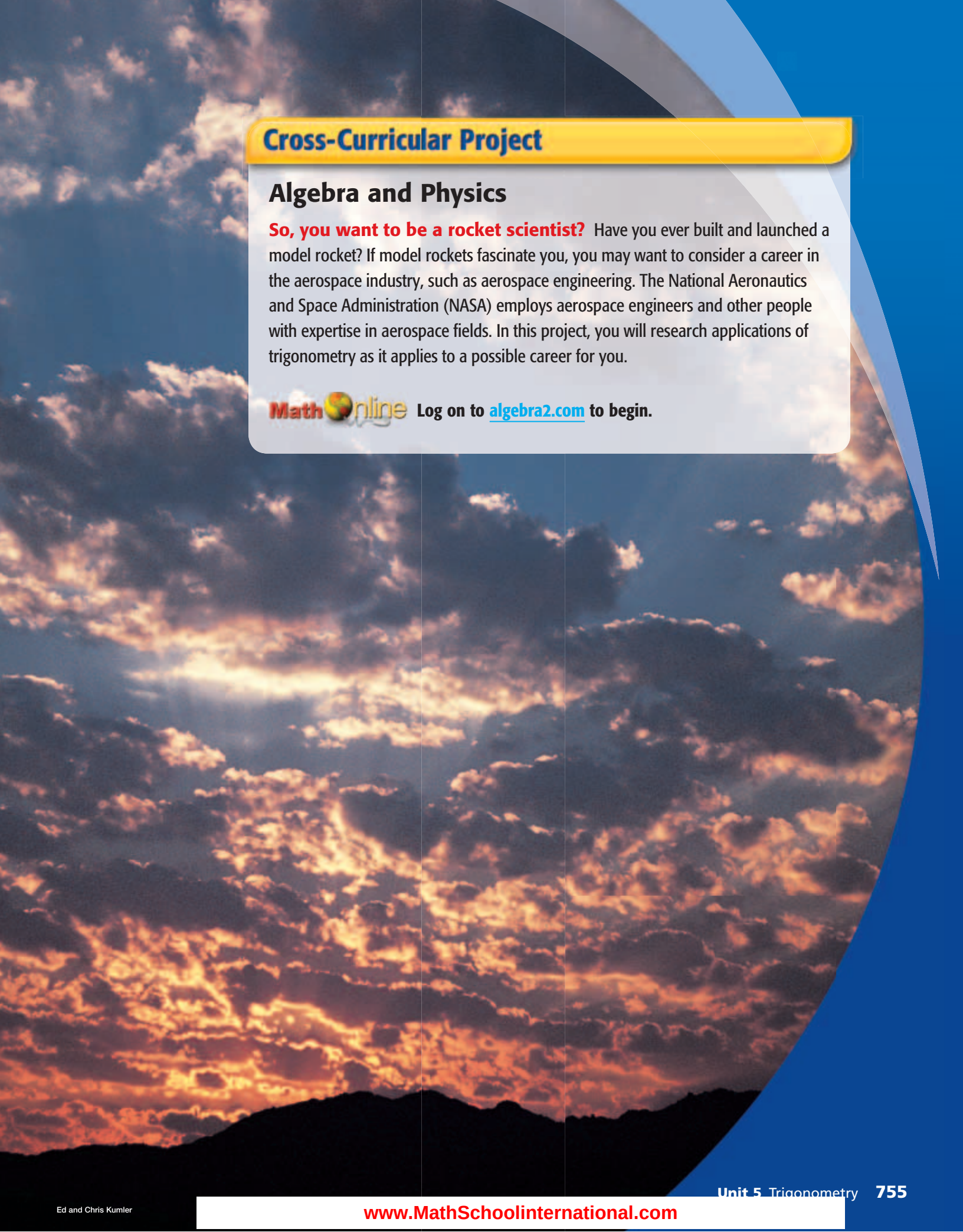

Cross-Curricular Project

Algebra and Physics

So, you want to be a rocket scientist? Have you ever built and launched a model rocket? If model rockets fascinate you, you may want to consider a career in the aerospace industry, such as aerospace engineering. The National Aeronautics and Space Administration (NASA) employs aerospace engineers and other people with expertise in aerospace fields. In this project, you will research applications of trigonometry as it applies to a possible career for you.

MathOnline **Log on to algebra2.com to begin.**

Ed and Chris Kumler

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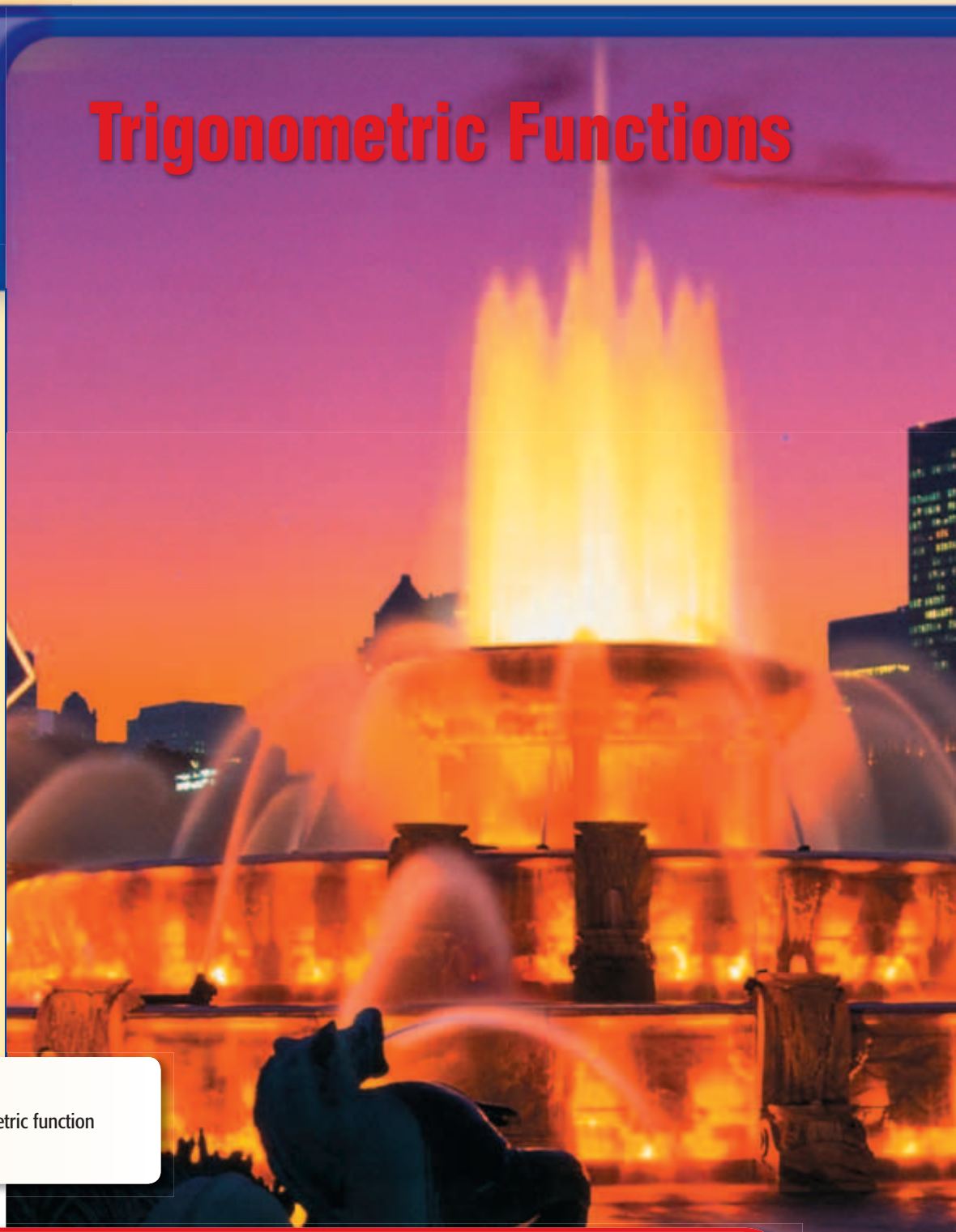
CHAPTER 13

Trigonometric Functions



BIG Ideas

- Find values of trigonometric functions.
- Solve problems by using right triangle trigonometry.
- Solve triangles by using the Law of Sines and Law of Cosines.

Key Vocabulary

solve a right triangle (p. 762)
radian (p. 769)
Law of Sines (p. 786)
Law of Cosines (p. 793)
circular function (p. 800)

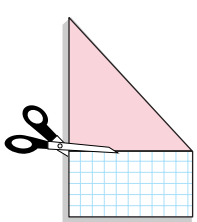
Real-World Link

Buildings Surveyors use a trigonometric function to find the heights of buildings.

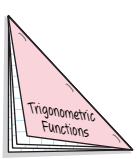
FOLDABLES Study Organizer

Trigonometric Functions Make this Foldable to help you organize your notes. Begin with one sheet of construction paper and two pieces of grid paper.

1 **Stack and Fold** on the diagonal. Cut to form a triangular stack.



2 **Staple** edge to form a book. Label Trigonometric Functions.



Bill Ross/CORBIS

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GET READY for Chapter 13

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

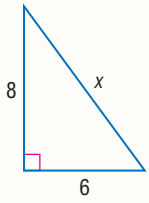
Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKQuiz

Find the value of x to the nearest tenth. (Prerequisite Skills, p. 881)

1.



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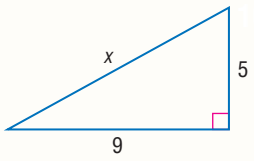

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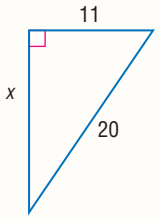
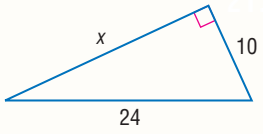


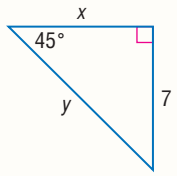
4.



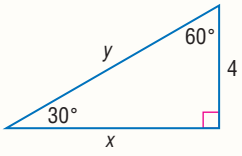
5. LADDER There is a window that is 10 feet high. You want to use a ladder to get up to the window; you decide to put the ladder 3 feet away from the wall. How long should the ladder be? (Prerequisite Skills, p. 881)

Find each missing measure. Write all radicals in simplest form. (Prerequisite Skill)

6.



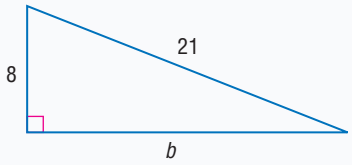
7.



8. KITES A kite is being flown at a 45° angle. The string of the kite is 20 feet long. How high is the kite? (Prerequisite Skill)

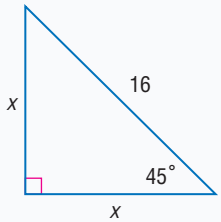
QUICKReview

Example 1 **Find the missing measure of the right triangle.**



$c^2 = a^2 + b^2$	Pythagorean Theorem
$21^2 = 8^2 + b^2$	Replace c with 21 and a with 8.
$441 = 64 + b^2$	Simplify.
$377 = b^2$	Subtract 64 from each side.
$19.4 \approx b$	Take the square root of each side.

Example 2 **Find the missing measures. Write all radicals in simplest form.**



$x^2 + x^2 = 16^2$	Pythagorean Theorem
$2x^2 = 16^2$	Combine like terms.
$2x^2 = 256$	Simplify.
$x^2 = 128$	Divide each side by 2.
$x = \sqrt{128}$	Take the square root of each side.
$x = 8\sqrt{2}$	Simplify.

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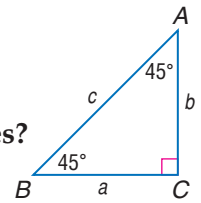
EXPLORE 13-1

Spreadsheet Lab

Special Right Triangles

ACTIVITY

The legs of a 45°-45°-90° triangle, a and b, are equal in measure. Use a spreadsheet to investigate the dimensions of 45°-45°-90° triangles. What patterns do you observe in the ratios of the side measures of these triangles?



=SQRT(A2^2+B2^2) =B2/A2 =B2/C2 =A2/C2

45-45-90 Triangles

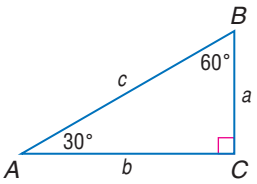
	A	B	C	D	E	F
1	a	b		a/b	b/c	a/c
2	1	1	1.41421356	1	0.70710678	0.70710678
3	2	2	2.82842712	1	0.70710678	0.70710678
4	3	3	4.24264069	1	0.70710678	0.70710678
5	4	4	5.65685425	1	0.70710678	0.70710678
6	5	5	7.07106781	1	0.70710678	0.70710678
7						

Sheet 1 Sheet 2 Sheet 3

The spreadsheet shows the formula that will calculate the length of side c. The formula uses the Pythagorean Theorem in the form $c = \sqrt{a^2 + b^2}$. Since 45°-45°-90° triangles share the same angle measures, the triangles listed in the spreadsheet are all similar triangles. Notice that all of the ratios of side b to side a are 1. All of the ratios of side b to side c and of side a to side c are approximately 0.71.

Model and Analyze

For Exercises 1–3, use the spreadsheet for 30°-60°-90° triangles.
If the measure of one leg of a right triangle and the measure of the hypotenuse are in a ratio of 1 to 2, then the acute angles of the triangle measure 30° and 60°.



30-60-90 Triangles

	A	B	C	D	E	F
1	a	b	c	b/a	b/c	a/c
2	1		2			
3	2		4			
4	3		6			
5	4		8			
6	5		10			
7						

Sheet 1 Sheet 2 Sheet 3

1. Copy and complete the spreadsheet above.
2. Describe the relationship among the 30°-60°-90° triangles with the dimensions given.
3. What patterns do you observe in the ratios of the side measures of these triangles?

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13-1 Right Triangle Trigonometry

Main Ideas

- Find values of trigonometric functions for acute angles.
- Solve problems involving right triangles.

New Vocabulary

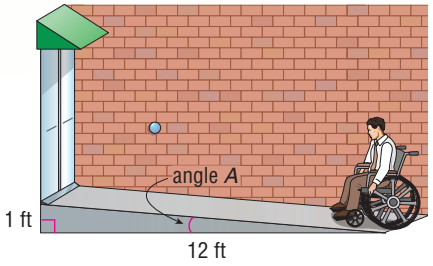
trigonometry
trigonometric functions
sine
cosine
tangent
cosecant
secant
cotangent
solve a right triangle
angle of elevation
angle of depression

Reading Math

Trigonometry
The word *trigonometry* is derived from two Greek words–*trigon* meaning triangle and *metra* meaning measurement.

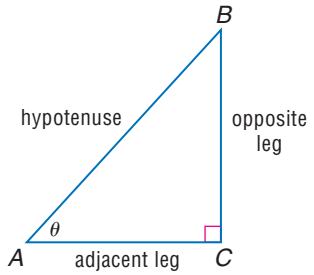
GET READY for the Lesson

The Americans with Disabilities Act (ADA) provides regulations designed to make public buildings accessible to all. Under this act, the slope of an entrance ramp designed for those with mobility disabilities must not exceed a ratio of 1 to 12. This means that for every 12 units of horizontal run, the ramp can rise or fall no more than 1 unit.



When viewed from the side, a ramp forms a right triangle. The slope of the ramp can be described by the *tangent* of the angle the ramp makes with the ground. In this example, the tangent of angle A is $\frac{1}{12}$.

Trigonometric Values The tangent of an angle is one of the ratios used in trigonometry. **Trigonometry** is the study of the relationships among the angles and sides of a right triangle.



Consider right triangle ABC in which the measure of acute angle A is identified by the Greek letter *theta*, θ. The sides of the triangle are the *hypotenuse*, the *leg opposite* θ, and the *leg adjacent to* θ.

Using these sides, you can define six **trigonometric functions: sine, cosine, tangent, cosecant, secant,** and **cotangent**. These functions are abbreviated sin, cos, tan, csc, sec, and cot, respectively.

KEY CONCEPT — Trigonometric Functions

If θ is the measure of an acute angle of a right triangle, *opp* is the measure of the leg opposite θ, *adj* is the measure of the leg adjacent to θ, and *hyp* is the measure of the hypotenuse, then the following are true.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Notice that the sine, cosine, and tangent functions are reciprocals of the cosecant, secant, and cotangent functions, respectively. Thus, the following are also true.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

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Study Tip

Memorize Trigonometric Ratios

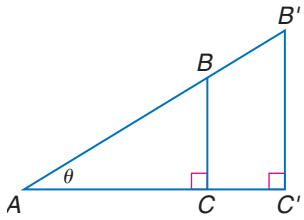
SOH-CAH-TOA is a mnemonic device for remembering the first letter of each word in the ratios for sine, cosine, and tangent.

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

The domain of each of these trigonometric functions is the set of all acute angles θ of a right triangle. The values of the functions depend only on the measure of θ and not on the size of the right triangle. For example, consider $\sin\theta$ in the figure at the right.



Using $\triangle ABC$: $\sin\theta = \frac{BC}{AB}$

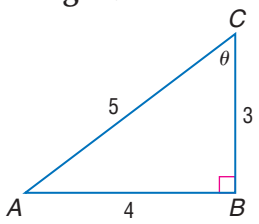
Using $\triangle AB'C'$: $\sin\theta = \frac{B'C'}{AB'}$

The right triangles are similar because they share angle θ. Since they are similar, the ratios of corresponding sides are equal. That is, $\frac{BC}{AB} = \frac{B'C'}{AB'}$. Therefore, you will find the same value for $\sin\theta$ regardless of which triangle you use.

EXAMPLE 1 Find Trigonometric Values

Find the values of the six trigonometric functions for angle θ.

For this triangle, the leg opposite θ is $\overline{AB}$, and the leg adjacent to θ is $\overline{CB}$. Recall that the hypotenuse is always the longest side of a right triangle, in this case $\overline{AC}$.



Use opp = 4, adj = 3, and hyp = 5 to write each trigonometric ratio.

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$ $\quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$ $\quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$ $\quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$ $\quad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

CHECK Your Progress

1. Find the values of the six trigonometric functions for angle A in $\triangle ABC$ above.

Throughout Unit 5, a capital letter will be used to denote both a vertex of a triangle and the measure of the angle at that vertex. The same letter in lowercase will be used to denote the side opposite that angle and its measure.

STANDARDIZED TEST EXAMPLE 2 Use One Trigonometric Ratio to Find Another

If $\cos A = \frac{2}{5}$, find the value of $\tan A$.

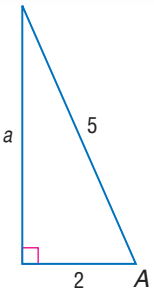
A $\frac{5}{2}$ $\quad$ B $\frac{2\sqrt{21}}{21}$ $\quad$ C $\frac{\sqrt{21}}{2}$ $\quad$ D $\sqrt{21}$

Test-Taking Tip

Whenever necessary or helpful, draw a diagram of the situation.

Read the Test Item

Begin by drawing a right triangle and labeling one acute angle A. Since $\cos\theta = \frac{\text{adj}}{\text{hyp}}$ and $\cos A = \frac{2}{5}$ in this case, label the adjacent leg 2 and the hypotenuse 5. This represents the simplest triangle for which $\cos A = \frac{2}{5}$.



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Solve the Test Item

Use the Pythagorean Theorem to find a.

$a^2 + b^2 = c^2$ Pythagorean Theorem

$a^2 + 2^2 = 5^2$ Replace b with 2 and c with 5.

$a^2 + 4 = 25$ Simplify.

$a^2 = 21$ Subtract 4 from each side.

$a = \sqrt{21}$ Take the square root of each side.

Now find $\tan A$.

$\tan A = \frac{\text{opp}}{\text{adj}}$ Tangent ratio

$= \frac{\sqrt{21}}{2}$ Replace *opp* with $\sqrt{21}$ and *adj* with 2.

The answer is C.

CHECK Your Progress

2. If $\tan B = \frac{3}{7}$, find the value of $\sin B$.

F $\frac{7}{3}$ **G** $\frac{\sqrt{58}}{3}$ **H** $\frac{3\sqrt{58}}{58}$ **J** $\frac{\sqrt{58}}{7}$

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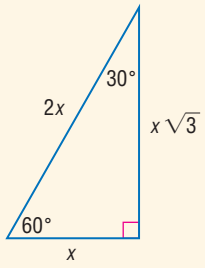
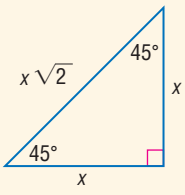
Angles that measure 30°, 45°, and 60° occur frequently in trigonometry. The table below gives the values of the six trigonometric functions for these angles. To remember these values, use the properties of 30°-60°-90° and 45°-45°-90° triangles.

KEY CONCEPT *Trigonometric Values for Special Angles*

30°-60°-90° Triangle



45°-45°-90° Triangle



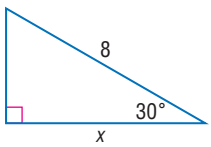
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

You will verify some of these values in Exercises 39 and 40.

Right Triangle Problems You can use trigonometric functions to solve problems involving right triangles.

EXAMPLE Find a Missing Side Length of a Right Triangle

3 **Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.**



The measure of the hypotenuse is 8. The side with the missing length is *adjacent* to the angle measuring 30°. The trigonometric function relating the adjacent side of a right triangle and the hypotenuse is the cosine function.

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$\cos \theta = \frac{adj}{hyp}$ cosine ratio

$\cos 30° = \frac{x}{8}$ Replace θ with 30°, *adj* with x, and *hyp* with 8.

$\frac{\sqrt{3}}{2} = \frac{x}{8}$ $\cos 30° = \frac{\sqrt{3}}{2}$

$4\sqrt{3} = x$ Multiply each side by 8. The value of x is $4\sqrt{3}$ or about 6.9.

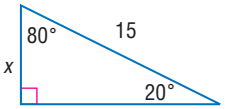
Study Tip

Common Misconception

The $\cos^{-1} x$ on a graphing calculator does *not* find $\frac{1}{\cos x}$. To find sec x or $\frac{1}{\cos x}$, find cos x and then use the x^{-1} key.

CHECK Your Progress

3. Write an equation involving sin, cos, or tan that can be used to find the value of x. Then solve the equation. Round to the nearest tenth.



A calculator can be used to find the value of trigonometric functions for *any* angle, not just the special angles mentioned. Use SIN, COS, and TAN for sine, cosine, and tangent. Use these keys and the reciprocal key, x^{-1}, for cosecant, secant, and cotangent. Be sure your calculator is in degree mode.

Here are some calculator examples.

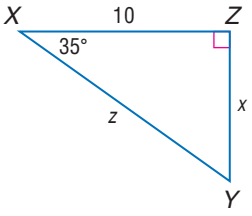
cos 46° **KEYSTROKES:** COS 46 ENTER 0.6946583705

cot 20° **KEYSTROKES:** TAN 20 ENTER x^{-1} ENTER 2.747477419

If you know the measures of any two sides of a right triangle or the measures of one side and one acute angle, you can determine the measures of all the sides and angles of the triangle. This process of finding the missing measures is known as **solving a right triangle.**

EXAMPLE Solve a Right Triangle

4 **Solve $\triangle XYZ$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



Find x and z.

$\tan 35° = \frac{x}{10}$

$10 \tan 35° = x$

$7.0 \approx x$

$\sec 35° = \frac{z}{10}$

$\frac{1}{\cos 35°} = \frac{z}{10}$

$\frac{1}{\cos 35°} = z$

$12.2 \approx z$

Find Y. $35° + Y = 90°$ Angles X and Y are complementary.

$Y = 55°$ Therefore, $Y = 55°$, $x \approx 7.0$, and $z \approx 12.2$.

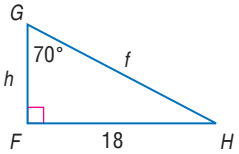
Study Tip

Error in Measurement

The value of z in Example 4 is found using the secant instead of using the Pythagorean Theorem. This is because the secant uses values given in the problem rather than calculated values.

CHECK Your Progress

4. Solve $\triangle FGH$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



Use the inverse capabilities of your calculator to find the measure of an angle when one of its trigonometric ratios is known. For example, use the $\sin^{-1}$ function to find the measure of an angle when the sine of the angle is known. *You will learn more about inverses of trigonometric functions in Lesson 13-7.*

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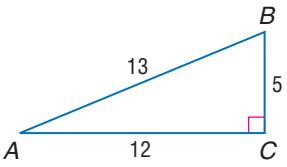
EXAMPLE Find Missing Angle Measures of Right Triangles

5 **Solve $\triangle ABC$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.**



You know the measures of the sides. You need to find A and B.

Find A. $\sin A = \frac{5}{13}$ $\sin A = \frac{\text{opp}}{\text{hyp}}$

Use a calculator and the [SIN^{-1}] function to find the angle whose sine is $\frac{5}{13}$.

KEYSTROKES: 2nd [SIN^{-1}] 5 ÷ 13) ENTER 22.61986495

To the nearest degree, $A \approx 23°$.

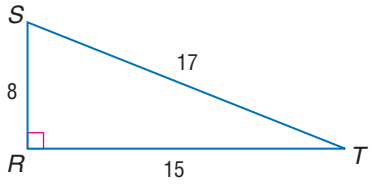
Find B. $23° + B \approx 90°$ Angles A and B are complementary.

$B \approx 67°$ Solve for B.

Therefore, $A \approx 23°$ and $B \approx 67°$.

CHECK Your Progress

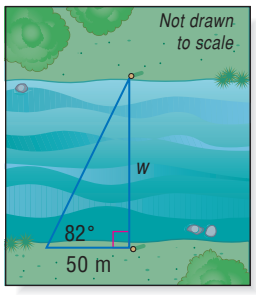
5. Solve $\triangle RST$. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



Trigonometry has many practical applications. Among the most important is the ability to find distances that either cannot or are not easily measured directly.

Real-World EXAMPLE Indirect Measurement

6 **BRIDGE CONSTRUCTION** **In order to construct a bridge, the width of the river must be determined. Suppose a stake is planted on one side of the river directly across from a second stake on the opposite side. At a distance 50 meters to the left of the stake, an angle of 82° is measured between the two stakes. Find the width of the river.**



Let w represent the width of the river at that location. Write an equation using a trigonometric function that involves the ratio of the distance w and 50.

$\tan 82° = \frac{w}{50}$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$

$50 \tan 82° = w$ Multiply each side by 50.

$355.8 \approx w$ The width of the river is about 355.8 meters.

Real-World Link

There are an estimated 595,625 bridges in use in the United States.

Source: betterroads.com

CHECK Your Progress

6. John found two trees directly across from each other in a canyon. When he moved 100 feet from the tree on his side (parallel to the edge of the canyon), the angle formed by the tree on his side, John, and the tree on the other side was 70°. Find the distance across the canyon.

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Getty Images
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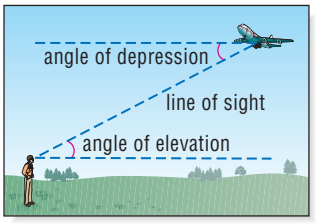
Study Tip

Angle of Elevation and Depression

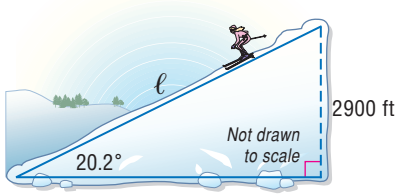
The angle of elevation and the angle of depression are congruent since they are alternate interior angles of parallel lines.

Some applications of trigonometry use an angle of elevation or depression. In the figure at the right, the angle formed by the line of sight from the observer and a line parallel to the ground is called the **angle of elevation.** The angle formed by the line of sight from the plane and a line parallel to the ground is called the **angle of depression.**



EXAMPLE 7 Use an Angle of Elevation

SKIING **The Aerial run in Snowbird, Utah, has an angle of elevation of 20.2°. Its vertical drop is 2900 feet. Estimate the length of this run.**



Let ℓ represent the length of the run. Write an equation using a trigonometric function that involves the ratio of ℓ and 2900.

$$\sin 20.2° = \frac{2900}{\ell} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\ell = \frac{2900}{\sin 20.2°} \qquad \text{Solve for } \ell.$$

$$\ell \approx 8398.5 \qquad \text{Use a calculator.}$$

The length of the run is about 8399 feet.

Real-World Link

The average annual snowfall in Alpine Meadows, California, is 495 inches. The longest designated run there is 2.5 miles.

Source: www.onthesnow.com

CHECK Your Progress

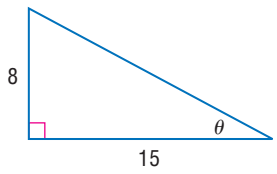
7. A ramp for unloading a moving truck has an angle of elevation of 32°. If the top of the ramp is 4 feet above the ground, estimate the length of the ramp.

CHECK Your Understanding

Example 1 (p. 760)

Find the values of the six trigonometric functions for angle θ.

1.



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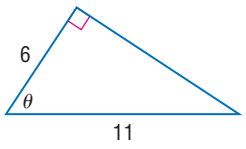
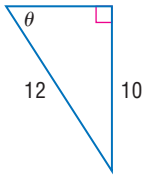


3.



Example 2 (pp. 760–761)

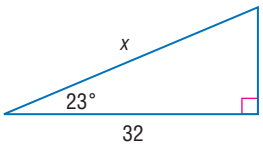
4. STANDARDIZED TEST PRACTICE If $\tan\theta = 3$, find the value of $\sin\theta$. **B**

A $\frac{3}{10}$ **B** $\frac{3\sqrt{10}}{10}$ **C** $\frac{10}{3}$ **D** $\frac{1}{3}$

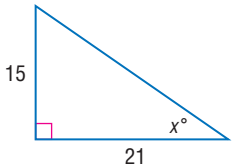
Examples 3, 5 (pp. 761–763)

Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

5.



6.



John P. Kelly/Getty Images

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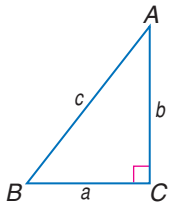
Examples 4, 5 (pp. 762–763)

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



7. $A = 45°, b = 6$

8. $B = 56°, c = 6$

9. $b = 7, c = 18$

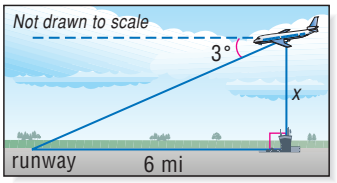
10. $a = 14, b = 13$

Example 6 (p. 763)

11. BRIDGES Tom wants to build a rope bridge between his tree house and Roy's tree house. Suppose Tom's tree house is directly behind Roy's tree house. At a distance of 20 meters to the left of Tom's tree house, an angle of 52° is measured between the two tree houses. Find the length of the rope bridge.

Example 7 (p. 764)

12. AVIATION When landing, a jet will average a 3° angle of descent. What is the altitude x, to the nearest foot, of a jet on final descent as it passes over an airport beacon 6 miles from the start of the runway?



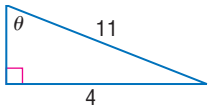
Exercises

HOMEWORK HELP

For Exercises	See Examples
12–14	1, 2
15–18	3
21–26	4
19, 20	5
27, 28	6, 7

Find the values of the six trigonometric functions for angle θ.

13.



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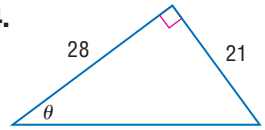

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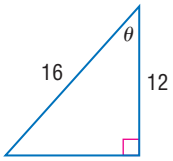
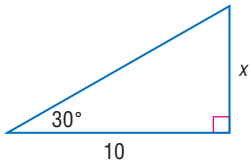


Write an equation involving sin, cos, or tan that can be used to find x. Then solve the equation. Round measures of sides to the nearest tenth and angles to the nearest degree.

16.



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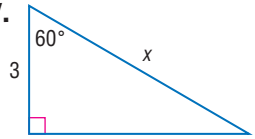

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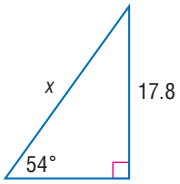
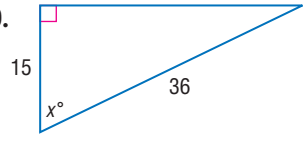


19.

20.



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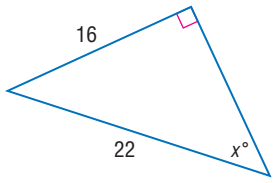
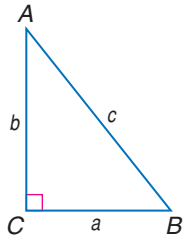


Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



22. $A = 16°, c = 14$

23. $B = 27°, b = 7$

24. $A = 34°, a = 10$

25. $B = 15°, c = 25$

26. $B = 30°, b = 11$

27. $A = 45°, c = 7\sqrt{2}$

28. SURVEYING A surveyor stands 100 feet from a building and sights the top of the building at a 55° angle of elevation. Find the height of the building.

Real-World Career

Surveyor

Land surveyors manage survey parties that measure distances, directions, and angles between points, lines, and contours on Earth's surface.

MathOnline

For more information, go to algebra2.com.

SuperStock

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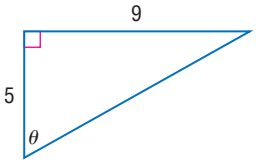
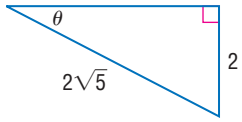
29. TRAVEL In a sightseeing boat near the base of the Horseshoe Falls at Niagara Falls, a passenger estimates the angle of elevation to the top of the falls to be 30°. If the Horseshoe Falls are 173 feet high, what is the distance from the boat to the base of the falls?

Find the values of the six trigonometric functions for angle θ.

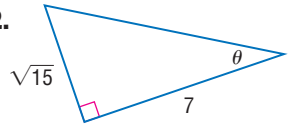
30.



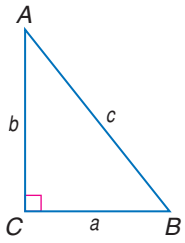
31.



32.



Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



33. $B = 18°, a = \sqrt{15}$

34. $A = 10°, b = 15$

35. $b = 6, c = 13$

36. $a = 4, c = 9$

37. $\tan B = \frac{7}{8}, b = 7$

38. $\sin A = \frac{1}{3}, a = 5$

39. Using the 30°-60°-90° triangle shown in the lesson, verify each value.

a. $\sin 30° = \frac{1}{2}$ **b.** $\cos 30° = \frac{\sqrt{3}}{2}$ **c.** $\sin 60° = \frac{\sqrt{3}}{2}$

40. Using the 45°-45°-90° triangle shown in the lesson, verify each value.

a. $\sin 45° = \frac{\sqrt{2}}{2}$ **b.** $\cos 45° = \frac{\sqrt{2}}{2}$ **c.** $\tan 45° = 1$

Cross-Curricular Project

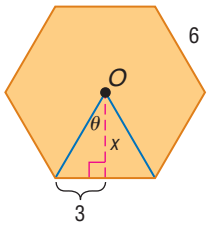
You can use the tangent ratio to determine the maximum height of a rocket. Visit algebra2.com to continue work on your project.

EXERCISE **For Exercises 41 and 42, use the following information.**
A preprogrammed workout on a treadmill consists of intervals walking at various rates and angles of incline. A 1% incline means 1 unit of vertical rise for every 100 units of horizontal run.

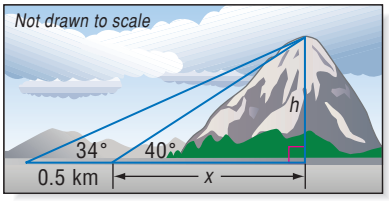
41. At what angle, with respect to the horizontal, is the treadmill bed when set at a 10% incline? Round to the nearest degree.

42. If the treadmill bed is 40 inches long, what is the vertical rise when set at an 8% incline?

43. GEOMETRY Find the area of the regular hexagon with point O as its center. (*Hint*: First find the value of x.)



44. GEOLOGY A geologist measured a 40° of elevation to the top of a mountain. After moving 0.5 kilometer farther away, the angle of elevation was 34°. How high is the top of the mountain? (*Hint*: Write a system of equations in two variables.)



EXTRA PRACTICE
See pages 920, 938.

Math Online
Self-Check Quiz at algebra2.com

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H.O.T. Problems

45. OPEN ENDED Draw two right triangles $\triangle ABC$ and $\triangle DEF$ for which $\sin A = \sin D$. What can you conclude about $\triangle ABC$ and $\triangle DEF$? Justify your reasoning.

46. REASONING Find a counterexample to the statement *It is always possible to solve a right triangle.*

47. CHALLENGE Explain why the sine and cosine of an acute angle are never greater that 1, but the tangent of an acute angle may be greater than 1.

48. *Writing in Math* Use the information on page 759 to explain how trigonometry is used in building construction. Include an explanation as to why the ratio of vertical rise to horizontal run on an entrance ramp is the tangent of the angle the ramp makes with the horizontal.

STANDARDIZED TEST PRACTICE

49. ACT/SAT If the secant of angle θ is $\frac{25}{7}$, what is the sine of angle θ?

A $\frac{5}{25}$

B $\frac{7}{25}$

C $\frac{24}{25}$

D $\frac{25}{7}$

50. REVIEW A person holds one end of a rope that runs through a pulley and has a weight attached to the other end. Assume the weight is directly beneath the pulley. The section of rope between the pulley and the weight is 12 feet long. The rope bends through an angle of 33 degrees in the pulley. How far is the person from the weight?

F 7.8 ft

G 10.5 ft

H 12.9 ft

J 14.3 ft

Spiral Review

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer (Lesson 12-9)

51. surveying band members to find the most popular type of music at your school

52. surveying people coming into a post office to find out what color cars are most popular

Find each probability if a coin is tossed 4 times (Lesson 12-8)

53. P(exactly 2 heads)

54. P(4 heads)

55. P(at least 1 head)

Solve each equation (Lesson 6-6)

56. $y^4 - 64 = 0$

57. $x^5 - 5x^3 + 4x = 0$

58. $d + \sqrt{d} - 132 = 0$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each product. Include the appropriate units with your answer.** (Lesson 6-1)

59. 5 gallons $\left(\frac{4 \text{ quarts}}{1 \text{ gallon}}\right)$

60. 6.8 miles $\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)$

61. $\left(\frac{2 \text{ square meters}}{5 \text{ dollars}}\right)$30 dollars

62. $\left(\frac{4 \text{ liters}}{5 \text{ minutes}}\right)$60 minutes

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13-2 Angles and Angle Measure

Main Ideas

- Change radian measure to degree measure and vice versa.
- Identify coterminal angles.

New Vocabulary

initial side
terminal side
standard position
unit circle
radian
coterminal angles

GET READY for the Lesson

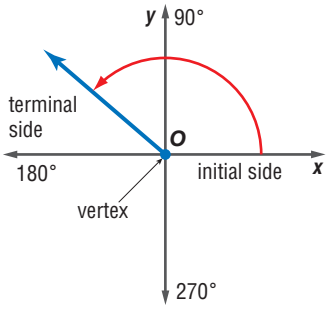
The Ferris wheel at Navy Pier in Chicago has a 140-foot diameter and 40 gondolas equally spaced around its circumference. The average angular velocity ω of one of the gondolas is given by $\omega = \frac{\theta}{t}$ where θ is the angle through which the gondola has revolved after a specified amount of time t. For example, if a gondola revolves through an angle of 225° in 40 seconds, then its average angular velocity is 225° ÷ 40 or about 5.6° per second.

ANGLE MEASUREMENT What does an angle measuring 225° look like? In Lesson 13-1, you worked only with acute angles, those measuring between 0° and 90°, but angles can have *any* real number measurement.

Reading Math

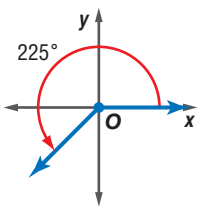
Angle of Rotation
In trigonometry, an angle is sometimes referred to as an *angle of rotation.*

On a coordinate plane, an angle may be generated by the rotation of two rays that share a fixed endpoint at the origin. One ray, called the **initial side** of the angle, is fixed along the positive x-axis. The other ray, called the **terminal side** of the angle, can rotate about the center. An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis is said to be in **standard position**.

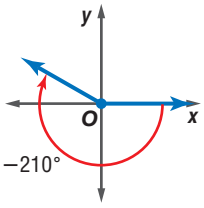


The measure of an angle is determined by the amount and direction of rotation from the initial side to the terminal side.

Positive Angle Measure
counterclockwise



Negative Angle Measure
clockwise



Concepts in Motion
Animation algebra2.com

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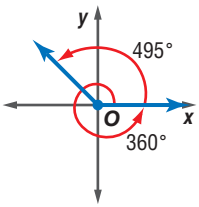
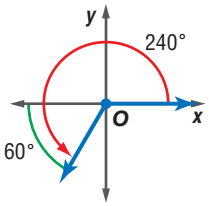
When terminal sides rotate, they may sometimes make one or more revolutions. An angle whose terminal side has made exactly one revolution has a measure of 360°.



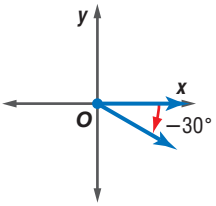
EXAMPLE Draw an Angle in Standard Position

1 **Draw an angle with the given measure in standard position.**

a. 240° $240° = 180° + 60°$
Draw the terminal side of the angle 60° counterclockwise past the negative x-axis.



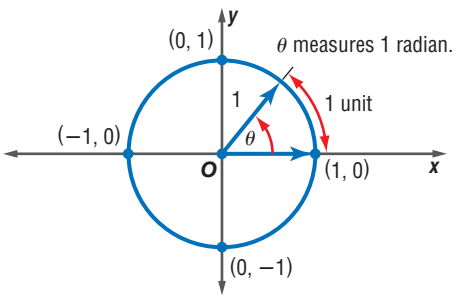
b. −30° The angle is negative.
Draw the terminal side of the angle 30° clockwise from the positive x-axis.

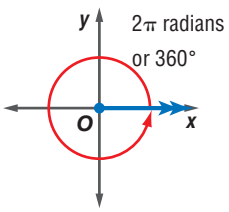


CHECK Your Progress

1A. 450° **1B.** −110°

Another unit used to measure angles is a radian. The definition of a radian is based on the concept of a **unit circle**, which is a circle of radius 1 unit whose center is at the origin of a coordinate system. One **radian** is the measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.





The circumference of any circle is $2\pi r$, where r is the radius measure. So the circumference of a unit circle is $2\pi(1)$ or 2π units. Therefore, an angle representing one complete revolution of the circle measures 2π radians. This same angle measures 360°. Therefore, the following equation is true.

$$2\pi \text{ radians} = 360°$$

As with degrees, the measure of an angle in radians is positive if its rotation is counterclockwise. The measure is negative if the rotation is clockwise.

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To change angle measures from radians to degrees or vice versa, solve the equation above in terms of both units.

$$2\pi \text{ radians} = 360°$$

$$\frac{2\pi \text{ radians}}{2\pi} = \frac{360°}{2\pi}$$

$$1 \text{ radian} = \frac{180°}{\pi}$$

1 radian is about 57 degrees.

$$2\pi \text{ radians} = 360°$$

$$\frac{2\pi \text{ radians}}{360} = \frac{360°}{360}$$

$$\frac{\pi \text{ radians}}{180} = 1°$$

1 degree is about 0.0175 radian.

These equations suggest a method for converting between radian and degree measure.

Reading Math

Radian Measure The word *radian* is usually omitted when angles are expressed in radian measure. Thus, when no units are given for an angle measure, radian measure is implied.

KEY CONCEPT *Radian and Degree Measure*

- To rewrite the radian measure of an angle in degrees, multiply the number of radians by $\frac{180°}{\pi \text{ radians}}$.
- To rewrite the degree measure of an angle in radians, multiply the number of degrees by $\frac{\pi \text{ radians}}{180°}$.

EXAMPLE **Convert Between Degree and Radian Measure**

2 **Rewrite the degree measure in radians and the radian measure in degrees.**

a. $60°$

$$60° = 60° \left(\frac{\pi \text{ radians}}{180°}\right)$$

$$= \frac{60\pi}{180} \text{ or } \frac{\pi}{3} \text{ radians}$$

b. $-\frac{7\pi}{4}$

$$-\frac{7\pi}{4} = \left(-\frac{7\pi}{4} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right)$$

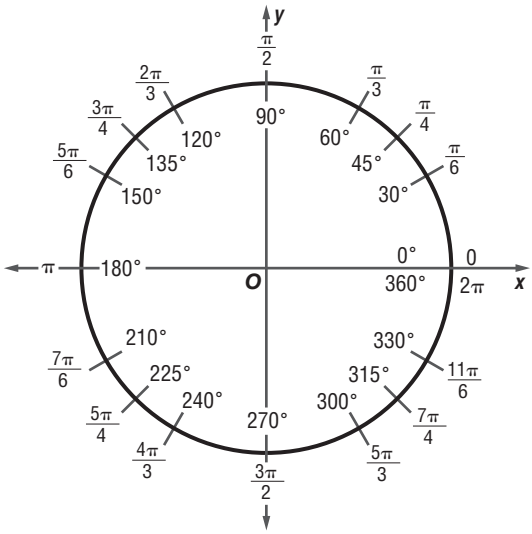
$$= -\frac{1260°}{4} \text{ or } -315°$$

CHECK Your Progress

2A. $190°$

2B. $\frac{3\pi}{8}$

You will find it useful to learn equivalent degree and radian measures for the special angles shown in the diagram at the right. This diagram is more easily learned by memorizing the equivalent degree and radian measures for the first quadrant and for 90°. All of the other special angles are multiples of these angles.



Concepts in Motion

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EXAMPLE Measure an Angle in Degrees and Radians

3 **TIME** **Find both the degree and radian measures of the angle through which the hour hand on a clock rotates from 1:00 P.M. to 3:00 P.M.**

The numbers on a clock divide it into 12 equal parts with 12 equal angles. The angle from 1 to 3 on the clock represents $\frac{2}{12}$ or $\frac{1}{6}$ of a complete rotation of 360°. $\frac{1}{6}$ of 360° is 60°.

Since the rotation is clockwise, the angle through which the hour hand rotates is negative. Therefore, the angle measures −60°.

60° has an equivalent radian measure of $\frac{\pi}{3}$. So the equivalent radian measure of −60° is $-\frac{\pi}{3}$.

Real-World Link

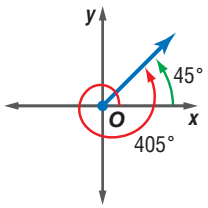
The clock tower in the United Kingdom Parliament House was opened in 1859. The copper minute hand in each of the four clocks of the tower is 4.2 meters long, 100 kilograms in mass, and travels a distance of about 190 kilometers a year.

Source: parliament.uk/index.cfm

CHECK Your Progress

3. How long does it take for a minute hand on a clock to pass through 2.5π radians?

COTERMINAL ANGLES If you graph a 405° angle and a 45° angle in standard position on the same coordinate plane, you will notice that the terminal side of the 405° angle is the same as the terminal side of the 45° angle. When two angles in standard position have the same terminal sides, they are called **coterminal angles**.



Notice that 405° − 45° = 360°. In degree measure, coterminal angles differ by an integral multiple of 360°. You can find an angle that is coterminal to a given angle by adding or subtracting a multiple of 360°. In radian measure, a coterminal angle is found by adding or subtracting a multiple of 2π.

EXAMPLE Find Coterminal Angles

4 **Find one angle with positive measure and one angle with negative measure coterminal with each angle.**

a. 240°

A positive angle is 240° + 360° or 600°.

A negative angle is 240° − 360° or −120°.

b. $\frac{9\pi}{4}$

A positive angle is $\frac{9\pi}{4} + 2\pi$ or $\frac{17\pi}{4}$. $\frac{9\pi}{4} + \frac{8\pi}{4} = \frac{17\pi}{4}$

A negative angle is $\frac{9\pi}{4} - 2(2\pi)$ or $-\frac{7\pi}{4}$. $\frac{9\pi}{4} + \left(\frac{-16\pi}{4}\right) = \frac{-7\pi}{4}$

Study Tip

Coterminal Angles

Notice in Example 4b that it is necessary to subtract a multiple of 2π to find a coterminal angle with negative measure.

CHECK Your Progress

4A. 15° **4B.** $-\frac{\pi}{4}$

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CHECK Your Understanding

Example 1 (p. 769) **Draw an angle with the given measure in standard position.**

1. 70° **2.** 300° **3.** 570° **4.** −45°

Example 2 (p. 770) **Rewrite each degree measure in radians and each radian measure in degrees.**

5. 130° **6.** −10° **7.** 485°

8. $\frac{3\pi}{4}$ **9.** $-\frac{\pi}{6}$ **10.** $\frac{19\pi}{3}$

Example 3 (pp. 770–771) **ASTRONOMY** **For Exercises 11 and 12, use the following information.**
Earth rotates on its axis once every 24 hours.

11. How long does it take Earth to rotate through an angle of 315°?

12. How long does it take Earth to rotate through an angle of $\frac{\pi}{6}$?

Example 4 (p. 771) **Find one angle with positive measure and one angle with negative measure coterminal with each angle.**

13. 60° **14.** 425° **15.** $\frac{\pi}{3}$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–19	1
20–27	2
28–33	4
34, 35	3

Draw an angle with the given measure in standard position.

16. 235° **17.** 270° **18.** 790° **19.** 380°

Rewrite each degree measure in radians and each radian measure in degrees.

20. 120° **21.** 60° **22.** −15° **23.** −225°

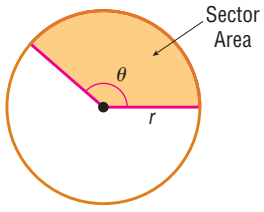
24. $\frac{5\pi}{6}$ **25.** $\frac{11\pi}{4}$ **26.** $-\frac{\pi}{4}$ **27.** $-\frac{\pi}{3}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

28. 225° **29.** 30° **30.** −15°

31. $\frac{3\pi}{4}$ **32.** $\frac{7\pi}{6}$ **33.** $-\frac{5\pi}{4}$

GEOMETRY **For Exercises 34 and 35, use the following information.**
A *sector* is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.



34. Find the area of a sector with a central angle of $\frac{4\pi}{3}$ radians in a circle whose radius measures 10 inches.

35. Find the area of a sector with a central angle of 150° in a circle whose radius measures 12 meters.

Draw an angle with the given measure in standard position.

36. −150° **37.** −50° **38.** π **39.** $-\frac{2\pi}{3}$

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Rewrite each degree measure in radians and each radian measure in degrees.

40. 660° **41.** 570° **42.** 158° **43.** 260°

44. $\frac{29\pi}{4}$ **45.** $\frac{17\pi}{6}$ **46.** 9 **47.** 3

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

48. −140° **49.** 368° **50.** 760°

51. $-\frac{2\pi}{3}$ **52.** $\frac{9\pi}{2}$ **53.** $\frac{17\pi}{4}$

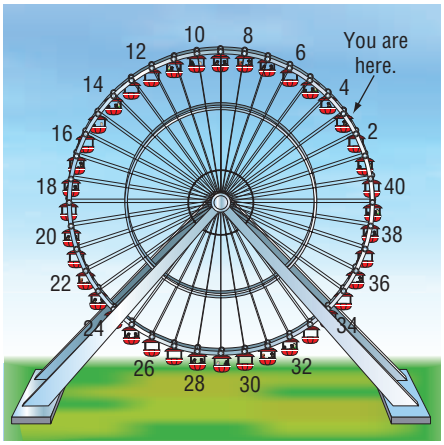
54. DRIVING Some sport-utility vehicles (SUVs) use 15-inch radius wheels. When driven 40 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and the nearest radian.

Real-World Link

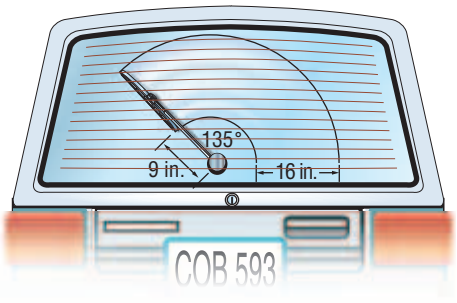
Vehicle tires are marked with numbers and symbols that indicate the specifications of the tire, including its size and the speed the tire can safely travel.

Source: usedtire.com

55. ENTERTAINMENT Suppose the gondolas on the Navy Pier Ferris Wheel were numbered from 1 through 40 consecutively in a counterclockwise fashion. If you were sitting in gondola number 3 and the wheel were to rotate counterclockwise through $\frac{47\pi}{10}$ radians, which gondola used to be in the position that you are in now?



56. CARS Use the Area of a Sector Formula in Exercises 34 and 35 to find the area swept by the rear windshield wiper of the car shown at the right.



EXTRA PRACTICE
See pages 920, 938.

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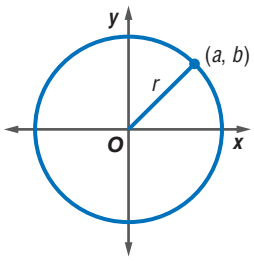
H.O.T. Problems

57. OPEN ENDED Draw and label an example of an angle with negative measure in standard position. Then find an angle with positive measure that is coterminal with this angle.

58. CHALLENGE A line with positive slope makes an angle of $\frac{\pi}{2}$ radians with the positive x-axis at the point (2, 0). Find an exact equation for this line.

59. CHALLENGE If (a, b) is on a circle that has radius r and center at the origin, prove that each of the following points is also on this circle.

a. $(a, -b)$ **b.** (b, a) **c.** $(b, -a)$



60. REASONING Express $\frac{1}{8}$ of a revolution in degrees.

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61. ***Writing in Math*** Use the information on page 768 to explain how angles can be used to describe circular motion. Include an explanation of the significance of angles of more than 180° in terms of circular motion, an explanation of the significance of angles with negative measure in terms of circular motion, and an interpretation of a rate of more than 360° per minute.

STANDARDIZED TEST PRACTICE

62. **ACT/SAT** Choose the radian measure that is equal to 56°.

A $\frac{\pi}{15}$

B $\frac{7\pi}{45}$

C $\frac{14\pi}{45}$

D $\frac{\pi}{3}$

63. **REVIEW** Angular velocity is defined by the equation $\omega = \frac{\theta}{t}$, where θ is usually expressed in radians and t represents time. Find the angular velocity in radians per second of a point on a bicycle tire if it completes 2 revolutions in 3 seconds.

F $\frac{\pi}{3}$

G $\frac{\pi}{2}$

H $\frac{2\pi}{3}$

J $\frac{4\pi}{3}$

Spiral Review

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)

64. $A = 34°, b = 5$

65. $B = 68°, b = 14.7$

66. $B = 55°, c = 16$

67. $a = 0.4, b = 0.4\sqrt{3}$

Find the margin of sampling error. (Lesson 12-9)

68. $p = 72\%, n = 100$

69. $p = 50\%, n = 200$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities. (Lesson 12-2)

70. choosing an arrangement of 5 CDs from your 30 favorite CDs

71. choosing 3 different types of snack foods out of 7 at the store to take on a trip

Find $[g \circ h](x)$ and $[h \circ g](x)$. (Lesson 7-1)

72. $g(x) = 2x$
$h(x) = 3x - 4$

73. $g(x) = 2x + 5$
$h(x) = 2x^2 - 3x + 9$

GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.** (Lesson 7-5)

74. $\frac{2}{\sqrt{3}}$

75. $\frac{3}{\sqrt{5}}$

76. $\frac{4}{\sqrt{6}}$

77. $\frac{5}{\sqrt{10}}$

78. $\frac{\sqrt{7}}{\sqrt{2}}$

79. $\frac{\sqrt{5}}{\sqrt{8}}$

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EXTEND 13-2

Algebra Lab

Investigating Regular Polygons Using Trigonometry

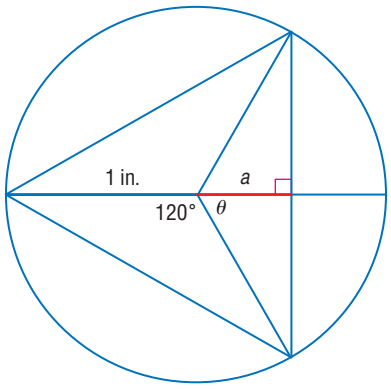
ACTIVITY

- Use a compass to draw a circle with a radius of one inch. Inscribe an equilateral triangle inside of the circle. To do this, use a protractor to measure three angles of 120° at the center of the circle, since $\frac{360°}{3} = 120°$. Then connect the points where the sides of the angles intersect the circle using a straightedge.
- The **apothem** of a regular polygon is a segment that is drawn from the center of the polygon perpendicular to a side of the polygon. Use the cosine of angle θ to find the length of an apothem, labeled a in the diagram below.

Analyze the Results

1. Make a table like the one shown below and record the length of the apothem of the equilateral triangle.

Number of Sides, n	θ	a
3	60	
4	45	
5		
6		
7		
8		
9		
10		



Inscribe each regular polygon named in the table in a circle of radius one inch. Copy and complete the table.

2. What do you notice about the measure of θ as the number of sides of the inscribed polygon increases?
3. What do you notice about the values of a?
4. **MAKE A CONJECTURE** Suppose you inscribe a 20-sided regular polygon inside a circle. Find the measure of angle θ.
5. Write a formula that gives the measure of angle θ for a polygon with n sides.
6. Write a formula that gives the length of the apothem of a regular polygon inscribed in a circle of radius one inch.
7. How would the formula you wrote in Exercise 6 change if the radius of the circle was not one inch?

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13-3 Trigonometric Functions of General Angles

Main Ideas

- Find values of trigonometric functions for general angles.
- Use reference angles to find values of trigonometric functions.

New Vocabulary

quadrantal angle
reference angle

GET READY for the Lesson

A skycoaster consists of a large arch from which two steel cables hang and are attached to riders suited together in a harness. A third cable, coming from a larger tower behind the arch, is attached with a ripcord. Riders are hoisted to the top of the larger tower, pull the ripcord, and then plunge toward Earth. They swing through the arch, reaching speeds of more than 60 miles per hour. After the first several swings of a certain skycoaster, the angle θ of the riders from the center of the arch is given by $\theta = 0.2 \cos (1.6t)$, where t is the time in seconds after leaving the bottom of their swing.

Trigonometric Functions and General Angles In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and $\frac{\pi}{2}$, of a right triangle. For $t > 0$ in the equation above, you must find the cosine of an angle greater than $\frac{\pi}{2}$. In this lesson, we will extend the domain of trigonometric functions to include angles of *any* measure.

KEY CONCEPT *Trigonometric Functions, θ in Standard Position*

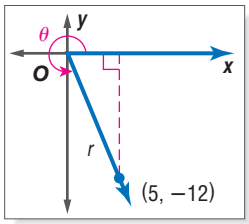
Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. Using the Pythagorean Theorem, the distance r from the origin to P is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

$\sin \theta = \frac{y}{r}$ $\quad \cos \theta = \frac{x}{r}$ $\quad \tan \theta = \frac{y}{x}, x \neq 0$

$\csc \theta = \frac{r}{y}, y \neq 0$ $\quad \sec \theta = \frac{r}{x}, x \neq 0$ $\quad \cot \theta = \frac{x}{y}, y \neq 0$

EXAMPLE 1 Evaluate Trigonometric Functions for a Given Point

Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point (5, −12).



From the coordinates, you know that $x = 5$ and $y = -12$. Use the Pythagorean Theorem to find r.

courtesy of Skycoaster of Florida

www.MathSchoolinternational.com

$r = \sqrt{x^2 + y^2}$ Pythagorean Theorem

$= \sqrt{5^2 + (-12)^2}$ Replace x with 5 and y with 2−12.

$= \sqrt{169}$ or 13 Simplify.

Now, use $x = 5$, $y = -12$, and $r = 13$ to write the ratios.

$\sin \theta = \frac{y}{r} = \frac{-12}{13}$ or $-\frac{12}{13}$ $\quad \cos \theta = \frac{x}{r} = \frac{5}{13}$ $\quad \tan \theta = \frac{y}{x} = -\frac{12}{5}$ or $-\frac{12}{5}$

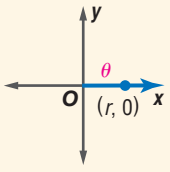
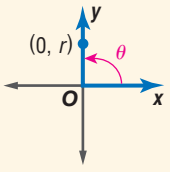
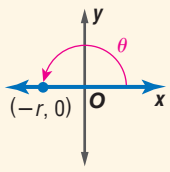
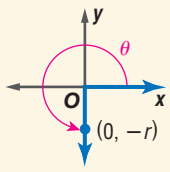
$\csc \theta = \frac{r}{y} = \frac{13}{-12}$ or $-\frac{13}{12}$ $\quad \sec \theta = \frac{r}{x} = \frac{13}{5}$ $\quad \cot \theta = \frac{x}{y} = \frac{5}{-12}$ or $-\frac{5}{12}$

CHECK Your Progress

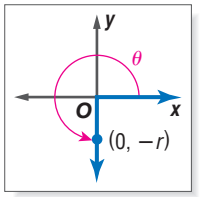
1. Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(-8, -15)$.

If the terminal side of angle θ lies on one of the axes, θ is called a **quadrantal angle.** The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either x or y is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

KEY CONCEPT *Quadrantal Angles*

$\theta = 0°$ or 0 radians	$\theta = 90°$ or $\frac{\pi}{2}$ radians	$\theta = 180°$ or π radians	$\theta = 270°$ or $\frac{3\pi}{2}$ radians
			

EXAMPLE 2 **Quadrantal Angles**



Find the values of the six trigonometric functions for an angle in standard position that measures 270°.

When $\theta = 270°$, $x = 0$ and $y = -r$.

$\sin \theta = \frac{y}{r} = \frac{-r}{r}$ or -1 $\quad \cos \theta = \frac{x}{r} = \frac{0}{r}$ or 0 $\quad \tan \theta = \frac{y}{x} = \frac{-r}{0}$ or undefined

$\csc \theta = \frac{r}{y} = \frac{r}{-r}$ or -1 $\quad \sec \theta = \frac{r}{x} = \frac{r}{0}$ or undefined $\quad \cot \theta = \frac{x}{y} = \frac{0}{-r}$ or 0

CHECK Your Progress

2. Find the values of the six trigonometric functions for an angle in standard position that measures 180°.

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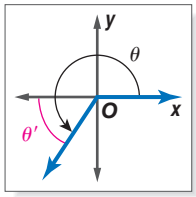
Reading Math

Theta Prime θ' is read *theta prime*.

Reference Angles To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If θ is a nonquadrantal angle in standard position, its **reference angle**, θ, is defined as the acute angle formed by the terminal side of θ and the x-axis.



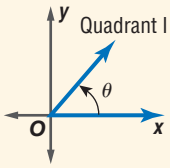
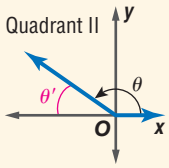
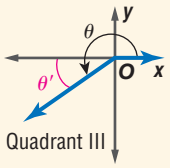
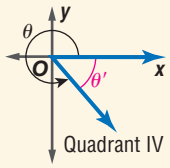
Concepts in Motion

Animation
algebra2.com

You can use the rule below to find the reference angle for any nonquadrantal angle θ where $0° < \theta < 360°$ (or $0 < \theta < 2\pi$).

KEY CONCEPT — ***Reference Angle Rule***

For any nonquadrantal angle θ, $0° < \theta < 360°$ (or $0 < \theta < 2\pi$), its reference angle θ' is defined as follows.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
			
$\theta' = \theta$	$\theta = 180° - \theta$ $(\theta' = \pi - \theta)$	$\theta' = \theta - 180°$ $(\theta' = \theta - \pi)$	$= 360° - \theta$ $(\theta' = 2\pi - \theta)$

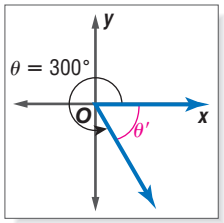
If the measure of θ is greater than 360° or less than 0°, its reference angle can be found by associating it with a coterminal angle of positive measure between 0° and 360°.

EXAMPLE **Find the Reference Angle for a Given Angle**

3 **Sketch each angle. Then find its reference angle.**

a. 300°

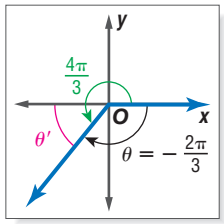
Because the terminal side of 300° lies in Quadrant IV, the reference angle is $360° - 300°$ or 60°



b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.



CHECK Your Progress

3A. $-200°$ **3B.** $\frac{2\pi}{3}$

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To use the reference angle θ' to find a trigonometric value of θ, you need to know the sign of that function for an angle θ. From the function definitions, these signs are determined by x and y, since r is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of θ lies.

The chart summarizes the signs of the trigonometric functions for each quadrant.

	Quadrant			
Function	**I**	**II**	**III**	**IV**
sin θ or csc θ	+	+	–	–
cos θ or sec θ	+	–	–	+
tan θ or cot θ	+	–	+	–

Use the following steps to find the value of a trigonometric function of any angle θ.

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.

Study Tip

Look Back

To review **trigonometric values of angles measuring 30°, 45°, and 60°**, see Lesson 13-1.

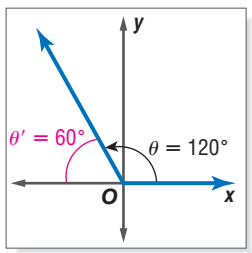
EXAMPLE **4** **Use a Reference Angle to Find a Trigonometric Value**

Find the exact value of each trigonometric function.

a. sin 120°

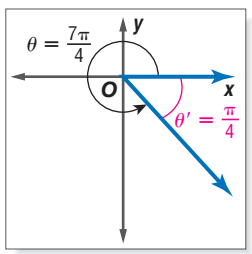
Because the terminal side of 120° lies in Quadrant II, the reference angle θ' is 180° – 120° or 60°. The sine function is positive in Quadrant II, so

$\sin 120° = \sin 60° \text{ or } \frac{\sqrt{3}}{2}$.



b. $\cot \frac{7\pi}{4}$

Because the terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV, the reference angle θ' is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. The cotangent function is negative in Quadrant IV.



$$\begin{aligned} \cot \frac{7\pi}{4} &= -\cot \frac{\pi}{4} \\ &= -\cot 45° && \tfrac{\pi}{4} \text{ radians} = 45° \\ &= -1 && \cot 45° = 1 \end{aligned}$$

CHECK Your Progress

4A. cos 135°

4B. $\tan \frac{5\pi}{6}$

If you know the quadrant that contains the terminal side of θ in standard position and the exact value of one trigonometric function of θ, you can find the values of the other trigonometric functions of θ using the function definitions.

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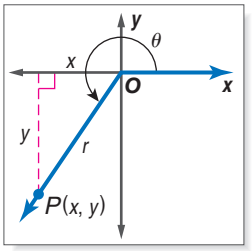
EXAMPLE 5 Quadrant and One Trigonometric Value of θ

Suppose θ is an angle in standard position whose terminal side is in Quadrant III and $\sec\theta = -\frac{4}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Draw a diagram of this angle, labeling a point $P(x, y)$ on the terminal side of θ. Use the definition of secant to find the values of x and r.



$\sec\theta = -\frac{4}{3}$ Given

$\frac{r}{x} = -\frac{4}{3}$ Definition of secant

Since x is negative in Quadrant III and r is always positive, $x = -3$ and $r = 4$. Use these values and the Pythagorean Theorem to find y.

$x^2 + y^2 = r^2$ Pythagorean Theorem

$(-3)^2 + y^2 = 4^2$ Replace x with -3 and r with 4.

$y^2 = 16 - 9$ Simplify. Then subtract 9 from each side.

$y = \pm\sqrt{7}$ Simplify. Then take the square root of each side.

$y = -\sqrt{7}$ y is negative in Quadrant III.

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write the remaining trigonometric ratios.

$\sin\theta = \frac{y}{r}$

$= \frac{-\sqrt{7}}{4}$ or $-\frac{\sqrt{7}}{4}$

$\cos\theta = \frac{x}{r}$

$= -\frac{3}{4}$

$\tan\theta = \frac{y}{x}$

$= \frac{-\sqrt{7}}{-3}$ or $\frac{\sqrt{7}}{3}$

$\csc\theta = \frac{r}{y}$

$= \frac{4}{-\sqrt{7}}$ or $-\frac{4\sqrt{7}}{7}$

$\cot\theta = \frac{x}{y}$

$= \frac{-3}{-\sqrt{7}}$ or $\frac{3\sqrt{7}}{7}$

CHECK Your Progress

5. Suppose θ is an angle in standard position whose terminal side is in Quadrant IV and $\tan\theta = -\frac{2}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.

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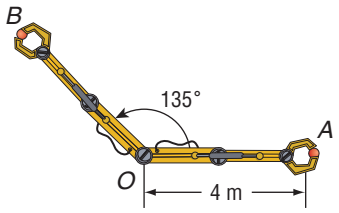
Real-World Link

RoboCup is an annual event in which teams from all over the world compete in a series of soccer matches in various classes according to the size and intellectual capacity of their robot. The robots are programmed to react to the ball and communicate with each other.

Source: www.robocup.org

Real-World EXAMPLE Find Coordinates Given a Radius and an Angle

6 ROBOTICS **In a robotics competition, a robotic arm 4 meters long is to pick up an object at point *A* and release it into a container at point *B*. The robot's arm is programmed to rotate through an angle of precisely 135° to accomplish this task. What is the new position of the object relative to the pivot point *O*?**



With the pivot point at the origin and the angle through which the arm rotates in standard position, point *A* has coordinates (4, 0). The reference angle θ' for 135° is 180° – 135° or 45°.

Let the position of point *B* have coordinates (x, y). Then, use the definitions of sine and cosine to find the value of x and y. The value of r is the length of the robotic arm, 4 meters. Because *B* is in Quadrant II, the cosine of 135° is negative.

$\cos 135° = \frac{x}{r}$	cosine ratio	$\sin 135° = \frac{y}{r}$	sine ratio
$-\cos 45° = \frac{x}{4}$	180° – 135° = 45°	$\sin 45° = \frac{y}{4}$	180° – 35° = 45°
$-\frac{\sqrt{2}}{2} = \frac{x}{4}$	$\cos 45° = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} = \frac{y}{4}$	$\sin 45° = \frac{\sqrt{2}}{2}$
$-2\sqrt{2} = x$	Solve for *x*.	$2\sqrt{2} = y$	Solve for *y*.

The exact coordinates of *B* are $(-2\sqrt{2}, 2\sqrt{2})$. Since $2\sqrt{2}$ is about 2.83, the object is about 2.83 meters to the left of the pivot point and about 2.83 meters in front of the pivot point.

CHECK Your Progress

6. After releasing the object in the container at point *B*, the arm must rotate another 75°. What is the new position of the end of the arm relative to the pivot point *O*?

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Example 1 (pp. 776–777)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. (−15, 8) **2.** (−3, 0) **3.** (4, 4)

Examples 2, 4 (pp. 777, 779)

Find the exact value of each trigonometric function.

4. sin 300° **5.** cos 180° **6.** $\tan \frac{5\pi}{3}$ **7.** $\sec \frac{7\pi}{6}$

Example 3 (p. 778)

Sketch each angle. Then find its reference angle.

8. 235° **9.** $\frac{7\pi}{4}$ **10.** −240°

Example 5 (p. 780)

Suppose θ is an a ngle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

11. $\cos \theta = -\frac{1}{2}$, Quadrant II **12.** $\cot \theta = -\frac{\sqrt{2}}{2}$, Quadrant IV

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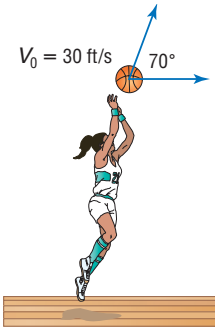
Example 6 (p. 781)

13. BASKETBALL The maximum height H in feet that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 (\sin \theta)^2}{64}$, where V_0 represents the initial velocity and θ represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18–25	2, 4
26–29	3
30–33	5
34–36	6

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

14. (7, 24) **15.** (2, 1) **16.** (5, −8) **17.** (4, −3)

18. (0, −6) **19.** (−1, 0) **20.** $(\sqrt{2}, -\sqrt{2})$ **21.** $(-\sqrt{3}, -\sqrt{6})$

Find the exact value of each trigonometric function.

22. sin 240° **23.** sec 120° **24.** tan 300° **25.** cot 510°

26. csc 5400° **27.** $\cos \frac{11\pi}{3}$ **28.** $\cot \left(-\frac{5\pi}{6}\right)$ **29.** $\sin \frac{3\pi}{4}$

30. $\sec \frac{3\pi}{2}$ **31.** $\csc \frac{17\pi}{6}$ **32.** cos (−30°) **33.** $\tan \left(-\frac{5\pi}{4}\right)$

Sketch each angle. Then find its reference angle.

34. 315° **35.** 240° **36.** $\frac{5\pi}{4}$ **37.** $\frac{5\pi}{6}$

38. −210° **39.** −125° **40.** $\frac{13\pi}{7}$ **41.** $-\frac{2\pi}{3}$

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

42. $\cos \theta = \frac{3}{5}$, Quadrant IV **43.** $\tan \theta = -\frac{1}{5}$, Quadrant II

44. $\sin \theta = \frac{1}{3}$, Quadrant II **45.** $\cot \theta = \frac{1}{2}$, Quadrant III

Real-World Link

If a major league pitcher throws a pitch at 95-miles per hour, it takes only about 4-tenths of a second for the ball to travel the 60-feet, 6-inches from the pitcher's mound to home plate. In that time, the hitter must decide whether to swing at the ball and if so, when to swing.

Source: exploratorium.edu

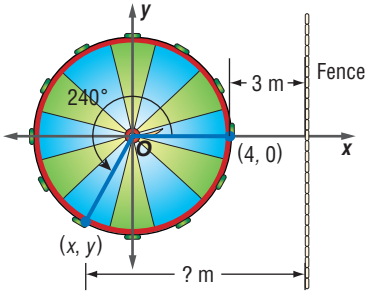
BASEBALL For Exercises 46 and 47, use the following information.

The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground.

46. If the ball was hit with an initial velocity of 80 feet per second at an angle of 30°, how far was it hit?

47. Which angle will result in the greatest distance? Explain your reasoning.

48. CAROUSELS Anthony's little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates 240°?



Otto Greule/Allsport

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49. SKYCOASTING Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with θ measured in radians and t measured in seconds. Determine the measure of the angle for $t = 0$, 0.5, 1, 1.5, 2, 2.5, and 3 in both radians and degrees.

EXTRA PRACTICE
See pages 920, 938
Math Online
Self-Check Quiz at algebra2.com

50. NAVIGATION Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 60°.

H.O.T. Problems

51. OPEN ENDED Give an example of an angle for which the sine is negative and the tangent is positive.

52. REASONING Determine whether the following statement is *true* or *false*. If true, explain your reasoning. If false, give a counterexample.

The values of the secant and tangent functions for any quadrantal angle are undefined.

53. *Writing in Math* Use the information on page 776 to explain how you can model the position of riders on a skycoaster.

STANDARDIZED TEST PRACTICE

54. ACT/SAT If the cotangent of angle θ is 1, then the tangent of angle θ is

A -1. **C** 1.

B 0. **D** 3.

55. REVIEW Which angle has a tangent and cosine that are both negative?

F 110° **H** 210°

G 180° **J** 340°

Spiral Review

Rewrite each degree measure in radians and each radian measure in degrees. (Lesson 13–2)

56. 90° **57.** $\frac{5\pi}{3}$ **58.** 5

59. LITERATURE In one of *Grimm's Fairy Tales*, Rumpelstiltskin has the ability to spin straw into gold. Suppose on the first day, he spun 5 pieces of straw into gold, and each day thereafter he spun twice as much. How many pieces of straw would he have spun into gold by the end of the week? (Lesson 11-4)

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

60. $3x - 4y = 13$
$-2x + 5y = -4$

61. $5x + 7y = 1$
$3x + 5y = 3$

62. $2x + 3y = -2$
$-6x + y = -34$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

63. $\frac{a}{\sin 32°} = \frac{8}{\sin 65°}$ **64.** $\frac{b}{\sin 45°} = \frac{21}{\sin 100°}$ **65.** $\frac{c}{\sin 60°} = \frac{3}{\sin 75°}$

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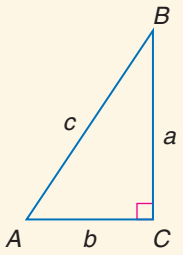
CHAPTER 13

Mid-Chapter Quiz

Lessons 13-1 through 13-3

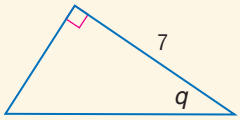
Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-1)



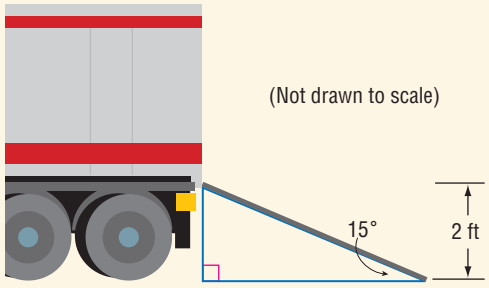
1. $A = 48°$, b 12
2. $a = 18$, $c = 21$

3. Draw an angle measuring $-60°$ in standard position. (Lesson 13-1)

4. Find the values of the six trigonometric functions for angle θ in the triangle at the right. (Lesson 13-1)



5. **TRUCKS** The tailgate of a moving truck is 2 feet above the ground. The incline of the ramp used for loading the truck is 15° as shown. Find the length of the ramp to the nearest tenth of a foot. (Lesson 13-1)



Rewrite each degree measure in radians and each radian measure in degrees. (Lesson 13-2)

6. 190°
7. 450°
8. $\frac{7\pi}{6}$
9. $-\frac{11\pi}{5}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

10. $-55°$
11. $\frac{11\pi}{3}$

SUNDIAL For Exercises 12 and 13, use the following information. (Lesson 13-2)

A sector is a region of a circle that is bounded by a central angle θ and its intercepted arc. The area A of a sector with radius r and central angle θ is given by $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

12. Find the shaded area of a sundial with a central angle of $\frac{3\pi}{4}$ radians and a radius that measures 6 inches.
13. Find the sunny area of a sundial with a central angle of 270° with a radius measuring 10 inches.

14. Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the point $(-2, 3)$. (Lesson 13-3)

15. Find the exact value of $\csc \frac{5\pi}{3}$. (Lesson 13-3)

16. **NAVIGATION** Airplanes and ships measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 120°. (Lesson 13-3)

17. **MULTIPLE CHOICE** Suppose θ is an angle in standard position with $\sin \theta > 0$. In which quadrant(s) does the terminal side of θ lie? (Lesson 13-3)

A I
C III
B II
D I and II

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13-4 Law of Sines

Main Ideas

- Solve problems by using the Law of Sines.
- Determine whether a triangle has one, two, or no solutions.

New Vocabulary

Law of Sines

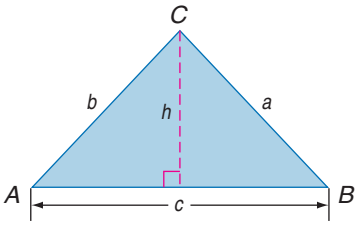
GET READY for the Lesson

You know how to find the area of a triangle when the base and the height are known. Using this formula, the area of $\triangle ABC$ below is $\frac{1}{2}ch$. If the height h of this triangle were not known, you could still find the area given the measures of angle A and the length of side b.

$$\sin A = \frac{h}{b} \rightarrow h = b \sin A$$

By combining this equation with the area formula, you can find a new formula for the area of the triangle.

$$\text{Area} = \frac{1}{2}ch \rightarrow \text{Area} = \frac{1}{2}c(b \sin A)$$



Law of Sines You can find two other formulas for the area of the triangle above in a similar way.

Study Tip

Area Formulas

These formulas allow you to find the area of any triangle when you know the measures of two sides and the included angle.

KEY CONCEPT *Area of a Triangle*

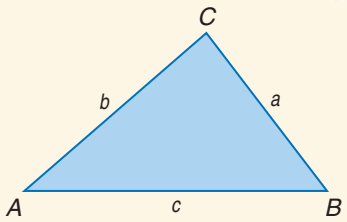
Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

Symbols

$$\text{area} = \frac{1}{2}bc \sin A$$

$$\text{area} = \frac{1}{2}ac \sin B$$

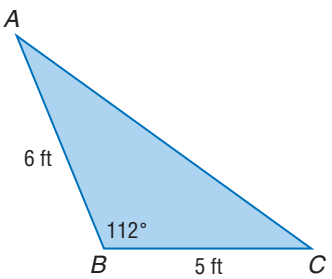
$$\text{area} = \frac{1}{2}ab \sin C$$



EXAMPLE **Find the Area of a Triangle**

1 **Find the area of $\triangle ABC$ to the nearest tenth.**

In this triangle, $a = 5$, $c = 6$, and $B = 112°$. Choose the second formula because you know the values of its variables.



$$\text{Area} = \frac{1}{2}ac \sin B \qquad \text{Area formula}$$

$$= \frac{1}{2}(5)(6) \sin 112° \qquad \text{Replace } a \text{ with 5, } c \text{ with 6, and } B \text{ with } 112°.$$

$$\approx 13.9$$

To the nearest tenth, the area is 13.9 square feet.

CHECK Your Progress

1. Find the area of $\triangle ABC$ to the nearest tenth if $A = 31°$, $b = 18$ m, and $c = 22$ m.

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All of the area formulas for $\triangle ABC$ represent the area of the same triangle. So, $\frac{1}{2}bc \sin A$, $\frac{1}{2}ac \sin B$, and $\frac{1}{2}ab \sin C$ are all equal. You can use this fact to derive the **Law of Sines**.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C \qquad \text{Set area formulas equal to each other.}$$

$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc} \qquad \text{Divide each expression by } \frac{1}{2}abc.$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Simplify.}$$

Study Tip

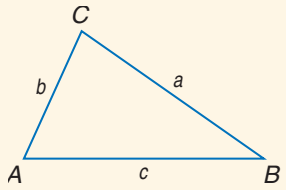
Alternate Representations

The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

KEY CONCEPT — **Law of Sines**

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C respectively. Then,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



The Law of Sines can be used to write three different equations.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In Lesson 13-1, you learned how to solve right triangles. To solve *any* triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

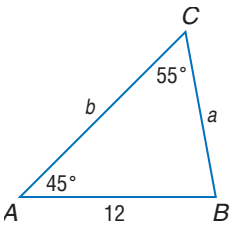
EXAMPLE **Solve a Triangle Given Two Angles and a Side**

2 **Solve $\triangle ABC$.**

You are given the measures of two angles and a side. First, find the measure of the third angle.

$45° + 55° + B = 180°$ The sum of the angle measures of a triangle is 180°.

$B = 80°$ $180 - (45 + 55) = 80$



Now use the Law of Sines to find a and b. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 45°}{a} = \frac{\sin 55°}{12}$	Replace A with 45°, B with 80°, C with 55°, and c with 12.	$\frac{\sin 80°}{b} = \frac{\sin 55°}{12}$
$a = \frac{12 \sin 45°}{\sin 55°}$	Solve for the variable.	$b = \frac{12 \sin 80°}{\sin 55°}$
$a \approx 10.4$	Use a calculator.	$b \approx 14.4$

Therefore, $B = 80°$, $a \approx 10.4$, and $b \approx 14.4$.

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CHECK Your Progress

2. Solve $\triangle FGH$ if $m\angle G = 80°$, $m\angle H = 40°$, and $g = 14$.

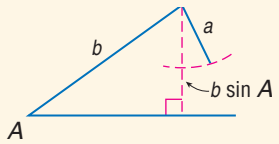
One, Two, or No Solutions When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

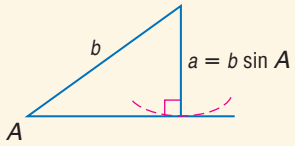
KEY CONCEPT ***Possible Triangles Given Two Sides and One Opposite Angle***

Suppose you are given a, b, and A for a triangle.

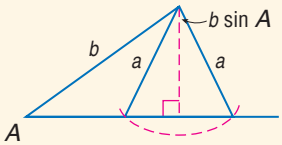
A Is Acute ($A < 90°$).



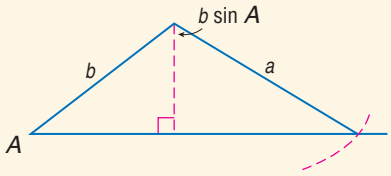
$a < b \sin A$
no solution



$a = b \sin A$
one solution

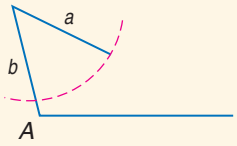


$b > a > b \sin A$
two solutions

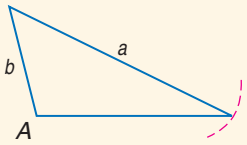


$a \geq b$
one solution

A Is Right or Obtuse ($A \geq 90°$).



$a \leq b$
no solution



$a > b$
one solution

EXAMPLE **One Solution**

3 **In $\triangle ABC$, $A = 118°$, $a = 20$, and $b = 17$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.**

Because angle A is obtuse and $a > b$, you know that one solution exists.

Use the Law of Sines to find B.

$\frac{\sin B}{17} = \frac{\sin 118°}{20}$ Law of Sines

$\sin B = \frac{17 \sin 118°}{20}$ Multiply each side by 17.

$\sin B \approx 0.7505$ Use a calculator.

$B \approx 49°$ Use the $\sin^{-1}$ function.

Use the Law of Sines again to find c.

$\frac{\sin 13}{c} = \frac{\sin 118°}{20}$ Law of Sines

$c = \frac{20 \sin 13°}{\sin 118°}$ or about 5.1

Therefore, $B \approx 49°$, $C \approx 13°$, and $c \approx 5.1$.

The measure of angle C is approximately $180 - (118 + 49)$ or 13°.

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CHECK Your Progress

3. In $\triangle ABC$, $B = 95°$, $b = 19$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

EXAMPLE No Solution

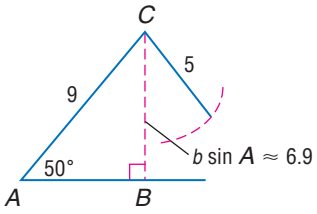
4 **In $\triangle ABC$, $A = 50°$, $a = 5$, and $b = 9$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.**

Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 9 \sin 50°$ Replace b with 9 and A with 50°.

≈ 6.9 Use a calculator.

Since $5 < 6.9$, there is no solution.



Study Tip

A Is Acute

We compare $b \sin A$ to a because $b \sin A$ is the minimum distance from C to $\overline{AB}$ when A is acute.

CHECK Your Progress

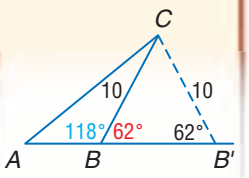
4. In $\triangle ABC$, $B = 95°$, $b = 10$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

When two solutions for a triangle exist, it is called the *ambiguous case.*

Study Tip

Alternate Method

Another way to find the obtuse angle in Case 2 of Example 5 is to notice in the figure below that $\triangle CBB'$ is isosceles. Since the base angles of an isosceles triangle are always congruent and $m\angle B' = 62°$, $m\angle CBB' = 62°$. Also, $\angle ABC$ and $m\angle CBB'$ are supplementary. Therefore, $m\angle ABC = 180° - 62°$ or $118°$.



EXAMPLE Two Solutions

5 **In $\triangle ABC$, $A = 39°$, $a = 10$, and $b = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.**

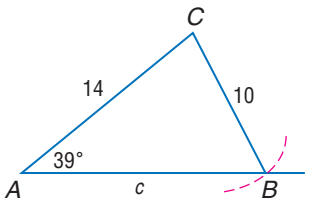
Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 14 \sin 39°$ Replace b with 14 and A with 39°.

≈ 8.81 Use a calculator.

Since $14 > 10 > 8.81$, there are two solutions. Thus, there are two possible triangles to be solved.

Case 1 Acute Angle B



First, use the Law of Sines to find B.

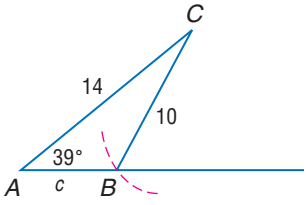
$$\frac{\sin B}{14} = \frac{\sin 39°}{10}$$

$$\sin B = \frac{14 \sin 39°}{10}$$

$$\sin B = 0.8810$$

$$B \approx 62°$$

Case 2 Obtuse Angle B



To find B, you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So B is approximately $180 - 62$ or 118°.

The measure of angle C is approximately $180 - (39 + 118)$ or 23°.

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The measure of angle C is approximately $180 - (39 + 62)$ or $79°$.

$$\frac{\sin 79°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 79°}{\sin 39°}$$
$$c \approx 15.6$$

Therefore, $B \approx 62°$, $C \approx 79°$, and $c \approx 15.6$.

Use the Law of Sines to find c.

$$\frac{\sin 23°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 23°}{\sin 39°}$$
$$c \approx 6.2$$

Therefore, $B \approx 118°$, $C \approx 23°$, and $c \approx 6.2$.

CHECK Your Progress

5. In $\triangle ABC$, $A = 44°$, $b = 19$, and $a = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

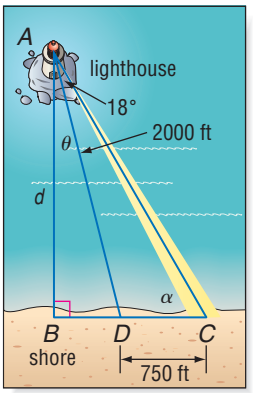
Real-World Link

Standing 208 feet tall, the Cape Hatteras Lighthouse in North Carolina is the tallest lighthouse in the United States.

Source: www.oldcapehatteras lighthouse.com

Real-World EXAMPLE **Use the Law of Sines to Solve a Problem**

6 LIGHTHOUSES **The light on a lighthouse revolves counterclockwise at a steady rate of one revolution per minute. The beam strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?**



Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is $\frac{3}{60}(360°)$ or $18°$.

Use the Law of Sines to find the measure of angle α.

$$\frac{\sin \alpha}{2000} = \frac{\sin 18°}{750}$$ Law of Sines

$$\sin \alpha = \frac{2000 \sin 18°}{750}$$ Multiply each side by 2000.

$$\sin \alpha \approx 0.8240$$ Use a calculator.

$$\alpha \approx 55°$$ Use the $\sin^{-1}$ function.

Use this angle measure to find the measure of angle θ.

$$\alpha + m\angle BAC = 90°$$ Angles α and $\angle BAC$ are complementary.

$$55° + (\theta + 18°) \approx 90°$$ $\alpha < 55°$ and $m\angle BAC = \theta + 18°$

$$\theta \approx 17°$$ Solve for θ.

To find the distance from the lighthouse to the shore, solve $\triangle ABD$ for d.

$$\cos \theta = \frac{AB}{AD}$$ Cosine ratio

$$\cos 17° \approx \frac{d}{2000}$$ $\theta = 17°$ and $AD = 2000$

$$d \approx 2000 \cos 17°$$ Solve for d.

$$d \approx 1913$$ Use a calculator.

To the nearest foot, it is 1913 feet from the lighthouse to the shore.

Peter Miller/Photo Researchers
www.MathSchoolinternational.com

CHECK Your Progress

6. The beam of light from another lighthouse strikes the shore 3000 feet away. Three seconds later, the beam strikes 1200 feet farther down the shore. To the nearest foot, how far is this lighthouse from the shore?

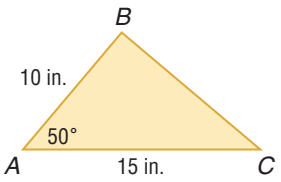
Online **Personal Tutor at algebra2.com**

CHECK Your Understanding

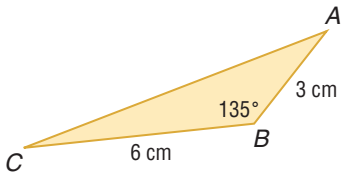
Example 1 (p. 785)

Find the area of $\triangle ABC$ to the nearest tenth.

1.



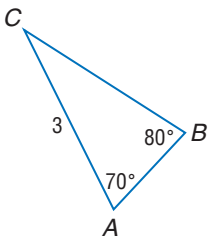
2.



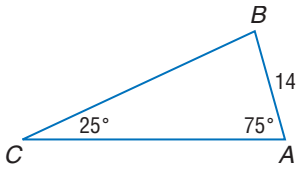
Example 2 (pp. 786–787)

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

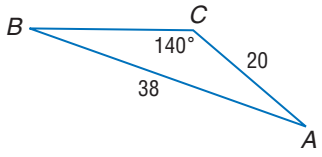
3.



4.



5.



Examples 3–5 (pp. 787–789)

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

6. $A = 123°, a = 12, b = 23$

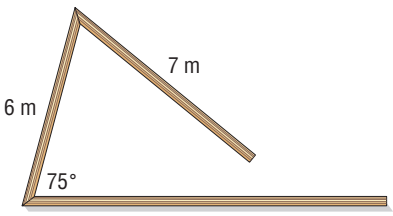
7. $A = 30°, a = 3, b = 4$

8. $A = 55°, a = 10, b = 5$

9. $A = 145°, a = 18, b = 10$

Example 6 (p. 789)

10. WOODWORKING Latisha is to join a 6-meter beam to a 7-meter beam so the angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form a triangular brace? Round to the nearest tenth.

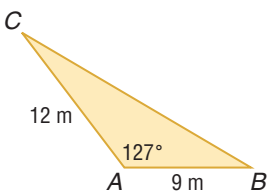


Exercises

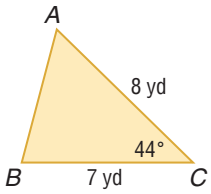
HOMEWORK HELP	
For Exercises	See Examples
11–16	1
17–30	2–5
31, 32	6

Find the area of $\triangle ABC$ to the nearest tenth.

11.



12.



13. $B = 85°, c = 23$ ft, $a = 50$ ft

14. $A = 60°, b = 12$ cm, $c = 12$ cm

15. $C = 136°, a = 3$ m, $b = 4$ m

16. $B = 32°, a = 11$ mi, $c = 5$ mi

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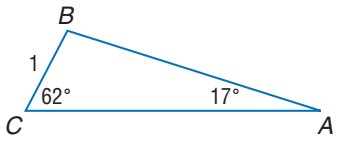
Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

17.



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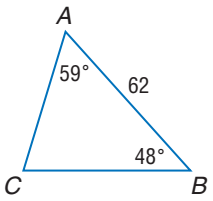

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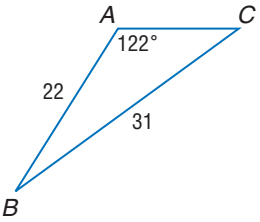
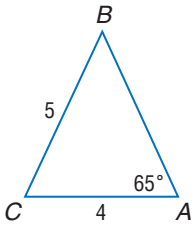
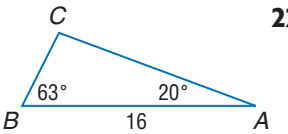


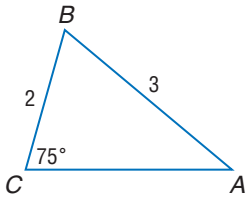
20.



21.



22.



Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

23. $A = 124°, a = 1, b = 2$

24. $A = 99°, a = 2.5, b = 1.5$

25. $A = 33°, a = 2, b = 3.5$

26. $A = 68°, a = 3, b = 5$

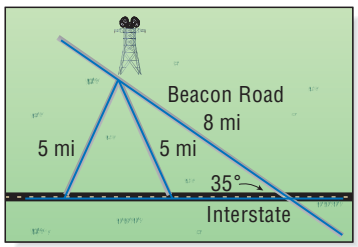
27. $A = 30°, a = 14, b = 28$

28. $A = 61°, a = 23, b = 8$

29. $A = 52°, a = 190, b = 200$

30. $A = 80°, a = 9, b = 9.1$

31. RADIO A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate tune in to hear this information?



32. FORESTRY Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 38° with the road, and the second ranger's line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger?

Real-World Link

Hot-air balloons range in size from approximately 54,000 cubic feet to over 250,000 cubic feet.

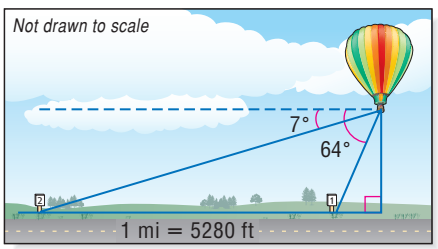
Source: www.unicorn-ballon.com

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

33. $A = 50°, a = 2.5, c = 3$

34. $B = 18°, C = 142°, b = 20$

35. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?



EXTRA PRACTICE

See pages 921, 938.

MathOnline

Self-Check Quiz at algebra2.com

SuperStock

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H.O.T. Problems

36. OPEN ENDED Give an example of a triangle that has two solutions by listing measures for A, a, and b, where a and b are in centimeters. Then draw both cases using a ruler and protractor.

37. FIND THE ERROR Dulce and Gabe are finding the area of $\triangle ABC$ for $A = 64°$, $a = 15$ meters, and $b = 8$ meters using the sine function. Who is correct? Explain your reasoning.

Dulce	Gabe
$\text{Area} = \frac{1}{2}(15)(8)\sin 64°$	$\text{Area} = \frac{1}{2}(15)(8)\sin 87.4°$
$\approx 53.9 \text{ m}^2$	$\approx 59.9 \text{ m}^2$

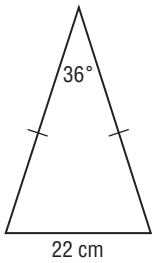
38. REASONING Determine whether the following statement is *sometimes*, *always* or *never* true. Explain your reasoning.

If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.

39. *Writing in Math* Use the information on page 785 to explain how trigonometry can be used to find the area of a triangle.

STANDARDIZED TEST PRACTICE

40. ACT/SAT Which of the following is the perimeter of the triangle shown?



A 49.0 cm **C** 91.4 cm

B 66.0 cm **D** 93.2 cm

41. REVIEW The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. What is the length of the shortest leg of the triangle?

F 50.1 in. **H** 60.1 in.

G 56.1 in. **J** 62.3 in.

Spiral Review

Find the exact value of each trigonometric function. (Lesson 13-3)

42. $\cos 30°$ **43.** $\cot\left(\frac{\pi}{3}\right)$ **44.** $\csc\left(\frac{\pi}{4}\right)$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

45. $300°$ **46.** $47°$ **47.** $\frac{5\pi}{3}$

48. AERONAUTICS A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? (Lesson 11-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

49. $a^2 = 3^2 + 5^2 - 2(3)(5)\cos 85°$ **50.** $c^2 = 12^2 + 10^2 - 2(12)(10)\cos 40°$

51. $7^2 = 11^2 + 9^2 - 2(11)(9)\cos B°$ **52.** $13^2 = 8^2 + 6^2 - 2(8)(6)\cos A°$

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13-5 Law of Cosines

Main Ideas

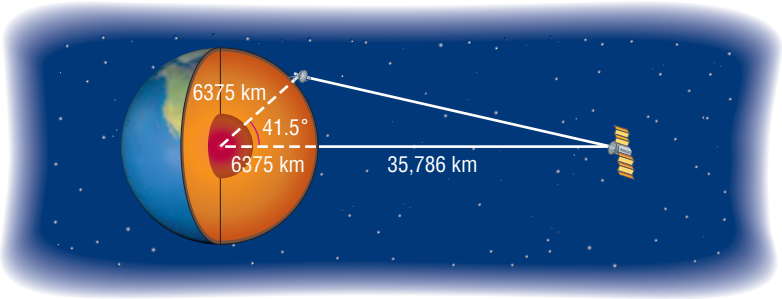
- Solve problems by using the Law of Cosines.
- Determine whether a triangle can be solved by first using the Law of Sines or the Law of Cosines.

New Vocabulary

Law of Cosines

GET READY for the Lesson

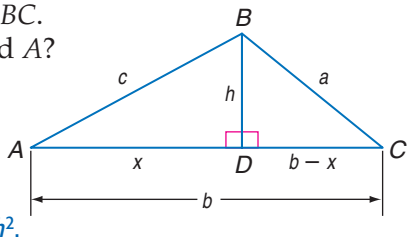
A satellite in a *geosynchronous orbit* about Earth appears to remain stationary over one point on the equator. A receiving dish for the satellite can be directed at one spot in the sky. The satellite orbits 35,786 kilometers above the equator at 87°W longitude. The city of Valparaiso, Indiana, is located at approximately 87°W longitude and 41.5°N latitude.



If the radius of Earth is about 6375 kilometers, you can use trigonometry to determine the angle at which to direct the receiver.

Law of Cosines Problems such as this, in which you know the measures of two sides and the included angle of a triangle, cannot be solved using the Law of Sines. You can solve problems such as this by using the **Law of Cosines**.

To derive the Law of Cosines, consider $\triangle ABC$. What relationship exists between a, b, c, and A?



$a^2 = (b - x)^2 + h^2$ Use the Pythagorean Theorem for $\triangle DBC$.

$= b^2 - 2bx + x^2 + h^2$ Expand $(b - x)^2$.

$= b^2 - 2bx + c^2$ In $\triangle ADB$, $c^2 = x^2 + h^2$.

$= b^2 - 2b(c \cos A) + c^2$ $\cos A = \frac{x}{c}$, so $x = c \cos A$.

$= b^2 + c^2 - 2bc \cos A$ Commutative Property

Study Tip

You can apply the Law of Cosines to a triangle if you know the measures of two sides and the included angle, or the measures of three sides.

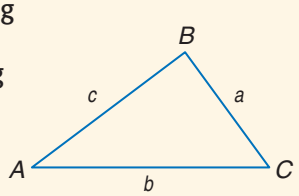
KEY CONCEPT *Law of Cosines*

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$



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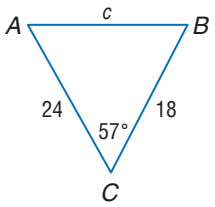
EXAMPLE Solve a Triangle Given Two Sides and Included Angle

1 **Solve $\triangle ABC$.**



Begin by using the Law of Cosines to determine c.

$c^2 = a^2 + b^2 - 2ab \cos C$ Law of Cosines

$c^2 = 18^2 + 24^2 - 2(18)(24) \cos 57°$ $a = 18$, $b = 24$, and $C = 57°$

$c^2 \approx 429.4$ Simplify using a calculator.

$c \approx 20.7$ Take the square root of each side.

Study Tip

Alternative Method

After finding the measure of c in Example 1, the Law of Cosines could be used again to find a second angle.

Next, you can use the Law of Sines to find the measure of angle A.

$\frac{\sin A}{a} = \frac{\sin C}{c}$ Law of Sines

$\frac{\sin A}{18} \approx \frac{\sin 57°}{20.7}$ $a = 18$, $C = 57°$, and $c < 20.7$

$\sin A \approx \frac{18 \sin 57°}{20.7}$ Multiply each side by 18.

$\sin A \approx 0.7293$ Use a calculator.

$A \approx 47°$ Use the $\sin^{-1}$ function.

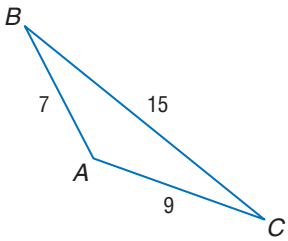
The measure of angle B is approximately $180° - (57° + 47°)$ or $76°$. Therefore, $c \approx 20.7$, $A \approx 47°$, and $B \approx 76°$.

CHECK Your Progress

1. Solve $\triangle FGH$ if $m\angle G = 82°$, $f = 6$, and $h = 4$.

EXAMPLE Solve a Triangle Given Three Sides

2 **Solve $\triangle ABC$.**



Use the Law of Cosines to find the measure of the largest angle first, angle A.

$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$15^2 = 9^2 + 7^2 - 2(9)(7) \cos A$ $a = 15$, $b = 9$, and $c = 7$

$15^2 - 9^2 - 7^2 = -2(9)(7) \cos A$ Subtract 9^2 and 7^2 from each side.

$\frac{5^2 - 9^2 - 7^2}{-2(9)(7)} = \cos A$ Divide each side by $-2(9)(7)$.

$-0.7540 \approx \cos A$ Use a calculator.

$139° \approx A$ Use the $\cos^{-1}$ function.

You can use the Law of Sines to find the measure of angle B.

$\frac{\sin B}{b} = \frac{\sin A}{a}$ Law of Sines

$\frac{\sin B}{9} \approx \frac{\sin 139°}{15}$ $b = 9$, $A \approx 139°$, and $a = 15$

$\sin B \approx \frac{9 \sin 139°}{15}$ Multiply each side by 9.

$\sin B \approx 0.3936$ Use a calculator.

$B \approx 23°$ Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180° - (139° + 23°)$ or $18°$. Therefore, $A \approx 139°$, $B \approx 23°$, and $C \approx 18°$.

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Study Tip

Sides and Angles

When solving triangles, remember that the angle with the greatest measure is always opposite the longest side. The angle with the least measure is always opposite the shortest side.

CHECK Your Progress

2. Solve $\triangle FGH$ if $f = 2$, $g = 11$, and $h = 1$.

Online Personal Tutor at algebra2.com

Choose the Method To solve a triangle that is *oblique*, or having no right angle, you need to know the measure of at least one side and any two other parts. If the triangle has a solution, then you must decide whether to begin solving by using the Law of Sines or the Law of Cosines. Use the chart to help you choose.

CONCEPT SUMMARY *Solving an Oblique Triangle*

Given	Begin by Using
two angles and any side	Law of Sines
two sides and an angle opposite one of them	Law of Sines
two sides and their included angle	Law of Cosines
three sides	Law of Cosines

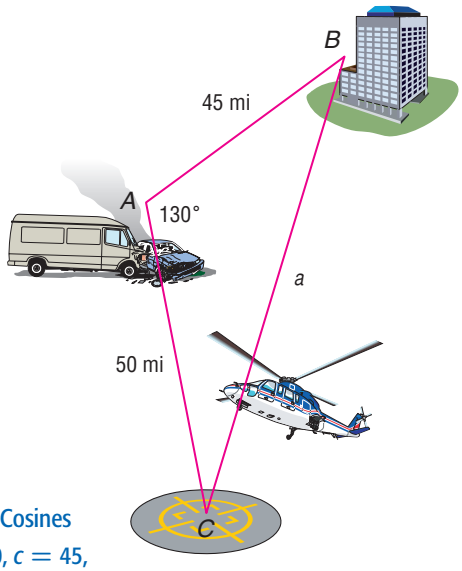
Real-World Link

Medical evacuation (Medevac) helicopters provide quick transportation from areas that are difficult to reach by any other means. These helicopters can cover long distances and are primary emergency vehicles in locations where there are few hospitals.

Source: The Helicopter Education Center

Real-World EXAMPLE **Apply the Law of Cosines**

3 **EMERGENCY MEDICINE** **A medical rescue helicopter has flown from its home base at point C to pick up an accident victim at point A and then from there to the hospital at point B. The pilot needs to know how far he is now from his home base so he can decide whether to refuel before returning. How far is the hospital from the helicopter's base?**



You are given the measures of two sides and their included angle, so use the Law of Cosines to find a.

$a^2 = b^2 + c^2 - 2bc \cos A$ Law of Cosines

$a^2 = 50^2 + 45^2 - 2(50)(45) \cos 130°$ $b = 50$, $c = 45$, and $A = 130°$.

$a^2 \approx 7417.5$ Use a calculator to simplify.

$a \approx 86.1$ Take the square root of each side.

The distance between the hospital and the helicopter base is approximately 86.1 miles.

CHECK Your Progress

3. As part of training to run a marathon, Amelia ran 6 miles in one direction. She then turned and ran another 9 miles. The two legs of her run formed an angle of 79°. How far was Amelia from her starting point at the end of the 9-mile leg of her run?

Roy Ooms/Masterfile

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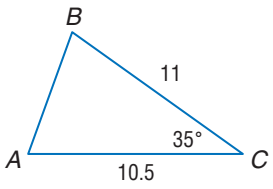
CHECK Your Understanding

Examples 1, 2 (pp. 794–795)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

1.



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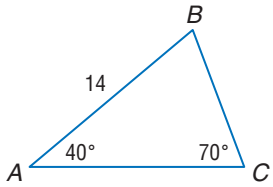


3. $A = 42°, b = 57, a = 63$

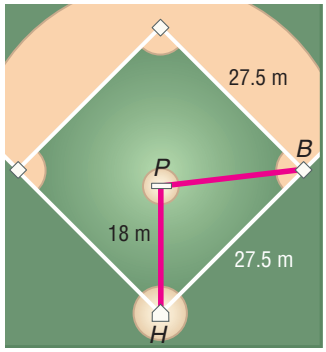
4. $a = 5, b = 12, c = 13$

Example 3 (p. 795)

BASEBALL For Exercises 5 and 6, use the following information.
In Australian baseball, the bases lie at the vertices of a square 27.5 meters on a side and the pitcher's mound is 18 meters from home plate.

5. Find the distance from the pitcher's mound to first base.

6. Find the angle between home plate, the pitcher's mound, and first base.

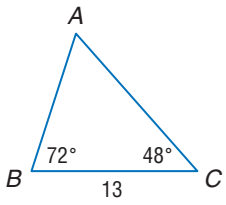


Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–18	1, 2
19, 20	3

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

7.



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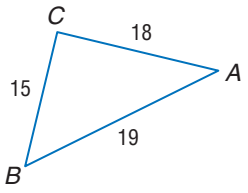

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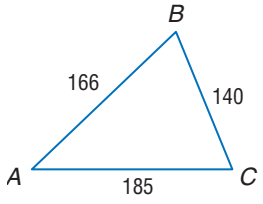

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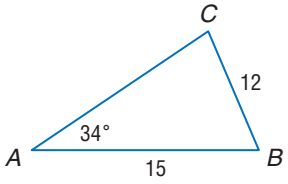

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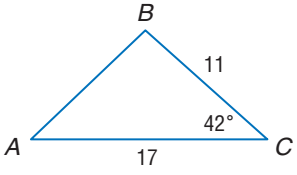

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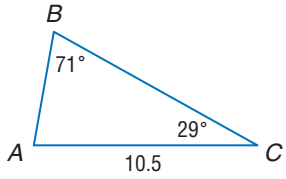


13. $a = 20, c = 24, B = 47°$

14. $a = 345, b = 648, c = 442$

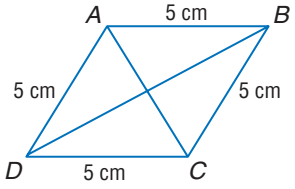
15. $A = 36°, a = 10, b = 19$

16. $A = 25°, B = 78°, a = 13.7$

17. $a = 21.5, b = 16.7, c = 10.3$

18. $a = 16, b = 24, c = 41$

19. GEOMETRY In rhombus $ABCD$, the measure of $\angle ADC$ is 52°. Find the measures of diagonals $\overline{AC}$ and $\overline{BD}$ to the nearest tenth.



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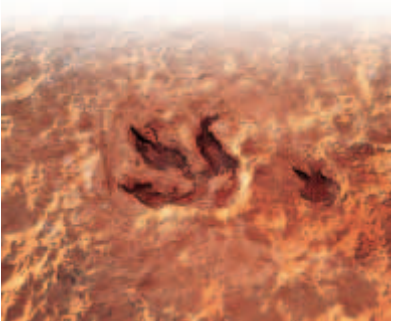


Real-World Link

At digs such as the one at the Glen Rose formation in Texas, anthropologists study the footprints made by dinosaurs millions of years ago. *Locomoter* parameters, such as pace and stride, taken from these prints can be used to describe how a dinosaur once moved.

Source: Mid-America Paleontology Society

20. SURVEYING Two sides of a triangular plot of land have lengths of 425 feet and 550 feet. The measure of the angle between those sides is 44.5°. Find the perimeter and area of the plot.

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21. $a = 8, b = 24, c = 18$

22. $B = 19°, a = 51, c = 61$

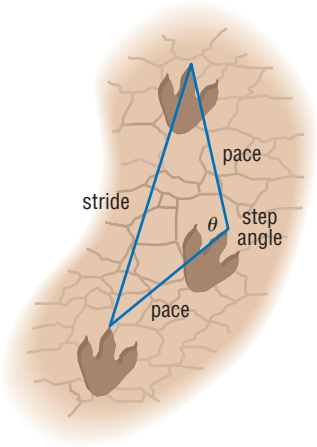
23. $A = 56°, B = 22°, a = 12.2$

24. $a = 4, b = 8, c = 5$

25. $a = 21.5, b = 13, C = 38°$

26. $A = 40°, b = 7, a = 6$

DINOSAURS For Exercises 27–29, use the diagram at the right.



27. An anthropologist examining the footprints made by a bipedal (two-footed) dinosaur finds that the dinosaur's average pace was about 1.60 meters and average stride was about 3.15 meters. Find the step angle θ for this dinosaur.

28. Find the step angle θ made by the hindfeet of a herbivorous dinosaur whose pace averages 1.78 meters and stride averages 2.73 meters.

29. An efficient walker has a step angle that approaches 180°, meaning that the animal minimizes "zig-zag" motion while maximizing forward motion. What can you tell about the motion of each dinosaur from its step angle?

30. AVIATION A pilot typically flies a route from Bloomington to Rockford, covering a distance of 117 miles. In order to avoid a storm, the pilot first flies from Bloomington to Peoria, a distance of 42 miles, then turns the plane and flies 108 miles on to Rockford. Through what angle did the pilot turn the plane over Peoria?

EXTRA PRACTICE
See pages 921, 938.
MathOnline
Self-Check Quiz at algebra2.com

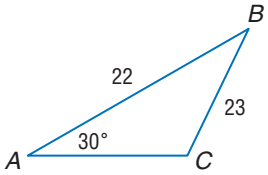
H.O.T. Problems

31. REASONING Explain how to solve a triangle by using the Law of Cosines if the lengths of

a. three sides are known.

b. two sides and the measure of the angle between them are known.

32. FIND THE ERROR Mateo and Amy are deciding which method, the Law of Sines or the Law of Cosines, should be used first to solve $\triangle ABC$.



Mateo	Amy
Begin by using the Law of Sines, since you are given two sides and an angle opposite one of them.	Begin by using the Law of Cosines, since you are given two sides and their included angle.

Who is correct? Explain your reasoning.

John T. Carbone/Photonica/Getty Ima
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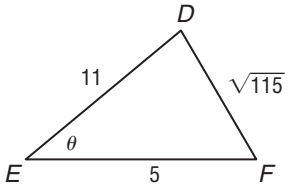
33. OPEN ENDED Give an example of a triangle that can be solved by first using the Law of Cosines.

34. CHALLENGE Explain how the Pythagorean Theorem is a special case of the Law of Cosines.

35. *Writing in Math* Use the information on page 793 to explain how you can determine the angle at which to install a satellite dish. Include an explanation of how, given the latitude of a point on Earth's surface, you can determine the angle at which to install a satellite dish at the same longitude.

STANDARDIZED TEST PRACTICE

36. ACT/SAT In $\triangle DEF$, what is the value of θ to the nearest degree?



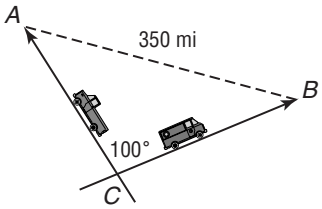
A 26°

B 74°

C 80°

D 141°

37. REVIEW Two trucks, A and B, start from the intersection C of two straight roads at the same time. Truck A is traveling twice as fast as truck B and after 4 hours, the two trucks are 350 miles apart. Find the approximate speed of truck B in miles per hour.



F 35 **G** 37 **H** 57 **J** 73

Spiral Review

38. SANDBOX Mr. Blackwell is building a triangular sandbox. He is to join a 3-meter beam to a 4 meter beam so the angle opposite the 4-meter beam measures 80°. To what length should Mr. Blackwell cut the third beam in order to form the triangular sandbox? Round to the nearest tenth. (Lesson 13-4)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point. (Lesson 13-3)

39. (5, 12) **40.** (4, 7) **41.** $(\sqrt{10}, \sqrt{6})$

Solve each equation or inequality. (Lesson 9-5)

42. $e^x + 5 = 9$ **43.** $4e^x - 3 > -1$ **44.** $\ln(x + 3) = 2$

GET READY for the Next Lesson

PREREQUISITE SKILL Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

45. 45° **46.** 30° **47.** 180°

48. $\frac{\pi}{2}$ **49.** $\frac{7\pi}{6}$ **50.** $\frac{4\pi}{3}$

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13-6 Circular Functions

Main Ideas

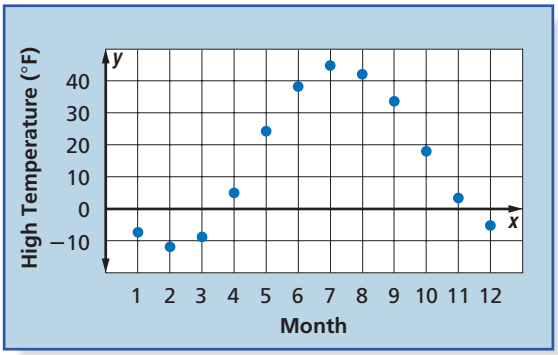
- Define and use the trigonometric functions based on the unit circle.
- Find the exact values of trigonometric functions of angles.

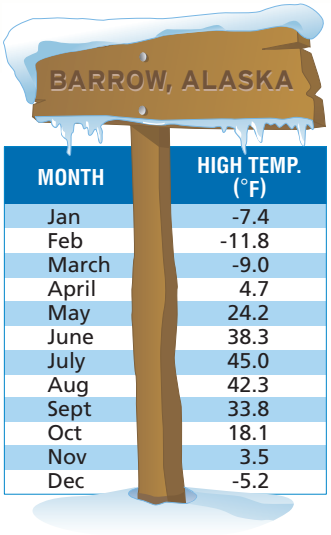
New Vocabulary

circular function
periodic
period

GET READY for the Lesson

The average high temperatures, in degrees Fahrenheit, for Barrow, Alaska, are given in the table at the right. With January assigned a value of 1, February a value of 2, March a value of 3, and so on, these data can be graphed as shown below. This pattern of temperature fluctuations repeats after a period of 12 months.

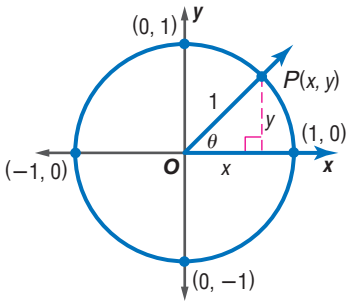




MONTH	HIGH TEMP. (°F)
Jan	-7.4
Feb	-11.8
March	-9.0
April	4.7
May	24.2
June	38.3
July	45.0
Aug	42.3
Sept	33.8
Oct	18.1
Nov	3.5
Dec	-5.2

Source: www.met.utah.edu

Unit Circle Definitions From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, $\frac{1}{2}$. In this lesson, we will further generalize the functions by defining them in terms of the unit circle.

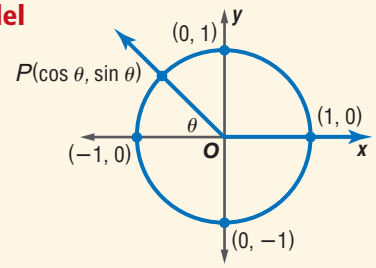


Consider an angle θ in standard position. The terminal side of the angle intersects the unit circle at a unique point, $P(x, y)$. Recall that $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. Since $P(x, y)$ is on the unit circle, $r = 1$. Therefore, $\sin \theta = y$ and $\cos \theta = x$.

KEY CONCEPT *Definition of Sine and Cosine*

Words If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.

Model



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Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos\theta = x$ and $\sin\theta = y$ are functions of θ. Because they are both defined using a unit circle, they are often called **circular functions**.

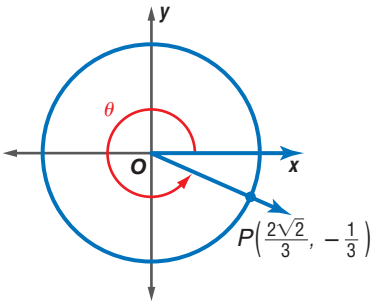
Study Tip

Remembering Relationships

To help you remember that $x = \cos\theta$ and $y = \sin\theta$, notice that alphabetically x comes before y and cosine comes before sine.

EXAMPLE Find Sine and Cosine Given Point on Unit Circle

1 **Given an angle θ in standard position, if $P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ lies on the terminal side and on the unit circle, find $\sin\theta$ and $\cos\theta$.**



$P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = P(\cos\theta, \sin\theta)$,

so $\sin\theta = -\frac{1}{3}$ and $\cos\theta = \frac{2\sqrt{2}}{3}$.

CHECK Your Progress

1. Given an angle θ in standard position, if $P\left(\frac{\sqrt{6}}{5}, \frac{\sqrt{19}}{5}\right)$ lies on the terminal side and on the unit circle, find $\sin\theta$ and $\cos\theta$.

GRAPHING CALCULATOR LAB

Sine and Cosine on the Unit Circle

Press MODE and highlight Degree and Par. Then use the following range values to set up a viewing window: TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = −2.4, XMAX = 2.35, XSCL = 0.5, YMIN = −1.5, YMAX = 1.55, YSCL = 0.5.

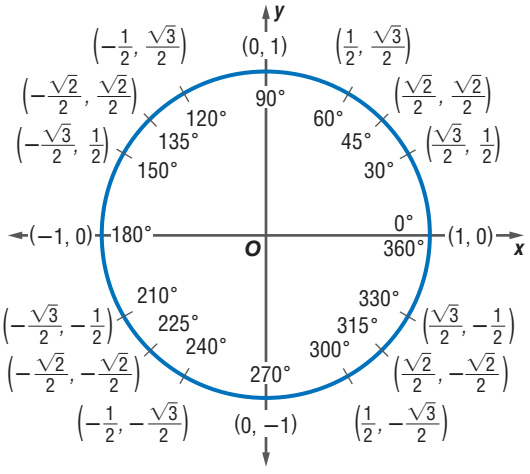
Press Y= to define the unit circle with $X_{1T} = \cos T$ and $Y_{1T} = \sin T$.

Press GRAPH. Use the TRACE function to move around the circle.

THINK AND DISCUSS

1. What does T represent? What do the x- and y-values represent?
2. Determine the sine and cosine of the angles whose terminal sides lie at 0°, 90°, 180°, and 270°.
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle at the right.



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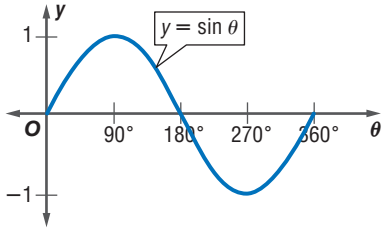
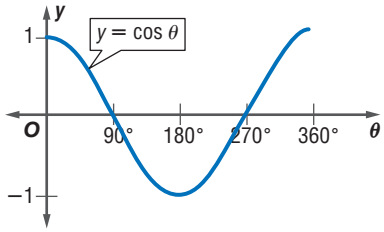
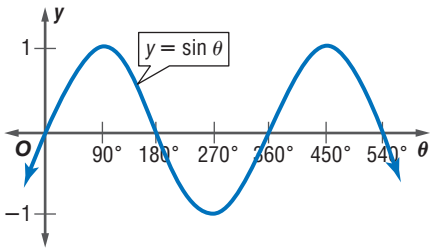
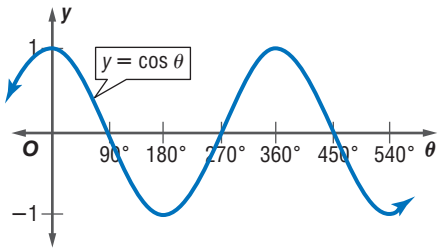
This same information is presented on the graphs of the sine and cosine functions below, where the horizontal axis shows the values of θ and the vertical axis shows the values of $\sin \theta$ or $\cos \theta$.





Periodic Functions Notice in the graph above that the values of sine for the coterminal angles 0° and 360° are both 0. The values of cosine for these angles are both 1. Every 360° or 2π radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are **periodic**, each having a **period** of 360° or 2π radians.





KEY CONCEPT *Periodic Function*

A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function. The least positive value of a for which $f(x) = f(x + a)$ is called the period of the function.

For the sine and cosine functions, $\cos (x + 360°) = \cos x$, and $\sin (x + 360°) = \sin x$. In radian measure, $\cos (x + 2\pi) = \cos x$, and $\sin (x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is 360° or 2π.

EXAMPLE **Find the Value of a Trigonometric Function**

2 **Find the exact value of each function.**

a. $\cos 675°$

$$\begin{aligned}\cos 675° &= \cos (315° + 360°)\\ &= \cos 315°\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

b. $\sin\left(-\frac{5\pi}{6}\right)$

$$\begin{aligned}\sin\left(-\frac{5\pi}{6}\right) &= \sin\left(-\frac{5\pi}{6} + 2\pi\right)\\ &= \sin \frac{7\pi}{6}\\ &= -\frac{1}{2}\end{aligned}$$

CHECK Your Progress

2A. $\cos\left(-\frac{3\pi}{4}\right)$ **2B.** $\sin 420°$

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When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

Many real-world situations have characteristics that can be described with periodic functions.

Real-World EXAMPLE **Find the Value of a Trigonometric Function**

3 **FERRIS WHEEL** **As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.**

Real-World Link

The Ferris Wheel was designed by bridge builder George W. Ferris in 1893. It was designed to be the landmark of the World's Fair in Chicago in 1893.

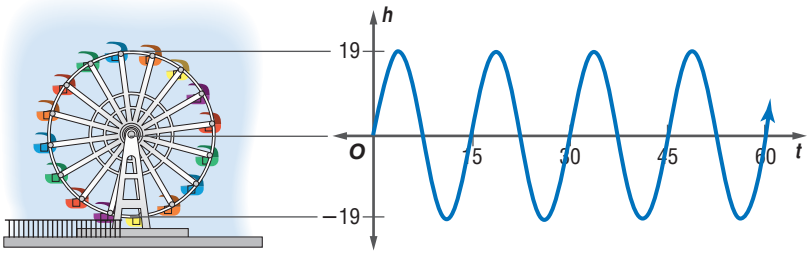
Source: National Academy of Sciences

a. Identify the period of this function.

Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is $\frac{1}{4}$ of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time t in seconds and the vertical axis represents the height h in feet in relation to the starting point.

Your height is 0 feet at the starting point. Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of $\frac{38}{2}$ or 19 feet above the starting point and a minimum of 19 feet below the starting point.



Because the period of the function is 15 seconds, the pattern of the graph repeats in intervals of 15 seconds on the x-axis.

CHECK Your Progress

A new model of the Ferris wheel travels at a rate of 5 revolutions per minute and has a diameter of 44 feet.

3A. What is the period of this function?

3B. Graph the function.

Bettman/CORBIS

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CHECK Your Understanding

Example 1 (p. 800)

If the given point P is located on the unit circle, find $\sin\theta$ and $\cos\theta$.

1. $P\left(\frac{5}{13}, -\frac{12}{13}\right)$

2. $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Example 2 (p. 801)

Find the exact value of each function.

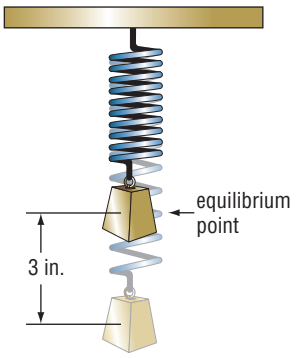
3. $\sin -240^\circ$

4. $\cos \frac{10\pi}{3}$

Example 3 (p. 802)

PHYSICS For Exercises 5 and 6, use the following information.

The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.



5. Find the period of this function.

6. Graph the height of the spring as a function of time.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–12	1
13–18	2
19–38	3

The given point P is located on the unit circle. Find $\sin\theta$ and $\cos\theta$.

7. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$

8. $P\left(-\frac{12}{13}, -\frac{5}{13}\right)$

9. $P\left(\frac{8}{17}, \frac{15}{17}\right)$

10. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

11. $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

12. $P(0.6, 0.8)$

Find the exact value of each function.

13. $\sin 690^\circ$

14. $\cos 750^\circ$

15. $\cos 5\pi$

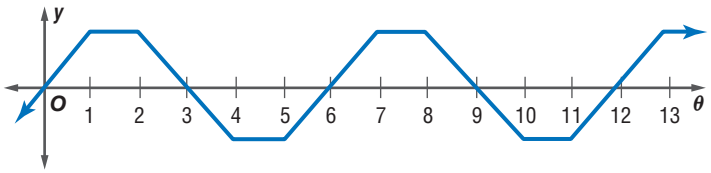
16. $\sin\left(\frac{14\pi}{6}\right)$

17. $\sin\left(-\frac{3\pi}{2}\right)$

18. $\cos(-225^\circ)$

Determine the period of each function.

19.



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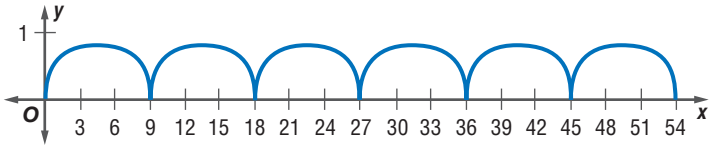

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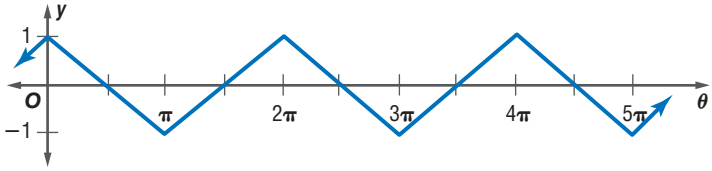

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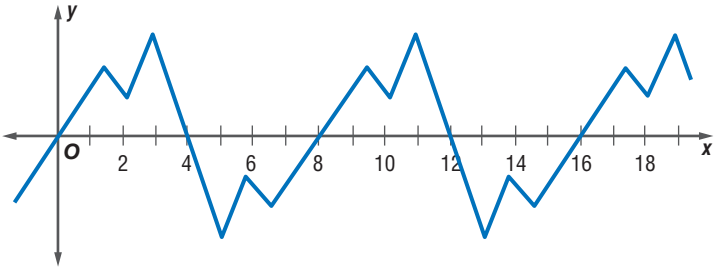
Determine the period of the function.

22.



Real-World Link

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and the springiness of the strings' material.

Source: www.howstuffworks.com

GUITAR **For Exercises 23 and 24, use the following information.**
When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz.

23. Find the period of this function.

24. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit and the minimum distance below this position have a value of 1 unit.

Find the exact value of each function.

25. $\frac{\cos 60^\circ + \sin 30^\circ}{4}$

26. $3(\sin 60^\circ)(\cos 30^\circ)$

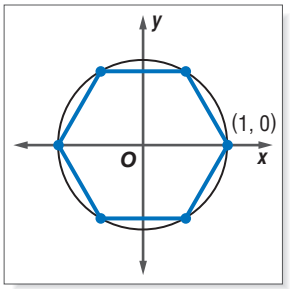
27. $\sin 30^\circ - \sin 60^\circ$

28. $\frac{4 \cos 330^\circ + 2 \sin 60^\circ}{3}$

29. $12(\sin 150^\circ)(\cos 150^\circ)$

30. $(\sin 30^\circ)^2 + (\cos 30^\circ)^2$

31. **GEOMETRY** A regular hexagon is inscribed in a unit circle centered at the origin. If one vertex of the hexagon is at (1, 0), find the exact coordinates of the remaining vertices.



32. **BIOLOGY** In a certain area of forested land, the population of rabbits R increases and decreases periodically throughout the year. If the population can be modeled by $R = 425 + 200 \sin \left[\frac{\pi}{365}(d - 60)\right]$, where d represents the dth day of the year, describe what happens to the population throughout the year.

SLOPE **For Exercises 33–38, use the following information.**
Suppose the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$.

33. What is the slope of $\overline{OP}$?

34. Which of the six trigonometric functions is equal to the slope of $\overline{OP}$?

35. What is the slope of any line perpendicular to $\overline{OP}$?

36. Which of the six trigonometric functions is equal to the slope of any line perpendicular to $\overline{OP}$?

37. Find the slope of $\overline{OP}$ when $\theta = 60^\circ$.

38. If $\theta = 60^\circ$, find the slope of the line tangent to circle O at point P.

EXTRA PRACTICE
See pages 921, 938.
MathOnline
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H.O.T. Problems

39. OPEN ENDED Give an example of a situation that could be described by a periodic function. Then state the period of the function.

40. WHICH ONE DOESN'T BELONG? Identify the expression that does not belong with the other three. Explain your reasoning.

$\sin 90°$ | $\tan \frac{\pi}{4}$ | $\cos 180°$ | $\csc \frac{\pi}{2}$

41. CHALLENGE Determine the domain and range of the functions $y = \sin \theta$ and $y = \cos \theta$.

42. Writing in Math If the formula for the temperature T in degrees Fahrenheit of a city t months into the year is given by $T = 50 + 25 \sin \left(\frac{\pi}{6}t\right)$, explain how to find the average temperature and the maximum and minimum predicted over the year.

STANDARDIZED TEST PRACTICE

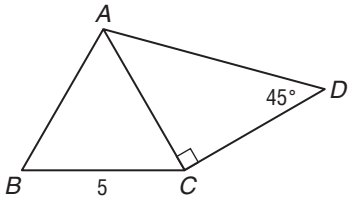
43. ACT/SAT If $\triangle ABC$ is an equilateral triangle, what is the length of $\overline{AD}$, in units?

A 5

B $5\sqrt{2}$

C 10

D $10\sqrt{2}$



44. REVIEW For which measure of θ is $\theta = \frac{\sqrt{3}}{3}$?

F 135°

G 270°

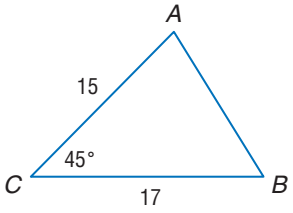
H 1080°

J 1830°

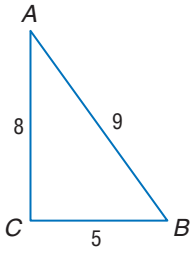
Spiral Review

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

45.



46.



Find the area of $\triangle ABC$. Round to the nearest tenth. (Lesson 13-4)

47. $a = 11$ in., $c = 5$ in., $B = 79°$

48. $b = 4$ m, $c = 7$ m, $A = 63°$

48. BULBS The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. How many light bulbs will last between 260 and 340 days? (Lesson 12-7)

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

49. $a_1 = 3$, $r = 1.2$

50. $16, 4, 1, \frac{1}{4}, \ldots$

51. $\sum_{n=1}^{\infty} 13(-0.625)^{n-1}$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find each value of θ. Round to the nearest degree.** (Lesson 13-1)

52. $\sin \theta = 0.3420$

53. $\cos \theta = -0.3420$

54. $\tan \theta = 3.2709$

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13-7 Inverse Trigonometric Functions

Main Ideas

- Solve equations by using inverse trigonometric functions.
- Find values of expressions involving trigonometric functions.

New Vocabulary

principal values
Arcsine function
Arccosine function
Arctangent function

GET READY for the Lesson

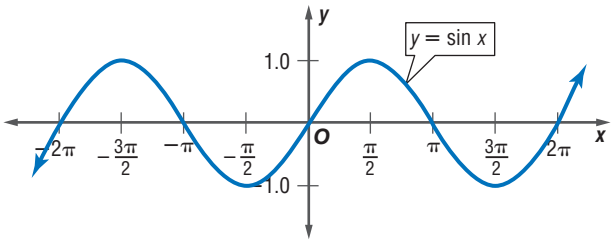
When a car travels a curve on a horizontal road, the friction between the tires and the road keeps the car on the road. Above a certain speed, however, the force of friction will not be great enough to hold the car in the curve. For this reason, civil engineers design banked curves.

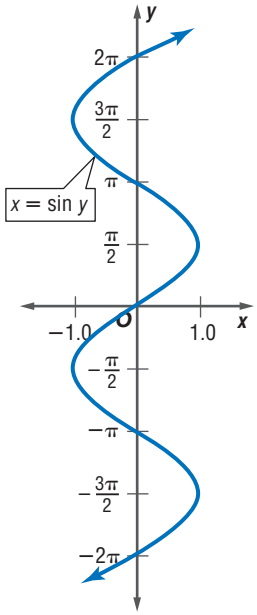
The proper banking angle θ for a car making a turn of radius r feet at a velocity v in feet per second is given by the equation $\tan \theta = \frac{v^2}{32r}$. In order to determine the appropriate value of θ for a specific curve, you need to know the radius of the curve, the maximum allowable velocity of cars making the curve, and how to determine the angle θ given the value of its tangent.

Doug Plummer/Photonica

Solve Equations Using Inverses Sometimes the value of a trigonometric function for an angle is known and it is necessary to find the measure of the angle. The concept of inverse functions can be applied to find the inverse of trigonometric functions.

In Lesson 8-8, you learned that the inverse of a function is the relation in which all the values of x and y are reversed. The graphs of $y = \sin x$ and its inverse, $x = \sin y$, are shown below.





Notice that the inverse is not a function, since it fails the vertical line test. None of the inverses of the trigonometric functions are functions.

We must restrict the domain of trigonometric functions so that their inverses are functions. The values in these restricted domains are called **principal values.** Capital letters are used to distinguish trigonometric functions with restricted domains from the usual trigonometric functions.

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KEY CONCEPT *Principal Values of Sine, Cosine, and Tangent*

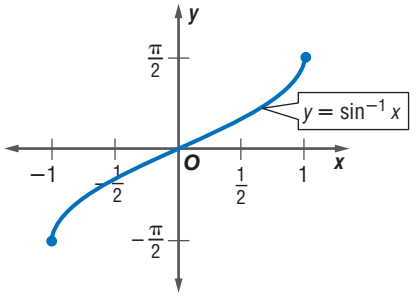
$y = \text{Sin } x$ if and only if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

$y = \text{Cos } x$ if and only if $y = \cos x$ and $0 \le x \le \pi$.

$y = \text{Tan } x$ if and only if $y = \tan x$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse of the Sine function is called the **Arcsine function** and is symbolized by **Sin^{-1}** or **Arcsin**. The Arcsine function has the following characteristics.

- Its domain is the set of real numbers from -1 to 1.
- Its range is the set of angle measures from $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $\text{Sin } x = y$ if and only if $\text{Sin}^{-1}\, y = x$.
- $[\text{Sin}^{-1} \circ \text{Sin}](x) = [\text{Sin} \circ \text{Sin}^{-1}](x) = x$.



Study Tip
Look Back
To review **composition and functions**, see Lesson 7-1.

The definitions of the **Arccosine** and **Arctangent** functions are similar to the definition of the Arcsine function.

CONCEPT SUMMARY *Inverse Sine, Cosine, and Tangent*

- Given $y = \text{Sin } x$, the inverse Sine function is defined by $y = \text{Sin}^{-1}\, x$ or $y = \text{Arcsin } x$.
- Given $y = \text{Cos } x$, the inverse Cosine function is defined by $y = \text{Cos}^{-1}\, x$ or $y = \text{Arccos } x$.
- Given $y = \text{Tan } x$, the inverse Tangent function is defined by $y = \text{Tan}^{-1}\, x$ or $y = \text{Arctan } x$.

The expressions in each row of the table below are equivalent. You can use these expressions to rewrite and solve trigonometric equations.

$y = \text{Sin } x$	$x = \text{Sin}^{-1}\, y$	$x = \text{Arcsin } y$
$y = \text{Cos } x$	$x = \text{Cos}^{-1}\, y$	$x = \text{Arccos } y$
$y = \text{Tan } x$	$x = \text{Tan}^{-1}\, y$	$x = \text{Arctan } y$

EXAMPLE **Solve an Equation**

1 **Solve $\text{Sin } x = \frac{\sqrt{3}}{2}$ by finding the value of x to the nearest degree.**

If $\text{Sin } x = \frac{\sqrt{3}}{2}$, then x is the least value whose sine is $\frac{\sqrt{3}}{2}$. So, $x = \text{Arcsin } \frac{\sqrt{3}}{2}$.

Use a calculator to find x.

KEYSTROKES: 2nd [SIN^{-1}] 2nd [$\sqrt{\ }$] 3) ÷ 2) ENTER 60

Therefore, $x = 60°$.

CHECK Your Progress

1. Solve $\text{Cos } x = -\frac{\sqrt{2}}{2}$ by finding the value of x to the nearest degree.

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Many application problems involve finding the inverse of a trigonometric function.

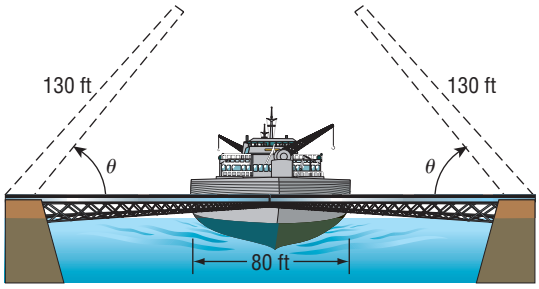
Real-World Link

Bascule bridges have spans (leaves) that pivot upward utilizing gears, motors, and counterweights.

Source: www.multnomah.lib.or.us

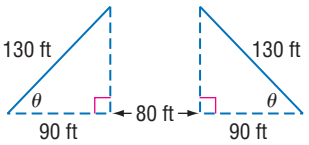
Real-World EXAMPLE Apply an Inverse to Solve a Problem

2 **DRAWBRIDGE** **Each leaf of a certain double-leaf drawbridge is 130 feet long. If an 80-foot wide ship needs to pass through the bridge, what is the minimum angle θ, to the nearest degree, which each leaf of the bridge should open so that the ship will fit?**



When the two parts of the bridge are in their lowered position, the bridge spans 130 + 130 or 260 feet. In order for the ship to fit, the distance between the leaves must be at least 80 feet.

This leaves a horizontal distance of $\frac{260 - 80}{2}$ or 90 feet from the pivot point of each leaf to the ship as shown in the diagram at the right.



To find the measure of angle θ, use the cosine ratio for right triangles.

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$ Cosine ratio

$\cos \theta = \frac{90}{130}$ Replace *adj* with 90 and *hyp* with 130.

$\theta = \cos^{-1}\left(\frac{90}{130}\right)$ Inverse cosine function

$\theta \approx 46.2°$ Use a calculator.

Thus, the minimum angle each leaf of the bridge should open is 47°.

CHECK Your Progress

2. If each leaf of another drawbridge is 150 feet long, what is the minimum angle θ, to the nearest degree, that each leaf should open to allow a 90-foot-wide ship to pass?

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Study Tip

Angle Measure

Remember that when evaluating an inverse trigonometric function the result is an angle measure.

Trigonometric Values You can use a calculator to find the values of trigonometric expressions.

EXAMPLE Find a Trigonometric Value

3 **Find each value. Write angle measures in radians. Round to the nearest hundredth.**

a. ArcSin $\frac{\sqrt{3}}{2}$

KEYSTROKES: 2nd [SIN^{-1}] 2nd [$\sqrt{\ }$] 3) ÷ 2) ENTER 1.047197551

Therefore, ArcSin $\frac{\sqrt{3}}{2} \approx 1.05$ radians.

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b. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

KEYSTROKES: TAN 2nd [COS^{-1}] 6 ÷ 7) ENTER 0.6009252126

Therefore, $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right) \approx 0.60$.

CHECK Your Progress

3A. $\text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$

3B. $\cos\left(\text{Arcsin}\frac{4}{5}\right)$

CHECK Your Understanding

Example 1 (p. 807)

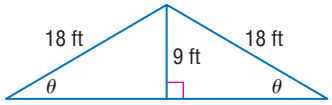
Solve each equation by finding the value of x to the nearest degree.

1. $x = \text{Cos}^{-1}\frac{\sqrt{2}}{2}$

2. $\text{Arctan } 0 = x$

Example 2 (p. 808)

3. ARCHITECTURE The support for a roof is shaped like two right triangles as shown at the right. Find θ.



Example 3 (pp. 808–809)

Find each value. Write degree measures in radians. Round to the nearest hundredth.

4. $\text{Tan}^{-1}\left(\frac{\sqrt{3}}{3}\right)$

5. $\text{Cos}^{-1}(-1)$

6. $\cos\left(\text{Cos}^{-1}\frac{2}{9}\right)$

7. $\sin\left(\text{Sin}^{-1}\frac{3}{4}\right)$

8. $\sin\left(\text{Cos}^{-1}\frac{3}{4}\right)$

9. $\tan\left(\text{Sin}^{-1}\frac{1}{2}\right)$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
10–24	1
25–35	3
36, 37	2

Solve each equation by finding the value of x to the nearest degree.

10. $x = \text{Cos}^{-1}\frac{1}{2}$

11. $\text{Sin}^{-1}\frac{1}{2} = x$

12. $\text{Arctan } 1 = x$

13. $x = \text{Arctan}\frac{\sqrt{3}}{3}$

14. $x = \text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15. $x = \text{Cos}^{-1} 0$

Find each value. Write angle measures in radians. Round to the nearest hundredth.

16. $\text{Cos}^{-1}\left(-\frac{1}{2}\right)$

17. $\text{Sin}^{-1}\frac{\pi}{2}$

18. $\text{Arctan}\frac{\sqrt{3}}{3}$

19. $\text{Arccos}\frac{\sqrt{3}}{2}$

20. $\sin\left(\text{Sin}^{-1}\frac{1}{2}\right)$

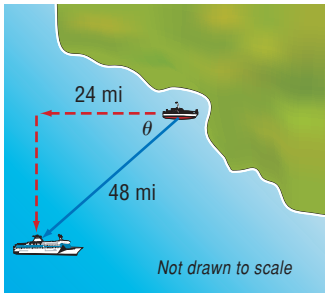
21. $\cot\left(\text{Sin}^{-1}\frac{5}{6}\right)$

22. $\tan\left(\text{Cos}^{-1}\frac{6}{7}\right)$

23. $\sin\left(\text{Arctan}\frac{\sqrt{3}}{3}\right)$

24. $\cos\left(\text{Arcsin}\frac{3}{5}\right)$

25. TRAVEL The cruise ship *Reno* sailed due west 24 miles before turning south. When the *Reno* became disabled and radioed for help, the rescue boat found that the fastest route to her covered a distance of 48 miles. The cosine of the angle at which the rescue boat should sail is 0.5. Find the angle θ, to the nearest tenth of a degree, at which the rescue boat should travel to aid the *Reno*.



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EXTRA PRACTICE
See pages 922, 938.

MathOnline
Self-Check Quiz at
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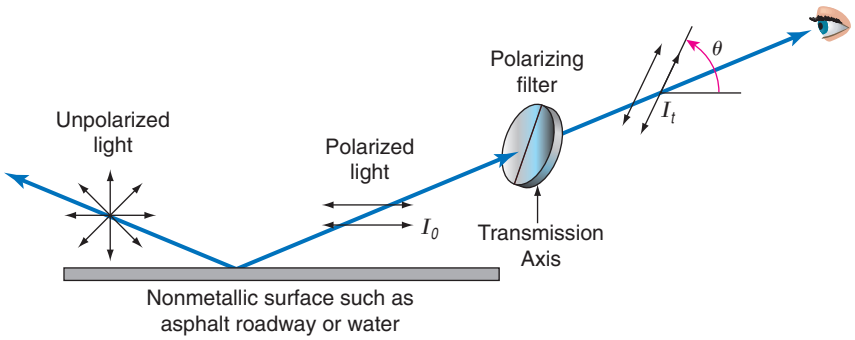
26. OPTICS You may have polarized sunglasses that eliminate glare by polarizing the light. When light is polarized, all of the waves are traveling in parallel planes. Suppose horizontally-polarized light with intensity I_0 strikes a polarizing filter with its axis at an angle of θ with the horizontal. The intensity of the transmitted light I_t and θ are related by the equation $\cos \theta = \sqrt{\frac{I_t}{I_0}}$. If one fourth of the polarized light is transmitted through the lens, what angle does the transmission axis of the filter make with the horizontal?



Real-World Link
The shot is a metal sphere that can be made out of solid iron. Shot putters stand inside a seven-foot circle and must "put" the shot from the shoulder with one hand.

Source: www.coolrunning.com.au

Find each value. Write angle measures in radians. Round to the nearest hundredth.

27. $\cot\left(\text{Sin}^{-1}\frac{7}{9}\right)$

28. $\cos\left(\text{Tan}^{-1}\sqrt{3}\right)$

29. $\tan(\text{Arctan } 3)$

30. $\cos\left[\text{Arccos}\left(-\frac{1}{2}\right)\right]$

31. $\text{Sin}^{-1}\left(\tan\frac{\pi}{4}\right)$

32. $\cos\left(\text{Cos}^{-1}\frac{\sqrt{2}}{2} - \frac{\pi}{2}\right)$

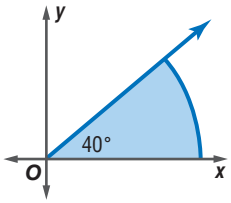
33. $\text{Cos}^{-1}(\text{Sin}^{-1}\,90)$

34. $\sin\left(2\,\text{Cos}^{-1}\frac{3}{5}\right)$

35. $\sin\left(2\,\text{Sin}^{-1}\frac{1}{2}\right)$

36. FOUNTAINS Architects who design fountains know that both the height and distance that a water jet will project is dependent on the angle θ at which the water is aimed. For a given angle θ, the ratio of the maximum height H of the parabolic arc to the horizontal distance D it travels is given by $\frac{H}{D} = \frac{1}{4}\tan\theta$. Find the value of θ, to the nearest degree, that will cause the arc to go twice as high as it travels horizontally.

37. TRACK AND FIELD A shot put must land in a 40° sector. The vertex of the sector is at the origin and one side lies along the x-axis. An athlete puts the shot at a point with coordinates (18, 17), did the shot land in the required region? Explain your reasoning.



For Exercises 38–40, consider $f(x) = \text{Sin}^{-1}x + \text{Cos}^{-1}x$.

38. Make a table of values, recording x and $f(x)$ for $x = \left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}, -1\right\}$.

39. Make a conjecture about $f(x)$.

40. Considering only positive values of x, provide an explanation of why your conjecture might be true.

H.O.T. Problems

41. OPEN ENDED Write an equation giving the value of the Cosine function for an angle measure in its domain. Then, write your equation in the form of an inverse function.

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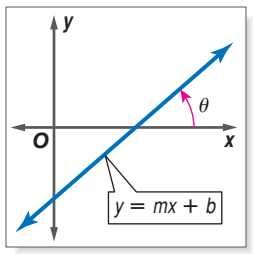
CHALLENGE **For Exercises 42–44, use the following information.**
If the graph of the line $y = mx + b$ intersects the x-axis such that an angle of θ is formed with the positive x-axis, then $\tan \theta = m$.



42. Find the acute angle that the graph of $3x + 5y = 7$ makes with the positive x-axis to the nearest degree.

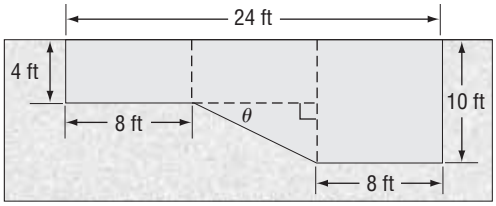
43. Determine the obtuse angle formed at the intersection of the graphs of $2x + 5y = 8$ and $6x - y = -8$. State the measure of the angle to the nearest degree.

44. Explain why this relationship, $\tan \theta = m$, holds true.

45. ***Writing in Math*** Use the information on page 806 to explain how inverse trigonometric functions are used in road design. Include a few sentences describing how to determine the banking angle for a road and a description of what would have to be done to a road if the speed limit were increased and the banking angle was not changed.

STANDARDIZED TEST PRACTICE

46. **ACT/SAT** To the nearest degree, what is the angle of depression θ between the shallow end and the deep end of the swimming pool?



Side View of Swimming Pool

A 25°
B 37°
C 53°
D 73°

47. **REVIEW** If $\sin \theta = 23$ and $-90° \leq \theta \leq 90°$, then $\cos (2\theta) =$

F $-\frac{1}{9}$.
G $-\frac{1}{3}$.
H $\frac{1}{3}$.
J $\frac{1}{9}$.

Spiral Review

Find the exact value of each function. (Lesson 13-6)

48. $\sin -660°$ **49.** $\cos 25\pi$ **50.** $(\sin 135°)^2 + (\cos -675°)^2$

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

51. $a = 3.1, b = 5.8, A = 30°$ **52.** $a = 9, b = 40, c = 41$

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function. (Lesson 6-7)

53. $f(x) = 5x^2 + 6x - 17$ **54.** $f(x) = -3x^2 + 2x - 1$ **55.** $f(x) = 4x^2 - 10x + 5$

56. **PHYSICS** A toy rocket is fired upward from the top of a 200-foot tower at a velocity of 80 feet per second. The height of the rocket t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$. Find the time at which the rocket reaches its maximum height of 300 feet. (Lesson 5-7)

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CHAPTER 13

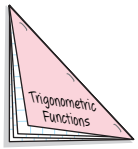
Study Guide and Review

Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Right Triangle Trigonometry (Lesson 13-1)

- $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$, $\csc\theta = \frac{\text{hyp}}{\text{opp}}$, $\sec\theta = \frac{\text{hyp}}{\text{adj}}$, $\cot\theta = \frac{\text{adj}}{\text{opp}}$

Angles and Angle Measure (Lesson 13-2)

- An angle in standard position has its vertex at the origin and its initial side along the positive x-axis.
- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side.

Trigonometric Functions of General Angles (Lesson 13-3)

- You can find the exact values of the six trigonometric functions of θ, given the coordinates of a point $P(x, y)$ on the terminal side of the angle.

Law of Sines and Law of Cosines (Lesson 13-4 and 13-5)

- $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Circular and Inverse Trigonometric Functions (Lesson 13-6 and 13-7)

- If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos\theta = x$ and $\sin\theta = y$.
- $y = \text{Sin}\, x$ if $y = \sin x$ and $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Key Vocabulary

- angle of depression (p. 764)
- angle of elevation (p. 764)
- arccosine function (p. 807)
- arcsine function (p. 807)
- arctangent function (p. 807)
- circular function (p. 800)
- cosecant (p. 759)
- cosine (p. 759)
- cotangent (p. 759)
- coterminal angles (p. 771)
- initial side (p. 768)
- law of cosines (p. 793)
- law of sines (p. 786)
- period (p. 801)
- periodic (p. 801)
- principal values (p. 806)
- quadrantal angles (p. 777)
- radian (p. 769)
- reference angle (p. 778)
- secant (p. 759)
- sine (p. 759)
- solve a right triangle (p. 762)
- standard position (p. 768)
- tangent (p. 759)
- terminal side (p. 768)
- trigonometric functions (p. 759)
- trigonometry (p. 759)
- unit circle (p. 769)

Vocabulary Check

State whether each sentence is *true* or *false*. If false, replace the underlined word(s) or number to make a true sentence.

1. When two angles in standard position have the same terminal side, they are called quadrantal angles. false; coterminal
2. The Law of Sines is used to solve a triangle when the measure of two angles and the measure of any side are known. true
3. Trigonometric functions can be defined by using a unit circle. true
4. For all values of θ, $\csc\theta = \frac{1}{\cos\theta}$. false; $\sec\theta$
5. A radian is the measure of an angle on the unit circle where the rays of the angle intercept an arc with length 1 unit. true
6. In a coordinate plane, the initial side of an angle is the ray that rotates about the center. false; terminal

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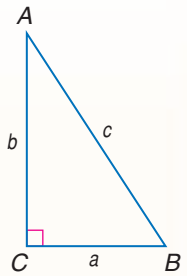
Lesson-by-Lesson Review

13-1 Right Triangle Trigonometry (pp. 759–767)

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



7. $c = 16, a = 7$

8. $A = 25°, c = 6$

9. $B = 45°, c = 12$

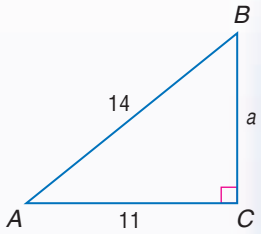
10. $B = 83°, b = \sqrt{31}$

11. $a = 9, B = 49°$

12. $\cos A = \frac{1}{4}, a = 4$

13. SKATEBOARDING A skateboarding ramp has an angle of elevation of 15.7°. Its vertical drop is 159 feet. Estimate the length of this ramp.

Example 1 Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



Find a.
$$a^2 + b^2 = c^2$$
$$a^2 + 11^2 = 14^2$$
$$a = \sqrt{14^2 - 11^2}$$
$$a \approx 8.7$$

Find A. $\cos A = \frac{11}{14}$

Use a calculator.

To the nearest degree $A \approx 38°$.

Find B. $38° + B \approx 90°$

$B \approx 52°$

Therefore, $a \approx 8.7$, $A \approx 38°$, and $B \approx 52°$.

13-2 Angles and Angle Measure (pp. 768–774)

Rewrite each degree measure in radians and each radian measure in degrees.

14. 255° **15.** −210°

16. $\frac{7\pi}{4}$ **17.** -4π

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

18. 205° **19.** −40°

20. $\frac{4\pi}{3}$ **21.** $-\frac{7\pi}{4}$

22. BICYCLING A bicycle tire has a 12-inch radius. When riding at a speed of 18 miles per hour, determine the measure of the angle through which a point on the wheel travels every second. Round to both the nearest degree and nearest radian.

Example 2 Rewrite the degree measure in radians and the radian measure in degrees.

a. 240°

$$240° = 240°\left(\frac{\pi \text{ radians}}{180°}\right)$$
$$= \frac{240\pi}{180} \text{ radians or } \frac{4\pi}{3}$$

b. $\frac{\pi}{12}$

$$\frac{\pi}{12} = \left(\frac{\pi}{12} \text{ radians}\right)\left(\frac{180°}{\pi \text{ radians}}\right)$$
$$= \frac{180°}{12} \text{ or } 15°$$

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CHAPTER 13

Study Guide and Review

13-3 Trigonometric Functions of General Angles (pp. 776–783)

Find the exact value of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

23. $P(2, 5)$ **24.** $P(15, -8)$

Find the exact value of each trigonometric function.

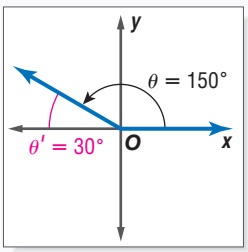
25. $\cos 3\pi$ **26.** $\tan 120°$

27. BASEBALL The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground. If the ball was hit with an initial velocity of 60 feet per second at an angle of 25°, how far was it hit?

about 86.2 ft

Example 3 **Find the exact value of $\cos 150°$.**

Because the terminal side of 150° lies in Quadrant II, the reference angle θ' is $180° - 150°$ or 30°. The cosine function is negative in Quadrant II, so $\cos 150° = -\cos 30°$ or $-\frac{\sqrt{3}}{2}$.



13-4 Law of Sines (pp. 785–792)

Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

28. $a = 24, b = 36, A = 64°$

29. $A = 40°, b = 10, a = 8$

30. $b = 10, c = 15, C = 66°$

31. $A = 82°, a = 9, b = 12$

32. $A = 105°, a = 18, b = 14$

33. NAVIGATION Two fishing boats, A, and B, are anchored 4500 feet apart in open water. A plane flies at a constant speed in a straight path directly over the two boats, maintaining a constant altitude. At one point duing the flight, the angle of depression to A is 85°, and the angle of depression to B is 25°. Ten seconds later the plane has passed over A and spots B at a 35° angle of depression. How fast is the plane flying?

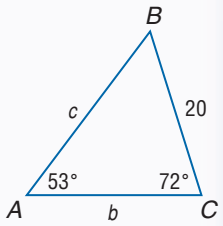
317.7 mph

Example 4 **Solve $\triangle ABC$.**

First, find the measure of the third angle.

$53° + 72° + B = 180°$

$B = 55°$



Now use the law of Sines to find b and c. Write two equations, each with one variable.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 53°}{20} = \frac{\sin 72°}{c} \qquad \frac{\sin 55°}{b} = \frac{20 \sin 53°}{20}$$

$$c = \frac{20 \sin 72°}{\sin 53°} \qquad b = \frac{20 \sin 55°}{\sin 53°}$$

$$c \approx 23.8 \qquad b \approx 20.5$$

Therefore, $B = 55°$, $b \approx 20.5$, and $c \approx 23.8$.

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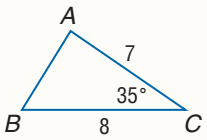
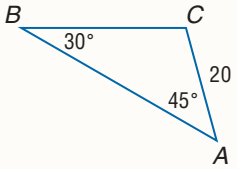
13-5 Law of Cosines (pp. 793–798)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

34.



35.



36. $C = 65°$, $a = 4$, $b = 7$

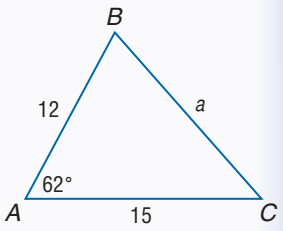
37. $A = 36°$, $a = 6$, $b = 8$

38. $b = 7.6$, $c = 14.1$, $A = 29°$

39. SURVEYING Two sides of a triangular plot of land have lengths of 320 feet and 455 feet. The measure of the angle between those sides is 54.3°. Find the perimeter of the plot.

Example 5 **$\triangle ABC$ for $A = 62°$, $b = 15$, and $c = 12$.**

You are given the measure of two sides and the included angle. Begin by drawing a diagram and using the Law of Cosines to determine a.



$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 15^2 + 12^2 - 2(15)(12) \cos 62°$

$a^2 \approx 200$

$a \approx 14.1$

Next, you can use the Law of Sines to find the measure of angle C.

$$\frac{\sin 62°}{14.1} \approx \frac{\sin C}{12}$$

$$\sin C \approx \frac{12 \sin 62°}{14.1} \text{ or about } 48.7°$$

The measure of the angle B is approximately $180 - (62 + 48.7)$ or $69.3°$. Therefore, $a \approx 14.1$, $C \approx 48.7°$, $B \approx 69.3°$.

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CHAPTER 13

Study Guide and Review

13-6 Circular Functions (pp. 799–805)

Find the exact value of each function.

40. $\sin(-150°)$

41. $\cos 300°$

42. $(\sin 45°)(\sin 225°)$

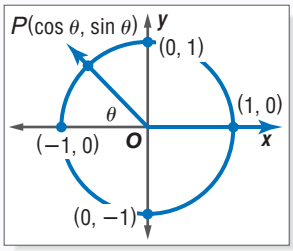
43. $\sin \frac{5\pi}{4}$

44. $(\sin 30°)^2 + (\cos 30°)^2$

45. $\frac{4\cos 150° + 2\sin 300°}{3}$

46. FERRIS WHEELS A Ferris wheel with a diameter of 100 feet completes 2.5 revolutions per minute. What is the period of the function that describes the height of a seat on the outside edge of the Ferris wheel as a function of time?

Example 6 Find the exact value of $\cos\left(-\frac{7\pi}{4}\right)$.



$$\cos -\frac{7\pi}{4} = \cos\left(-\frac{7\pi}{4} + 2\pi\right)$$
$$= \cos \frac{\pi}{4} \text{ or } \frac{\sqrt{2}}{2}$$

13-7 Inverse Trigonometric Functions (pp. 806–811)

Find each value. Write angle measures in radians. Round to the nearest hundredth.

47. $\text{Sin}^{-1}(-1)$

48. $\text{Tan}^{-1}\sqrt{3}$

49. $\tan\left(\text{Arcsin}\frac{3}{5}\right)$

50. $\cos(\text{Sin}^{-1} 1)$

51. FLYWHEELS The equation $y = \text{Arctan } 1$ describes the counterclockwise angle through which a flywheel rotates in 1 millisecond. Through how many degrees has the flywheel rotated after 25 milliseconds?

Example 7 Find the value of $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]$ in radians. Round to the nearest hundredth.

KEYSTROKES: 2nd [COS^{-1}] TAN (–) 2nd [π] ÷ 6))
ENTER 2.186276035

Therefore, $\text{Cos}^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right] \approx 2.19$ radians.

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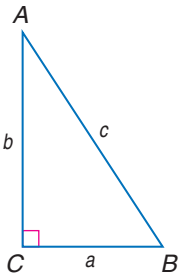
CHAPTER 13

Practice Test

Solve $\triangle ABC$ by using the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.



1. $a = 7, A = 49°$

2. $B = 75°, b = 6$

3. $A = 22°, c = 8$

4. $a = 7, c = 16$

Rewrite each degree measure in radians and each radian measure in degrees.

5. $275°$

6. $-\frac{\pi}{6}$

7. $\frac{11\pi}{2}$

8. $330°$

9. $-600°$

10. $-\frac{7\pi}{4}$

Find the exact value of each expression. Write angle measures in degrees.

11. $\cos(-120°)$

12. $\sin\frac{7\pi}{4}$

13. $\cot 300°$

14. $\sec\left(-\frac{7\pi}{6}\right)$

15. $\text{Sin}^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

16. Arctan 1

17. $\tan 135°$

18. $\csc\frac{5\pi}{6}$

19. Determine the number of possible solutions for a triangle in which $A = 40°$, $b = 10$, and $a = 14$. If a solution exists, solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

20. Determine whether $\triangle ABC$, with $A = 22°$, $a = 15$, and $b = 18$, has *no* solution, *one* solution, or *two* solutions. Then solve the triangle, if possible. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

21. Suppose θ is an angle in standard position whose terminal side lies in Quadrant II. Find the exact values of the remaining five trigonometric functions for θ for $\cos\theta = -\frac{\sqrt{3}}{2}$.

22. GEOLOGY From the top of the cliff, a geologist spots a dry riverbed. The measurement of the angle of depression to the riverbed is 70°. The cliff is 50 meters high. How far is the riverbed from the base of the cliff?

23. MULTIPLE CHOICE Triangle ABC has a right angle at C, angle $B = 30°$, and $BC = 6$. Find the area of triangle ABC.

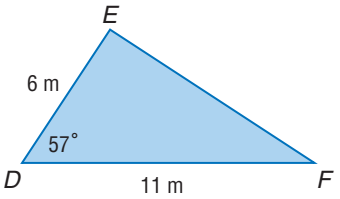
A 6 units^2

B $\sqrt{3}$ units^2

C $6\sqrt{3}$ units^2

D 12 units^2

24. Find the area of $\triangle DEF$ to the nearest tenth.



25. Determine whether $\triangle ABC$, with $b = 11$, $c = 14$, and $A = 78°$, should be solved by beginning with the Law of Sines or Law of Cosines. Then solve the triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

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CHAPTER 13

Standardized Test Practice

Cumulative, Chapters 1–13

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. If $3n + k = 30$ and n is a positive even integer, then which of the following statements must be true?

I. k is divisible by 3.
II. k is an even integer.
III. k is less than 20.

A I only

B II only

C I and II only

D I, II, and III

2. If $4x^2 + 5x = 80$ and $4x^2 - 5y = 30$, then what is the value of $6x + 6y$?

F 10

G 50

H 60

J 110

3. If $a = b + cb$, then what does $\frac{b}{a}$ equal in terms of c?

A $\frac{1}{c}$

B $\frac{1}{1+c}$

C $1 - c$

D $1 + c$

4. GRIDDABLE What is the value of $\sum_{n=1}^{5} 3n^2$?

5. GRIDDABLE When six consecutive integers are multiplied, their product is 0. What is their greatest possible sum?

6. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4

G 6

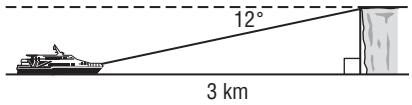
H 8

J 12

TEST-TAKING TIP

Question 6 The answer choices for multiple-choice questions can provide clues to help you solve a problem. In Question 6, you can add the values in the answer choices to the number of yellow marbles and the total number of marbles to find which is the correct answer.

7. From a lookout point on a cliff above a lake, the angle of depression to a boat on the water is 12°. The boat is 3 kilometers from the shore just below the cliff. What is the height of the cliff from the surface of the water to the lookout point?



A $\frac{3}{\sin 12°}$

B $\frac{3}{\tan 12°}$

C $\frac{3}{\cos 12°}$

D $3 \tan 12°$

8. If $x + y = 90°$ and x and y are positive, then $\frac{\cos x}{\sin y} =$

F 0.

G $\frac{1}{2}$.

H 1.

J cannot be determined

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9. A child flying a kite holds the string 4 feet above the ground. The taut string is 40 feet long and makes an angle of 35° with the horizontal. How high is the kite off the ground?

A $4 + 40 \sin 35°$

B $4 + 40 \cos 35°$

C $4 + 40 \tan 35°$

D $4 + \frac{40}{\sin 35°}$

10. If $\sin \theta = -\frac{1}{2}$ and $180° < \theta < 270°$, then $\theta =$

F 200°.

G 210°.

H 225°.

J 240°.

11. If $\cos \theta = \frac{8}{17}$ and the terminal side of the angle is in quadrant IV, then $\sin \theta =$

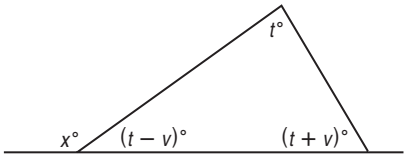
A $-\frac{15}{8}$.

B $-\frac{17}{15}$.

C $-\frac{15}{17}$.

D $-\frac{15}{17}$.

12. GRIDDABLE In the figure, if $t = 2v$, what is the value of x?



13. The variables a, b, c, d, and e are integers in a sequence, where $a = 2$ and $b = 12$. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, $c = 25$, because $2(12) = 24$, $2 - 1 = 1$, and $24 + 1 = 25$. What is the value of e?

F 74

G 144

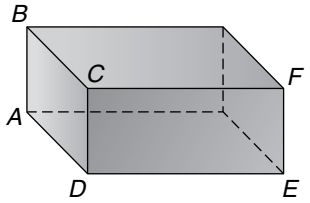
H 146

J 256

Pre-AP

Record your answers on a sheet of paper. Show your work.

14. GEOMETRY The length, width, and height of the rectangular box illustrated below are each integers greater than 1. If the area of $ABCD$ is 18 square units and the area of $CDEF$ is 21 square units, what is the volume of the box?



NEED EXTRA HELP?														
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Go to Lesson...	1-2	5-3	1-3	11-4	Prior Course	12-3	13-1	13-2	13-1	13-3	13-2	11-1	1-3	Prior Course

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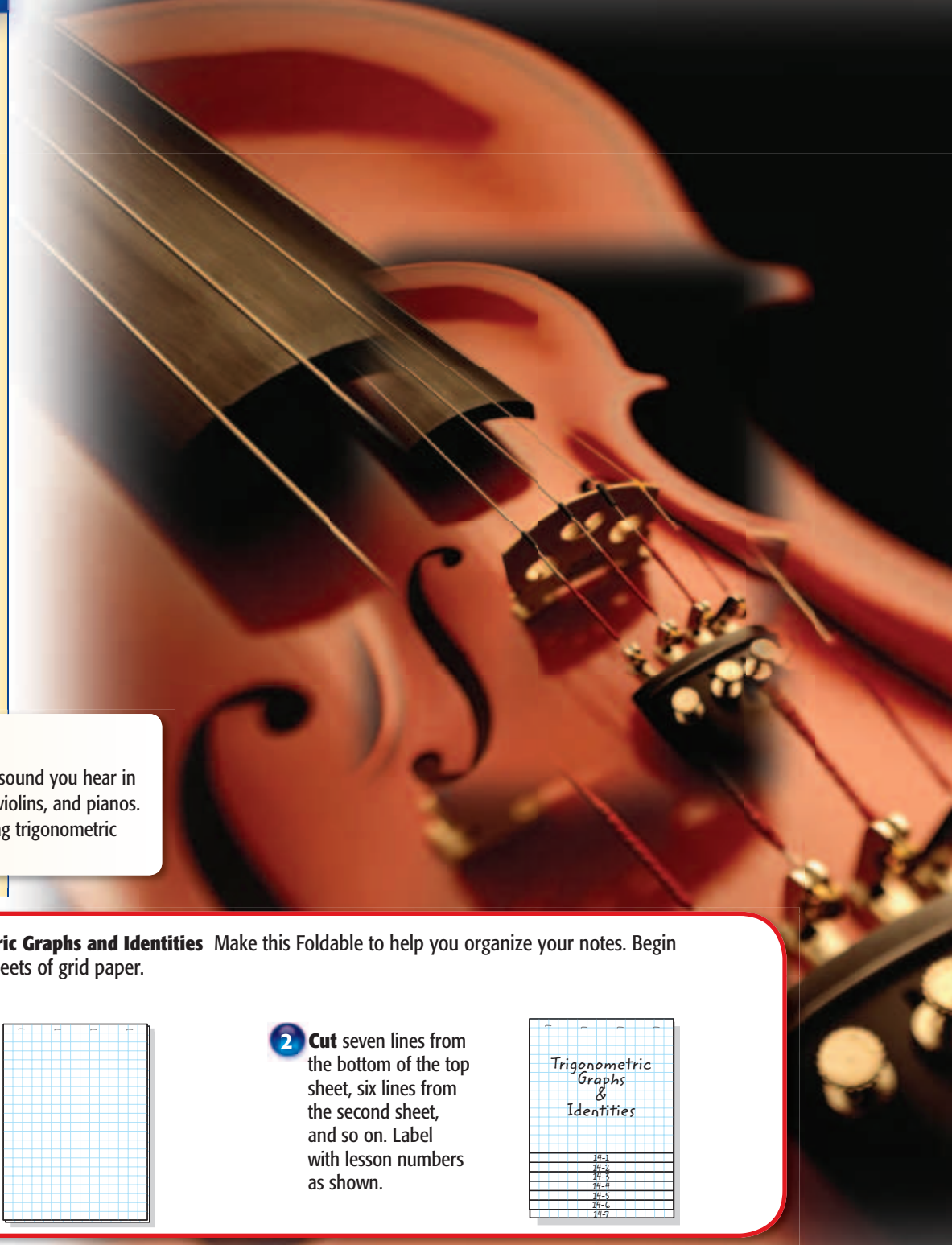
CHAPTER 14

Trigonometric Graphs and Identities

BIG Ideas

- Graph trigonometric functions and determine period, amplitude, phase shifts, and vertical shifts.
- Use and verify trigonometric identities.
- Solve trigonometric equations.

Key Vocabulary

amplitude (p. 823)
phase shift (p. 829)
vertical shift (p. 831)
trigonometric identity (p. 837)
trigonometric equation (p. 861)

Real-World Link

Music String vibrations produce the sound you hear in stringed instruments such as guitars, violins, and pianos. These vibrations can be modeled using trigonometric functions.

FOLDABLES Study Organizer

Trigonometric Graphs and Identities Make this Foldable to help you organize your notes. Begin with eight sheets of grid paper.

1. **Staple** the stack of grid paper along the top to form a booklet.

2. **Cut** seven lines from the bottom of the top sheet, six lines from the second sheet, and so on. Label with lesson numbers as shown.

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GET READY for Chapter 14

Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2

MathOnline Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

QUICKCheck

Find the exact value of each trigonometric function. (Lesson 13-3)

1. $\sin 135°$ **2.** $\tan 315°$ **3.** $\cos 90°$

4. $\tan 45°$ **5.** $\sin \frac{5\pi}{4}$ **6.** $\cos \frac{7\pi}{6}$

7. $\cos(-150°)$ **8.** $\cot \frac{9\pi}{4}$ **9.** $\sec \frac{13\pi}{6}$

10. $\tan\left(-\frac{3\pi}{2}\right)$ **11.** $\tan \frac{8\pi}{3}$ **12.** $\csc(-720°)$

13. AMUSEMENT The distance from the highest point of a Ferris wheel to the ground can be found by multiplying 60 ft by $\sin 90°$. What is the height of the Ferris wheel at the highest point? (Lesson 13-3)

Factor completely. If the polynomial is not factorable, write *prime*. (Lesson 6-6)

14. $-15x^2 - 5x$ **15.** $2x^4 - 4x^2$

16. $x^3 + 4$ **17.** $2x^2 - 3x - 2$

18. PARKS The rectangular wooded area of a park covers $x^2 - 6x + 8$ square feet of land. If the area is $(x - 2)$ feet long, what is the width? (Lesson 6-3)

Solve each equation by factoring. (Lesson 5-3)

19. $x^2 - 5x - 24 = 0$ **20.** $x^2 - 2x - 48 = 0$

21. $x^2 - 12x = 0$ **22.** $x^2 - 16 = 0$

23. HOME IMPROVEMENT You are putting new flooring in your laundry room, which is 40 square feet. The expression $x^2 + 3x$ can be used to represent the product of the length and the width of the room. Find the possible values for x. (Lesson 5-3)

QUICKReview

Example 1 **Find the exact value of $\sin \frac{11\pi}{6}$.**

The terminal side of $\frac{11\pi}{6}$ lies in Quadrant IV, so the reference angle θ is $2\pi - \frac{11\pi}{6}$ or $\frac{\pi}{6}$. The sine function is negative in the Quadrant IV.

$$\sin \frac{11\pi}{6} = -\sin \frac{\pi}{6}$$

$$= -\sin 30° \quad \frac{\pi}{6} \text{ radians} = 30°$$

$$= -\frac{1}{2} \quad \sin 30° = \frac{1}{2}$$

Example 2 **Factor $x^3 - 4x^2 - 21x$ completely.**

$$x^3 - 4x^2 - 21x = x(x^2 - 4x - 21)$$

The product of the coefficients of the x-terms must be -21, and their sum must be -4. The product of 7 and 3 is 21 and their difference is 4. Since the sum must be negative, the coefficients of the x-terms are -7 and 3.

$$x(x^2 - 4x - 21) = x(x - 7)(x + 3)$$

Example 3 **Solve the equation factored in Example 2.**

From Example 2,

$$x^3 - 4x^2 - 21x = x(x - 7)(x + 3).$$

Apply the Zero Product Property and solve.

$$x = 0 \quad \text{or} \quad x - 7 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 7 \qquad\qquad x = -3$$

The solution set is $\{-3, 0, 7\}$.

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14-1 Graphing Trigonometric Functions

Main Ideas

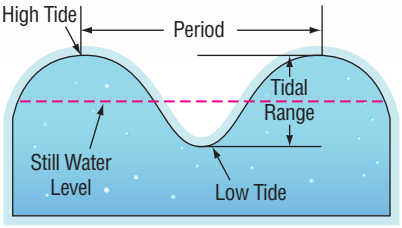
- Graph trigonometric functions.
- Find the amplitude and period of variation of the sine, cosine, and tangent functions.

New Vocabulary

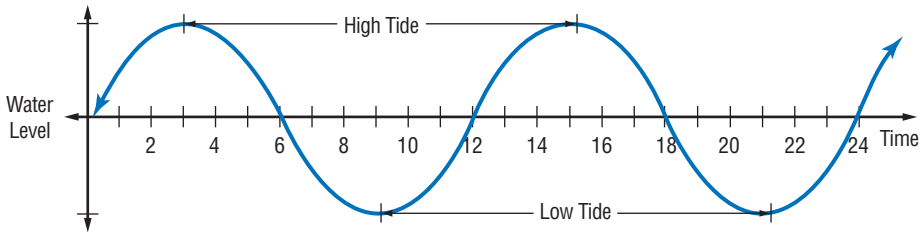
amplitude

GET READY for the Lesson

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.



Graph Trigonometric Functions The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides.



Review Vocabulary

Period The least possible value of a for which $f(x) = f(x + a)$.

In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is said to be *periodic*.

To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

To graph the functions $y = \sin \theta$, $y = \cos \theta$, or $y = \tan \theta$, use values of θ expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form $(\theta, \sin \theta)$, $(\theta, \cos \theta)$, and $(\theta, \tan \theta)$, respectively.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
nearest tenth	0	0.5	0.7	0.9	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
nearest tenth	1	0.9	0.7	0.5	0	−0.5	−0.7	−0.9	−1	−0.9	−0.7	−0.5	0	0.5	0.7	0.9	1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	nd	$-\sqrt{3}$	−1	$-\frac{\sqrt{3}}{3}$	0
nearest tenth	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0	0.6	1	1.7	nd	−1.7	−1	−0.6	0
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

nd = not defined

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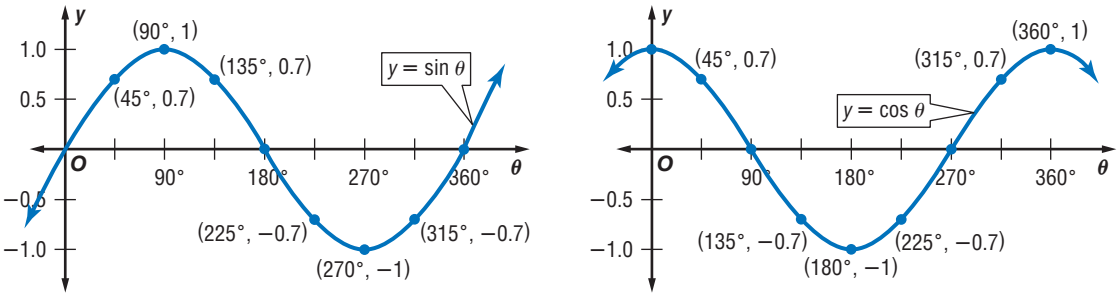
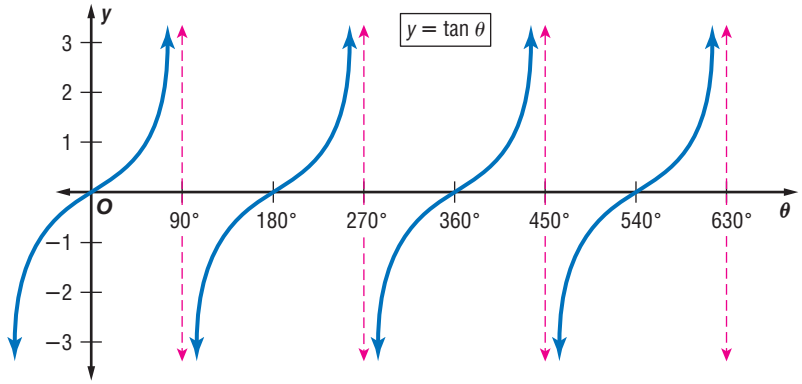
After plotting several points, complete the graphs of $y = \sin \theta$ and $y = \cos \theta$ by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or 2π radians. That is, the graph of each function repeats itself every 360° or 2π radians.



Notice that both the sine and cosine have a maximum value of 1 and a minimum value of −1. The **amplitude** of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is $\left|\frac{1-(-1)}{2}\right|$ or 1.

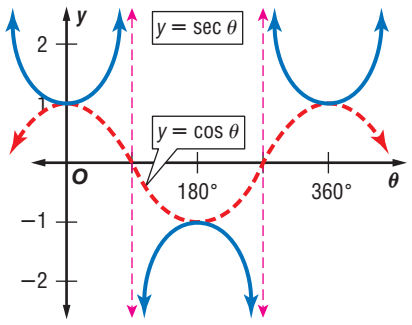
Concepts in Motion
Animation
algebra2.com

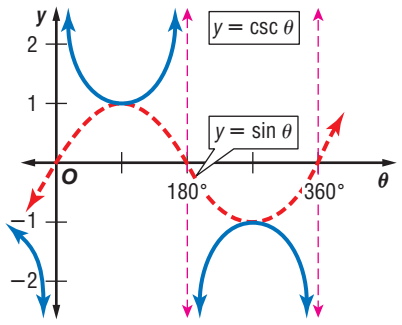
By examining the values for tan θ in the table, you can see that the tangent function is not defined for 90°, 270°, …, 90° + k • 180°, where k is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which $y = \tan \theta$ is not defined.

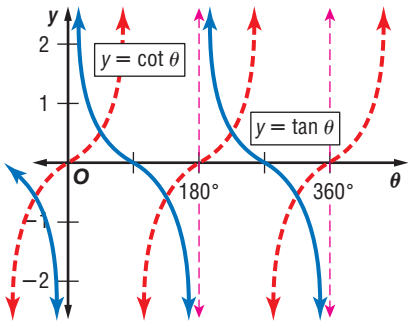


The period of the tangent function is 180° or π radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

Compare the graphs of the secant, cosecant, and cotangent functions to the graphs of the cosine, sine, and tangent functions, shown below.







Notice that the period of the secant and cosecant functions is 360° or 2π radians. The period of the cotangent is 180° or π radians. Since none of these functions have a maximum or minimum value, they have no amplitude.

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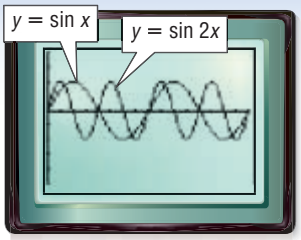
Variations of Trigonometric Functions Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

GRAPHING CALCULATOR LAB

Period and Amplitude

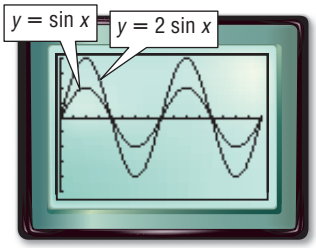
On a TI-83/84 Plus, set the MODE to degrees.

THINK AND DISCUSS



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

1. Graph $y = \sin x$ and $y = \sin 2x$. What is the maximum value of each function?
2. How many times does each function reach a maximum value?
3. Graph $y = \sin\left(\frac{x}{2}\right)$. What is the maximum value of this function? How many times does this function reach its maximum value?
4. Use the equations $y = \sin bx$ and $y = \cos bx$. Repeat Exercises 1–3 for maximum values and the other values of b. What conjecture can you make about the effect of b on the maximum values and the periods of these functions?



[0, 720] scl: 45 by [−2.5, 2.5] scl: 0.5

5. Graph $y = \sin x$ and $y = 2 \sin x$. What is the maximum value of each function? What is the period of each function?
6. Graph $y = \frac{1}{2} \sin x$. What is the maximum value of this function? What is the period of this function?
7. Use the equations $y = a \sin x$ and $y = a \cos x$. Repeat Exercises 5 and 6 for other values of a. What conjecture can you make about the effect of a on the amplitudes and periods of $y = a \sin x$ and $y = a \cos x$?

Study Tip

Amplitude and Period

Note that the amplitude affects the graph along the vertical axis and the period affects it along the horizontal axis.

The results of the investigation suggest the following generalization.

KEY CONCEPT *Amplitudes and Periods*

Words For functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$.

For functions of the form $y = a \tan b$, the amplitude is not defined, and the period is $\frac{180°}{|b|}$ or $\frac{\pi}{|b|}$.

Examples

$y = 3 \sin 4\theta$	amplitude 3 and period $\frac{360°}{4}$ or 90°
$y = -6 \cos 5\theta$	amplitude $\lvert -6 \rvert$ or 6 and period $\frac{2\pi}{5}$
$y = 2 \tan \frac{1}{3}\theta$	no amplitude and period 3π

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You can use the amplitude and period of a trigonometric function to help you graph the function.

EXAMPLE Graph Trigonometric Functions

1 **Find the amplitude, if it exists, and period of each function. Then graph the function.**

a. $y = \cos 3\theta$

First, find the amplitude.

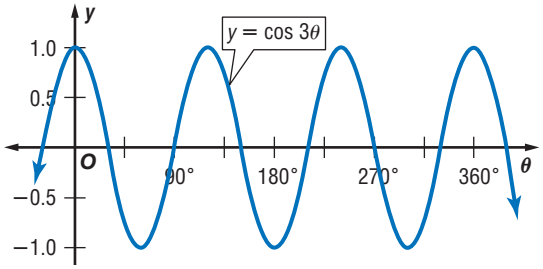
$|a| = |1|$ The coefficient of $\cos 3\theta$ is 1.

Next, find the period.

$\frac{360°}{|b|} = \frac{360°}{|3|}$ $b = 3$

$= 120°$

Use the amplitude and period to graph the function.

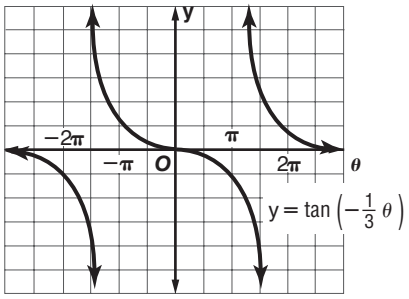


b. $y = \tan\left(-\frac{1}{3}\theta\right)$

Amplitude: This function does not have an amplitude because it has no maximum or minimum value.

Period: $\frac{\pi}{|b|} = \frac{\pi}{\left|-\frac{1}{3}\right|}$

$= 3\pi$



Study Tip

Amplitude

Notice that the graph of the longest function has no amplitude, because the tangent function has no minimum or maximum value.

CHECK Your Progress

1A. $y = \frac{1}{4}\sin\theta$

1B. $y = -2\sec\left(\frac{1}{4}\theta\right)$

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Real-World EXAMPLE Use Trigonometric Functions

2 OCEANOGRAPHY **Refer to the application at the beginning of the lesson. Suppose the tidal range of a city on the Atlantic coast is 18 feet. A tide is at *equilibrium* when it is at its normal level, halfway between its highest and lowest points. Write a function to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$ and that the high tide is beginning. Then graph the function.**

Real-World Link

Lake Superior has one of the smallest tidal ranges. It can be measured in inches, while the tidal range in the Bay of Fundy in Canada measures up to 50 feet.

Source: Office of Naval Research

Since the height of the tide is 0 at $t = 0$, use the sine function $h = a \sin bt$, where a is the amplitude of the tide and t is time in hours.

Find the amplitude. The difference between high tide and low tide is the tidal range or 18 feet.

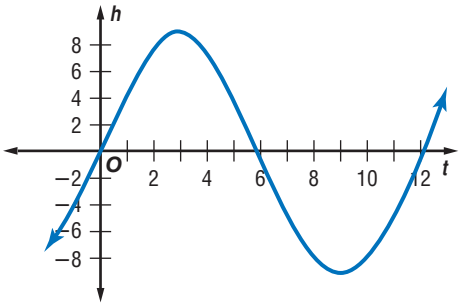
$a = \frac{18}{2}$ or 9

Find the value of b. Each tide cycle lasts about 12 hours.

$\frac{2\pi}{|b|} = 12$ period $= \frac{2\pi}{|b|}$

$b = \frac{2\pi}{12}$ or $\frac{\pi}{6}$ Solve for b.

Thus, an equation to represent the height of the tide is $h = 9 \sin \frac{\pi}{6}t$.



CHECK Your Progress

2A. Assume that the tidal range is 13 feet. Write a function to represent the height h of the tide. Assume the tide is at equilibrium at $t = 0$ and that the high tide is beginning.

2B. Graph the tide function.

Online Personal Tutor at algebra2.com

CHECK Your Understanding

Example 1 (p. 825)

Find the amplitude, if it exists, and period of each function. Then graph each function.

1. $y = \frac{1}{2} \sin \theta$ **2.** $y = 2 \sin \theta$ **3.** $y = \frac{2}{3} \cos \theta$

4. $y = \frac{1}{4} \tan \theta$ **5.** $y = \csc 2\theta$ **6.** $y = 4 \sin 2\theta$

Example 2 (p. 826)

7. $y = 4 \cos \frac{3}{4}\theta$ **8.** $y = \frac{1}{2} \sec 3\theta$ **9.** $y = \frac{3}{4} \cos \frac{1}{2}\theta$

BIOLOGY **For Exercises 10 and 11, use the following information.**

In a certain wildlife refuge, the population of field mice can be modeled by $y = 3000 + 1250 \sin \frac{\pi}{6}t$, where y represents the number of mice and t represents the number of months past March 1 of a given year.

10. Determine the period of the function. What does this period represent?

11. What is the maximum number of mice, and when does this occur?

Larry Hamill

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Exercises

HOMEWORK HELP

For Exercises	See Examples
12–23	1
24–26	2

Find the amplitude, if it exists, and period of each function. Then graph each function.

12. $y = 3 \sin \theta$ **13.** $y = 5 \cos \theta$ **14.** $y = 2 \csc \theta$

15. $y = 2 \tan \theta$ **16.** $y = \frac{1}{5} \sin \theta$ **17.** $y = \frac{1}{3} \sec \theta$

18. $y = \sin 4\theta$ **19.** $y = \sin 2\theta$ **20.** $y = \sec 3\theta$

21. $y = \cot 5\theta$ **22.** $y = 4 \tan \frac{1}{3}\theta$ **23.** $y = 2 \cot \frac{1}{2}\theta$

MEDICINE **For Exercises 24 and 25, use the following information.**
Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

24. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

25. How do the periods of the tuning forks compare?

Find the amplitude, if it exists, and period of each function. Then graph each function.

26. $y = 6 \sin \frac{2}{3}\theta$ **27.** $y = 3 \cos \frac{1}{2}\theta$ **28.** $y = 3 \csc \frac{1}{2}\theta$

29. $y = \frac{1}{2} \cot 2\theta$ **30.** $2y = \tan \theta$ **31.** $\frac{3}{4}y = \frac{2}{3} \sin \frac{3}{5}\theta$

32. Draw a graph of a sine function with an amplitude $\frac{3}{5}$ and a period of 90°. Then write an equation for the function.

33. Draw a graph of a cosine function with an amplitude of $\frac{7}{8}$ and a period of $\frac{2\pi}{5}$. Then write an equation for the function.

34. Graph the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is measured in radians, for x between 0 and 2π. Identify the points of intersection of the two graphs.

35. Identify all asymptotes to the graph of $g(x) = \sec x$.

BOATING **For Exercises 36–38, use the following information.**
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

36. Write an equation for the motion of the buoy. Assume that it is at equilibrium at $t = 0$ and that it is on the way up from the normal water level.

37. Draw a graph showing the height of the buoy as a function of time.

38. What is the height of the buoy after 12 seconds?

EXTRA PRACTICE
See pages 922, 939.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. **OPEN ENDED** Write a trigonometric function that has an amplitude of 3 and a period of π. Graph the function.

40. **REASONING** Explain what it means to say that the period of a function is 180°.

41. **CHALLENGE** A function is called *even* if the graphs of $y = f(x)$ and $y = f(-x)$ are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.

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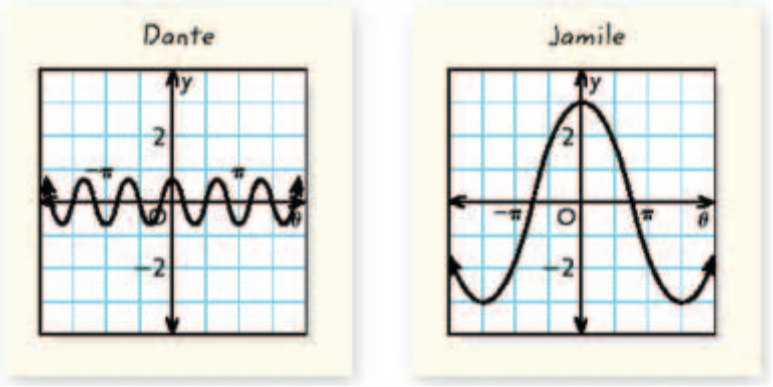
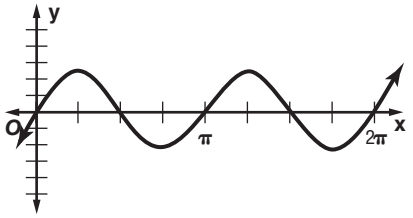
42. FIND THE ERROR Dante and Jamile graphed $y = 3 \cos \frac{2}{3}\theta$. Who is correct? Explain your reasoning.



43. *Writing in Math* Use the information on page 822 to explain how you can predict the behavior of tides. Explain why certain tidal characteristics follow the patterns seen in the graph of the sine function.

STANDARDIZED TEST PRACTICE

44. ACT/SAT Identify the equation of the graphed function.



A $y = \frac{1}{2} \sin 4\theta$

B $y = \frac{1}{4} \sin 2\theta$

C $y = 2 \sin \frac{1}{4}\theta$

D $y = 4 \sin \frac{1}{2}\theta$

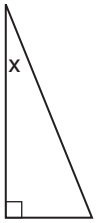
45. REVIEW Refer to the figure below. If $\tan x = \frac{10}{24}$, what are $\sin x$ and $\cos x$?

F $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$

G $\sin x = \frac{10}{26}$ and $\cos x = \frac{24}{26}$

H $\sin x = \frac{26}{10}$ and $\cos x = \frac{26}{24}$

J $\sin x = \frac{26}{10}$ and $\cos x = \frac{24}{26}$



Spiral Review

Solve each equation. (Lesson 13-7)

46. $x = \text{Sin}^{-1} 1$

47. $\text{Arcsin}(-1) = y$

48. $\text{Arccos} \frac{\sqrt{2}}{2} = x$

Find the exact value of each function. (Lesson 13-6)

49. sin 390°

50. sin (−315°)

51. cos 405°

52. PROBABILITY There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. (Lesson 12-5)

53. Find the first five terms of the sequence in which $a_1 = 3$, $a_{n+1} = 2a_n + 5$. (Lesson 11-5)

GET READY for the Next Lesson

PREREQUISITE SKILL **Graph each pair of functions on the same set of axes.** (Lesson 5-7)

54. $y = x^2, y = 3x^2$

55. $y = 3x^2, y = 3x^2 - 4$

56. $y = 2x^2, y = 2(x + 1)^2$

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14-2 Translations of Trigonometric Graphs

Main Ideas

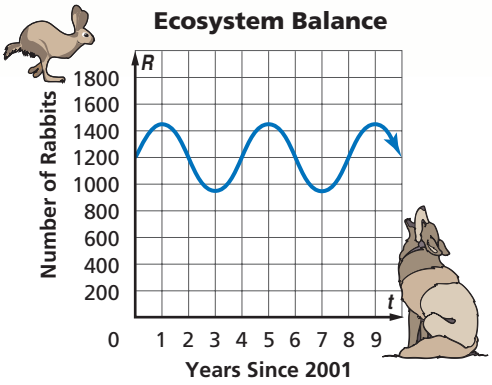
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

New Vocabulary

phase shift
vertical shift
midline

GET READY for the Lesson

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population R can be modeled by the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$, where t is the time in years since January 1, 2001.



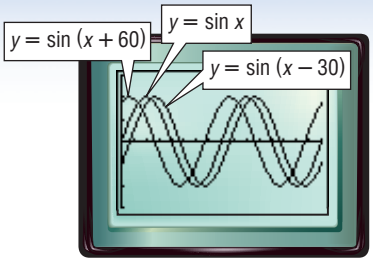
Horizontal Translations Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

GRAPHING CALCULATOR

Horizontal Translations

On a TI-83/84 Plus, set the MODE to degrees.

THINK AND DISCUSS



[0, 720] scl: 45 by [−1.5, 1.5] scl: 0.5

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of h in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
   - $y = \cos x$ and $y = \cos (x + 30)$
   - $y = \tan x$ and $y = \tan (x - 45)$
   - $y = \sec x$ and $y = \sec (x + 75)$

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If (x, y) are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \mp h)$. A horizontal translation of a trigonometric function is called a **phase shift.**

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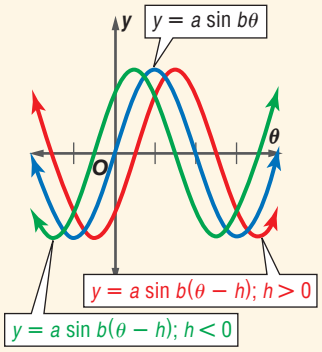
KEY CONCEPT — Phase Shift

Words The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$.

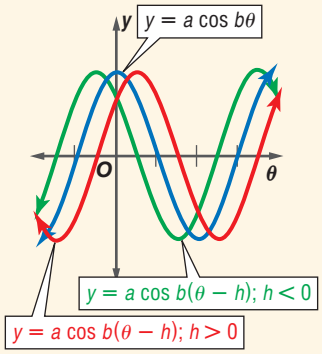
If $h > 0$, the shift is to the right.
If $h < 0$, the shift is to the left.

Models:

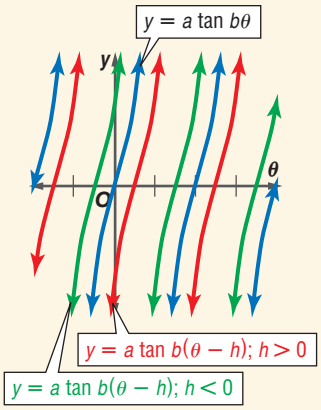
Sine



Cosine



Tangent



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The secant, cosecant, and cotangent can be graphed using the same rules.

Study Tip

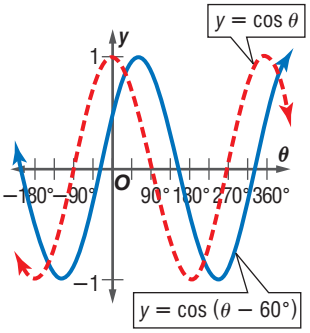
Verifying a Graph
After drawing the graph of a trigonometric function, select values of θ and evaluate them in the equation to verify your graph.

EXAMPLE — Graph Horizontal Translations

1 **State the amplitude, period, and phase shift for $y = \cos(\theta - 60°)$. Then graph the function.**

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos \theta$. However, $h = 60°$, so the phase shift is 60°. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos(\theta - 60°)$, consider the graph of $y = \cos \theta$. Graph this function and then shift the graph 60° to the right. The graph $y = \cos(\theta - 60°)$ is the graph of $y = \cos \theta$ shifted to the right.



CHECK Your Progress

1. State the amplitude, period, and phase shift for $y = 2 \sin\left(\theta + \frac{\pi}{4}\right)$. Then graph the function.

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Study Tip

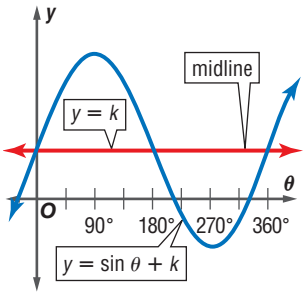
Notation

Pay close attention to trigonometric functions for the placement of parentheses. Note that $\sin(\theta + x) \neq \sin\theta + x$. The first expression represents a phase shift while the second expression represents a vertical shift.

Vertical Translations In Chapter 5, you learned that the graph of $y = x^2 + 4$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If (x, y) are coordinates of $y = \sin x$, then $(x, y \pm k)$ are coordinates of $y = \sin x \pm k$.

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of $y = \sin\theta + k$, the midline is the graph of $y = k$.



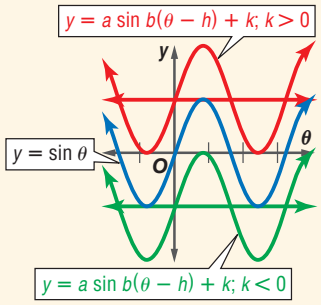
KEY CONCEPT — **Vertical Shift**

Words The vertical shift of the functions $y = a\sin b(\theta - h) + k$, $y = a\cos b(\theta - h) + k$, and $y = a\tan b(\theta - h) + k$ is k.

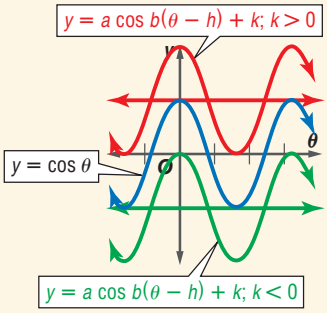
If $k > 0$, the shift is up. If $k < 0$, the shift is down. The midline is $y = k$.

Models:

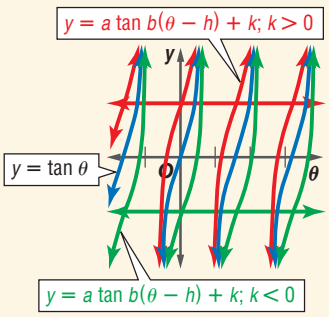
Sine



Cosine



Tangent



The secant, cosecant, and cotangent can be graphed using the same rules.

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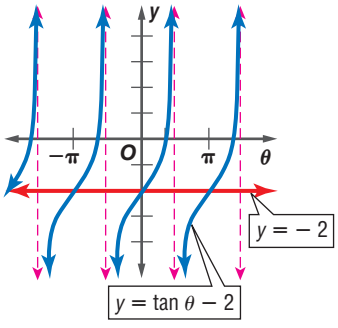
EXAMPLE Graph Vertical Translations

2 **State the vertical shift, equation of the midline, amplitude, and period for $y = \tan\theta - 2$. Then graph the function.**

Since $\tan\theta - 2 = \tan\theta + (-2)$, $k = -2$, and the vertical shift is -2. Draw the midline, $y = -2$. The tangent function has no amplitude and the period is the same as that of $\tan\theta$.

Draw the graph of the function relative to the midline.



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Study Tip

Graphing

It may be helpful to first graph the parent graph $y = \sin \theta$ in one color. Then apply the vertical shift and graph the function in another color. Then apply the change in amplitude and graph the function in the final color.

CHECK Your Progress

2. State the vertical shift, equation of the midline, amplitude, and period for $y = \frac{1}{2}\sin\theta + 1$. Then graph the function.

In general, use the following steps to graph any trigonometric function.

CONCEPT SUMMARY *Graphing Trigonometric Functions*

Step 1 Determine the vertical shift, and graph the midline.

Step 2 Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.

Step 3 Determine the period of the function and graph the appropriate function.

Step 4 Determine the phase shift and translate the graph accordingly.

EXAMPLE Graph Transformations

3 **State the vertical shift, amplitude, period, and phase shift of $y = 4\cos\left[\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)\right] - 6$. Then graph the function.**

The function is written in the form $y = a\cos[b(\theta - h)] + k$. Identify the values of k, a, b, and h.

$k = -6$, so the vertical shift is -6.

$a = 4$, so the amplitude is $|4|$ or 4.

$b = \frac{1}{2}$, so the period is $\frac{2\pi}{\left|\frac{1}{2}\right|}$ or 4π.

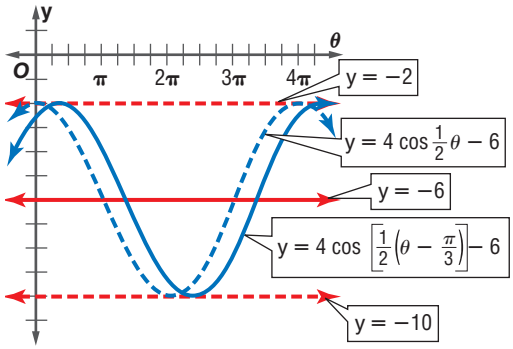
$h = \frac{\pi}{3}$, so the phase shift is $\frac{\pi}{3}$ to the right.

Step 1 The vertical shift is -6. Graph the midline $y = -6$.

Step 2 The amplitude is 4. Draw dashed lines 4 units above and below the midline at $y = -2$ and $y = -10$.

Step 3 The period is 4π, so the graph will be stretched. Graph $y = 4\cos\frac{1}{2}\theta - 6$ using the midline as a reference.

Step 4 Shift the graph $\frac{\pi}{3}$ to the right.



CHECK Your Progress **Graph each equation.**

3. State the vertical shift, amplitude, period, and phase shift of $y = 3\sin\left[\frac{1}{3}\left(\theta - \frac{\pi}{2}\right)\right] + 2$. Then graph the function.

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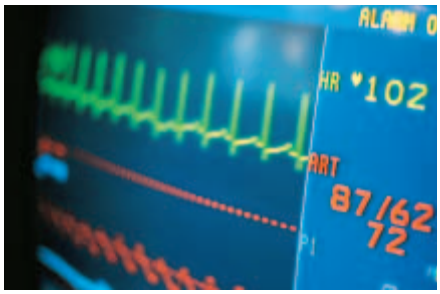
Real-World EXAMPLE

4 **HEALTH** **Suppose a person's resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person's resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure at time t seconds. Then graph the function.**



Real-World Link

Blood pressure can change from minute to minute and can be affected by the slightest of movements, such as tapping your fingers or crossing your arms.

Source: American Heart Association

Explore You know that the function is periodic and can be modeled using sine.

Plan Let P represent blood pressure and let t represent time in seconds. Use the equation $P = a \sin [b(t - h)] + k$.

Solve

- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.

  $P = \frac{120 + 80}{2}$ or 100

- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.

  $a = |120 - 100|$ $\qquad$ $a = |80 - 100|$

  $= |20|$ or 20 $\qquad$ $= |-20|$ or 20

  Thus, $a = 20$.

- Determine the period of the function and solve for b. Recall that the period of a function can be found using the expression $\frac{2\pi}{|b|}$.

  Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.

  $1 = \frac{2\pi}{|b|}$ Write an equation.

  $|b| = 2\pi$ Multiply each side by $|b|$.

  $b = \pm 2\pi$ Solve.

  For this example, let $b = 2\pi$. The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).

- There is no phase shift, so $h = 0$. So, the equation is $P = 20 \sin 2\pi t + 100$.

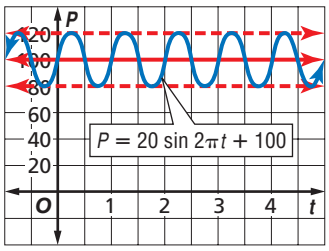
- Graph the function.

  Step 1 Draw the midline $P = 100$.

  Step 2 Draw maximum and minimum reference lines.

  Step 3 Use the period to draw the graph of the function.

  Step 4 There is no phase shift.



Check Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.

Ben Edwards/Getty Images
www.MathSchoolinternational.com

CHECK Your Progress

4. Suppose that while doing some moderate physical activity, the person's blood pressure is 130 over 90 and that the person has a heart rate of 90 beats per minute. Write a sine function that represents the person's blood pressure at time t seconds. Then graph the function.

CHECK Your Understanding

Example 1 (p. 830)

State the amplitude, period, and phase shift for each function. Then graph the function.

1. $y = \sin\left(\theta - \frac{\pi}{2}\right)$

2. $y = \tan(\theta + 60°)$

3. $y = \cos(\theta - 45°)$

4. $y = \sec\left(\theta + \frac{\pi}{3}\right)$

Example 2 (pp. 831–832)

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

5. $y = \cos\theta + \frac{1}{4}$

6. $y = \sec\theta - 5$

7. $y = \tan\theta + 4$

8. $y = \sin\theta + 0.25$

Example 3 (p. 832)

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

9. $y = 3\sin[2(\theta - 30°)] + 10$

10. $y = 2\cot(3\theta + 135°) - 6$

11. $y = \frac{1}{2}\sec\left[4\left(\theta - \frac{\pi}{4}\right)\right] + 1$

12. $y = \frac{2}{3}\cos\left[\frac{1}{2}\left(\theta + \frac{\pi}{6}\right)\right] - 2$

Example 4 (p. 833)

PHYSICS **For Exercises 13–15, use the following information.**

A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.

13. Determine the vertical shift, amplitu de, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds.

14. Write the equation for the height h of the weight above the floor as a function of time t seconds.

15. Draw a graph of the function you wrote in Exercise 14.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
16–21	1
22–27	2
28–35	3
36–38	4

State the amplitude, period, and phase shift for each function. Then graph the function.

16. $y = \cos(\theta + 90°)$

17. $y = \cot(\theta - 30°)$

18. $y = \sin\left(\theta - \frac{\pi}{4}\right)$

19. $y = \cos\left(\theta + \frac{\pi}{3}\right)$

20. $y = \frac{1}{4}\tan(\theta + 22.5°)$

21. $y = 3\sin(\theta - 75°)$

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

22. $y = \sin\theta - 1$

23. $y = \sec\theta + 2$

24. $y = \cos\theta - 5$

25. $y = \csc\theta - \frac{3}{4}$

26. $y = \frac{1}{2}\sin\theta + \frac{1}{2}$

27. $y = 6\cos\theta + 1.5$

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State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

28. $y = 2 \sin [3(\theta - 45°)] + 1$

29. $y = 4 \cos [2(\theta + 30°)] - 5$

30. $y = 3 \csc \left[\frac{1}{2}(\theta + 60°)\right] - 3.5$

31. $y = 6 \cot \left[\frac{2}{3}(\theta - 90°)\right] + 0.75$

32. $y = \frac{1}{4} \cos (2\theta - 150°) + 1$

33. $y = \frac{2}{5} \tan (6\theta + 135°) - 4$

34. $y = 3 + 2 \sin \left[\left(2\theta + \frac{\pi}{4}\right)\right]$

35. $y = 4 + 5 \sec \left[\frac{1}{3}\left(\theta + \frac{2\pi}{3}\right)\right]$

ZOOLOGY **For Exercises 36–38, use the following information.**
The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls O can be represented by $O = 150 + 30 \sin \left(\frac{\pi}{10}t\right)$ where t is the time in years since January 1, 2001. In that same system, the population of mice M can be represented by $M = 600 + 300 \sin \left(\frac{\pi}{10}t + \frac{\pi}{20}\right)$.

Real-World Link
The average weight of a male Cactus Ferruginous Pygmy-Owl is 2.2 ounces.
Source: www.kidsplanet.org

36. Find the maximum number of owls. After how many years does this occur?

37. What is the minimum number of mice? How long does it take for the population of mice to reach this level?

38. Why would the maximum owl population follow behind the population of mice?

39. Graph $y = 3 - \frac{1}{2} \cos \theta$ and $y = 3 + \frac{1}{2} \cos (\theta + \pi)$. How do the graphs compare?

40. Compare the graphs of $y = -\sin \left[\frac{1}{4}\left(\theta - \frac{\pi}{2}\right)\right]$ and $y = \cos \left[\frac{1}{4}\left(\theta + \frac{3\pi}{2}\right)\right]$.

41. Graph $y = 5 + \tan \left(\theta + \frac{\pi}{4}\right)$. Describe the transformation to the parent graph $y = \tan \theta$.

42. Draw a graph of the function $y = \frac{2}{3}\cos (\theta - 50°) + 2$. How does this graph compare to the graph of $y = \cos \theta$?

43. **MUSIC** When represented on oscilloscope, the note A above middle C has a period of $\frac{1}{440}$. Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is K.
a. $y = K \sin 220\pi t$ **b.** $y = K \sin 440\pi t$ **c.** $y = K \sin 880\pi t$

EXTRA PRACTICE
See pages 922, 939.
MathOnline
Self-Check Quiz at algebra2.com

44. **TIDES** The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day.

H.O.T. Problems

45. **OPEN ENDED** Write the equation of a trigonometric function with a phase shift of $-45°$. Then graph the function, and its parent graph.

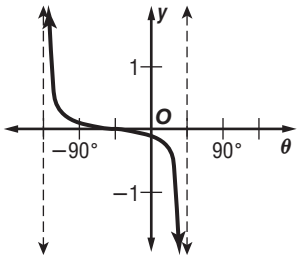
46. **CHALLENGE** The graph of $y = \cot \theta$ is a transformation of the graph of $y = \tan \theta$. Determine a, b, and h so that $\cot \theta = a \tan [b(\theta - h)]$ for all values of θ for which each function is defined.

Art Wolfe/Getty Images
www.MathSchoolinternational.com

47. ***Writing in Math*** Use the information on page 829 to explain how translations of trigonometric graphs can be used to show animal populations. Include a description of what each number in the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$ represents.

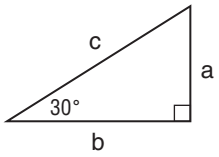
STANDARDIZED TEST PRACTICE

48. ACT/SAT Which equation is represented by the graph?



A $y = \cot(\theta + 45°)$
B $y = \cot(\theta - 45°)$
C $y = \tan(\theta + 45°)$
D $y = \tan(\theta - 45°)$

49. REVIEW Refer to the figure below. If $c = 14$, find the value of b.



F $\frac{\sqrt{3}}{2}$
G $14\sqrt{3}$
H 7
J $7\sqrt{3}$

Spiral Review

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = 3 \csc \theta$ **51.** $y = \sin \frac{\theta}{2}$ **52.** $y = 3 \tan \frac{2}{3}\theta$

Find each value. (Lesson 13-7)

53. $\sin\left(\text{Cos}^{-1} \frac{2}{3}\right)$ **54.** $\cos\left(\text{Cos}^{-1} \frac{4}{7}\right)$ **55.** $\text{Sin}^{-1}\left(\sin \frac{5}{6}\right)$ **56.** $\cos\left(\text{Tan}^{-1} \frac{3}{4}\right)$

57. GEOMETRY Find the total number of diagonals that can be drawn in a decagon. (Lesson 12-2)

Solve each equation. Round to the nearest hundredth. (Lesson 9-4)

58. $4^x = 24$ **59.** $4.3^{3x+1} = 78.5$ **60.** $7^{x-2} = 53^{-x}$

Simplify each expression. (Lesson 8-4)

61. $\frac{3}{a-2} + \frac{2}{a-3}$ **62.** $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$ **63.** $\frac{3y+1}{2y-10} + \frac{1}{y^2-2y-15}$

GET READY for the Next Lesson

PREREQUISITE SKILL **Find the value of each function.** (Lessons 13-3)

64. $\cos 150°$ **65.** $\tan 135°$ **66.** $\sin \frac{3\pi}{2}$ **67.** $\cos\left(-\frac{\pi}{3}\right)$

68. $\sin(-\pi)$ **69.** $\tan\left(-\frac{5\pi}{6}\right)$ **70.** $\cos 225°$ **71.** $\tan 405°$

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14-3 Trigonometric Identities

Main Ideas

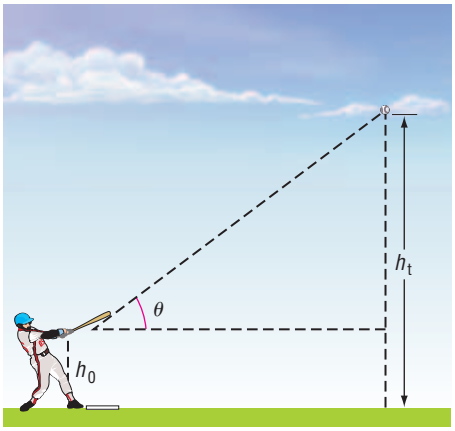
- Use identities to find trigonometric values.
- Use trigonometric identities to simplify expressions.

New Vocabulary

trigonometric identity

GET READY for the Lesson

A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of v feet per second at an angle of θ from the horizontal, then the height h of the ball after t seconds can be represented by $h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0$, where h_0 is the height of the ball in feet the moment it is hit.



Find Trigonometric Values In the equation above, the second term $\left(\frac{\sin\theta}{\cos\theta}\right)t$ can also be written as $(\tan\theta)t$. $\left(\frac{\sin\theta}{\cos\theta}\right)t = (\tan\theta)t$ is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is true except for angle measures such as 90°, 270°, 450°, …, $90° + 180° \cdot k$. The cosine of each of these angle measures is 0, so none of the expressions tan 90°, tan 270°, tan 450°, and so on, are defined. An identity similar to this is $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

These identities are sometimes called *quotient identities*. These and other basic trigonometric identities are listed below.

KEY CONCEPT — ***Basic Trigonometric Identities***

Quotient Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$	
Reciprocal Identities	$\csc\theta = \frac{1}{\sin\theta}$, $\sin\theta \neq 0$	$\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta \neq 0$	$\cot\theta = \frac{1}{\tan\theta}$, $\tan\theta \neq 0$
Pythagorean Identities	$\cos^2\theta + \sin^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$

You can use trigonometric identities to find values of trigonometric functions.

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EXAMPLE 1 Find a Value of a Trigonometric Function

a. Find $\cos \theta$ if $\sin \theta = -\frac{3}{5}$ and $90° < \theta < 180°$.

$\cos^2 \theta + \sin^2 \theta = 1$ Trigonometric identity

$\cos^2 \theta = 1 - \sin^2 \theta$ Subtract $\sin^2 \theta$ from each side.

$\cos^2 \theta = 1 - \left(\frac{3}{5}\right)^2$ Substitute $\frac{3}{5}$ for $\sin \theta$.

$\cos^2 \theta = 1 - \frac{9}{25}$ Square $\frac{3}{5}$.

$\cos^2 \theta = \frac{16}{25}$ Subtract.

$\cos \theta = \pm\frac{4}{5}$ Take the square root of each side.

Since θ is in the second quadrant, $\cos \theta$ is negative.

Thus, $\cos \theta = -\frac{4}{5}$.

b. Find $\csc \theta$ if $\cot \theta = -\frac{1}{4}$ and $270° < \theta < 360°$.

$\cot^2 \theta + 1 = \csc^2 \theta$ Trigonometric identity

$\left(-\frac{1}{4}\right)^2 + 1 = \csc^2 \theta$ Substitute $-\frac{1}{4}$ for $\cot \theta$.

$\frac{1}{16} + 1 = \csc^2 \theta$ Square $-\frac{1}{4}$.

$\frac{17}{16} = \csc^2 \theta$ Add.

$\pm\frac{\sqrt{17}}{4} = \csc \theta$ Take the square root of each side.

Since θ is in the fourth quadrant, $\csc \theta$ is negative.

Thus, $\csc \theta = -\frac{\sqrt{17}}{4}$.

CHECK Your Progress

1A. Find $\sin \theta$ if $\cos \theta = \frac{1}{3}$ and $270° < \theta < 360°$.

1B. Find $\sec \theta$ if $\sin \theta = -\frac{2}{7}$ and $180° < \theta < 270°$.

SIMPLIFY EXPRESSIONS Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

Study Tip

It is often easiest to write all expressions in terms of sine and/or cosine.

EXAMPLE 2 Simplify an Expression

Simplify $\frac{\csc^2 \theta - \cot^2 \theta}{\cos \theta}$.

$\frac{\csc^2 \theta - \cot^2 \theta}{\cos \theta} = \frac{\frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}}{\cos \theta}$ $\csc^2 \theta = \frac{1}{\sin^2 \theta}$, $\cot^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta}$

$= \frac{\frac{1 - \cos^2 \theta}{\sin^2 \theta}}{\cos \theta}$ Add.

MathOnline Extra Examples at algebra2.com

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$$= \frac{\frac{\sin^2\theta}{\sin^2\theta}}{\cos\theta} \qquad 1 - \cos^2\theta = \sin^2\theta$$

$$= \frac{1}{\cos\theta} \qquad \frac{\sin^2\theta}{\sin^2\theta} = 1$$

$$= \sec\theta \qquad \frac{1}{\cos\theta} = \sec\theta$$

CHECK Your Progress **Simplify each expression.**

2A. $\dfrac{\tan^2\theta\csc^2\theta - 1}{\sec^2\theta}$ **2B.** $\dfrac{\sec\theta}{\sin\theta}(1 - \cos^2\theta)$

EXAMPLE Simplify and Use an Expression

3 **BASEBALL** **Refer to the application at the beginning of the lesson. Rewrite the equation in terms of tan θ.**

$$h = \left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0 \qquad \text{Original equation}$$

$$= -\frac{16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0 \qquad \text{Factor.}$$

$$= -\frac{16}{v^2}\left(\frac{1}{\cos^2\theta}\right)t^2 + (\tan\theta)t + h_0 \qquad \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$= -\frac{16}{v^2}(\sec^2\theta)t^2 + (\tan\theta)t + h_0 \qquad \text{Since } \frac{1}{\cos\theta} = \sec\theta, \frac{1}{\cos^2\theta} = \sec^2\theta.$$

$$= -\frac{16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0 \qquad \sec^2\theta = 1 + \tan^2\theta$$

Thus, $\left(\frac{-16}{v^2\cos^2\theta}\right)t^2 + \left(\frac{\sin\theta}{\cos\theta}\right)t + h_0 = -\frac{16}{v^2}(1 + \tan^2\theta)t^2 + (\tan\theta)t + h_0$.

CHECK Your Progress

3. Rewrite the expression $\cot^2\theta - \tan^2\theta$ in terms of sin θ.

CHECK Your Understanding

Example 1 (p. 838) **Find the value of each expression.**

1. tan θ, if $\sin\theta = \frac{1}{2}$; $90° \le \theta < 180°$

2. csc θ, if $\cos\theta = -\frac{3}{5}$; $180° \le \theta < 270°$

3. cos θ, if $\sin\theta = \frac{4}{5}$; $0° \le \theta < 90°$

4. sec θ, if $\tan\theta = -1$; $270° < \theta < 360°$

Example 2 (pp. 838–839) **Simplify each expression.**

5. $\csc\theta\cos\theta\tan\theta$

6. $\sec^2\theta - 1$

7. $\dfrac{\tan\theta}{\sin\theta}$

8. $\sin\theta(1 + \cot^2\theta)$

Example 3 (p. 839)

9. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the *angle of inclination* and is represented by the equation $\tan\theta = \frac{v^2}{gR}$, where R is the radius of the circular path, v is the speed of the person in meters per second, and g is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using sin θ and cos θ.

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Exercises

HOMEWORK HELP

For Exercises	See Examples
10–17	1
18–26	2
27, 28	3

Find the value of each expression.

10. $\tan \theta$, if $\cot \theta = 2$; $0° \le \theta < 90°$

11. $\sin \theta$, if $\cos \theta = \frac{2}{3}$; $0° \le \theta < 90°$

12. $\sec \theta$, if $\tan \theta = -2$; $90° < \theta < 180°$

13. $\tan \theta$, if $\sec \theta = -3$; $180° < \theta < 270°$

14. $\csc \theta$, if $\cos \theta = -\frac{3}{5}$; $90° < \theta < 180°$

15. $\cos \theta$, if $\sec \theta = \frac{5}{3}$; $270° < \theta < 360°$

16. $\cos \theta$, if $\sin \theta = \frac{1}{2}$; $0° \le \theta < 90°$

17. $\csc \theta$, if $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

Simplify each expression.

18. $\cos \theta \csc \theta$

19. $\tan \theta \cot \theta$

20. $\sin \theta \cot \theta$

21. $\cos \theta \tan \theta$

22. $2(\csc^2 \theta - \cot^2 \theta)$

23. $3(\tan^2 \theta - \sec^2 \theta)$

24. $\dfrac{\cos \theta \csc \theta}{\tan \theta}$

25. $\dfrac{\sin \theta \csc \theta}{\cot \theta}$

26. $\dfrac{1 - \cos^2 \theta}{\sin^2 \theta}$

ELECTRONICS **For Exercises 27 and 28, use the following information.**

When an alternating current of frequency f and a peak current I pass through a resistance R, then the power delivered to the resistance at time t seconds is $P = I^2R - I^2R \cos^2 2ft\pi$.

27. Write an expression for the power in terms of $\sin^2 2ft\pi$.

28. Write an expression for the power in terms of $\tan^2 2ft\pi$.

Find the value of each expression.

29. $\tan \theta$, if $\cos \theta = \frac{4}{5}$; $0° \le \theta < 90°$

30. $\cos \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$

31. $\sec \theta$, if $\sin \theta = \frac{3}{4}$; $90° < \theta < 180°$

32. $\sin \theta$, if $\tan \theta = 4$; $180° < \theta < 270°$

Simplify each expression.

33. $\dfrac{1 - \sin^2 \theta}{\sin^2 \theta}$

34. $\dfrac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$

35. $\dfrac{\tan^2 \theta - \sin^2 \theta}{\tan^2 \theta \sin^2 \theta}$

Real-World Link

The oldest operational carousel in the United States is the Flying Horse Carousel at Martha's Vineyard, Massachusetts.

Source: Martha's Vineyard Preservation Trust

AMUSEMENT PARKS **For Exercises 36–38, use the following information.**

Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

36. Refer to Exercise 9. If the sine of the angle of inclination of the child is $\frac{1}{5}$, what is the angle of inclination made by the child?

37. What is the velocity of the merry-go-round?

38. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

LIGHTING **For Exercises 39 and 40, use the following information.**

The amount of light that a source provides to a surface is called the *illuminance*. The illuminance E in foot candles on a surface is related to the distance R in feet from the light source. The formula $\sec \theta = \dfrac{I}{ER^2}$, where I is the intensity of the light source measured in candles and θ is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

39. Solve the formula in terms of E.

40. Is the equation in Exercise 39 equivalent to $R^2 = \dfrac{I \tan \theta \cos \theta}{E}$? Explain.

EXTRA PRACTICE

See pages 923, 939.

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Self-Check Quiz at algebra2.com

James Schot/Martha's Vineyard Preserva

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H.O.T. Problems

41. REASONING Describe how you can determine the quadrant in which the terminal side of angle α lies if $\sin \alpha = -\frac{1}{4}$.

42. OPEN ENDED Write two expressions that are equivalent to $\tan \theta \sin \theta$.

43. REASONING If $\cot(x) = \cot\left(\frac{\pi}{3}\right)$ and $3\pi < x < 4\pi$, find x.

44. CHALLENGE If $\tan \beta = \frac{3}{4}$, find $\frac{\sin \beta \sec \beta}{\cot \beta}$.

45. *Writing in Math* Use the information on page 837 to explain how trigonometry can be used to model the path of a baseball. Include an explanation of why the equation at the beginning of the lesson is the same as $y = -\frac{16 \sec^2 \theta}{v^2}x^2 + (\tan \theta)x + h_0$.

STANDARDIZED TEST PRACTICE

46. ACT/SAT If $\sin x = m$ and $0 < x < 90°$, then $\tan x =$

A $\frac{1}{m^2}$.

B $\frac{1 - m^2}{m}$.

C $\frac{m}{\sqrt{1 - m^2}}$.

D $\frac{m}{1 - m^2}$.

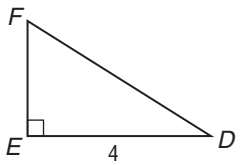
47. REVIEW Refer to the figure below. If $\cos D = 0.8$, what is length $\overline{DF}$?

F 5

G 4

H 3.2

J $\frac{4}{5}$



Spiral Review

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function. (Lesson 14-2)

48. $y = \sin \theta - 1$

49. $y = \tan \theta + 12$

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = \csc 2\theta$

51. $y = \cos 3\theta$

52. $y = \frac{1}{3}\cot 5\theta$

53. Find the sum of a geometric series for which $a_1 = 48$, $a_n = 3$, and $r = \frac{1}{2}$. (Lesson 11-4)

54. Write an equation of a parabola with focus at $(11, -1)$ and directrix $y = 2$. (Lesson 10-2)

55. TEACHING Ms. Granger has taught 288 students at this point in her career. If she has 30 students each year from now on, the function $S(t) = 288 + 30t$ gives the number of students $S(t)$ she will have taught after t more years. How many students will she have taught after 7 more years? (Lesson 2-1)

GET READY for the Next Lesson

PREREQUISITE SKILL Name the property illustrated by each statement. (Lesson1-3)

56. If $4 + 8 = 12$, then $12 = 4 + 8$.

57. If $7 + s = 21$, then $s = 14$.

58. If $4x = 16$, then $12x = 48$.

59. If $q + (8 + 5) = 32$, then $q + 13 = 32$.

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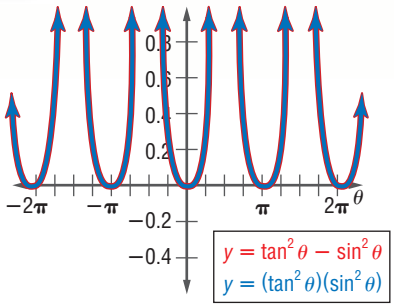
14-4 Verifying Trigonometric Identities

Main Ideas

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

GET READY for the Lesson

Examine the graphs at the right. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of θ and conclude that the statement is true for all values of θ. In order to show that the equation $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ for all values of θ, you must consider the general case.



Transform One Side of an Equation You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity, you need to show that it is true for all values of θ.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation *separately* until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

Study Tip

Common Misconception

You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

EXAMPLE 1 Transform One Side of an Equation

Verify that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity.

Transform the left side.

$$\tan^2 \theta - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Original equation}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Rewrite using the LCD, } \cos^2 \theta.$$

$$\frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Subtract.}$$

$$\frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Factor.}$$

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$$\frac{\sin^2\theta \sin^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta \sin^2\theta \qquad 1 - \cos^2\theta = \sin^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\sin^2\theta}{1} \stackrel{?}{=} \tan^2\theta \sin^2\theta \qquad \frac{ab}{c} = \frac{a}{c} \cdot \frac{b}{1}$$

$$\tan^2\theta \sin^2\theta = \tan^2\theta \sin^2\theta \qquad \frac{\sin^2\theta}{\cos^2\theta} = \tan\theta$$

CHECK Your Progress

1. Verify that $\cot^2\theta - \cos^2\theta = \cot^2\theta \cos^2\theta$ is an identity.

STANDARDIZED TEST EXAMPLE — Find an Equivalent Expression

2 $\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\cot\theta}\right) =$

A $\cos\theta$ **B** $\sin\theta$ **C** $\cos^2\theta$ **D** $\sin^2\theta$

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Transform the given expression to match one of the choices.

$$\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\cot\theta}\right) = \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\right) \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$= \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta\sin\theta}{\cos\theta}\right) \qquad \text{Simplify.}$$

$$= \sin\theta\left(\frac{1}{\sin\theta} - \sin\theta\right) \qquad \text{Simplify.}$$

$$= 1 - \sin^2\theta \qquad \text{Distributive Property}$$

$$= \cos^2\theta$$

The answer is C.

Test-Taking Tip

Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

CHECK Your Progress

2. $\tan^2\theta(\cot^2\theta - \cos^2\theta) =$

F $\cot^2\theta$ **G** $\tan^2\theta$ **H** $\cos^2\theta$ **J** $\sin^2\theta$

Transform Both Sides of an Equation Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.

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EXAMPLE Verify by Transforming Both Sides

3 **Verify that $\sec^2 \theta - \tan^2 \theta = \tan \theta \cot \theta$ is an identity.**

$\sec^2 \theta - \tan^2 \theta \stackrel{?}{=} \tan \theta \cot \theta$ Original equation

$\frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta}$ Express all terms using sine and cosine.

$1 - \frac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$ Subtract on the left. Multiply on the right.

$\frac{\cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$ $1 - \sin^2 \theta = \cos^2 \theta$

$1 = 1$ Simplify the left side.

CHECK Your Progress

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

CHECK Your Understanding

Examples 1, 3 (pp. 842, 844)

Verify that each of the following is an identity.

1. $\tan \theta (\cot \theta + \tan \theta) = \sec^2 \theta$

2. $\tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta$

3. $\frac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$

4. $\frac{1 + \tan^2 \theta}{\csc^2 \theta} = \tan^2 \theta$

5. $\frac{\sin \theta}{\sec \theta} = \frac{1}{\tan \theta + \cot \theta}$

6. $\frac{\sec \theta + 1}{\tan \theta} = \frac{\tan \theta}{\sec \theta - 1}$

Example 2 (p. 843)

7. **STANDARDIZED TEST PRACTICE** Which expression can be used to form an identity with $\frac{\sec \theta + \csc \theta}{1 + \tan \theta}$?

A $\sin \theta$ **B** $\cos \theta$ **C** $\tan \theta$ **D** $\csc \theta$

Exercises

HOMEWORK HELP

For Exercises	See Examples
8–21	1–3

Verify that each of the following is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$

9. $\cot \theta (\cot \theta + \tan^2 \theta) = \csc^2 \theta$

10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$

11. $\sin \theta \sec \theta \cot \theta = 1$

12. $\frac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$

13. $\frac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$

14. $\cot \theta \csc \theta = \frac{\cot \theta + \csc \theta}{\sin \theta + \tan \theta}$

15. $\sin \theta + \cos \theta = \frac{1 + \tan \theta}{\sec \theta}$

16. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$

17. $\frac{\sin \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta$

18. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity.

19. Show that $1 + \cos \theta$ and $\frac{\sin^2 \theta}{1 - \cos \theta}$ form an identity.

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Real-World Link

Model rocketry was developed during the "space-race" era. The rockets are constructed of cardboard, plastic, and balsa wood, and are fueled by single-use rocket motors.

PHYSICS **For Exercises 20 and 21, use the following information.**
If an object is propelled from ground level, the maximum height that it reaches is given by $h = \frac{v^2 \sin^2 \theta}{2g}$, where θ is the angle between the ground and the initial path of the object, v is the object's initial velocity, and g is the acceleration due to gravity, 9.8 meters per second squared.

20. Verify the identity $\frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta}$.

21. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket.

Verify that each of the following is an identity.

22. $\frac{1 + \sin \theta}{\sin \theta} = \frac{\cot^2 \theta}{\csc \theta - 1}$

23. $\frac{1 + \tan \theta}{1 + \cot \theta} = \frac{\sin \theta}{\cos \theta}$

24. $\frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1$

25. $1 + \frac{1}{\cos \theta} = \frac{\tan^2 \theta}{\sec \theta - 1}$

26. $1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta$

27. $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$

28. $\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$

29. $\frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta$

Graphing Calculator

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation *might* be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following *may be* or *is not* an identity.

30. $\cot x + \tan x = \csc x \cot x$

31. $\sec^2 x - 1 = \sin^2 x \sec^2 x$

32. $(1 + \sin x)(1 - \sin x) = \cos^2 x$

33. $\frac{1}{\sec x \tan x} = \csc x - \sin x$

34. $\frac{\sec^2 x}{\tan x} = \sec x \csc x$

35. $\frac{1}{\sec x} + \frac{1}{\csc x} = 1$

EXTRA PRACTICE
See pages 923, 939.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. OPEN ENDED Write a trigonometric equation that is *not* an identity. Explain how you know it is not an identity.

37. Which One Doesn't Belong? Identify the equation that does not belong with the other three. Explain your reasoning.

$\sin^2 \theta + \cos^2 \theta = 1$ $\quad$ $1 + \cot^2 \theta = \csc^2 \theta$

$\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta$ $\quad$ $\tan^2 \theta + 1 = \sec^2 \theta$

38. CHALLENGE Present a logical argument for why the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ is true when $0 \le x \le 1$.

39. ***Writing in Math*** Use the information on pages 842 and 843 to explain why you cannot perform operations to each side of an unverified identity and explain why you cannot use the graphs of two expressions to verify an identity.

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STANDARDIZED TEST PRACTICE

40. ACT/SAT Which of the following is not equivalent to $\cos \theta$?

A $\dfrac{\cos \theta}{\cos^2 \theta + \sin^2 \theta}$

B $\dfrac{1 - \sin^2 \theta}{\cos \theta}$

C $\cot \theta \sin \theta$

D $\tan \theta \csc \theta$

41. REVIEW Which of the following is equivalent to $\sin \theta + \cot \theta \cos \theta$?

F $2 \sin \theta$

G $\dfrac{1}{\sin \theta}$

H $\cos^2 \theta$

J $\dfrac{\sin \theta + \cos \theta}{\sin^2 \theta}$

Spiral Review

Find the value of each expression. (Lesson 14-3)

42. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$

43. $\cos \theta$, if $\sin \theta = -\frac{2}{3}$; $180° < \theta < 270°$

44. $\csc \theta$, if $\cot \theta = -\frac{7}{12}$; $90° < \theta < 180°$

45. $\sin \theta$, if $\cos \theta = \frac{3}{4}$; $270° < \theta < 360°$

State the amplitude, period, and phase shift of each function. Then graph each function. (Lesson 14-2)

46. $y = \cos (\theta - 30°)$

47. $y = \sin (\theta - 45°)$

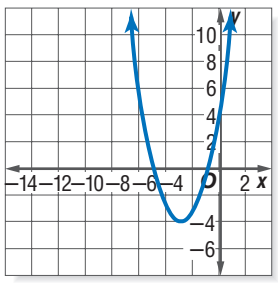
48. $y = 3 \cos \left(\theta + \frac{\pi}{2}\right)$

49. COMMUNICATIONS The carrier wave for a certain FM radio station can be modeled by the equation $y = A \sin (10^7 \cdot 2\pi t)$, where A is the amplitude of the wave and t is the time in seconds. Determine the period of the carrier wave. (Lesson 14-1)

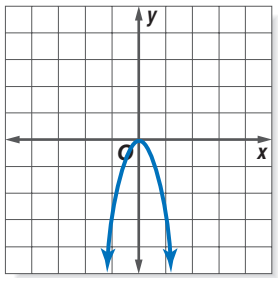
50. BUSINESS A company estimates that it costs $0.03x^2 + 4x + 1000$ dollars to produce x units of a product. Find an expression for the average cost per unit. (Lesson 6-3)

Use the related graph of each equation to determine its solutions. (Lesson 5-2)

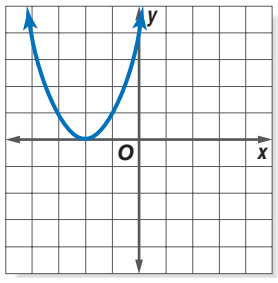
51. $y = x^2 + 6x + 5$



52. $y = -3x^2$



53. $y = x^2 + 4x - 4$



GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.** (Lessons 7-5)

54. $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

55. $\frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

56. $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}$

57. $\frac{1}{2} - \frac{\sqrt{3}}{4}$

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CHAPTER 14

Mid-Chapter Quiz

Lessons 14-1 through 14-4

1. Find the amplitude and period of $y = \frac{3}{4}\sin\frac{1}{2}\theta$. Then graph the function. (Lesson 14-1)

POPULATION **For Exercises 2–4 use the following information.**

The population of a certain species of deer can be modeled by the function $p = 30{,}000 + 20{,}000\cos\left(\frac{\pi}{10}t\right)$, where p is the population and t is the time in years. (Lesson 14-1)

2. What is the amplitude of the population and what does it represent?
3. What is the period of the function and what does it represent?
4. Graph the function.

5. **MULTIPLE CHOICE** Find the amplitude, if it exists, and period of $y = 3\cot\left(-\frac{1}{4}\theta\right)$. (Lesson 14-1)

A $3; \frac{\pi}{4}$ C not defined; 4π

B $3; 4\pi$ D not defined; $\frac{\pi}{4}$

For Exercises 6–9, consider the function $y = 2\cos\left[\frac{1}{4}\left(\theta - \frac{\pi}{4}\right)\right] - 5$. (Lesson 14-2)

6. State the vertical shift.
7. State the amplitude and period.
8. State the phase shift.
9. Graph the function.

10. **PENDULUM** The position of the pendulum on a particular clock can be modeled using a sine equation. The period of the pendulum is 2 seconds and the phase shift is 0.5 second. The pendulum swings 6 inches to either side of the center position. Write an equation to represent the position of the pendulum p at time t seconds. Assume that the x-axis represents the center line of the pendulum's path, that the area above the x-axis represents a swing to the right, and that the pendulum swings to the right first. (Lesson 14-2)

Find the value of each expression. (Lesson 14-3)

11. $\cos\theta$, if $\sin\theta = \frac{4}{5}$; $90^\circ < \theta < 180^\circ$
12. $\csc\theta$, if $\cot\theta = -\frac{2}{3}$; $270^\circ < \theta < 360^\circ$
13. $\sec\theta$, if $\tan\theta = \frac{1}{2}$; $0^\circ < \theta < 90^\circ$

14. **SWINGS** Amy takes her cousin to the park to swing while she is babysitting. The horizontal force that Amy uses to push her cousin can be found using the formula $F = Mg\tan\theta$, where F is the force, M is the mass of the child, g is gravity, and θ is the angle that the swing makes with it's resting position. Write an equivalent expressing using $\sin\theta$ and $\sec\theta$. (Lesson 14-3)

15. **MULTIPLE CHOICE** Which of the following is equivalent to $\frac{1-\sin^2\theta}{1-\cos^2\theta}\cdot\tan\theta$? (Lesson 14-3)

F $\tan\theta$ H $\sin\theta$

G $\cot\theta$ J $\cos\theta$

Verify that each of the following is an identity. (Lesson 14-4)

16. $\tan^2\theta + 1 = \frac{\tan\theta}{\cos\theta\cdot\sin\theta}$
17. $\frac{\sin\theta\cdot\sec\theta}{\sec\theta - 1} = (\sec\theta + 1)\cot\theta$
18. $\sin^2\theta\cdot\tan^2\theta = \tan^2\theta - \sin^2\theta$
19. $\cot\theta(1-\cos\theta) = \frac{\cos\theta\cdot\sin\theta}{1+\cos\theta}$
20. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using the formula $z = 2p\cos\theta$ where z is the combined power of the prisms, p is the power of the individual prisms, and θ is the angle between the two prisms. Verify the identity $2p\cos\theta = 2p(1-\sin^2\theta)\sec\theta$. (Lesson 14-4)

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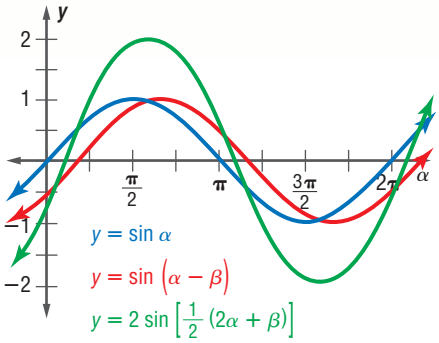
14-5 Sum and Differences of Angles Formulas

Main Ideas

- Find values of sine and cosine involving sum and difference formulas.
- Verify identities by using sum and difference formulas.

GET READY for the Lesson

Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude.



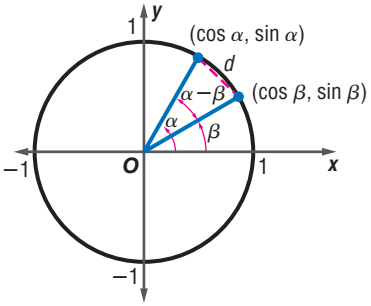
Sum and Difference Formulas Notice that the third equation shown above involves the sum of α and β. It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find sin 15° by evaluating sin (60° − 45°). Formulas can be developed that can be used to evaluate expressions like sin $(\alpha - \beta)$ or cos $(\alpha + \beta)$.

Reading Math

Greek Letters The Greek letter *beta*, β, can be used to denote the measure of an angle.

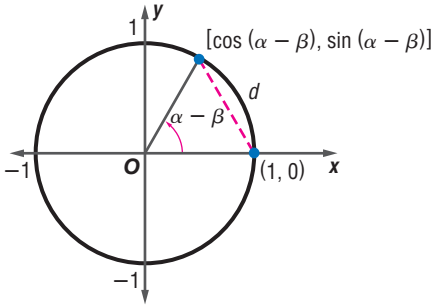
It is important to realize that sin $(\alpha \pm \beta)$ is not the same as sin $\alpha \pm$ sin β.

The figure at the right shows two angles α and β in standard position on the unit circle. Use the Distance Formula to find d, where $(x_1, y_1) = (\cos \beta, \sin \beta)$ and $(x_2, y_2) = (\cos \alpha, \sin \alpha)$.



$$d = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2}$$
$$d^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$
$$d^2 = (\cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta)$$
$$d^2 = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta$$
$$d^2 = 1 + 1 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \qquad \sin^2 \alpha + \cos^2 \alpha = 1 \text{ and}$$
$$d^2 = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \qquad \sin^2 \beta + \cos^2 \beta = 1$$

Now find the value of d^2 when the angle having measure $\alpha - \beta$ is in standard position on the unit circle, as shown in the figure at the left.



$$d = \sqrt{[\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2}$$
$$d^2 = [\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2$$
$$= [\cos^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1] + \sin^2 (\alpha - \beta)$$
$$= \cos^2 (\alpha - \beta) + \sin^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1$$
$$= 1 - 2 \cos (\alpha - \beta) + 1$$
$$= 2 - 2 \cos (\alpha - \beta)$$

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By equating the two expressions for d^2, you can find a formula for $\cos(\alpha - \beta)$.

$$\begin{aligned} d^2 &= d^2 \\ 2 - 2\cos(\alpha - \beta) &= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ -1 + \cos(\alpha - \beta) &= -1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Divide each side by } -2. \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta && \text{Add 1 to each side.} \end{aligned}$$

Use the formula for $\cos(\alpha - \beta)$ to find a formula for $\cos(\alpha + \beta)$.

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha - (-\beta)] \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta && \cos(-\beta) = \cos\beta;\ \sin(-\beta) = -\sin\beta \end{aligned}$$

You can use a similar method to find formulas for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$.

KEY CONCEPT — ***Sum and Difference of Angles Formulas***

The following identities hold true for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

Notice the symbol $\mp$ in the formula for $\cos(\alpha \pm \beta)$. It means "minus or plus." In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

EXAMPLE 1 Use Sum and Difference of Angles Formulas

Find the exact value of each expression.

a. $\cos 75°$

Use the formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\begin{aligned} \cos 75° &= \cos(30° + 45°) && \alpha = 30°,\ \beta = 45° \\ &= \cos 30°\cos 45° - \sin 30°\sin 45° \\ &= \left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2} \cdot \frac{\sqrt{2}}{2}\right) && \text{Evaluate each expression.} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} && \text{Multiply.} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} && \text{Simplify.} \end{aligned}$$

b. $\sin(-210°)$

Use the formula $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$$\begin{aligned} \sin(-210°) &= \sin(60° - 270°) && \alpha = 60°,\ \beta = 270° \\ &= \sin 60°\cos 270° - \cos 60°\sin 270° \\ &= \left(\frac{\sqrt{3}}{2}\right)(0) - \left(\frac{1}{2}\right)(-1) && \text{Evaluate each expression.} \\ &= 0 - \left(-\frac{1}{2}\right) \text{ or } \frac{1}{2} && \text{Simplify.} \end{aligned}$$

Extra Examples at algebra2.com

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CHECK Your Progress

1A. $\sin 15°$ **1B.** $\cos (-15°)$

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Reading Math

Greek Letters The symbol ϕ is the lowercase Greek letter *phi*.

Real-World EXAMPLE

2 PHYSICS **On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by $E \sin (113.5° - \phi)$, where ϕ is the latitude of the location and E is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.**

Real-World Link

In the northern hemisphere, the day with the least number of hours of daylight is December 21 or 22, the first day of winter.

Source: www.infoplease.com

Use the difference formula for sine.

$$\begin{aligned}\sin (113.5° - \phi) &= \sin 113.5° \cos \phi - \cos 113.5° \sin \phi \\ &= \sin 113.5° \cos 43.1° - \cos 113.5° \sin 43.1° \\ &= 0.9171 \cdot 0.7302 - (-0.3987) \cdot 0.6833 \\ &= 0.9420\end{aligned}$$

In Rochester, New York, the maximum light energy per square foot is $0.9420E$.

CHECK Your Progress

2. Determine the amount of light energy in West Hollywood, California, which is located at a latitude of 34.1° N.

Verify Identities You can also use the sum and difference formulas to verify identities.

EXAMPLE Verify Identities

3 Verify that each of the following is an identity.

a. $\sin (180° + \theta) = -\sin \theta$

$$\begin{aligned}\sin (180° + \theta) &\stackrel{?}{=} -\sin \theta && \text{Original equation} \\ \sin 180° \cos \theta + \cos 180° \sin \theta &\stackrel{?}{=} -\sin \theta && \text{Sum of angles formula} \\ 0 \cos \theta + (-1) \sin \theta &\stackrel{?}{=} -\sin \theta && \text{Evaluate each expression.} \\ -\sin \theta &= -\sin \theta && \text{Simplify.}\end{aligned}$$

b. $\cos (180° + \theta) = -\cos \theta$

$$\begin{aligned}\cos (180° + \theta) &\stackrel{?}{=} -\cos \theta && \text{Original equation} \\ \cos 180° \cos \theta - \sin 180° \sin \theta &\stackrel{?}{=} -\cos \theta && \text{Sum of angles formula} \\ (-1) \cos \theta - 0 \sin \theta &\stackrel{?}{=} -\cos \theta && \text{Evaluate each expression.} \\ -\cos \theta &= -\cos \theta && \text{Simplify.}\end{aligned}$$

CHECK Your Progress

3A. $\sin (90° - \theta) = \cos \theta$ **3B.** $\cos (90° + \theta) = -\sin \theta$

Cosmo Condina/Getty Images

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CHECK Your Understanding

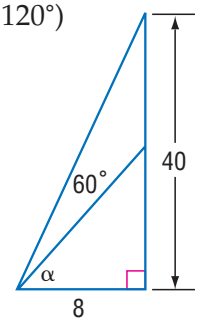
Example 1 (pp. 849–850)

Find the exact value of each expression.

1. sin 75°
2. sin 165°
3. cos 255°
4. cos (−30°)
5. sin (−240°)
6. cos (−120°)

Example 2 (p. 850)

7. GEOMETRY Determine the exact value of tan α in the figure.



Example 3 (p. 850)

Verify that each of the following is an identity.

8. $\cos (270° - \theta) = -\sin \theta$
9. $\sin \left(\theta + \frac{\pi}{2}\right) = \cos \theta$
10. $\sin (\theta + 30°) + \cos (\theta + 60°) = \cos \theta$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
11–24	1
25–28	2
29–36	3

Find the exact value of each expression.

11. sin 135°
12. cos 105°
13. sin 285°
14. cos 165°
15. cos 195°
16. sin 255°
17. cos 225°
18. sin 315°
19. sin (−15°)
20. cos (−45°)
21. cos (−150°)
22. sin (−165°)

PHYSICS For Exercises 23–26, use the following information.

On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by $E \sin (113.5° + \phi)$, where ϕ is the latitude of the location. Find the amount of light energy, in terms of E, for each location.

23. Salem, OR (Latitude: 44.9° N)
24. Chicago, IL (Latitude: 41.8° N)
25. Charleston, SC (Latitude: 28.5° N)
26. San Diego, CA (Latitude 32.7° N)

Verify that each of the following is an identity.

27. $\sin (270° - \theta) = -\cos \theta$
28. $\cos (90° + \theta) = -\sin \theta$
29. $\cos (90° - \theta) = \sin \theta$
30. $\sin (90° - \theta) = \cos \theta$
31. $\sin \left(\theta + \frac{3\pi}{2}\right) = -\cos \theta$
32. $\cos (\pi - \theta) = -\cos \theta$
33. $\cos (2\pi + \theta) = \cos \theta$
34. $\sin (\pi - \theta) = \sin \theta$

COMMUNICATION For Exercises 35 and 36, use the following information.

A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is $v = 10 \sin (2t - 30°)$ and the equation of the data wave is $d = 10 \cos (2t + 60°)$.

35. Draw a graph of the waves when they are combined.

36. Refer to the application at the beginning of the lesson. What type of interference results? Explain.

EXTRA PRACTICE
See pages 923, 939.
MathOnline
Self-Check Quiz at algebra2.com

Verify that each of the following is an identity.

37. $\sin (60° + \theta) + \sin (60° - \theta) = \sqrt{3} \cos \theta$
38. $\sin \left(\theta + \frac{\pi}{3}\right) - \cos \left(\theta + \frac{\pi}{6}\right) = \sin \theta$
39. $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$
40. $\cos (\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\sec \alpha \sec \beta}$

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H.O.T. Problems

41. OPEN ENDED Give a counterexample to the statement that $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ is an identity.

42. REASONING Determine whether $\cos(\alpha - \beta) < 1$ is *sometimes*, *always*, or *never* true. Explain your reasoning.

43. CHALLENGE Use the sum and difference formulas for sine and cosine to derive formulas for $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$.

44. *Writing in Math* Use the information on page 848 to explain how the sum and difference formulas are used to describe communication interference. Include an explanation of the difference between constructive and destructive interference.

STANDARDIZED TEST PRACTICE

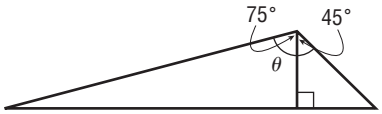
45. ACT/SAT Find the exact value of $\sin\theta$.

A $\frac{\sqrt{3}}{2}$

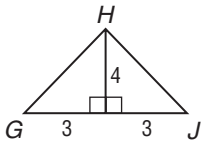
B $\frac{\sqrt{2}}{2}$

C $\frac{1}{2}$

D $\frac{\sqrt{3}}{3}$



46. REVIEW Refer to the figure below. Which equation could be used to find $m\angle G$?



F $\sin G = \frac{3}{4}$

G $\cos G = \frac{3}{4}$

H $\cot G = \frac{3}{4}$

J $\tan G = \frac{3}{4}$

Spiral Review

Verify that each of the following is an identity. (Lesson 14-4)

47. $\cot\theta + \sec\theta = \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

48. $\sin^2\theta + \tan^2\theta = (1 - \cos^2\theta) + \frac{\sec^2\theta}{\csc^2\theta}$

49. $\sin\theta(\sin\theta + \csc\theta) = 2 - \cos^2\theta$

50. $\frac{\sec\theta}{\tan\theta} = \csc\theta$

Simplify each expression. (Lesson 14-3)

51. $\frac{\tan\theta\csc\theta}{\sec\theta}$

52. $4\left(\sec^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\right)$

53. $(\cot\theta + \tan\theta)\sin\theta$

54. $\csc\theta\tan\theta + \sec\theta$

55. AVIATION A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? (Lesson 13-5)

56. Write $6y^2 - 34x^2 = 204$ in standard form. (Lesson 10-6)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation.** (Lesson 5-5)

57. $x^2 = \frac{20}{16}$

58. $x^2 = \frac{9}{25}$

59. $x^2 = \frac{5}{25}$

60. $x^2 = \frac{18}{32}$

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14-6 Double-Angle and Half-Angle Formulas

Main Ideas

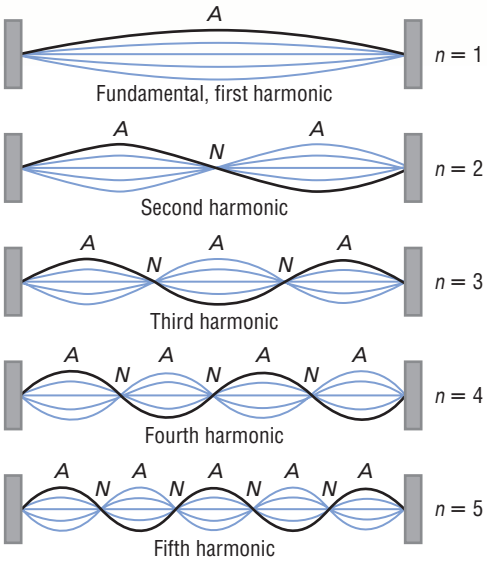
- Find values of sine and cosine involving double-angle formulas.
- Find values of sine and cosine involving half-angle formulas.

New Vocabulary

double-angle formulas
half-angle formula

GET READY for the Lesson

Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called *nodes*, that do not move under the vibration. Halfway between each pair of consecutive nodes are *antinodes* that undergo the maximum vibration. The nodes and antinodes form *harmonics*. These harmonics can be represented using variations of the equations $y = \sin 2\theta$ and $y = \sin \frac{1}{2}\theta$.



Double-Angle Formulas You can use the formula for $\sin(\alpha + \beta)$ to find the sine of twice an angle θ, $\sin 2\theta$, and the formula for $\cos(\alpha + \beta)$ to find the cosine of twice an angle θ, $\cos 2\theta$.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

You can find alternate forms for $\cos 2\theta$ by making substitutions into the expression $\cos^2\theta - \sin^2\theta$.

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= (1 - \sin^2\theta) - \sin^2\theta && \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta. \\ &= 1 - 2\sin^2\theta && \text{Simplify.}\end{aligned}$$

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= \cos^2\theta - (1 - \cos^2\theta) && \text{Substitute } 1 - \cos^2\theta \text{ for } \sin^2\theta. \\ &= 2\cos^2\theta - 1 && \text{Simplify.}\end{aligned}$$

These formulas are called the **double-angle formulas**.

KEY CONCEPT *Double-Angle Formulas*

The following identities hold true for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\sin^2\theta - 1$$

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EXAMPLE Double-Angle Formulas

1 **Find the exact value of each expression if $\sin \theta = \frac{4}{5}$ and θ is between 90° and 180°.**

a. sin 2θ

Use the identity $\sin 2\theta = 2 \sin \theta \cos \theta$.

First, find the value of $\cos \theta$.

$\cos^2 \theta = 1 - \sin^2 \theta$ $\quad \cos^2 \theta + \sin^2 \theta = 1$

$\cos^2 \theta = 1 - \left(\frac{4}{5}\right)^2$ $\quad \sin \theta = \frac{4}{5}$

$\cos^2 \theta = \frac{9}{25}$ $\quad$ Subtract.

$\cos \theta = \pm\frac{3}{5}$ $\quad$ Find the square root of each side.

Since θ is in the second quadrant, cosine is negative. Thus, $\cos \theta = -\frac{3}{5}$.

Now find $\sin 2\theta$.

$\sin 2\theta = 2 \sin \theta \cos \theta$ $\quad$ Double-angle formula

$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)$ $\quad \sin \theta = \frac{4}{5}, \cos \theta = -\frac{3}{5}$

$= -\frac{24}{25}$ $\quad$ Multiply.

The value of $\sin 2\theta$ is $-\frac{24}{25}$.

b. sin 2θ

Use the identity $\cos 2\theta = 1 - 2 \sin^2 \theta$.

$\cos 2\theta = 1 - 2 \sin^2 \theta$ $\quad$ Double-angle formula

$= 1 - 2\left(\frac{4}{5}\right)^2$ $\quad \sin \theta = \frac{4}{5}$

$= -\frac{7}{25}$ $\quad$ Simplify.

The value of $\cos 2\theta$ is $-\frac{7}{25}$.

CHECK Your Progress

Find the exact value of each expression if $\cos = -\frac{1}{3}$ and $90° < \theta < 180°$.

1A. $\sin 2\theta$ **1B.** $\cos 2\theta$

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Half-Angle Formulas You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find $\sin \frac{\alpha}{2}$.

$1 - 2 \sin^2 \theta = \cos 2\theta$ $\quad$ Double-angle formula

$1 - 2 \sin^2 \frac{\alpha}{2} = \cos \alpha$ $\quad$ Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2}$ $\quad$ Solve for $\sin^2 \frac{\alpha}{2}$.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ $\quad$ Take the square root of each side.

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Find $\cos \frac{\alpha}{2}$.

$2\cos^2 \theta - 1 = \cos 2\theta$ Double-angle formula

$2\cos^2 \frac{\alpha}{2} - 1 = \cos \alpha$ Substitute $\frac{\alpha}{2}$ for θ and α for 2θ.

$\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2}$ Solve for $\cos^2 \frac{\alpha}{2}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ Take the square root of each side.

These are called the **half-angle formulas**. The signs are determined by the function of $\frac{\alpha}{2}$.

KEY CONCEPT *Half-Angle Formulas*

The following identities hold true for all values of α.

$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$

EXAMPLE Half-Angle Formulas

2 **Find $\cos \frac{\alpha}{2}$ if $\sin \alpha = -\frac{3}{4}$ and α is in the third quadrant.**

Since $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$, we must find $\cos \alpha$ first.

$\cos^2 \alpha = 1 - \sin^2 \alpha$ $\cos^2 \alpha + \sin^2 \alpha = 1$

$\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2$ $\sin \alpha = -\frac{3}{4}$

$\cos^2 \alpha = \frac{7}{16}$ Simplify.

$\cos \alpha = \pm\frac{\sqrt{7}}{4}$ Take the square root of each side.

Since α is in the third quadrant, $\cos \alpha = \frac{\sqrt{7}}{4}$.

$\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos \alpha}{2}}$ Half-angle formula

$= \pm\sqrt{\frac{1 - \frac{\sqrt{7}}{4}}{2}}$ $\cos \alpha = -\frac{\sqrt{7}}{4}$

$= \pm\sqrt{\frac{4 - \sqrt{7}}{8}}$ Simplify the radicand.

$= \pm\frac{\sqrt{4 - \sqrt{7}}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$ Rationalize.

$= \pm\frac{\sqrt{8 - 2\sqrt{7}}}{4}$ Multiply.

Since α is between 180° and 270°, $\frac{\alpha}{2}$ is between 90° and 135°. Thus, $\cos \frac{\alpha}{2}$ is negative and equals $-\frac{\sqrt{8 - 2\sqrt{7}}}{4}$.

Study Tip

Choosing the Sign

You may want to determine the quadrant in which the terminal side of $\frac{\alpha}{2}$ will lie in the first step of the solution. Then you can use the correct sign from the beginning.

CHECK Your Progress

2. Find $\sin \frac{\alpha}{2}$ if $\sin \alpha = \frac{2}{3}$ and α is in the 2nd quadrant.

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EXAMPLE Evaluate Using Half-Angle Formulas

3 **Find the exact value of each expression by using the half-angle formulas.**

a. sin 105°

$$\begin{aligned}
\sin 105° &= \sin \frac{210°}{2} \\
&= \sqrt{\frac{1 - \cos 210°}{2}} && \sin \tfrac{\alpha}{2} = \pm\sqrt{\tfrac{1 - \cos \alpha}{2}} \\
&= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{2}} && \cos 210° = -\tfrac{\sqrt{3}}{2} \\
&= \sqrt{\frac{2 + \sqrt{3}}{4}} && \text{Simplify the radicand.} \\
&= \frac{\sqrt{2 + \sqrt{3}}}{2} && \text{Simplify the denominator.}
\end{aligned}$$

b. $\cos \frac{\pi}{8}$

$$\begin{aligned}
\cos \frac{\pi}{8} &= \frac{\frac{\pi}{4}}{2} \\
&= \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} && \cos \tfrac{\alpha}{2} = \pm\sqrt{\tfrac{1 + \cos \alpha}{2}} \\
&= \sqrt{\frac{1 + \frac{\sqrt{2}}{2}}{2}} && \cos \tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2} \\
&= \sqrt{\frac{2 + \sqrt{2}}{4}} && \text{Simplify the radicand.} \\
&= \frac{\sqrt{2 + \sqrt{2}}}{2} && \text{Simplify the denominator.}
\end{aligned}$$

CHECK Your Progress

3A. sin 135°

3B. $\cos \frac{7\pi}{8}$

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

EXAMPLE Verify Identities

4 **Verify that $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$ is an identity.**

$$\begin{aligned}
(\sin \theta + \cos \theta)^2 &\stackrel{?}{=} 1 + \sin 2\theta && \text{Original equation} \\
\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta &\stackrel{?}{=} 1 + \sin 2\theta && \text{Multiply.} \\
1 + 2 \sin \theta \cos \theta &\stackrel{?}{=} 1 + \sin 2\theta && \sin^2 \theta + \cos^2 \theta = 1 \\
1 + \sin 2\theta &= 1 + \sin 2\theta && \text{Double-angle formula}
\end{aligned}$$

CHECK Your Progress

4. Verify that $4 \cos^2 x - \sin^2 2x = 4 \cos^4 x$

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CHECK Your Understanding

Examples 1, 2 (pp. 854–855)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

1. $\cos \theta = \frac{3}{5}$; $0° < \theta < 90°$

2. $\cos \theta = -\frac{2}{3}$; $180° < \theta < 270°$

3. $\sin \theta = \frac{1}{2}$; $0° < \theta < 90°$

4. $\sin \theta = -\frac{3}{4}$; $270° < \theta < 360°$

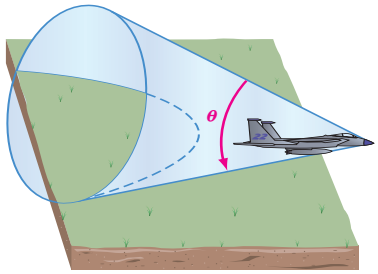
Example 3 (p. 856)

Find the exact value of each expression by using the half-angle formulas.

5. $\sin 195°$

6. $\cos \frac{19\pi}{12}$

7. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If θ is the measure of the angle at the vertex of the cone, then the Mach number M can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of 75°.



Example 4 (p. 856)

Verify that each of the following is an identity.

8. $\cot x = \frac{\sin 2x}{1 - \cos 2x}$

9. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Exercises

HOMEWORK	
For Exercises	See Examples
10–15	1, 2
16–21	3
22–27	4

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

10. $\sin \theta = \frac{5}{13}$; $90° < \theta < 180°$

11. $\cos \theta = \frac{1}{5}$; $270° < \theta < 360°$

12. $\cos \theta = -\frac{1}{3}$; $180° < \theta < 270°$

13. $\sin \theta = -\frac{3}{5}$; $180° < \theta < 270°$

14. $\sin \theta = -\frac{3}{8}$; $270° < \theta < 360°$

15. $\cos \theta = -\frac{1}{4}$; $90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

16. $\cos 165°$

17. $\sin 22\frac{1}{2}°$

18. $\cos 157\frac{1}{2}°$

19. $\sin 345°$

20. $\sin \frac{7\pi}{8}$

21. $\cos \frac{7\pi}{12}$

Verify that each of the following is an identity.

22. $\sin 2x = 2 \cot x \sin^2 x$

23. $2 \cos^2 \frac{x}{2} = 1 + \cos x$

24. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$

25. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

26. $\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$

27. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$

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PHYSICS **For Exercises 28 and 29, use the following information.**

An object is propelled from ground level with an initial velocity of v at an angle of elevation θ.

28. The horizontal distance d it will travel can be determined using the formula $d = \frac{v^2 \sin 2\theta}{g}$, where g is the acceleration due to gravity. Verify that this expression is the same as $\frac{2}{g}v^2(\tan \theta - \tan \theta \sin^2 \theta)$.

29. The maximum height h the object will reach can be determined using the formula $d = \frac{v^2 \sin^2 \theta}{2g}$. Find the ratio of the maximum height attained to the horizontal distance traveled.

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

30. $\cos \theta = \frac{1}{6}$; $0° < \theta < 90°$

31. $\cos \theta = -\frac{12}{13}$; $180° < \theta < 270°$

32. $\sin \theta = -\frac{1}{3}$; $270° < \theta < 360°$

33. $\sin \theta = -\frac{1}{4}$; $180° < \theta < 270°$

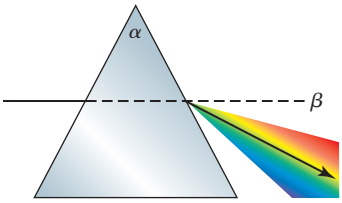
34. $\cos \theta = \frac{2}{3}$; $0° < \theta < 90°$

35. $\sin \theta = \frac{2}{5}$; $90° < \theta < 180°$

Real-World Link

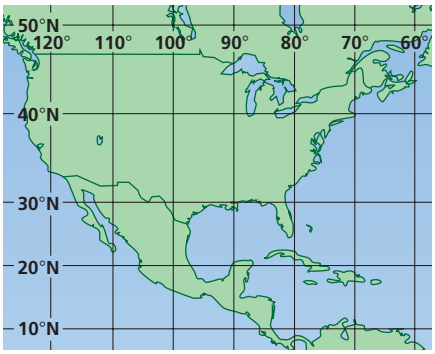
A rainbow appears when the sun shines through water droplets that act as a prism.

36. OPTICS If a glass prism has an apex angle of measure α and an angle of deviation of measure β, then the index of refraction n of the prism is given by $n = \frac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin \frac{\alpha}{2}}$. What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2?



GEOGRAPHY **For Exercises 37 and 38, use the following information.**

A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.



37. Write this expression in terms of a trigonometric function of L.

38. Find the exact value of the expression if $L = 60°$.

EXTRA PRACTICE
See pages 924, 939.
MathOnline
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. REASONING Explain how to find $\cos \frac{x}{2}$ if x is in the third quadrant.

40. REASONING Describe the conditions under which you would use each of the three identities for $\cos 2\theta$.

41. OPEN ENDED Find a counterexample to show that $\cos 2\theta = 2 \cos \theta$ is not an identity.

42. *Writing in Math* Use the information on page 853 to explain how trigonometric functions can be used to describe music. Include a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next and an explanation of what happens to the period of the function as you move from the nth harmonic to the $(n + 1)$th harmonic.

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STANDARDIZED TEST PRACTICE

43. ACT/SAT Find the exact value of $\cos 2\theta$ if $\sin \theta = \frac{-\sqrt{5}}{3}$ and $180° < \theta < 270°$.

A $\frac{-\sqrt{6}}{6}$

B $\frac{-\sqrt{30}}{6}$

C $\frac{-4\sqrt{5}}{9}$

D $\frac{-1}{9}$

44. REVIEW Which of the following is equivalent to $\frac{\cos\theta\,(\cot^2\theta + 1)}{\csc\theta}$?

F $\tan\theta$

G $\cot\theta$

H $\sec\theta$

J $\csc\theta$

Spiral Review

Find the exact value of each expression. (Lesson 14-5)

45. $\cos 15°$ **46.** $\sin 15°$ **47.** $\sin(-135°)$

48. $\cos 150°$ **49.** $\sin 105°$ **50.** $\cos(-300°)$

Verify that each of the following is an identity. (Lesson 14-4)

51. $\cot^2\theta - \sin^2\theta = \frac{\cos^2\theta\csc^2\theta - \sin^2\theta}{\sin^2\theta\csc^2\theta}$

52. $\cos\theta(\cos\theta + \cot\theta) = \cot\theta\cos\theta(\sin\theta + 1)$

ANALYZE TABLES For Exercises 53 and 54, use the following information.

The magnitude of an earthquake M measured on the Richter scale is given by $M = \log_{10} x$, where x represents the amplitude of the seismic wave causing ground motion. (Lesson 9-2)

Strongest Earthquakes in 20th Century

Location, Year	Magnitude
Chile, 1960	9.5
Alaska, 1964	9.2
Russia, 1952	9.0
Ecuador, 1906	8.8
Alaska, 1957	8.8
Kuril Islands, 1958	8.7
Alaska, 1965	8.7
India, 1950	8.6
Chile, 1922	8.5
Indonesia, 1938	8.5

Source: U.S. Geological Survey

53. How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?

54. The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock?

Write each expression in quadratic form, if possible. (Lesson 6-6)

55. $a^8 - 7a^4 + 13$ **56.** $5n^7 + 3n - 3$ **57.** $d^6 + 2d^3 + 10$

Find each value if $f(x) = x^2 - 7x + 5$. (Lesson 2-1)

58. $f(2)$ **59.** $f(0)$ **60.** $f(-3)$ **61.** $f(n)$

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each equation. (Lesson 5-3)

62. $(x + 6)(x - 5) = 0$ **63.** $(x - 1)(x + 1) = 0$

64. $x(x + 2) = 0$ **65.** $(2x - 5)(x + 2) = 0$

66. $(2x + 1)(2x - 1) = 0$ **67.** $x^2(2x + 1) = 0$

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EXPLORE 14-7

Graphing Calculating Lab

Solving Trigonometric Equations

The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83/84 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

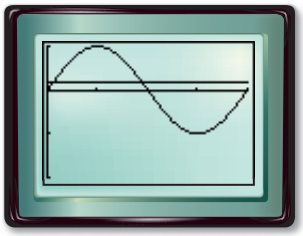
ACTIVITY 1

Use a graphing calculator to solve $\sin x = 0.2$ if $0° \le x < 360°$.

Rewrite the equation as two functions, $y = \sin x$ and $y = 0.2$. Then graph the two functions. Look for the point of intersection.

Make sure that your calculator is in degree mode to get the correct viewing window.

KEYSTROKES: MODE ▼ ▼ ▶ ENTER WINDOW 0 ENTER 360 ENTER 90 ENTER −2 ENTER 1 ENTER 1 ENTER Y= SIN X,T,θ,*n* ENTER 0.2 ENTER GRAPH



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Based on the graph, you can see that there are two points of intersection in the interval $0° \le x < 360°$. Use ZOOM or 2nd [CALC] 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

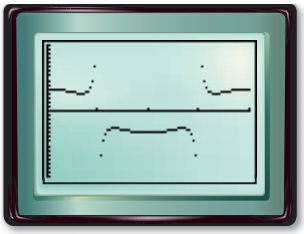
Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

ACTIVITY 2

Use a graphing calculator to solve $\tan^2 x \cos x + 5 \cos x = 0$ if $0° \le x < 360°$.

Because the tangent function is not continuous, place the calculator in Dot mode. The related functions to be graphed are $y = \tan^2 x \cos x + 5 \cos x$ and $y = 0$.

These two functions do not intersect. Therefore, the equation $\tan^2 x \cos x + 5 \cos x = 0$ has no real solutions.



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EXERCISES

Use a graphing calculator to solve each equation for the values of x indicated.

1. $\sin x = 0.8$ if $0° \le x < 360°$
2. $\tan x = \sin x$ if $0° \le x < 360°$
3. $2 \cos x + 3 = 0$ if $0° \le x < 360°$
4. $0.5 \cos x = 1.4$ if $-720° \le x < 720°$
5. $\sin 2x = \sin x$ if $0° \le x < 360°$
6. $\sin 2x - 3 \sin x = 0$ if $-360° \le x < 360°$

Other Calculator Keystrokes at algebra2.com

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14-7 Solving Trigonometric Equations

Main Ideas

- Solve trigonometric equations.
- Use trigonometric equations to solve real-world problems.

New Vocabulary

trigonometric equations

GET READY for the Lesson

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function $T = 11.56 \sin (0.4516x - 1.641) + 80.89$, where T represents the average daily high temperature in degrees Fahrenheit and x represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

Solve Trigonometric Equations You have seen that trigonometric identities are true for *all* values of the variable for which the equation is defined. However, most **trigonometric equations**, like some algebraic equations, are true for *some* but not *all* values of the variable.

EXAMPLE **Solve Equations for a Given Interval**

1 **Find all solutions of $\sin 2\theta = 2 \cos \theta$ for the interval $0 \leq \theta < 360°$.**

$\sin 2\theta = 2 \cos \theta$ Original equation

$2 \sin \theta \cos \theta = 2 \cos \theta$ $\sin 2\theta = 2 \sin \theta \cos \theta$

$2 \sin \theta \cos \theta - 2 \cos \theta = 0$ Solve for 0.

$2 \cos \theta (\sin \theta - 1) = 0$ Factor.

Use the Zero Product Property.

$2 \cos \theta = 0$ or $\sin \theta - 1 = 0$

$\cos \theta = 0$ $\sin \theta = 1$

$\theta = 90°$ or $270°$ $\theta = 90°$

The solutions are 90° and 270°.

CHECK Your Progress

1. Find all solutions of $\cos^2 \theta = 1$ for the interval $0° \leq \theta < 360°$.

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Trigonometric equations are usually solved for values of the variable between 0° and 360° or 0 radians and 2π radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

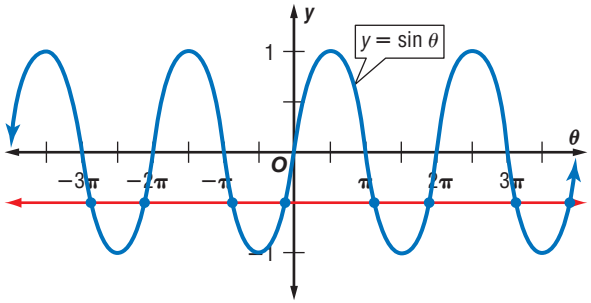
EXAMPLE 2 Solve Trigonometric Equations

Solve $2 \sin \theta = -1$ for all values of θ if θ is measured in radians.

$2 \sin \theta = -1$ Original equation

$\sin \theta = -\frac{1}{2}$ Divide each side by 2.

Look at the graph of $y = \sin \theta$ to find solutions of $\sin \theta = -\frac{1}{2}$.



The solutions are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{19\pi}{6}$, $\frac{23\pi}{6}$, and so on, and $\frac{-7\pi}{6}$, $\frac{-11\pi}{6}$, $\frac{-19\pi}{6}$, $\frac{-23\pi}{6}$, and so on. The only solutions in the interval 0 to 2π are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is 2π radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where k is any integer.

Study Tip

Expressing Solutions as Multiples

The expression $\frac{\pi}{2} + k \cdot \pi$ includes $\frac{3\pi}{2}$ and its multiples, so it is not necessary to list them separately,

CHECK Your Progress

2. Solve for $\cos 2\theta + \cos \theta + 1 = 0$ for all values of θ if θ is measured in degrees.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

EXAMPLE 3 Solve Trigonometric Equations Using Identities

Solve $\cos \theta \tan \theta - \sin^2 \theta = 0$.

$\cos \theta \tan \theta - \sin^2 \theta = 0$ Original equation

$\cos \theta \left(\frac{\sin \theta}{\cos \theta}\right) - \sin^2 \theta = 0$ $\tan \theta = \frac{\sin \theta}{\cos \theta}$

$\sin \theta - \sin^2 \theta = 0$ Multiply.

$\sin \theta (1 - \sin \theta) = 0$ Factor.

$\sin \theta = 0$ or $1 - \sin \theta = 0$

$\theta = 0°, 180°, \text{or } 360°$ $\sin \theta = 1$

$\theta = 90°$

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CHECK

$\cos\theta \tan\theta - \sin^2\theta = 0$

$\cos 0° \tan 0° - \sin^2 0° \stackrel{?}{=} 0$ $\theta = 0°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta \tan\theta - \sin^2\theta = 0$

$\cos 360° \tan 360° - \sin^2 360° \stackrel{?}{=} 0$ $\theta = 360°$

$1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta \tan\theta - \sin^2\theta = 0$

$\cos 180° \tan 180° - \sin^2 180° \stackrel{?}{=} 0$ $\theta = 180°$

$-1 \cdot 0 - 0 \stackrel{?}{=} 0$

$0 = 0$ true

$\cos\theta \tan\theta - \sin^2\theta = 0$

$\cos 90° \tan 90° - \sin^2 90° \stackrel{?}{=} 0$ $\theta = 90°$

$\tan 90°$ is undefined.

Thus, 90° is not a solution.

The solution is $0° + k \cdot 180°$.

CHECK Your Progress

Solve each equation.

3A. $\sin\theta \cot\theta - \cos^2\theta = 0$

3B. $\frac{\cos\theta}{\cot\theta} + 2\sin^2\theta = 0$

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Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between -1 and 1, inclusive. Thus, the solution set for $\cos x = 4$ is empty.

EXAMPLE **Determine Whether a Solution Exists**

4 **Solve $3\cos 2\theta - 5\cos\theta = 1$.**

$3\cos 2\theta - 5\cos\theta = 1$ Original equation

$3(2\cos^2\theta - 1) - 5\cos\theta = 1$ $\cos 2\theta = 2\cos^2\theta - 1$

$6\cos^2\theta - 3 - 5\cos\theta = 1$ Multiply.

$6\cos^2\theta - 5\cos\theta - 4 = 0$ Subtract 1 from each side.

$(3\cos\theta - 4)(2\cos\theta + 1) = 0$ Factor.

$3\cos\theta - 4 = 0$ or $2\cos\theta + 1 = 0$

$3\cos\theta = 4$ $2\cos\theta = -1$

$\cos\theta = \frac{4}{3}$ $\cos\theta = -\frac{1}{2}$

Not possible since $\cos\theta$ cannot be greater than 1.

$\theta = 120°$ or $240°$

Thus, the solutions are $120° + k \cdot 360°$ and $240° + k \cdot 360°$.

CHECK Your Progress **Solve each equation.**

4A. $\sin^2\theta + 2\cos^2\theta = 4$

4B. $\cos^2\theta - 3 = 4 - \sin^2\theta$

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Use Trigonometric Equations Trigonometric equations are often used to solve real-world situations.

Real-World EXAMPLE

5 **GARDENING** **Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight H in her town can be represented by $H = 11.45 + 6.5 \sin (0.0168d - 1.333)$, where d is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?**

H	$= 11.45 + 6.5 \sin (0.0168d - 1.333)$	Original equation
14	$= 11.45 + 6.5 \sin (0.0168d - 1.333)$	$H = 14$
2.55	$= 6.5 \sin (0.0168d - 1.333)$	Subtract 11.45 from each side.
0.392	$= \sin (0.0168d - 1.333)$	Divide each side by 6.5.
0.403	$= 0.0168d - 1.333$	$\sin^{-1} 0.392 = 0.403$
1.736	$= 0.0168d$	Add 1.333 to each side.
103.333	$= d$	Divide each side by 0.0168.

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

CHECK Your Progress

5. If Rhonda decides to wait only until there are 12 hours of daylight, on what day is it safe for her to plant her flowers?

CHECK Your Understanding

Example 1 (p. 861)

Find all solutions of each equation for the given interval.

1. $4 \cos^2 \theta = 1; 0° \le \theta < 360°$

2. $2 \sin^2 \theta - 1 = 0; 90° < \theta < 270°$

3. $\sin 2\theta = \cos \theta; 0 \le \theta < 2\pi$

4. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \le \theta < \frac{\pi}{2}$

Example 2 (p. 862)

Solve each equation for all values of θ if θ is measured in radians.

5. $\cos 2\theta = \cos \theta$

6. $\sin \theta + \sin \theta \cos \theta = 0$

Solve each equation for all values of θ if θ is measured in degrees.

7. $\sin \theta = 1 + \cos \theta$

8. $2 \cos^2 \theta + 2 = 5 \cos \theta$

Examples 3, 4 (pp. 862–863)

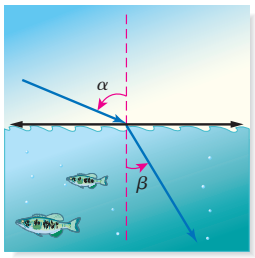
Solve each equation for all values of θ.

9. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

10. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

Example 5 (p. 864)

11. **PHYSICS** According to Snell's law, the angle at which light enters water α is related to the angle at which light travels in water β by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of 23° through the water?



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Exercises

HOMEWORK HELP	
For Exercises	See Examples
12–15	1
16–23	2
24–27	3, 4
28, 29	5

Find all solutions of each equation for the given interval.

12. $2 \cos \theta - 1 = 0; 0° \leq \theta < 360°$

13. $2 \sin \theta = -\sqrt{3}; 180° < \theta < 360°$

14. $4 \sin^2 \theta = 1; 180° < \theta < 360$

15. $4 \cos^2 \theta = 3; 0° \leq \theta < 360°$

Solve each equation for all values of θ if θ is measured in radians.

16. $\cos 2\theta + 3 \cos \theta - 1 = 0$

17. $2 \sin^2 \theta - \cos \theta - 1 = 0$

18. $\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0$

19. $\cos \theta = 3 \cos \theta - 2$

Solve each equation for all values of θ if θ is measured in degrees.

20. $\sin \theta = \cos \theta$

21. $\tan \theta = \sin \theta$

22. $\sin^2 \theta - 2 \sin \theta - 3 = 0$

23. $4 \sin^2 \theta - 4 \sin \theta + 1 = 0$

Solve each equation for all values of θ.

24. $\sin^2 \theta + \cos 2\theta - \cos \theta = 0$

25. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

26. $\sin^2 \theta = \cos^2 \theta - 1$

27. $2 \cos^2 \theta + \cos \theta = 0$

WAVES **For Exercises 28 and 29, use the following information.**

After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where h is the maximum height of the wave in feet, P is the period in seconds, and t is the propagation of the wave in seconds.

28. If $h = 3$ and $P = 2$, write the equation for the wave and draw its graph over a 10-second interval.

29. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

Find all solutions of each equation for the given interval.

30. $2 \cos^2 \theta = \sin \theta + 1; 0 \leq \theta < 2\pi$

31. $\sin^2 \theta - 1 = \cos^2 \theta; 0 \leq \theta < \pi$

32. $2 \sin^2 \theta + \sin \theta = 0; \pi < \theta < 2\pi$

33. $2 \cos^2 \theta = -\cos \theta; 0 \leq \theta < 2\pi$

Solve each equation for all values of θ if θ is measured in radians.

34. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$

35. $\cos 2\theta = 1 - \sin \theta$

36. $(\cos \theta)(\sin 2\theta) - 2 \sin \theta + 2 = 0$

37. $2 \sin^2 \theta + (\sqrt{2} - 1) \sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation for all values of θ if θ is measured in degrees.

38. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$

39. $\cos^2 \theta - \frac{7}{2} \cos \theta - 2 = 0$

40. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3} \sin \theta + \cos \theta$

41. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

Solve each equation for all values of θ.

42. $\sin \frac{\theta}{2} + \cos \theta = 1$

43. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$

44. $2 \sin \theta = \sin 2\theta$

45. $\tan^2 \theta + \sqrt{3} = (1 + \sqrt{3}) \tan \theta$

Real-World Link

Fireflies are bioluminescent, which means that they produce light through a biochemical reaction. Almost 100% of a firefly's energy is given off as light.

Source: www.nfs.gov

LIGHT **For Exercises 46 and 47, use the following information.**

The height of the International Peace Memorial at Put-in-Bay, Ohio, is 352 feet.

46. The length of the shadow S of the Memorial depends upon the angle of inclination of the Sun, θ. Express S as a function of θ.

47. Find the angle of inclination θ that will produce a shadow 560 feet long.

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H.O.T. Problems

EXTRA PRACTICE
See pages 924, 939.

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48. OPEN ENDED Write an example of a trigonometric equation that has no solution.

49. REASONING Explain why the equation $\sec \theta = 0$ has no solutions.

50. CHALLENGE Computer games often use transformations to distort images on the screen. In one such transformation, an image is rotated counterclockwise using the equations $x' = x \cos \theta - y \sin \theta$ and $y' = x \sin \theta + y \cos \theta$. If the coordinates of an image point are (3, 4) after a 60° rotation, what are the coordinates of the preimage point?

51. REASONING Explain why the number of solutions to the equation $\sin \theta = \frac{\sqrt{3}}{2}$ is infinite.

52. *Writing in Math* Use the information on page 861 to explain how trigonometric equations can be used to predict temperature. Include an explanation of why the sine function can be used to model the average daily temperature and an explanation of why, during one period, you might find a specific average temperature twice.

STANDARDIZED TEST PRACTICE

53. ACT/SAT Which of the following is *not* a possible solution of $0 = \sin \theta + \cos \theta \tan^2 \theta$?

A $\frac{3\pi}{4}$

B $\frac{7\pi}{4}$

C 2π

D $\frac{5\pi}{2}$

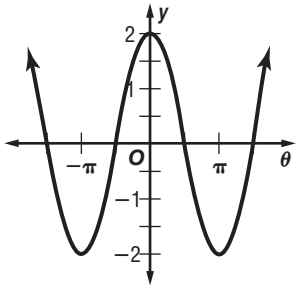
54. REVIEW The graph of the equation $y = 2 \cos \theta$ is shown. Which is a solution for $2 \cos \theta = 1$?

F $\frac{8\pi}{3}$

G $\frac{13\pi}{3}$

H $\frac{10\pi}{3}$

J $\frac{15\pi}{3}$



Spiral Review

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following. (Lesson 14-6)

55. $\sin \theta = \frac{3}{5}$; $0° < \theta < 90°$

56. $\cos \theta = \frac{1}{2}$; $0° < \theta < 90°$

57. $\cos \theta = \frac{5}{6}$; $0° < \theta < 90°$

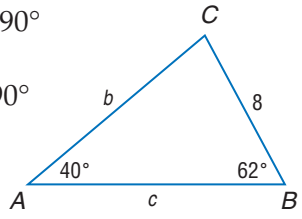
58. $\sin \theta = \frac{4}{5}$; $0° < \theta < 90°$

Find the exact value of each expression. (Lesson 14-5)

59. sin 240°

60. cos 315°

61. sin 150°



62. Solve $\triangle ABC$. Round measures of sides and angles to the nearest tenth. (Lesson 13-4)

Cross-Curricular Project

Algebra and Physics

So you want to be a rocket scientist? It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

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CHAPTER 14

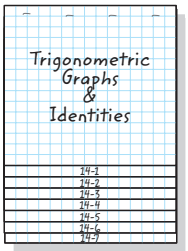
Study Guide and Review

STUDY TO GO Download Vocabulary Review from algebra2.com

FOLDABLES Study Organizer

GET READY to Study

Be sure the following Key Concepts are noted in your Foldable.



Key Concepts

Graphing Trigonometric Functions (Lesson 14-1)

- For trigonometric functions of the form $y = a \sin b\theta$ and $y = a \cos b\theta$, the amplitude is $|a|$, and the period is $\frac{360^\circ}{|b|}$ or $\frac{2\pi}{|b|}$.
- The period of $y = a \tan b\theta$ is $\frac{180^\circ}{|b|}$ or $\frac{\pi}{|b|}$.

Translations of Trigonometric Graphs (Lesson 14-2)

- For trigonometric functions of the form $y = a \sin(\theta - h) + k$, $y = a \cos(\theta - h) + k$, $y = a \tan(\theta - h) + k$, the phase shift is to the right when h is positive and to the left when h is negative. The vertical shift is up when k is positive and down when k is negative.

Trigonometric Identities

(Lessons 14-3, 14-4, and 14-7)

- Trigonometric identities describe the relationships between trigonometric functions.
- Trigonometric identities can be used to simplify, verify, and solve trigonometric equations and expressions.

Sum and Difference of Angles Formulas

(Lesson 14-5)

- For all values of α and β:

$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$

Double-Angle and Half-Angle Formulas

(Lesson 14-6)

- Double-angle formulas:

$\sin 2\theta = 2 \sin\theta \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

- Half-angle formulas:

$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos\alpha}{2}}$

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 + \cos\alpha}{2}}$

Key Vocabulary

amplitude (p. 823)
difference of angles formula (p. 849)
double-angle formula (p. 853)
half-angle formula (p. 855)
midline (p. 831)
phase shift (p. 829)
sum of angles formula (p. 849)
trigonometric equation (p. 861)
trigonometric identity (p. 837)
vertical shift (p. 831)

Vocabulary Check

Choose the correct term from the list above to compete each sentence.

1. The horizontal translation of a trigonometric function is a(n) _____.
2. A reference line about which a graph oscillates is a(n) _____.
3. The vertical translation of a trigonometric function is called a(n) _____.
4. The _____ formula can be used to find $\cos 22\frac{1}{2}^\circ$.
5. The _____ can be used to find $\sin 60^\circ$ using 30° as a reference.
6. The _____ can be used to find the sine or cosine of 75° if the sine and cosine of 45° and 30° are known.
7. A(n) _____ is an equation that is true for all values for which every expression in the equation is defined
8. The _____ can be used to find the sine or cosine of 65° if the sine and cosine of 90° and 25° are known.
9. The absolute value of half the difference between the maximum value and the minimum value of a periodic function is called the _____.

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CHAPTER 14

Study Guide and Review

Lesson-by-Lesson Review

14–1 Graphing Trigonometric Functions (pp. 822–828)

Find the amplitude, if it exists, and period of each function. Then graph each function.

10. $y = -\frac{1}{2}\cos\theta$

11. $y = 4\sin 2\theta$

12. $y = \sin\frac{1}{2}\theta$

13. $y = 5\sec\theta$

14. $y = \frac{1}{2}\csc\frac{2}{3}\theta$

15. $y = \tan 4\theta$

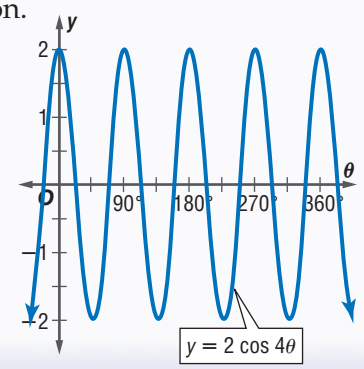
16. MECHANICS The position of a piston can be modeled using the equation $y = A\sin\left(\frac{1}{4}\cdot 2\pi t\right)$ where A is the amplitude of oscillation and t is the time in seconds. Determine the period of oscillation.

Example 1 **Find the amplitude and period of $y = 2\cos 4\theta$. Then graph.**

The amplitude is $|2|$ or 2.

The period is $\frac{360°}{|4|}$ or 90°.

Use the amplitude and period to graph the function.



14–2 Translations of Trigonometric Graphs (pp. 829–836)

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

17. $y = \frac{1}{2}\sin[2(\theta - 60°)] - 1$

18. $y = 2\tan\left[\frac{1}{4}(\theta - 90°)\right] + 3$

19. $y = 3\sec\left[\frac{1}{2}\left(\theta + \frac{\pi}{4}\right)\right] + 1$

20. $y = \frac{1}{3}\cos\left[\frac{1}{3}\left(\theta - \frac{2\pi}{3}\right)\right] - 2$

21. BIOLOGY The population of a species of bees varies periodically over the course of a year. The maximum population of bees occurs in March, and is 50,000. The minimum population of bees occurs in September and is 20,000. Assume that the population can be modeled using the sine function. Write an equation to represent the population of bees p, t months after January.

Example 2 **State the vertical shift, amplitude, period, and phase shift of $y = 3\sin\left[2\left(\theta - \frac{\pi}{2}\right)\right] - 2$. Then graph the function.**

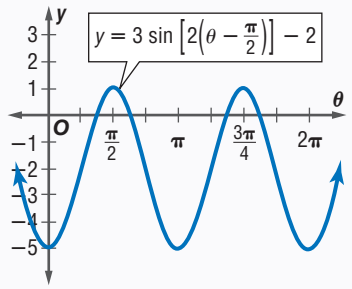
Identify the values of k, a, b, and h.

$k = -2$, so the vertical shift is –2.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{|2|}$ or π.

$h = \frac{\pi}{2}$, so the phase shift is $\frac{\pi}{2}$ to the right.



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14–3 Trigonometric Identities (pp. 837–841)

Find the value of each expression.

22. $\cot \theta$, if $\csc \theta = -\frac{5}{3}$; $270° < \theta < 360°$

23. $\sec \theta$, if $\sin \theta = \frac{1}{2}$; $0° \leq \theta < 90°$

Simplify each expression.

24. $\sin \theta \csc \theta - \cos^2 \theta$

25. $\cos^2 \theta \sec \theta \csc \theta$

26. $\cos \theta + \sin \theta \tan \theta$

27. $\sin \theta (1 + \cot^2 \theta)$

28. PHYSICS The magnetic force on a particle can be modeled by the equation $F = qvB \sin \theta$, where F is the magnetic force, q is the charge of the particle, B is the m agnetic field strength, and θ is the angle between the particle's path and the direction of the magnetic field. Write an equation for the magnetic force in terms of $\tan \theta$ and $\sec \theta$.

Example 3 **Find $\cos \theta$ if $\sin \theta = -\frac{3}{4}$ and $90° < \theta < 180°$.**

$\cos^2 \theta + \sin^2 \theta = 1$	Trigonometric identity
$\cos^2 \theta = 1 - \sin^2 \theta$	Subtract $\sin^2 \theta$ from each side.
$\cos^2 \theta = 1 - \left(\frac{3}{4}\right)^2$	Substitute $\frac{3}{4}$ for $\sin \theta$.
$\cos^2 \theta = 1 - \frac{9}{16}$	Square $\frac{3}{4}$.
$\cos^2 \theta = \frac{7}{16}$	Subtract.
$\cos \theta = \pm \frac{\sqrt{7}}{4}$	Take the square root of each side.

Since θ is in the second quadrant, $\cos \theta$ is negative. Thus, $\cos \theta = -\frac{\sqrt{7}}{4}$.

Example 4 **Simplify $\sin \theta \cot \theta \cos \theta$.**

$\sin \theta \cot \theta \cos \theta$	
$= \frac{\sin \theta}{1} \cdot \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{1}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$
$= \cos^2 \theta$	Multiply.

14–4 Verifying Trigonometric Identities (pp. 842–846)

Verify that each of the following is an identity.

29. $\frac{\sin \theta}{\tan \theta} + \frac{\cos \theta}{\cot \theta} = \cos \theta + \sin \theta$

30. $\frac{\sin \theta}{1 - \cos \theta} = \csc \theta + \cot \theta$

31. $\cot^2 \theta \sec^2 \theta = 1 + \cot^2 \theta$

32. $\sec \theta (\sec \theta - \cos \theta) = \tan^2 \theta$

33. OPTICS The amount of light passing through a polarization filter can be modeled using the equation $I = I_m \cos^2 \theta$, where I is the amount of light passing through the filter, I_m is the amount of light shined on the filter, and θ is the angle of rotation between the light source and the filter. Verify the identity $I_m \cos^2 \theta = I_m - \frac{I_m}{\cos^2 \theta + 1}$.

Example 5 **Verify that $\tan \theta + \cot \theta = \sec \theta \csc \theta$ is an identity.**

$\tan \theta + \cot \theta \stackrel{?}{=} \sec \theta \csc \theta$	Original equation
$\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \stackrel{?}{=} \sec \theta \csc \theta$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$
$\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} \stackrel{?}{=} \sec \theta \csc \theta$	Rewrite using the LCD, $\cos \theta \sin \theta$.
$\frac{1}{\cos \theta \sin \theta} \stackrel{?}{=} \sec \theta \csc \theta$	$\sin^2 \theta + \cos^2 \theta = 1$
$\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} \stackrel{?}{=} \sec \theta \csc \theta$	Rewrite as the product of two expressions.
$\sec \theta \csc \theta = \sec \theta \csc \theta$	$\frac{1}{\cos \theta} = \sec \theta$, $\frac{1}{\sin \theta} = \csc \theta$

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CHAPTER 14

Study Guide and Review

14–5 Sum and Difference of Angles Formula (pp. 848–852)

Find the exact value of each expression.

34. cos 15°
35. cos 285°
36. sin 150°
37. sin 195°
38. cos (−210°)
39. sin (−105°)

Verify that each of the following is an identity.

40. $\cos (90° + \theta) = -\sin \theta$
41. $\sin (30° - \theta) = \cos (60° + \theta)$
42. $\sin (\theta + \pi) = -\sin \theta$
43. $-\cos \theta = (\cos \pi + \theta)$

Example 6 **Find the exact value of sin 195°.**

$\sin 195° = \sin (150° + 45°)$ $195° = 150° + 45°$

$= \sin 150° \cos 45° + \cos 150° \sin 45°$ $\alpha = 150°, \beta = 45°$

$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$ Evaluate each expression.

$= \frac{\sqrt{2} - \sqrt{6}}{4}$ Simplify.

14–6 Double-Angle and Half-Angle Formulas (pp. 853–859)

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

44. $\sin \theta = \frac{1}{4}$; $0° < \theta < 90°$
45. $\sin \theta = -\frac{5}{13}$; $180° < \theta < 270°$
46. $\cos \theta = -\frac{5}{17}$; $90° < \theta < 180°$
47. $\cos \theta = \frac{12}{13}$; $270° < \theta < 360°$

Example 7 **Verify that $\csc 2\theta = \frac{\sec \theta}{2 \sin \theta}$ is an identity.**

$\csc 2\theta \stackrel{?}{=} \frac{\sec \theta}{2 \sin \theta}$ Original equation

$\frac{1}{\sin 2\theta} \stackrel{?}{=} \frac{\frac{1}{\cos \theta}}{2 \sin \theta}$ $\csc \theta = \frac{1}{\sin \theta}$, $\sec \theta = \frac{1}{\cos \theta}$

$\frac{1}{\sin 2\theta} \stackrel{?}{=} \frac{1}{2 \sin \theta \cos \theta}$ Simplify the complex fraction.

$\frac{1}{\sin 2\theta} = \frac{1}{\sin 2\theta}$ $2 \sin \theta \cos \theta = \sin 2\theta$

14–7 Solving Trigonometric Equations (pp. 861–866)

Find all solutions of each equation for the interval $0° \le \theta < 360°$.

48. $2 \sin 2\theta = 1$
49. $\cos^2 \theta + \sin^2 \theta = 2 \cos \theta$ **0°**

50. PRISMS The horizontal and vertical components of an oblique prism can be modeled using the equations $Z_x = P \cos \theta$ and $Z_y = P \sin \theta$ where Z_x is the horizontal component, Z_y is the vertical component, P is the power of the prism, and θ is the angle between the prism and the horizontal. For what values of θ will the vertical and horizontal components be equivalent?

Example 8 **Solve $\sin 2\theta + \sin \theta = 0$ if $0° \le \theta < 360°$.**

$\sin 2\theta + \sin \theta = 0$ Original equation

$2 \sin \theta \cos \theta + \sin \theta = 0$ $\sin 2\theta = 2 \sin \theta \cos \theta$

$\sin \theta (2 \cos \theta + 1) = 0$ Factor.

$\sin \theta = 0$ or $2 \cos \theta + 1 = 0$

$\theta = 0°$ or $180°$ $\cos \theta = -\frac{1}{2}$

$\theta = 120°$ or $240°$

The solutions are 0°, 120°, 180°, and 240°.

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CHAPTER 14

Practice Test

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

1. $y = \frac{2}{3}\sin 2\theta + 5$
2. $y = 4\cos\left[\frac{1}{2}(\theta + 30°)\right] - 1$
3. $y = 7\cos\left[4\left(\theta + \frac{\pi}{6}\right)\right]$
4. **AUTOMOTIVE** The pistons in a car oscillate according to a sine function. The amplitude of the oscillation is 2, the period is 6π, and the phase shift is $\frac{\pi}{2}$ to the left. Write a formula to model the position of the piston, p, at time t seconds. Graph the equation.

Find the value of each expression.

5. $\tan\theta$, if $\sin\theta = \frac{1}{2}$; $90° < \theta < 180°$
6. $\sec\theta$, if $\cot\theta = \frac{3}{4}$; $180° < \theta < 270°$
7. $\csc\theta$, if $\sec\theta = \frac{\sqrt{5}}{2}$; $270° < \theta < 360°$

Verify that each of the following is an identity.

8. $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$
9. $\frac{\cos\theta}{1 - \sin^2\theta} = \sec\theta$
10. $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$
11. $\frac{1 + \tan^2\theta}{\cos^2\theta} = \sec^4\theta$
12. **RACING** Race tracks are designed based on the average car velocity so that the angle of the track prevents sliding in the curves. The equation for the banking angle is $\tan\theta = \frac{v^2}{gr}$ where v is velocity, g is gravity, and r is the radius of the curve. Write an equivalent expression using $\sec\theta$ and $\csc\theta$.

Find the exact value of each expression.

13. $\cos 165°$
14. $\sin 255°$
15. $\sin(-225°)$
16. $\cos 480°$
17. $\cos 67.5°$
18. $\sin 75°$

Solve each equation for all values of θ if θ is measured in degrees.

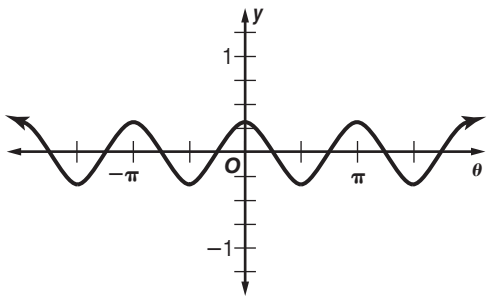
19. $\sec\theta = 1 + \tan\theta$
20. $\cos 2\theta = \cos\theta$
21. $\cos 2\theta + \sin\theta = 1$
22. $\sin\theta = \tan\theta$

GOLF For Exercises 23 and 24, use the following information.

A golf ball leaves the club with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula $d = \frac{v_0^2}{g}\sin 2\theta$, where v_0 is the initial velocity, g is the acceleration due to gravity, and θ is the measurement of the angle that the path of the ball makes with the ground. The acceleration due to gravity is 32 feet per second squared.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°.
24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree?

25. **MULTIPLE CHOICE** Identify the equation of the graphed function.



A $y = 3\cos 2\theta$
C $y = 3\cos\frac{1}{2}\theta$
B $y = \frac{1}{3}\cos 2\theta$
D $y = \frac{1}{3}\cos\frac{1}{2}\theta$

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CHAPTER 14

Standardized Test Practice

Cumulative, Chapters 1–14

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.

5, 14, 6, 8, 12

If the mean of this data is 9, what is the population standard deviation for these data? (Round to the nearest tenth.)

A 2.6

B 5.7

C 8.6

D 12.3

2. If $f(x) = 2x^3 + 5x - 8$, find $f(2a^2)$.

F $f(2a^2) = 16a^5 + 10a^2 - 8$

G $f(2a^2) = 64a^5 + 10a^2 - 8$

H $f(2a^2) = 16a^6 + 10a^2 - 8$

J $f(2a^2) = 64a^6 + 10a^2 - 8$

3. Simplify $128^{\frac{1}{4}}$.

A $2\sqrt[4]{2}$

B $2\sqrt[4]{8}$

C 4

D $4\sqrt[4]{2}$

4. Lisa is 6 years younger than Petra. Stella is twice as old as Petra. The total of their ages is 54. Which equation can be used to find Petra's age?

F $x + (x - 6) + 2(x - 6) = 54$

G $x - 6x + (x + 2) = 54$

H $x - 6 + 2x = 54$

J $x + (x - 6) + 2x = 54$

5. GRIDDABLE The mean of seven numbers is 0. The sum of three of the numbers is −9. What is the sum of the remaining four numbers?

6. Which of the following functions represents exponential decay?

A $y = 0.2(7)^x$

B $y = (0.5)^x$

C $y = 4(9)^x$

D $y = 5\left(\frac{4}{3}\right)^x$

7. Solve the following system of equations.

$$3y = 4x + 1$$

$$2y - 3x = 2$$

F (−4, −5)

G (−2, −3)

H (2, 3)

J (4, 5)

8. GRIDDABLE If k is a positive integer and $7k + 3$ equals a prime number that is less than 50, then what is one possible value of $7k + 3$?

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9. Find the center and radius of the circle with the equation $(x - 4)^2 + y^2 - 16 = 0$.

A $C(-4, 0)$; $r = 4$ units

B $C(-4, 0)$; $r = 16$ units

C $C(4, 0)$; $r = 4$ units

D $C(4, 0)$; $r = 16$ units

10. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4

G 6

H 8

J 12

11. What is the effect of the graph on the equation $y = 3x^2$ when the equation is changed to $y = 2x^2$?

A The graph of $y = 2x^2$ is a reflection of the graph of $y = 3x^2$ across the y-axis.

B The graph is rotated 90 degress about the origin.

C The graph is narrower.

D The graph is wider.

TEST-TAKING TIP

Question 11 If the question involves a graph but does not include the graph, draw one. A diagram can help you see relationships among the given values that will help you answer the question.

12. GEOMETRY The perimeter of a right triangle is 36 inches. Twice the length of the longer leg minus twice the length of the shorter leg exceeds the hypotenuse by 6 inches. What are the lengths of all three sides?

F 3 in., 4 in., 5 in.

G 6 in., 8 in., 10 in.

H 9 in., 12 in., 15 in.

J 12 in., 16 in., 20 in.

Pre-AP/Anchor Problem

13. The table below shows the cost of a pizza depending on the diameter of the pizza.

	Dimensions	Cost ($)
Round	10″ diameter	8.10
Round	20″ diameter	15.00
Square	10″ side	10.00
Square	20″ side	20.00

Which pizza should you buy if you want to get the most pizza per dollar?

NEED EXTRA HELP?													
If You Missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13
Go to Lesson...	12-6	6-4	7-5	2-4	1-4	9-1	3-1	11-8	10-3	12-4	5-7	Prior Course	1-3

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Student Handbook

Built-In Workbooks

Prerequisite Skills **876**

Extra Practice. **891**

Mixed Problem Solving **926**

Preparing for Standardized Tests **941**

Reference

English-Spanish Glossary **R2**

Selected Answers **R28**

Photo Credits **R103**

Index **R104**

Formulas and Symbols. . . . . **Inside Back Cover**

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How to Use the Student Handbook

The Student Handbook is the additional skill and reference material found at the end of the text. This handbook can help you answer these questions.

What if I Forget What I Learned Last Year?

Use the **Prerequisite Skills** section to refresh your memory about things you have learned in other math classes. Here's a list of the topics covered in your book.

1. The FOIL Method
2. Factoring Polynomials
3. Congruent and Similar Figures
4. Pythagorean Theorem
5. Mean, Median, and Mode
6. Bar and Line Graphs
7. Frequency Tables and Histograms
8. Stem-and-Leaf Plots
9. Box-and-Whisker Plots

What If I Need More Practice?

You, or your teacher, may decide that working through some additional problems would be helpful. The **Extra Practice** section provides these problems for each lesson so you have ample opportunity to practice new skills.

What If I Have Trouble with Word Problems?

The **Mixed Problem Solving** portion of the book provides additional word problems that use the skills presented in each lesson. These problems give you real-world situations where math can be applied.

What If I Need to Practice for a Standardized Test?

You can review the types of problems commonly used for standardized tests in the **Preparing for Standardized Tests** section. This section includes examples and practice with multiple-choice, griddable or grid-in, and extended-response test items.

What If I Forget a Vocabulary Word?

The **English-Spanish Glossary** provides a list of important or difficult words used throughout the textbook. It provides a definition in English and Spanish as well as the page number(s) where the word can be found.

What If I Need to Check a Homework Answer?

The answers to odd-numbered problems are included in **Selected Answers**. Check your answers to make sure you understand how to solve all of the assigned problems.

What If I Need to Find Something Quickly?

The **Index** alphabetically lists the subjects covered throughout the entire textbook and the pages on which each subject can be found.

What if I Forget a Formula?

Inside the back cover of your math book is a list of **Formulas and Symbols** that are used in the book.

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Prerequisite Skills

1 The FOIL Method

The product of two binomials is the sum of the products of **F** the *first* terms, **O** the *outer* terms, **I** the *inner* terms, and **L** the *last* terms.

EXAMPLE

1 **Find $(x + 3)(x - 5)$.**

F L

$(x + 3)(x - 5) = x \cdot x + (-5) \cdot x + 3 \cdot x + (-3) \cdot 5$

I

O

First Outer Inner Last

$= x^2 - 5x + 3x - 15$

$= x^2 - 2x - 15$

EXAMPLE

2 **Find $(3y + 2)(5y + 4)$.**

$(3y + 2)(5y + 4) = y \cdot y + 4 \cdot 3y + 2 \cdot 5y + 2 \cdot 4$

$= y^2 + 12y + 10y + 8$

$= y^2 + 22y + 8$

Exercises

Find each product.

1. $(a + 2)(a + 4)$
2. $(v - 7)(v - 1)$
3. $(h + 4)(h - 4)$
4. $(d - 1)(d + 1)$
5. $(b + 4)(b - 3)$
6. $(s - 9)(s + 11)$
7. $(r + 3)(r - 8)$
8. $(k - 2)(k + 5)$
9. $(p + 8)(p + 8)$
10. $(x - 15)(x - 15)$
11. $(2c + 1)(c - 5)$
12. $(7n - 2)(n + 3)$
13. $(3m + 4)(2m - 5)$
14. $(5g + 1)(6g + 9)$
15. $(2q - 17)(q + 2)$
16. $(4t - 7)(3t - 12)$

NUMBER **For Exercises 17 and 18, use the following information.**

I'm thinking of two integers. One is 7 less than a number, and the other is 2 greater than the same number.

17. Write expressions for the two numbers.
18. Write a polynomial expression for the product of the numbers.

OFFICE SPACE **For Exercises 19–21, use the following information.**

Monica's current office is square. Her office in the company's new building will be 3 feet wider and 5 feet longer.

19. Write expressions for the dimensions of Monica's new office.
20. Write a polynomial expression for the area of Monica's new office.
21. Suppose Monica's current office is 7 feet by 7 feet. How much larger will her new office be?

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2 Factoring Polynomials

Some polynomials can be factored using the Distributive Property.

EXAMPLE

1 **Factor $4a^2 + 8a$.**

Find the GCF of $4a^2$ and $8a$.

$4a^2 = 2 \cdot 2 \cdot a \cdot a$ $\quad 8a = 2 \cdot 2 \cdot 2 \cdot a$ $\quad$ GCF: $2 \cdot 2 \cdot a$ or $4a$

$4a^2 + 8a = 4a(a) + 4a(2)$ Rewrite each term using the GCF.

$= 4a(a + 2)$ Distributive Property

To factor quadratic trinomials of the form $x^2 + bx + c$, find two integers m and n with a product of c and with a sum of b. Then write $x^2 + bx + c$ using the pattern $(x + m)(x + n)$.

EXAMPLE

2 **Factor each polynomial.**

a. $x^2 + 5x + 6$ ← Both b and c are positive.

In this trinomial, b is 5 and c is 6. Find two numbers with a product of 6 and a sum of 5.

Factors of 6	Sum of Factors	
1, 6	7	
2, 3	5	The correct factors are 2 and 3.

$x^2 + 5x + 6 = (x + m)(x + n)$ Write the pattern.

$= (x + 2)(x + 3)$ $m = 2$ and $n = 3$

CHECK Multiply the binomials to check the factorization.

$(x + 2)(x + 3) = x^2 + 3x + 2x + 2(3)$ FOIL

$= x^2 + 5x + 6$ ✓

b. $x^2 - 8x + 12$ ← b is negative and c is positive.

In this trinomial, $b = -8$ and $c = 12$. This means that $m + n$ is negative and mn is positive. So m and n must both be negative.

Factors of 12	Sum of Factors	
−1, −12	−13	
−2, −6	−8	The correct factors are −2 and −6.

$x^2 - 8x + 12 = (x + m)(x + n)$ Write the pattern.

$= [x + (-2)][x + (-6)]$ $m = -2$ and $n = -6$

$= (x - 2)(x - 6)$ Simplify.

c. $x^2 + 14x - 15$ ← b is positive and c is negative.

In this trinomial, $b = 14$ and $c = -15$. This means that $m + n$ is positive and mn is negative. So either m or n must be negative, but not both.

Factors of 12	Sum of Factors	
1, −15	−14	
−1, 15	14	The correct factors are −1 and 15.

$x^2 + 14x - 15 = (x + m)(x + n)$ Write the pattern.

$= [x + (-1)](x + 15)$ $m = -1$ and $n = 15$

$= (x - 1)(x + 15)$ Simplify.

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To factor quadratic trinomials of the form $ax^2 + bx + c$, find two integers m and n whose product is equal to ac and whose sum is equal to b. Write $ax^2 + bx + c$ using the pattern $ax^2 + mx + nx + c$. Then factor by grouping.

EXAMPLE

3 **Factor $6x^2 + 7x - 3$.**

In this trinomial, $a = 6$, $b = 7$ and $c = -3$. Find two numbers with a product of $6 \cdot (-3)$ or -18 and a sum of 7.

Factors of -18	Sum of Factors
$1, -18$	-17
$-1, 18$	17
$2, -9$	-7
$-2, 9$	7

The correct factors are -2 and 9.

$6x^2 + 7x - 3 = 6x^2 + mx + nx - 3$ Write the pattern.

$= 6x^2 + (-2)x + 9x - 3$ $m = -2$ and $n = 9$

$= (6x^2 - 2x) + (9x - 3)$ Group terms with common factors.

$= 2x(3x - 1) + 3(3x - 1)$ Factor the GCF from each group.

$= (2x + 3)(3x - 1)$ Distributive Property

Here are some special products.

Perfect Square Trinomials

$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$

$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$

Difference of Squares

$a^2 - b^2 = (a + b)(a - b)$

EXAMPLE

4 **Factor each polynomial.**

a. $4x^2 + 20x + 25$

The first and last terms are perfect squares. The middle term is equal to $2(2x)(5)$. This is a perfect square trinomial of the form $(a + b)^2$.

$4x^2 + 20x + 25 = (2x)^2 + 2(2x)(5) + 5^2$ Write as $a^2 + 2ab + b^2$.

$= (2x + 5)^2$ Factor using the pattern.

b. $x^2 - 4$

This is a difference of squares.

$x^2 - 4 = x^2 - (2)^2$ Write in the form $a^2 - b^2$.

$= (x + 2)(x - 2)$ Factor the difference of squares.

Exercises

Factor the following polynomials.

1. $12x^2 + 4x$
2. $6x^2 y + 2x$
3. $8ab^2 - 12ab$
4. $x^2 + 5x + 4$
5. $y^2 + 12y + 27$
6. $x^2 + 6x + 8$
7. $3y^2 + 13y + 4$
8. $7x^2 + 51x + 14$
9. $3x^2 + 28x + 32$
10. $x^2 - 5x + 6$
11. $y^2 - 5y + 4$
12. $6x^2 - 13x + 5$
13. $6a^2 - 50ab + 16b^2$
14. $11x^2 - 78x + 7$
15. $18x^2 - 31xy + 6y^2$
16. $x^2 + 4xy + 4y^2$
17. $9x^2 - 24x + 16$
18. $4a^2 + 12ab + 9b^2$
19. $x^2 - 144$
20. $4c^2 - 9$
21. $16y^2 - 1$
22. $25x^2 - 4y^2$
23. $36y^2 - 16$
24. $9a^2 - 49b^2$

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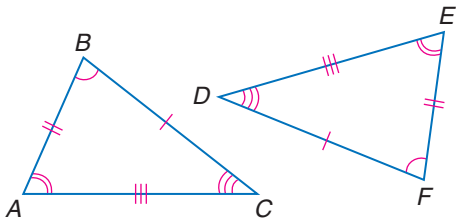
3 Congruent and Similar Figures

Congruent figures have the same size and the same shape.

Two polygons are congruent if their corresponding sides are congruent and their corresponding angles are congruent.



$\triangle ABC \cong \triangle EFD$

The order of the vertices indicates the corresponding parts.

Congruent Angles	Congruent Sides
$\angle A \cong \angle E$	$\overline{AB} \cong \overline{EF}$
$\angle B \cong \angle F$	$\overline{BC} \cong \overline{FD}$
$\angle C \cong \angle D$	$\overline{AC} \cong \overline{ED}$

Read the symbol $\cong$ as *is congruent to*.

EXAMPLE

1 **The corresponding parts of two congruent triangles are marked on the figure. Write a congruence statement for the two triangles.**

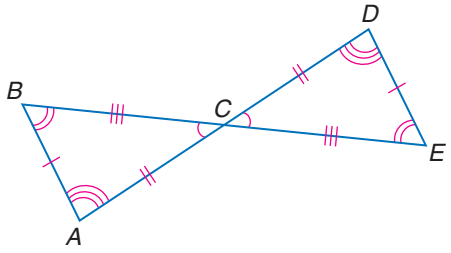
List the congruent angles and sides.

$\angle A \cong \angle D$ $\quad \overline{AB} \cong \overline{DE}$

$\angle B \cong \angle E$ $\quad \overline{AC} \cong \overline{DC}$

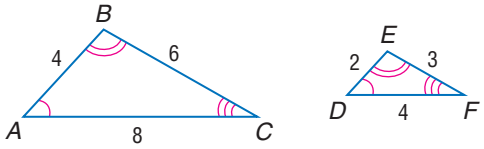
$\angle ACB \cong \angle DCE$ $\quad \overline{BC} \cong \overline{EC}$

Match the vertices of the congruent angles. Therefore, $\triangle ABC \cong \triangle DEC$.



Similar figures have the same shape, but not necessarily the same size.

In similar figures, corresponding angles are congruent, and the measures of corresponding sides are proportional. (They have equivalent ratios.)



Congruent Angles

$\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F$

Proportional Sides

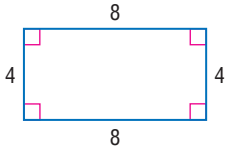
$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

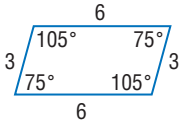
$\triangle ABC \sim \triangle DEF$ ← Read the symbol $\sim$ as *is similar to*.

EXAMPLE

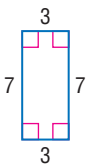
2 **Determine whether the polygons are similar. Justify your answer.**

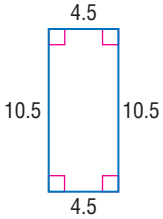
a. Since $\frac{4}{3} = \frac{8}{6} = \frac{4}{3} = \frac{8}{6}$, the measures of the sides of the polygons are proportional. However, the corresponding angles are not congruent. The polygons are not similar.





b. Since $\frac{7}{10.5} = \frac{3}{4.5} = \frac{7}{10.5} = \frac{3}{4.5}$, the measures of the sides of the polygons are proportional. The corresponding angles are congruent. Therefore, the polygons are similar.





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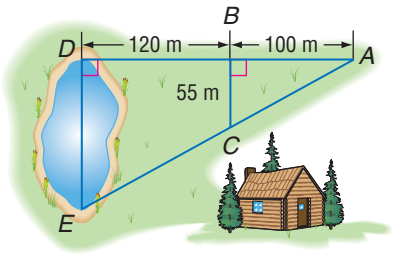
EXAMPLE

3 **CIVIL ENGINEERING** The city of Mansfield plans to build a bridge across Pine Lake. Use the information in the diagram to find the distance across Pine Lake.



$\triangle ABC \sim \triangle ADE$

$\frac{AB}{AD} = \frac{BC}{DE}$ Definition of similar polygons

$\frac{100}{220} = \frac{55}{DE}$ $AB = 100$, $AD = 100 + 120 = 220$, $BC = 55$

$100DE = 220(55)$ Cross products

$100DE = 12{,}100$ Simplify.

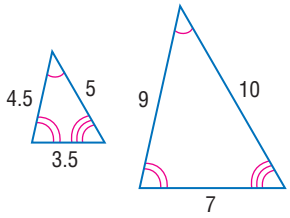
$DE = 121$ Divide each side by 100.

The distance across the lake is 121 meters.

Exercises

Determine whether each pair of figures is *similar*, *congruent*, or *neither*.

1.



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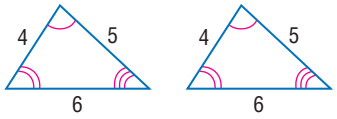

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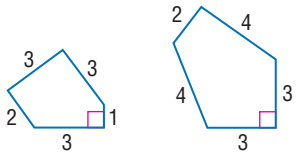

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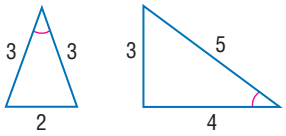

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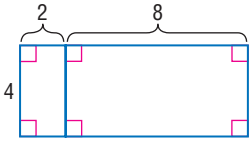

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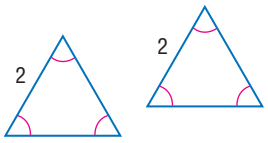
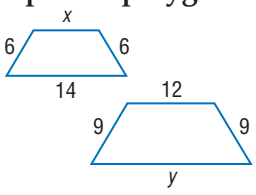


Each pair of polygons is similar. Find the values of x and y.

7.



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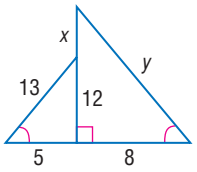

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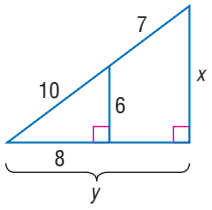
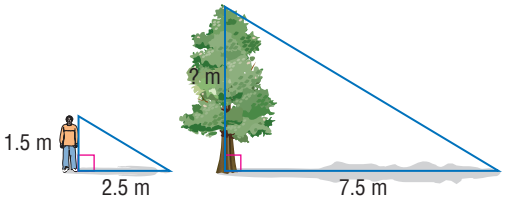


10. **SHADOWS** On a sunny day, Jason measures the length of his shadow and the length of a tree's shadow. Use the figures at the right to find the height of the tree.



11. **PHOTOGRAPHY** A photo that is 4 inches wide by 6 inches long must be reduced to fit in a space 3 inches wide. How long will the reduced photo be?

12. **SURVEYING** Surveyors use instruments to measure objects that are too large or too far away to measure by hand. They can use the shadows that objects cast to find the height of the objects without measuring them. A surveyor finds that a telephone pole that is 25 feet tall is casting a shadow 20 feet long. A nearby building is casting a shadow 52 feet long. What is the height of the building?

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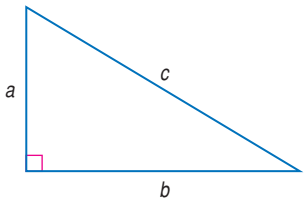
4 Pythagorean Theorem

The **Pythagorean Theorem** states that in a right triangle, the square of the length of the hypotenuse c is equal to the sum of the squares of the lengths of the legs a and b.

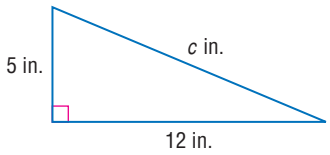
That is, in any right triangle, $c^2 = a^2 + b^2$.



EXAMPLE 1

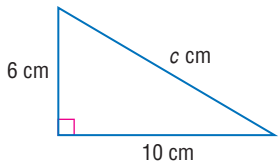
Find the length of the hypotenuse of each right triangle.

a.



$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = 5^2 + 12^2$	Replace a with 5 and b with 12.
$c^2 = 25 + 144$	Simplify.
$c^2 = 169$	Add.
$c = \sqrt{169}$	Take the square root of each side.
$c = 13$	The length of the hypotenuse is 13 inches.

b.

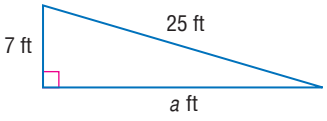


$c^2 = a^2 + b^2$	Pythagorean Theorem
$c^2 = 6^2 + 10^2$	Replace a with 6 and b with 10.
$c^2 = 36 + 100$	Simplify.
$c^2 = 136$	Add.
$c = \sqrt{136}$	Take the square root of each side.
$c \approx 11.7$	Use a calculator. To the nearest tenth, the length of the hypotenuse is 11.7 centimeters.

EXAMPLE 2

Find the length of the missing leg in each right triangle.

a.



$c^2 = a^2 + b^2$	Pythagorean Theorem
$25^2 = a^2 + 7^2$	Replace c with 25 and b with 7.
$625 = a^2 + 49$	Simplify.
$625 - 49 = a^2 + 49 - 49$	Subtract 49 from each side.
$576 = a^2$	Simplify.
$\sqrt{576} = a$	Take the square root of each side.
$24 = a$	The length of the leg is 24 feet.

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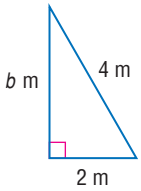
b.



$c^2 = a^2 + b^2$	Pythagorean Theorem
$4^2 = 2^2 + b^2$	Replace c with 4 and a with 2.
$16 = 4 + b^2$	Simplify.
$12 = b^2$	Subtract 4 from each side.
$\sqrt{12} = b$	Take the square root of each side.
$3.5 \approx b$	Use a calculator to find the square root of 12. Round to the nearest tenth.

To the nearest tenth, the length of the leg is 3.5 meters.

EXAMPLE

3 **The lengths of the three sides of a triangle are 5, 7, and 9 inches. Determine whether this triangle is a right triangle.**

Since the longest side is 9 inches, use 9 as c, the measure of the hypotenuse.

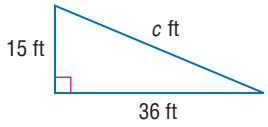
$c^2 = a^2 + b^2$	Pythagorean Theorem
$9^2 \stackrel{?}{=} 5^2 + 7^2$	Replace c with 9, a with 5, and b with 7.
$81 \stackrel{?}{=} 25 + 49$	Evaluate 9^2, 5^2, and 7^2.
$81 \neq 74$	Simplify.

Since $c^2 \neq a^2 + b^2$, the triangle is *not* a right triangle.

Exercises

Find each missing measure. Round to the nearest tenth, if necessary.

1.



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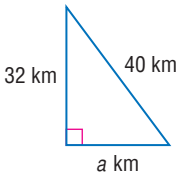

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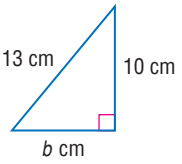

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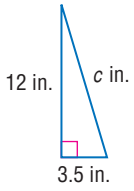

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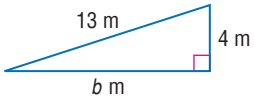

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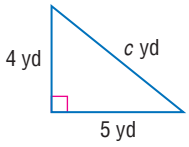


7. $a = 3, b = 4, c = ?$

8. $a = ?, b = 12, c = 13$

9. $a = 14, b = ?, c = 50$

10. $a = 2, b = 9, c = ?$

11. $a = 6, b = ?, c = 13$

12. $a = ?, b = 7, c = 11$

The lengths of three sides of a triangle are given. Determine whether each triangle is a right triangle.

13. 5 in., 7 in., 8 in.

14. 9 m, 12 m, 15 m

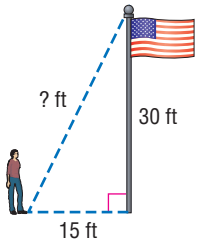
15. 6 cm, 7 cm, 12 cm

16. 11 ft, 12 ft, 16 ft

17. 10 yd, 24 yd, 26 yd

18. 11 km, 60 km, 61 km

19. **FLAGPOLES** Mai-Lin wants to find the distance from her feet to the top of the flagpole. If the flagpole is 30 feet tall and Mai-Lin is standing a distance of 15 feet from the flagpole, what is the distance from her feet to the top of the flagpole?



20. **CONSTRUCTION** The walls of the Downtown Recreation Center are being covered with paneling. The doorway into one room is 0.9 meter wide and 2.5 meters high. What is the width of the widest rectangular panel that can be taken through this doorway?

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5 Mean, Median, and Mode

Mean, median, and mode are measures of central tendency that are often used to represent a set of data.

- To find the **mean**, find the sum of the data and divide by the number of items in the data set. (The mean is often called the average.)
- To find the **median**, arrange the data in numerical order. The median is the middle number. If there is an even number of data, the median is the mean of the two middle numbers.
- The **mode** is the number (or numbers) that appears most often in a set of data. If no item appears most often, the set has no mode.

EXAMPLE

1 **Michelle is saving to buy a car. She saved $200 in June, $300 in July, $400 in August, and $150 in September. What was her mean (or average) monthly savings?**

mean = sum of monthly savings/number of months

$$= \frac{\$200 + \$300 + \$400 + \$150}{4}$$

$$= \frac{\$1050}{4} \text{ or } \$262.50$$

Michelle's mean monthly savings was $262.50.

EXAMPLE

2 **Find the median of the data.**

To find the median, order the numbers from least to greatest. The median is in the middle. The two middle numbers are 3.7 and 4.1.

$$\frac{3.7 + 4.1}{2} = 3.9$$

There is an even number of data. Find the mean of the middle two.

Peter's Best Running Times	
Week	Minutes to Run a Mile
1	4.5
2	3.7
3	4.1
4	4.1
5	3.6
6	3.4

EXAMPLE

3 **GOLF** **Four players tied for first in the 2001 PGA Tour Championship. The scores for each player for each round are shown in the table below. What is the mode score?**

Player	Round 1	Round 2	Round 3	Round 4
Mike Weir	68	66	68	68
David Toms	73	66	64	67
Sergio Garcia	69	67	66	68
Ernie Els	69	68	65	68

Source: ESPN

The mode is the score that occurred most often. Since the score of 68 occurred 6 times, it is the mode of these data.

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The range of a set of data is the difference between the greatest and the least values of the set. It describes how a set of data varies.

EXAMPLE

4 **Find the range of the data.** {6, 11, 18, 4, 9, 15, 6, 3}

The greatest value is 18 and the least value is 3. So, the range is 18 − 3 or 15.

Exercises **Find the mean, median, mode, and range for each set of data. Round to the nearest tenth if necessary.**

1. {2, 8, 12, 13, 15}
2. {66, 78, 78, 64, 34, 88}
3. {87, 95, 84, 89, 100, 82}
4. {99, 100, 85, 96, 94, 99}
5. {9.9, 9.9, 10, 9.9, 8.8, 9.5, 9.5}
6. {501, 503, 502, 502, 502, 504, 503, 503}
7. {7, 19, 15, 13, 11, 17, 9}
8. {6, 12, 21, 43, 1, 3, 13, 8}
9. {0.8, 0.04, 0.9, 1.1, 0.25}
10. $\left\{2\frac{1}{2}, 1\frac{7}{8}, 2\frac{5}{8}, 2\frac{3}{4}, 2\frac{1}{8}\right\}$

11. **CHARITY** The table shows the amounts collected by classes at Jackson High School. Find the mean, median, mode, and range of the data.

Amounts Collected for Charity			
Class	Amount	Class	Amount
A	$150	E	$10
B	$300	F	$25
C	$55	G	$200
D	$40	H	$100

12. **SCHOOL** The table shows Pilar's grades in chemistry class for the semester. Find her mean, median, and mode scores, and the range of her scores.

Chemistry Grades	
Assignment	Grade (out of 100)
Homework	100
Electron Project	98
Test I	87
Atomic Mass Project	95
Test II	88
Phase Change Project	90
Test III	95

13. **WEATHER** The table shows the precipitation for the month of July in Cape Hatteras, North Carolina, in various years. Find the mean, median, mode, and range of the data.

July Precipitation in Cape Hatteras, North Carolina												
Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Inches	4.22	8.58	5.28	2.03	3.93	1.08	9.54	4.94	10.85	2.66	6.04	3.26

Source: National Climatic Data Center

14. **SCHOOL** Kaitlyn's scores on her first five algebra tests are 88, 90, 91, 89, and 92. What test score must Kaitlyn earn on the sixth test so that her mean score will be at least 90?
15. **GOLF** Colin's average for three rounds of golf is 94. What is the highest score he can receive for the fourth round to have an average (mean) of 92?
16. **SCHOOL** Mika has a mean score of 21 on his first four Spanish quizzes. If each quiz is worth 25 points, what is the highest possible mean score he can have after the fifth quiz?
17. **SCHOOL** To earn a grade of B in math, Latisha must have an average (mean) score of at least 84 on five math tests. Her scores on the first three tests are 85, 89, and 82. What is the lowest total score that Latisha must have on the last two tests to earn a B test average?

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6 Bar and Line Graphs

A **bar graph** compares different categories of data by showing each as a bar whose length is related to the frequency. A **double bar graph** compares two sets of data. Another way to represent data is by using a **line graph**. A line graph usually shows how data changes over a period of time.

EXAMPLE

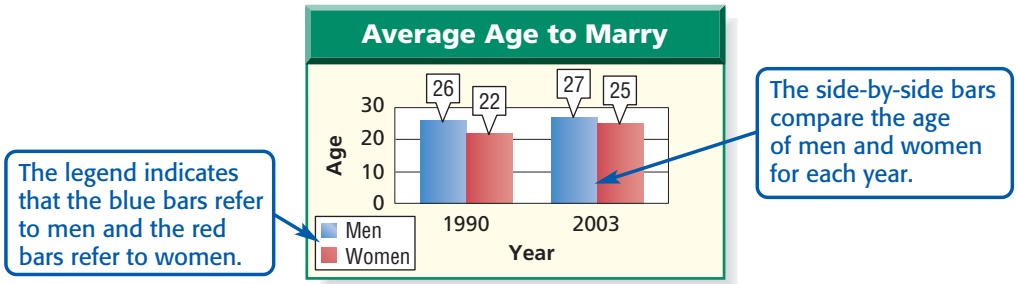
1 **MARRIAGE** **The table shows the average age at which Americans marry for the first time. Make a double bar graph to display the data.**

Average Age to Marry		
Year	1990	2003
Men	26	27
Women	22	25

Source: U.S. Census Bureau

Step 1 Draw a horizontal and a vertical axis and label them as shown.

Step 2 Draw side-by-side bars to represent each category.



EXAMPLE

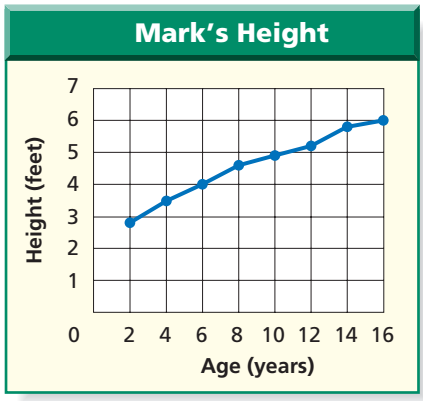
2 **HEALTH** **The table shows Mark's height at 2-year intervals. Make a line graph to display the data.**

Age	2	4	6	8	10	12	14	16
Height (feet)	2.8	3.5	4.0	4.6	4.9	5.2	5.8	6

Step 1 Draw a horizontal and a vertical axis. Label them as shown.

Step 2 Plot the points.

Step 3 Draw a line connecting each pair of consecutive points.



Exercises

1. **HEALTH** The table below shows the life expectancy for Americans born in each year listed. Make a double-bar graph to display the data.

Life Expectancy		
Year of Birth	Male	Female
1980	70.0	77.5
1985	71.2	78.2
1990	71.8	78.8
1995	72.5	78.9
1998	73.9	79.4

2. **MONEY** The amount of money in Becky's savings account from August through March is shown in the table below. Make a line graph to display the data.

Month	Amount	Month	Amount
August	$300	December	$780
September	$400	January	$800
October	$700	February	$950
November	$780	March	$900

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7 Frequency Tables and Histograms

A **frequency table** shows how often an item appears in a set of data. A tally mark is used to record each response. The total number of marks for a given response is the *frequency* of that response. Frequencies can be shown in a bar graph called a histogram. A **histogram** differs from other bar graphs in that no space is between the bars and the bars usually represent numbers grouped by intervals.

EXAMPLE

1 TELEVISION **Use the frequency table of Brad's data.**

Favorite Television Shows		
Program	Tally	Frequency
Sports	𝍸 II	7
Mysteries	IIII	4
Soap operas	𝍸	5
News	𝍸	5
Quiz shows	𝍸 I	6
Music videos	II	2
Adventure	𝍸 IIII	9
Comedies	𝍸 II	7

a. How many more chose sports programs than news?

b. Which two programs together have the same frequency as adventures?

a. Seven people chose sports. Five people chose news. $7 - 5 = 2$, so 2 more people chose sports than news.

b. As many people chose adventures as the following pairs of programs.

sports and music videos — mysteries and soap operas
mysteries and news — comedies and music videos

EXAMPLE

2 FITNESS **A gym teacher tested the number of sit-ups students in two classes could do in 1 minute. The results are shown.**

Number of Sit-Ups	Frequency
0–4	8
5–9	12
10–14	15
15–19	6
20–24	18
25–29	10

a. Make a histogram of the data. Title the histogram.

b. How many students were able to do 25–29 sit-ups in 1 minute?

c. How many students were unable to do 10 sit-ups in 1 minute?

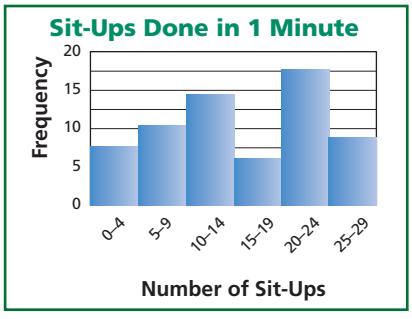
d. Between which two consecutive intervals does the greatest increase in frequency occur? What is the increase?

a. Use the same intervals as those in the frequency table on the horizontal axis. Label the vertical axis with a scale that includes the frequency numbers from the table.

b. Ten students were able to do 25–29 sit-ups in 1 minute.

c. Add the students who did 0–4 sit-ups and 5–9 sit-ups. So $8 + 12$, or 20, students were unable to do 10 sit-ups in 1 minute.

d. The greatest increase is between intervals 15–19 and 20–24. These frequencies are 6 and 18. So the increase is $18 - 6 = 12$.



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Exercises

ART **For Exercises 1–4, use the following information.**

The prices in dollars of paintings sold at an art auction are shown.

1800	750	600	600	1800	1350	300	1200	750	600	750	2700
600	750	300	750	600	450	2700	1200	600	450	450	300

1. Make a frequency table of the data.
2. What price was paid most often for the artwork?
3. What is the average price paid for artwork at this auction?
4. How many works of art sold for at least \$600 and no more than \$1200?

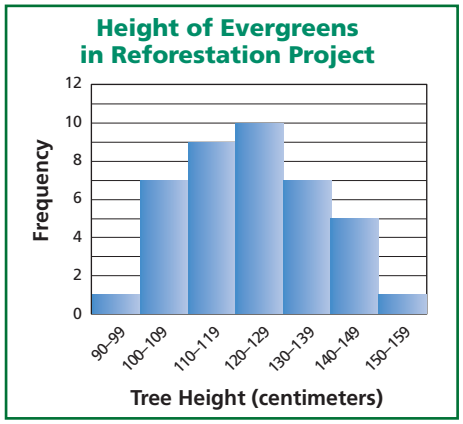
PETS **For Exercises 5–9, use the following information.**

Number of Pets per Family

1	2	3	1	0	2	1	0
1	0	1	4	1	2	0	0
0	1	1	2	2	5	1	0

5. Use a frequency table to make a histogram of the data.
6. How many families own two to three pets?
7. How many families own more than three pets?
8. To the nearest percent, what percent of families own no pets?
9. Name the median, mode, and range of the data.

TREES **For Exercises 10–12, use the histogram shown.**



10. Which interval contains the most evergreen seedlings?
11. Which intervals contain an equal number of trees?
12. Which intervals contain 95% of the data?
13. Between which two consecutive intervals does the greatest increase in frequency occur? What is the increase?

14. **MARKET RESEARCH** A civil engineer is studying traffic patterns. She counts the number of cars that make it through one rush hour green light cycle. Organize her data into a frequency table, and then make a histogram.

15 16 10 8 8 14 9 7 6 9 10 11 14 10 7 8 9 11 14 10

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8 Stem-and-Leaf Plots

In a **stem-and-leaf plot**, data are organized in two columns. The greatest place value of the data is used for the stems. The next greatest place value forms the leaves. Stem-and-leaf plots are useful for organizing long lists of numbers.

EXAMPLE

SCHOOL **Isabella has collected data on the GPAs (grade point average) of the 16 students in the art club. Display the data in a stem-and-leaf plot.**
{4.0, 3.9, 3.1, 3.9, 3.8, 3.7, 1.8, 2.6, 4.0, 3.9, 3.5, 3.3, 2.9, 2.5, 1.1, 3.5}

Step 1 Find the least and the greatest number. Then identify the greatest place-value digit in each number. In this case, ones.

least data: 1.1 — The least number has 1 in the ones place.

greatest data: 4.0 — The greatest number has 4 in the ones place.

Step 2 Draw a vertical line and write the stems from 1 to 4 to the left of the line.

Step 3 Write the leaves to the right of the line, with the corresponding stem. For example, write 0 to the right of 4 for 4.0.

Stem	Leaf
1	8 1
2	5 6 9
3	9 1 9 8 7 9 5 3 5
4	0 0

Step 4 Rearrange the leaves so they are ordered from least to greatest.

Stem	Leaf
1	1 8
2	5 6 9
3	1 3 5 5 7 8 9 9 9
4	0 0

Step 5 Include a key or an explanation. *3|1 = 3.1*

Exercises

GAMES **For Exercises 1–4, use the following information.**
The stem-and-leaf plot at the right shows Charmaine's scores for her favorite computer game.

Stem	Leaf
9	0 0 0 1 3 4 5 5 7 8 8 8 9 9
10	0 3 4 4 5 6 9
11	0 3 9 9
12	1 2 6
13	0

12|6 = 126

1. What are Charmaine's highest and lowest scores?
2. Which score(s) occurred most frequently?
3. How many scores were above 115?
4. Has Charmaine ever scored 123?
5. **SCHOOL** The class scores on a 50-item test are shown in the table at the right. Make a stem-and-leaf plot of the data.

Test Scores					
45	15	30	40	28	35
39	29	38	18	43	49
46	44	48	35	36	30

6. **GEOGRAPHY** The table shows the land area of each county in Wyoming. Round each area to the nearest hundred square miles and organize the data in a stem-and-leaf plot.

County	Area (mi)^2	County	Area (mi)^2	County	Area (mi)^2
Albany	4273	Hot Springs	2004	Sheridan	2523
Big Horn	3137	Johnson	4166	Sublette	4883
Campbell	4797	Laramie	2686	Sweetwater	10,425
Carbon	7896	Lincoln	4069	Teton	4008
Converse	4255	Natrona	5340	Unita	2082
Crook	2859	Niobrara	2626	Washakie	2240
Fremont	9182	Park	6942	Weston	2398
Goshen	2225	Platte	2085		

Source: *The World Almanac*

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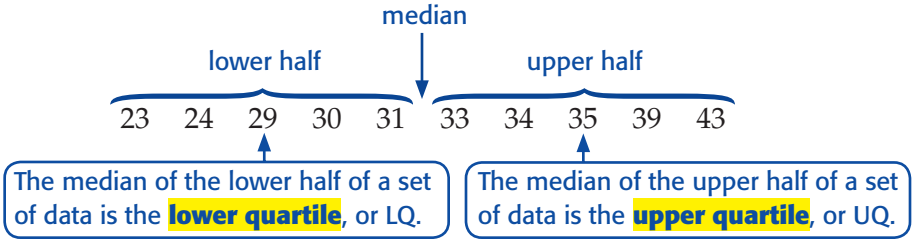
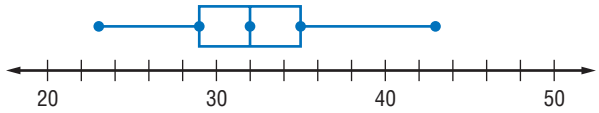
9 Box-and-Whisker Plots

In a set of data, **quartiles** are values that divide the data into four equal parts.



To make a **box-and-whisker plot**, draw a box around the quartile values, and lines or *whiskers* to represent the values in the lower fourth of the data and the upper fourth of the data.

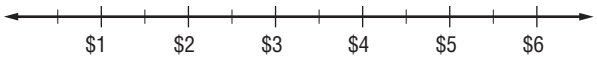


EXAMPLE

1 **MONEY** **The amount spent in the cafeteria by 20 students is shown. Display the data in a box-and-whisker plot.**

Amount Spent			
$2.00	$2.00	$1.00	$4.00
$1.00	$2.50	$2.50	$2.00
$2.50	$1.00	$4.00	$2.50
$3.50	$2.00	$3.00	$2.50
$4.00	$4.00	$5.50	$1.50

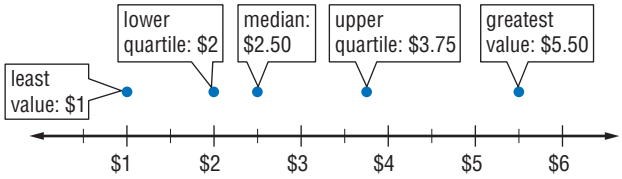
Step 1 Find the least and greatest number. Then draw a number line that covers the range of the data. In this case, the least value is 1 and the greatest value is 5.5.



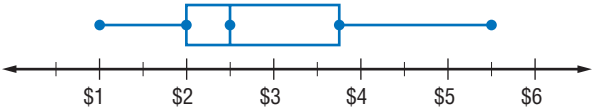
Step 2 Find the median, the extreme values, and the upper and lower quartiles. Mark these points above the number line.

1, 1, 1, 1.5, 2, 2, 2, 2, 2.5, 2.5, 2.5, 2.5, 2.5, 3, 3.5, 4, 4, 4, 4, 5.5

$LQ = \frac{2+2}{2}$ or 2 $M = \frac{2.5+2.5}{2}$ or 2.5 $UQ = \frac{3.5+4}{2}$ or 3.75



Step 3 Draw a box and the whiskers.



The **interquartile range (IQR)** is the range of the middle half of the data and contains 50% of the data in the set.

$$\text{Interquartile range} = UQ - LQ$$

The interquartile range of the data in Example 1 is 3.75 − 2 or 1.75.

An **outlier** is any element of a set that is at least 1.5 interquartile ranges less than the lower quartile or greater than the upper quartile. The whisker representing the data is drawn from the box to the least or greatest value that is not an outlier.

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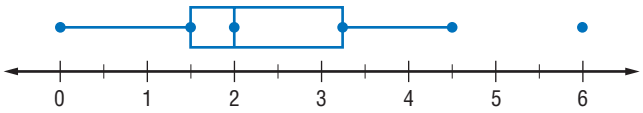
EXAMPLE

2 **SCHOOL** **The number of hours José studied each day for the last month is shown in the box-and-whisker plot below.**



a. **What percent of the data lies between 1.5 and 3.25?**
The value 1.5 is the lower quartile and 3.25 is the upper quartile. The values between the lower and upper quartiles represent 50% of the data.

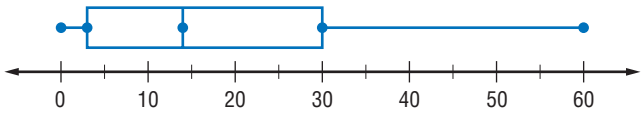
b. **What was the greatest amount of time José studied in a day?**
The greatest value in the plot is 6, so the greatest amount of time José studied in a day was 6 hours.

c. **What is the interquartile range of this box-and-whisker plot?**
The interquartile range is UQ − LQ. For this plot, the interquartile range is 3.25 − 1.5 or 1.75 hours.

d. **Identify any outliers in the data.**
An outlier is at least 1.5(1.75) less than the lower quartile or more than the upper quartile. Since 3.25 + (1.5)(1.75) = 5.875, and 6 > 5.875, the value 6 is an outlier, and was not included in the whisker.

Exercises

DRIVING **For Exercises 1–3, use the following information.**
Tyler surveyed 20 randomly chosen students at his school about how many miles they drive in an average day. The results are shown in the box-and-whisker plot.



1. What percent of the students drive more than 30 miles in a day?
2. What is the interquartile range of the box-and-whisker plot?
3. Does a student at Tyler's school have a better chance to meet someone who drives the same mileage they do if they drive 50 miles in a day or 15 miles in a day? Why?
4. **SOFT DRINKS** Carlos surveyed his friends to find the number of cans of soft drink they drink in an average week. Make a box-and-whisker plot of the data.
{0, 0, 0, 1, 1, 1, 2, 2, 3, 4, 4, 5, 5, 7, 10, 10, 10, 11, 11}
5. **BASEBALL** The table shows the number of sacrifice hits made by teams in the National Baseball League in one season. Make a box-and-whisker plot of the data.
6. **ANIMALS** The average life span of some animals commonly found in a zoo are as follows: {1, 7, 7, 10, 12, 12, 15, 15, 18, 20, 20, 20, 25, 40, 100}. Make a box-and-whisker plot of the data.

Team	Home Runs	Team	Home Runs
Arizona	71	Milwaukee	65
Atlanta	64	Montreal	64
Chicago	117	New York	52
Cincinnati	66	Philadelphia	67
Colorado	81	Pittsburgh	60
Florida	60	San Diego	29
Houston	71	San Francisco	67
Los Angeles	57	St. Louis	83

Source: ESPN

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Extra Practice

Lesson 1-1

(pages 6–10)

Evaluate each expression if $q = \frac{1}{2}$, $r = 1.2$, $s = -6$, and $t = 5$.

1. $qr - st$
2. $qr \div st$
3. $qrst$
4. $qr + st$
5. $\frac{3q}{4s}$
6. $\frac{5qr}{t}$
7. $\frac{2r(4s - 1)}{t}$
8. $\frac{4q^3s + 1}{t - 1}$

Evaluate each expression if $a = -0.5$, $b = 4$, $c = 5$, and $d = -3$.

9. $3b + 4d$
10. $ab^2 + c$
11. $bc + d \div a$
12. $7ab - 3d$
13. $ad + b^2 - c$
14. $\frac{4a + 3c}{3b}$
15. $\frac{3ab^2 - d^3}{a}$
16. $\frac{5a + ad}{bc}$

Lesson 1-2

(pages 11–17)

Name the sets of numbers to which each number belongs. (Use N, W, Z, Q, I, and R.)

1. 8.2
2. -9
3. $\sqrt{36}$
4. $-\frac{1}{3}$
5. $\sqrt{2}$
6. $-0.\overline{24}$

Name the property illustrated by each equation.

7. $(4 + 9a)2b = 2b(4 + 9a)$
8. $3\left(\frac{1}{3}\right) = 1$
9. $a(3 - 2) = a \cdot 3 - a \cdot 2$
10. $(-3b) + 3b = 0$
11. $jk + 0 = jk$
12. $(2a)b = 2(ab)$

Simplify each expression.

13. $7s + 9t + 2s - 7t$
14. $6(2a + 3b) + 5(3a - 4b)$
15. $4(3x - 5y) - 8(2x + y)$
16. $0.2(5m - 8) + 0.3(6 - 2m)$
17. $\frac{1}{2}(7p + 3q) + \frac{3}{4}(6p - 4q)$
18. $\frac{4}{5}(3v - 2w) - \frac{1}{5}(7v - 2w)$

Lesson 1-3

(pages 18–26)

Write an algebraic expression to represent each verbal expression.

1. twelve decreased by the square of a number
2. twice the sum of a number and negative nine
3. the product of the square of a number and 6
4. the square of the sum of a number and 11

Name the property illustrated by each statement.

5. If $a + 1 = 6$, then $3(a + 1) = 3(6)$.
6. If $x + (4 + 5) = 21$, then $x + 9 = 21$.
7. If $7x = 42$, then $7x - 5 = 42 - 5$.
8. If $3 + 5 = 8$ and $8 = 2 \cdot 4$, then $3 + 5 = 2 \cdot 4$.

Solve each equation. Check your solution.

9. $5t + 8 = 88$
10. $27 - x = -4$
11. $\frac{3}{4}y = \frac{2}{3}y + 5$
12. $8s - 3 = 5(2s + 1)$
13. $3(k - 2) = k + 4$
14. $0.5z + 10 = z + 4$
15. $8q - \frac{q}{3} = 46$
16. $-\frac{2}{7}r + \frac{3}{7} = 5$
17. $d - 1 = \frac{1}{2}(d - 2)$

Solve each equation or formula for the specified variable.

18. $C = \pi r$; for r
19. $I = Prt$, for t
20. $m = \frac{n - 2}{n}$, for n

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Lesson 1-4 (pages 27–31)

Evaluate each expression if $x = -5$, $y = 3$, and $z = -2.5$.

1. $|2x|$
2. $|-3y|$
3. $|2x + y|$
4. $|y + 5z|$
5. $-|x + z|$
6. $8 - |5y - 3|$
7. $2|x| - 4|2 + y|$
8. $|x + y| - 6|z|$

Solve each equation. Check your solutions.

9. $|d + 1| = 7$
10. $|a - 6| = 10$
11. $2|x - 5| = 22$
12. $|t + 9| - 8 = 5$
13. $|p + 1| + 10 = 5$
14. $6|g - 3| = 42$
15. $2|y + 4| = 14$
16. $|3b - 10| = 2b$
17. $|3x + 7| + 4 = 0$
18. $|2c + 3| - 15 = 0$
19. $7 - |m - 1| = 3$
20. $3 + |z + 5| = 10$
21. $2|2d - 7| + 1 = 35$
22. $|3t + 6| + 9 = 30$
23. $|d - 3| = 2d + 9$
24. $|4y - 5| + 4 = 7y + 8$
25. $|2b + 4| - 3 = 6b + 1$
26. $|5t| + 2 = 3t + 18$

Lesson 1-5 (pages 33–39)

Solve each inequality. Then graph the solution set on a number line.

1. $2z + 5 \leq 7$
2. $3r - 8 > 7$
3. $0.75b < 3$
4. $-3x > 6$
5. $2(3f + 5) \geq 28$
6. $-33 > 5g + 7$
7. $-3(y - 2) \geq -9$
8. $7a + 5 > 4a - 7$
9. $5(b - 3) \leq b - 7$
10. $3(2x - 5) < 5(x - 4)$
11. $8(2c - 1) > 11c + 22$
12. $2(d + 4) - 5 \geq 5(d + 3)$
13. $8 - 3t < 4(3 - t)$
14. $-x \geq \frac{x + 4}{7}$
15. $\frac{a + 8}{4} \leq \frac{7 + a}{3}$
16. $-y < \frac{y + 5}{2}$
17. $5(x - 1) - 4x \geq 3(3 - x)$
18. $6s - (4s + 7) > 5 - s$

Define a variable and write an inequality for each problem. Then solve.

19. The product of 7 and a number is greater than 42.
20. The difference of twice a number and 3 is at most 11.
21. The product of −10 and a number is greater than or equal to 20.
22. Thirty increased by a number is less than twice the number plus three.

Lesson 1-6 (pages 41–48)

Write an absolute value inequality for each of the following. Then graph the solution set on a number line.

1. all numbers less than −9 and greater than 9
2. all numbers between −5.5 and 5.5
3. all numbers greater than or equal to −2 and less than or equal to 2

Solve each inequality. Graph the solution set on a number line.

4. $3m - 2 < 7$ or $2m + 1 > 13$
5. $2 < n + 4 < 7$
6. $-3 \leq s - 2 \leq 5$
7. $5t + 3 \leq -7$ or $5t - 2 \geq 8$
8. $7 \leq 4x + 3 \leq 19$
9. $4x + 7 < 5$ or $2x - 4 > 12$
10. $|7x| \geq 21$
11. $|8p| \leq 16$
12. $|7d| \geq -42$
13. $|a + 3| < 1$
14. $|t - 4| > 1$
15. $|2y - 5| < 3$
16. $|3d + 6| \geq 3$
17. $|4x - 1| < 5$
18. $|6v + 12| > 18$
19. $|2r + 4| < 6$
20. $|5w - 3| \geq 9$
21. $|z + 2| \geq 0$
22. $12 + |2q| < 0$
23. $|3h| + 15 < 0$
24. $|5n - 16| \geq 4$

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Lesson 2-1

(pages 58–64)

State the domain and range of each relation. Then determine whether each relation is a function. Write *yes* or *no*.

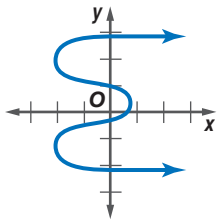
1.

Year	Population
1970	11,605
1980	13,468
1990	15,630
2000	18,140

2.

x	y
1	5
2	5
3	5
4	5

3.



Graph each relation or equation and find the domain and range. Then determine whether the relation or equation is a function and state whether *discrete* or *continuous*.

4. $\{(1, 2), (2, 3), (3, 4), (4, 5)\}$
5. $\{(0, 3), (0, 2), (0, 1), (0, 0)\}$
6. $y = -x$
7. $y = 2x - 1$
8. $y = 2x^2$
9. $y = -x^2$

Find each value if $f(x) = x + 7$ and $g(x) = (x + 1)^2$.

10. $f(2)$
11. $f(-4)$
12. $f(a + 2)$
13. $g(4)$
14. $g(-2)$
15. $f(0.5)$
16. $g(b - 1)$
17. $g(3c)$

Lesson 2-2

(pages 66–70)

State whether each equation or function is linear. Write *yes* or *no*. If no, explain your reasoning.

1. $\frac{x}{2} - y = 7$
2. $\sqrt{x} = y + 5$
3. $g(x) = \frac{2}{x - 3}$
4. $f(x) = 7$

Write each equation in standard form. Identify *A*, *B*, and *C*.

5. $x + 7 = y$
6. $x = -3y$
7. $5x = 7y + 3$
8. $y = \frac{2}{3}x + 8$
9. $-0.4x = 10$
10. $0.75y = -6$

Find the *x*-intercept and the *y*-intercept of the graph of each equation. Then graph the equation.

11. $2x + y = 6$
12. $3x - 2y = -12$
13. $y = -x$
14. $x = 3y$
15. $\frac{3}{4}y - x = 1$
16. $y = -3$

Lesson 2-3

(pages 71–77)

Find the slope of the line that passes through each pair of points.

1. $(0, 3), (5, 0)$
2. $(2, 3), (5, 7)$
3. $(2, 8), (2, -8)$
4. $(1.5, -1), (3, 1.5)$
5. $\left(-\frac{1}{2}, \frac{3}{5}\right), \left(\frac{3}{10}, -\frac{1}{4}\right)$
6. $(-3, c), (4, c)$

Graph the line passing through the given point with the given slope.

7. $(0, 3); 1$
8. $(2, 3); 0$
9. $(-1, 1); -\frac{1}{3}$

Graph the line that satisfies each set of conditions.

10. passes through $(0, 1)$, parallel to a line with a slope of -2
11. passes through $(4, -5)$, perpendicular to the graph of $-2x + 5y = 1$

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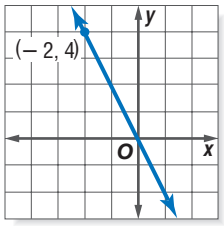
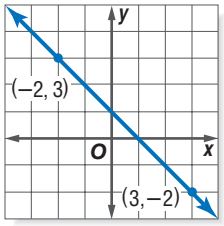
Lesson 2-4

(pages 79–84)

Write an equation in slope-intercept form for each graph.

1. 

2. 

Write an equation in slope-intercept form for the line that satisfies each set of conditions.

3. slope -1, passes through $(7, 2)$
4. slope $\frac{3}{4}$, passes through the origin
5. passes through $(1, -3)$ and $(-1, 2)$
6. x-intercept -5, y-intercept 2
7. passes through $(1, 1)$, parallel to the graph of $2x + 3y = 5$
8. passes through $(0, 0)$, perpendicular to the graph of $2y + 3x = 4$

Lesson 2-5

(pages 86–91)

Complete parts a–c for each set of data in Exercises 1–3.

a. **Draw a scatter plot and describe the correlation.**

b. **Use two ordered pairs to write a prediction equation.**

c. **Use your prediction equation to predict the missing value.**

1.

Telephone Costs	
Minutes	Cost ($)
1	0.20
3	0.52
4	0.68
6	1.00
9	1.48
15	?

2.

Washington	
Year	Population
1960	2,853,214
1970	3,413,244
1980	4,132,353
1990	4,866,669
2000	5,894,121
2010	?

Source: *The World Almanac*

3.

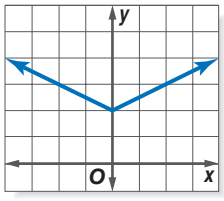
Federal Minimum Wage	
Year	Wage
1981	$3.35
1990	$3.80
1991	$4.25
1996	$4.75
1997	$5.15
2015	?

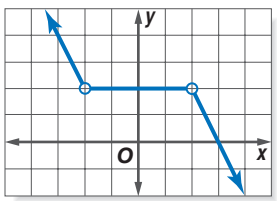
Source: *The World Almanac*

Lesson 2-6

(pages 95–101)

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise.

1. 

2. 

Graph each function. Identify the domain and range.

3. $f(x) = [\![x + 5]\!]$
4. $g(x) = [\![x]\!] - 2$
5. $f(x) = -2[\![x]\!]$
6. $h(x) = |x| - 3$
7. $h(x) = |x - 1|$
8. $g(x) = |2x| + 2$
9. $h(x) = \begin{cases} x \text{ if } x < -2 \\ 4 \text{ if } x \geq -2 \end{cases}$
10. $f(x) = \begin{cases} -3 \text{ if } x \leq 1 \\ -x \text{ if } x > 1 \end{cases}$

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Lesson 2-7

(pages 102–105)

Graph each inequality.

1. $y \geq x - 2$
2. $y < -3x - 1$
3. $4y \leq -3x + 8$
4. $3x > y$
5. $x + 2 \geq y - 7$
6. $2x < 5 - y$
7. $y > \frac{1}{5}x - 8$
8. $2y - 5x \leq 8$
9. $-2x + 5 \leq \frac{2}{3}y$
10. $3x + 2y \geq 0$
11. $x \leq 2$
12. $\frac{y}{2} \leq x - 1$
13. $y - 3 < 5$
14. $y \geq -|x|$
15. $|x| \leq y + 3$
16. $y > |5x - 3|$
17. $y \leq |8 - x|$
18. $y < |x + 3| - 1$
19. $y + |2x| \geq 4$
20. $y \geq |2x - 1| + 5$
21. $y < \left|\frac{2x}{3}\right| - 1$

Lesson 3-1

(pages 116–122)

Solve each system of equations by graphing or by completing a table.

1. $x + 3y = 18$
   $-x + 2y = 7$
2. $x - y = 2$
   $2x - 2y = 10$
3. $2x + 6y = 6$
   $\frac{1}{3}x + y = 1$
4. $x + 3y = 0$
   $2x + 6y = 5$
5. $2x - y = 7$
   $\frac{2}{5}x - \frac{4}{3}y = -2$
6. $y = \frac{1}{3}x + 1$
   $y = 4x + 1$

Graph each system of equations and describe it as *consistent and independent, consistent and dependent,* or *inconsistent*.

7. $2x + 3y = 5$
   $-6x - 9y = -15$
8. $x - 2y = 4$
   $y = x - 2$
9. $y = 0.5x$
   $2y = x + 4$
10. $9x - 5 = 7y$
    $4.5x - 3.5y = 2.5$
11. $\frac{3}{4}x - y = 0$
    $\frac{1}{3}y + \frac{1}{2}x = 6$
12. $\frac{2}{3}x = \frac{5}{3}y$
    $2x - 5y = 0$

Lesson 3-2

(pages 123–129)

Solve each system of equations by using substitution.

1. $2x + 3y = 10$
   $x + 6y = 32$
2. $x = 4y - 10$
   $5x + 3y = -4$
3. $3x - 4y = -27$
   $2x + y = -7$

Solve each system of equations by using elimination.

4. $7x + y = 9$
   $5x - y = 15$
5. $r + 5s = -17$
   $2r - 6s = -2$
6. $6p + 8q = 20$
   $5p - 4q = -26$

Solve each system of equations by using either substitution or elimination.

7. $2x - 3y = 7$
   $3x + 6y = 42$
8. $2a + 5b = -13$
   $3a - 4b = 38$
9. $3c + 4d = -1$
   $6c - 2d = 3$
10. $7x - y = 35$
    $y = 5x - 19$
11. $3m + 4n = 28$
    $5m - 3n = -21$
12. $x = 2y - 1$
    $4x - 3y = 21$
13. $2.5x + 1.5y = -2$
    $3.5x - 0.5y = 18$
14. $\frac{5}{2}x + \frac{1}{3}y = 13$
    $\frac{1}{2}x - y = -7$
15. $\frac{2}{7}c - \frac{4}{3}d = 16$
    $\frac{4}{7}c + \frac{8}{3}d = -16$

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Lesson 3-3

(pages 130–135)

Solve each system of inequalities.

1. $x \le 5$
   $y \ge -3$
2. $y < 3$
   $y - x \ge -1$
3. $x + y < 5$
   $x < 2$
4. $y + x < 2$
   $y \ge x$
5. $x + y \le 2$
   $y - x \le 4$
6. $y \le x + 4$
   $y - x \ge 1$
7. $y < \frac{1}{3}x + 5$
   $y > 2x + 1$
8. $y + x \ge 1$
   $y - x \ge -1$
9. $|x| > 2$
   $|y| \le 5$
10. $|x - 3| \le 3$
    $4y - 2x \le 6$
11. $4x + 3y \ge 12$
    $2y - x \ge -1$
12. $y \le -1$
    $3x - 2y \ge 6$

Find the coordinates of the vertices of the figure formed by each system of inequalities.

13. $y \le 3$
    $x \le 2$
    $y \ge -\frac{3}{2}x + 3$
14. $y \ge -1$
    $y \le x$
    $y \le -x + 4$
15. $y \le \frac{1}{3}x + \frac{7}{3}$
    $4x - y \le 5$
    $y \ge -\frac{3}{2}x + \frac{1}{2}$

Lesson 3-4

(pages 138–144)

A feasible region has vertices at (−3, 2), (1, 3), (6, 1), and (2, −2). Find the maximum and minimum values of each function.

1. $f(x, y) = 2x - y$
2. $f(x, y) = x + 5y$
3. $f(x, y) = y - 4x$
4. $f(x, y) = -x + 3y$
5. $f(x, y) = 3x - y$
6. $f(x, y) = 2y - 2x$

Graph each system of inequalities. Name the coordinates of the vertices of the feasible region. Find the maximum and minimum values of the given function for this region.

7. $4x - 5y \le -10$
   $y \le 6$
   $2x + y \ge 2$
   $f(x, y) = x + y$
8. $x \le 5$
   $y \ge 2$
   $2x - 5y \ge -10$
   $f(x, y) = 3x + y$
9. $x - 2y \ge -7$
   $x + y \le 8$
   $y \ge 5x + 8$
   $f(x, y) = 3x - 4y$
10. $y \le 4x + 6$
    $x + 4y \ge 7$
    $2x + y \le 7$
    $f(x, y) = 2x - y$
11. $y \ge 0$
    $y \le 5$
    $y \le -x + 7$
    $5x + 3y \ge 20$
    $f(x, y) = x + 2y$
12. $y \ge 0$
    $3x - 2y \ge 0$
    $x + 3y \le 11$
    $2x + 3y \le 16$
    $f(x, y) = 4x + y$

Lesson 3-5

(pages 145–152)

For each system of equations, an ordered triple is given. Determine whether or not it is a solution of the system.

1. $4x + 2y - 6z = -38$
   $5x - 4y + z = -18$
   $x + 3y + 7z = 38$;
   $(-3, 2, 5)$
2. $u + 3v + w = 14$
   $2u - v + 3w = -9$
   $4u - 5v - 2w = -2$;
   $(1, 5, -2)$
3. $x + y = -6$
   $x + z = -2$
   $y + z = 2$;
   $(-4, -2, 2)$

Solve each system of equations.

4. $5a = 5$
   $6b - 3c = 15$
   $2a + 7c = -5$
5. $s + 2t = 5$
   $7r - 3s + t = 20$
   $2t = 8$
6. $2u - 3v = 13$
   $3v + w = -3$
   $4u - w = 2$
7. $4a + 2b - c = 5$
   $2a + b - 5c = -11$
   $a - 2b + 3c = 6$
8. $x + 2y - z = 1$
   $x + 3y + 2z = 7$
   $2x + 6y + z = 8$
9. $2x + y - z = 7$
   $3x - y + 2z = 15$
   $x - 4y + z = 2$

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Lesson 4-1

(pages 162–167)

Solve each matrix equation.

1. $[2x \quad 3y \quad -z] = [2y \quad -z \quad 15]$

2. $\begin{bmatrix} x+y \\ 4x-3y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$

3. $-2\begin{bmatrix} w+5 & x-z \\ 3y & 8 \end{bmatrix} = \begin{bmatrix} -16 & -4 \\ 6 & 2x+8z \end{bmatrix}$

4. $y\begin{bmatrix} 2 & x \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -10 \\ 10 & 2z \end{bmatrix}$

5. $\begin{bmatrix} 2x \\ -y \\ 3z \end{bmatrix} = \begin{bmatrix} 16 \\ 18 \\ -21 \end{bmatrix}$

6. $\begin{bmatrix} x-3y \\ 4y-3x \end{bmatrix} = -5\begin{bmatrix} 2 \\ x \end{bmatrix}$

7. $\begin{bmatrix} x^2+4 & y+6 \\ x-y & 2-y \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} x+y & 3 \\ y & 6 \end{bmatrix} = \begin{bmatrix} 0 & 2y-x \\ z & 4-2x \end{bmatrix}$

Lesson 4-2

(pages 169–176)

Perform the indicated matrix operations. If the matrix does not exist, write *impossible*.

1. $\begin{bmatrix} 3 & 5 \\ -7 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 8 & -1 \end{bmatrix}$

2. $[0 \quad -1 \quad 3] + \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$

3. $\begin{bmatrix} 45 & 36 & 18 \\ 63 & 29 & 5 \end{bmatrix} - \begin{bmatrix} 45 & -2 & 36 \\ 18 & 9 & -10 \end{bmatrix}$

4. $4[-8 \quad 2 \quad 9] - 3[2 \quad -7 \quad 6]$

5. $5\begin{bmatrix} 6 & -2 \\ 5 & 4 \end{bmatrix} - 2\begin{bmatrix} 6 & -2 \\ 5 & 4 \end{bmatrix} + 4\begin{bmatrix} 7 & -6 \\ -4 & 2 \end{bmatrix}$

6. $1.3\begin{bmatrix} 3.7 \\ -5.4 \end{bmatrix} + 4.1\begin{bmatrix} 6.4 \\ -3.7 \end{bmatrix} - 6.2\begin{bmatrix} -0.8 \\ 7.4 \end{bmatrix}$

Use matrices *A*, *B*, *C*, *D*, and *E* to find the following.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}, D = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}, E = \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$

7. $A + B$
8. $C + D$
9. $A - B$
10. $4B$
11. $D - C$
12. $E + 2A$
13. $D - 2B$
14. $2A + 3E - D$

Lesson 4-3

(pages 177–184)

Find each product, if possible.

1. $[-3 \quad 4] \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 15 \end{bmatrix}$

5. $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 6 & 1 \\ 2 & -4 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 1 & -2 \\ 5 & 3 & -4 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 0 & 2 \\ -6 & 5 & -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$

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Lesson 4-4

(pages 185–192)

For Exercises 1–3, use the following information.

The vertices of quadrilateral $ABCD$ are $A(1, 1)$, $B(-2, 3)$, $C(-4, -1)$, and $D(2, -3)$. The quadrilateral is dilated so that its perimeter is 2 times the original perimeter.

1. Write the coordinates for $ABCD$ in a vertex matrix.
2. Find the coordinates of the image $A'B'C'D'$.
3. Graph $ABCD$ and $A'B'C'D'$.

For Exercises 4–10, use the following information.

The vertices of $\triangle MQN$ are $M(2, 4)$, $Q(3, -5)$, and $N(1, -1)$.

4. Write the coordinates of $\triangle MQN$ in a vertex matrix.
5. Write the reflection matrix for reflecting over the line $y = x$.
6. Find the coordinates of $\triangle M'Q'N'$ after the reflection.
7. Graph $\triangle MQN$ and $\triangle M'Q'N'$.
8. Write a rotation matrix for rotating $\triangle MQN$ 90° counterclockwise about the origin.
9. Find the coordinates of $\triangle M'Q'N'$ after the rotation.
10. Graph $\triangle MQN$ and $\triangle M'Q'N'$.

Lesson 4-5

(pages 194–200)

Evaluate each determinant using expansion by minors.

1. $\begin{vmatrix} 2 & -3 & 5 \\ 1 & -2 & -7 \\ -1 & 4 & -3 \end{vmatrix}$
2. $\begin{vmatrix} 0 & -1 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}$
3. $\begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & -8 \\ 6 & 4 & -1 \end{vmatrix}$
4. $\begin{vmatrix} -3 & 0 & 2 \\ 1 & -2 & -1 \\ 0 & 5 & 0 \end{vmatrix}$

Evaluate each determinant using diagonals.

5. $\begin{vmatrix} 3 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 3 \end{vmatrix}$
6. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
7. $\begin{vmatrix} 6 & 4 & -1 \\ 2 & 5 & -8 \\ 4 & 3 & -2 \end{vmatrix}$
8. $\begin{vmatrix} 6 & 12 & 15 \\ 9 & 3 & 14 \\ 5 & 6 & 3 \end{vmatrix}$

Lesson 4-6

(pages 201–207)

Use Cramer's Rule to solve each system of equations.

1. $5x - 3y = 19$
   $7x + 2y = 8$
2. $4p - 3q = 22$
   $2p + 8q = 30$
3. $-x + y = 5$
   $2x + 4y = 38$
4. $\frac{1}{3}x - \frac{1}{2}y = -8$
   $\frac{3}{5}x + \frac{5}{6}y = -4$
5. $\frac{1}{4}c + \frac{2}{3}d = 6$
   $\frac{3}{4}c - \frac{5}{3}d = -4$
6. $0.3a + 1.6b = 0.44$
   $0.4a + 2.5b = 0.66$
7. $x + y + z = 6$
   $2x - y - z = -3$
   $3x + y - 2z = -1$
8. $2a + b - c = -6$
   $a - 2b + c = 8$
   $-a - 3b + 2c = 14$
9. $r + 2s - t = 10$
   $-2r + 3s + t = 6$
   $3r - 2s + 2t = -19$

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Lesson 4-7

(pages 208–215)

Determine whether each pair of matrices are inverses.

1. $A = \begin{bmatrix} -7 & -6 \\ 8 & 7 \end{bmatrix}, B = \begin{bmatrix} -7 & -6 \\ 8 & 7 \end{bmatrix}$
2. $C = \begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}, D = \begin{bmatrix} -2 & -2 \\ -4 & -3 \end{bmatrix}$
3. $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4. $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Find the inverse of each matrix, if it exists.

5. $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 3 & 2 \\ 0 & -4 \end{bmatrix}$
7. $\begin{bmatrix} 3 & 8 \\ 0 & -1 \end{bmatrix}$
8. $\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$
9. $\begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}$
10. $\begin{bmatrix} 8 & -5 \\ -6 & 4 \end{bmatrix}$
11. $\begin{bmatrix} 10 & 3 \\ 5 & -2 \end{bmatrix}$
12. $\begin{bmatrix} -3 & 4 \\ -4 & 8 \end{bmatrix}$

Lesson 4-8

(pages 216–222)

Write a matrix equation for each system of equations.

1. $5a + 3b = 6$
   $2a - b = 9$
2. $3x + 4y = -8$
   $2x - 3y = 6$
3. $m + 3n = 1$
   $4m - n = -22$
4. $4c - 3d = -1$
   $5c - 2d = 39$
5. $x + 2y - z = 6$
   $-2x + 3y + z = 1$
   $x + y + 3z = 8$
6. $2a - 3b - c = 4$
   $4a + b + c = 15$
   $a - b - c = -2$

Solve each matrix equation or system of equations.

7. $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -1 \end{bmatrix}$
8. $\begin{bmatrix} -1 & 1 \\ 7 & -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
9. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -29 \\ 52 \end{bmatrix}$
10. $5x - y = 7$
    $8x + 2y = 4$
11. $3m + n = 4$
    $2m + 2n = 3$
12. $6c + 5d = 7$
    $3c - 10d = -4$
13. $3a - 5b = 1$
    $a + 3b = 5$
14. $2r - 7s = 24$
    $-r + 8s = -21$
15. $x + y = -3$
    $3x - 10y = 43$
16. $2m - 3n = 3$
    $-4m + 9n = -8$
17. $x + y = 1$
    $2x - 2y = -12$

Lesson 5-1

(pages 236–244)

For Exercises 1–12, complete parts a–c for each quadratic function.

a. **Find the y-intercept, the equation of the axis of symmetry, and the x-coordinate of the vertex.**

b. **Make a table of values that includes the vertex.**

c. **Use this information to graph the function.**

1. $f(x) = 6x^2$
2. $f(x) = -x^2$
3. $f(x) = x^2 + 5$
4. $f(x) = -x^2 - 2$
5. $f(x) = 2x^2 + 1$
6. $f(x) = -3x^2 + 6x$
7. $f(x) = x^2 + 6x - 3$
8. $f(x) = x^2 - 2x - 8$
9. $f(x) = -3x^2 - 6x + 12$
10. $f(x) = x^2 + 5x - 6$
11. $f(x) = 2x^2 + 7x - 4$
12. $f(x) = -5x^2 + 10x + 1$

Determine whether each function has a maximum or a minimum value and find the maximum or minimum value. Then state the domain and range of the function.

13. $f(x) = 9x^2$
14. $f(x) = 9 - x^2$
15. $f(x) = x^2 - 5x + 6$
16. $f(x) = 2 + 7x - 6x^2$
17. $f(x) = 4x^2 - 9$
18. $f(x) = x^2 + 2x + 1$
19. $f(x) = 8 - 3x - 4x^2$
20. $f(x) = x^2 - x + \frac{5}{4}$
21. $f(x) = -x^2 + \frac{14}{3}x + \frac{5}{3}$

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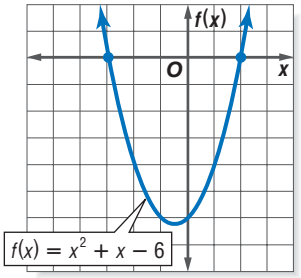
Lesson 5-2

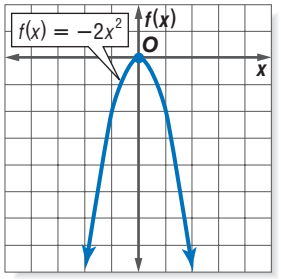
(pages 246–251)

Use the related graph of each equation to determine its solutions.

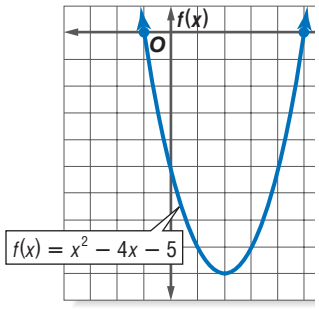
1. $x^2 + x - 6 = 0$



2. $-2x^2 = 0$



3. $x^2 - 4x - 5 = 0$



Solve each equation by graphing. If exact roots cannot be found, state the consecutive integers between which the roots are located.

4. $x^2 - 2x = 0$
5. $x^2 + 8x - 20 = 0$
6. $-2x^2 + 10x - 5 = 0$
7. $-5x + 2x^2 - 3 = 0$
8. $3x^2 - x + 8 = 0$
9. $-x^2 + 2 = 7x$
10. $4x^2 - 4x + 1 = 0$
11. $4x + 1 = 3x^2$
12. $x^2 = -9x$
13. $x^2 + 6x - 27 = 0$
14. $0.4x^2 + 1 = 0$
15. $0.5x^2 + 3x - 2 = 0$

Lesson 5-3

(pages 253–258)

Solve each equation by factoring.

1. $x^2 + 7x + 10 = 0$
2. $3x^2 = 75x$
3. $2x^2 + 7x = 9$
4. $8x^2 = 48 - 40x$
5. $5x^2 = 20x$
6. $16x^2 - 64 = 0$
7. $24x^2 - 15 = 2x$
8. $x^2 = 72 - x$
9. $4x^2 + 9 = 12x$
10. $2x^2 - 8x = 0$
11. $8x^2 + 10x = 3$
12. $12x^2 - 5x = 3$
13. $x^2 + 9x + 14 = 0$
14. $9x^2 + 1 = 6x$
15. $6x^2 + 7x = 3$
16. $x^2 - 4x = 21$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

17. 2, 1
18. -3, 4
19. -1, -7
20. $-1, \frac{1}{2}$
21. $-5, \frac{1}{4}$
22. $-\frac{1}{3}, -\frac{1}{2}$

Lesson 5-4

(pages 259–266)

Simplify.

1. $\sqrt{-289}$
2. $\sqrt{-\frac{25}{121}}$
3. $\sqrt{-625b^8}$
4. $\sqrt{-\frac{28t^6}{27s^5}}$
5. $(7i)^2$
6. $(6i)(-2i)(11i)$
7. $(\sqrt{-8})(\sqrt{-12})$
8. $-i^{22}$
9. $i^{17} \cdot i^{12} \cdot i^{26}$
10. $(14 - 5i) + (-8 + 19i)$
11. $(7i) - (2 + 3i)$
12. $(2 + 2i) - (5 + i)$
13. $(7 + 3i)(7 - 3i)$
14. $(8 - 2i)(5 + i)$
15. $(6 + 8i)^2$
16. $\frac{3}{6 - 2i}$
17. $\frac{5i}{3 + 4i}$
18. $\frac{3 - 7i}{5 + 4i}$

Solve each equation.

19. $x^2 + 8 = 3$
20. $\frac{4x^2}{49} + 6 = 3$
21. $8x^2 + 5 = 1$
22. $12 - 9x^2 = 38$
23. $9x^2 + 7 = 4$
24. $\frac{1}{2}x^2 + 1 = 0$

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Lesson 5-5

(pages 268–275)

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

1. $x^2 - 4x + c$
2. $x^2 + 20x + c$
3. $x^2 - 11x + c$
4. $x^2 - \frac{2}{3}x + c$
5. $x^2 + 30x + c$
6. $x^2 + \frac{3}{8}x + c$
7. $x^2 - \frac{2}{5}x + c$
8. $x^2 - 3x + c$

Solve each equation by completing the square.

9. $x^2 + 3x - 4 = 0$
10. $x^2 + 5x = 0$
11. $x^2 + 2x - 63 = 0$
12. $3x^2 - 16x - 35 = 0$
13. $x^2 + 7x + 13 = 0$
14. $5x^2 - 8x + 2 = 0$
15. $x^2 - 6x + 11 = 0$
16. $x^2 - 12x + 36 = 0$
17. $8x^2 + 13x - 4 = 0$
18. $3x^2 + 5x + 6 = 0$
19. $x^2 + 14x - 1 = 0$
20. $4x^2 - 32x + 15 = 0$
21. $3x^2 - 11x - 4 = 0$
22. $x^2 + 8x - 84 = 0$
23. $x^2 - 7x + 5 = 0$
24. $x^2 + 3x - 8 = 0$
25. $x^2 - 5x - 10 = 0$
26. $3x^2 - 12x + 4 = 0$
27. $x^2 + 20x + 75 = 0$
28. $x^2 - 5x - 24 = 0$
29. $2x^2 + x - 21 = 0$

Lesson 5-6

(pages 276–283)

For Exercises 1–16, complete parts a–c for each quadratic equation.

a. **Find the value of the discriminant.**

b. **Describe the number and type of roots.**

c. **Find the exact solutions by using the Quadratic Formula.**

1. $x^2 + 7x + 13 = 0$
2. $6x^2 + 6x - 21 = 0$
3. $5x^2 - 5x + 4 = 0$
4. $9x^2 + 42x + 49 = 0$
5. $4x^2 - 16x + 3 = 0$
6. $2x^2 = 5x + 3$
7. $x^2 + 81 = 18x$
8. $3x^2 - 30x + 75 = 0$
9. $24x^2 + 10x = 43$
10. $9x^2 + 4 = 2x$
11. $7x = 8x^2$
12. $18x^2 = 9x + 45$
13. $x^2 - 4x + 4 = 0$
14. $4x^2 + 16x + 15 = 0$
15. $x^2 - 6x + 13 = 0$

Solve each equation by using the method of your choice. Find exact solutions.

16. $x^2 + 4x + 29 = 0$
17. $4x^2 + 3x - 2 = 0$
18. $2x^2 + 5x = 9$
19. $x^2 = 8x - 16$
20. $7x^2 = 4x$
21. $2x^2 + 6x + 5 = 0$
22. $9x^2 - 30x + 25 = 0$
23. $3x^2 - 4x + 2 = 0$
24. $3x^2 = 108x$

Lesson 5-7

(pages 286–292)

Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

1. $y = (x + 6)^2 - 1$
2. $y = 2(x - 8)^2 - 5$
3. $y = -(x + 1)^2 + 7$
4. $y = -9(x - 7)^2 + 3$
5. $y = -x^2 + 10x - 3$
6. $y = -2x^2 + 16x + 7$

Graph each function.

7. $y = x^2 - 2x + 4$
8. $y = -3x^2 + 18x$
9. $y = -2x^2 - 4x + 1$
10. $y = 2x^2 - 8x + 9$
11. $y = \frac{1}{3}x^2 + 2x + 7$
12. $y = x^2 + 6x + 9$
13. $y = x^2 + 3x + 6$
14. $y = -0.5x^2 + 4x - 3$
15. $y = -2x^2 - 8x - 1$

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Lesson 5-8

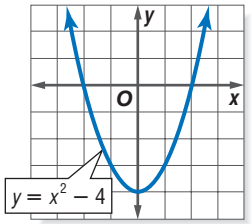
(pages 294–301)

Graph each inequality.

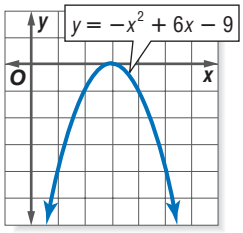
1. $y \leq 5x^2 + 3x - 2$
2. $y > -3x^2 + 2$
3. $y \geq x^2 - 8x$
4. $y \geq -x^2 - x + 3$
5. $y \leq 3x^2 + 4x - 8$
6. $y \leq -5x^2 + 2x - 3$
7. $y > 4x^2 + x$
8. $y \geq -x^2 - 3$

Use the graph of the related function of each inequality to write its solutions.

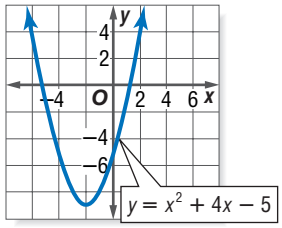
9. $x^2 - 4 \leq 0$



10. $-x^2 + 6x - 9 \geq 0$



11. $x^2 + 4x - 5 < 0$



Solve each inequality algebraically.

12. $x^2 - 1 < 0$
13. $10x^2 - x - 2 \geq 0$
14. $-x^2 - 5x - 6 > 0$
15. $-3x^2 \geq 5$
16. $x^2 - 2x - 8 \leq 0$
17. $2x^2 \geq 5x + 12$
18. $x^2 + 3x - 4 > 0$
19. $2x - x^2 \leq -15$

Lesson 6-1

(pages 312–318)

Simplify. Assume that no variable equals 0.

1. $x^7 \cdot x^3 \cdot x$
2. $m^8 \cdot m \cdot m^{10}$
3. $7^5 \cdot 7^2$
4. $(-3)^4(-3)$
5. $\frac{t^{12}}{t}$
6. $-\frac{16x^8}{8x^2}$
7. $\frac{6^5}{6^3}$
8. $\frac{p^5q^7}{p^2q^5}$
9. $-(m^3)^8$
10. $(3^5)^7$
11. -3^4
12. $(abc)^3$
13. $(x^2)^5$
14. $(b^4)^6$
15. $(-2y^5)^2$
16. $3x^0$
17. $(5x^4)^{-2}$
18. $(-3)^{-2}$
19. -3^{-2}
20. $\frac{x}{x^7}$
21. $-\left(\frac{x}{5}\right)^2$
22. $\left(\frac{5a^7}{2b^5c}\right)^3$
23. $\frac{1}{x^{-3}}$
24. $\frac{5^6a^{x+y}}{5^4a^{x-y}}$

Evaluate. Express the result in scientific notation.

25. $(8.95 \times 10^9)(1.82 \times 10^7)$
26. $(3.1 \times 10^5)(7.9 \times 10^{-8})$
27. $\frac{(2.38 \times 10^{13})(7.56 \times 10^{-5})}{(4.2 \times 10^{18})}$

Lesson 6-2

(pages 320–324)

Simplify.

1. $(4x^3 + 5x - 7x^2) + (-2x^3 + 5x^2 - 7y^2)$
2. $(2x^2 - 3x + 11) + (7x^2 + 2x - 8)$
3. $(-3x^2 + 7x + 23) + (-8x^2 - 5x + 13)$
4. $(-3x^2 + 7x + 23) - (-8x^2 - 5x + 13)$
5. $\frac{7}{uw}\left(4u^2w^3 - 5uw + \frac{w}{7u}\right)$
6. $-4x^5(-3x^4 - x^3 + x + 7)$
7. $(2x - 3)(4x + 7)$
8. $(3x - 5)(-2x - 1)$
9. $(3x - 5)(2x - 1)$
10. $(2x + 5)(2x - 5)$
11. $(3x - 7)(3x + 7)$
12. $(5 + 2w)(5 - 2w)$
13. $(2a^2 + 8)(2a^2 - 8)$
14. $(-5x + 10)(-5x - 10)$
15. $(4x - 3)^2$
16. $(5x + 6)^2$
17. $(-x + 1)^2$
18. $\frac{3}{4}x(x^2 + 4x + 14)$
19. $-\frac{1}{2}a^2(a^3 - 6a^2 + 5a)$

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Lesson 6-3

(pages 325–330)

Find $p(5)$ and $p(-1)$ for each function.

1. $p(x) = 7x - 3$
2. $p(x) = -3x^2 + 5x - 4$
3. $p(x) = 5x^4 + 2x^2 - 2x$
4. $p(x) = -13x^3 + 5x^2$
5. $p(x) = x^6 - 2$
6. $p(x) = \frac{2}{3}x^2 + 5x$
7. $p(x) = x^3 + x^2 - x + 1$
8. $p(x) = x^4 - x^2 - 1$
9. $p(x) = 1 - x^3$

If $p(x) = -2x^2 + 5x + 1$ and $q(x) = x^3 - 1$, find each value.

10. $q(n)$
11. $p(2b)$
12. $q(z^3)$
13. $p(3m^2)$
14. $q(x + 1)$
15. $p(3 - x)$
16. $q(a^2 - 2)$
17. $3q(h - 3)$
18. $5[p(c - 4)]$
19. $q(n - 2) + q(n^2)$
20. $-3p(4a) - p(a)$
21. $2[q(d^2 + 1)] + 3q(d)$

Lesson 6-4

(pages 331–338)

For Exercises 1–16, complete each of the following.

a. **Graph each function by making a table of values.**

b. **Determine the values of x between which the real zeros are located.**

c. **Estimate the x-coordinates at which the relative maxima and relative minima occur.**

1. $f(x) = x^3 + x^2 - 3x$
2. $f(x) = -x^4 + x^3 + 5$
3. $f(x) = x^3 - 3x^2 + 8x - 7$
4. $f(x) = 2x^5 + 3x^4 - 8x^2 + x + 4$
5. $f(x) = x^4 - 5x^3 + 6x^2 - x - 2$
6. $f(x) = 2x^6 + 5x^4 - 3x^2 - 5$
7. $f(x) = -x^3 - 8x^2 + 3x - 7$
8. $f(x) = -x^4 - 3x^3 + 5x$
9. $f(x) = x^5 - 7x^4 - 3x^3 + 2x^2 - 4x + 9$
10. $f(x) = x^4 - 5x^3 + x^2 - x - 3$
11. $f(x) = x^4 - 128x^2 + 960$
12. $f(x) = -x^5 + x^4 - 208x^2 + 145x + 9$
13. $f(x) = x^5 - x^3 - x + 1$
14. $f(x) = x^3 - 2x^2 - x + 5$
15. $f(x) = 2x^4 - x^3 + x^2 - x + 1$
16. $f(x) = -x^3 - x^2 - x - 1$

Lesson 6-5

(pages 339–345)

Factor completely. If the polynomial is not factorable, write *prime*.

1. $14a^3b^3c - 21a^2b^4c + 7a^2b^3c$
2. $10ax - 2xy - 15ab + 3by$
3. $x^2 + x - 42$
4. $2x^2 + 5x + 3$
5. $6x^2 + 71x - 12$
6. $6x^4 - 12x^3 + 3x^2$
7. $x^2 - 6x + 2$
8. $x^2 - 2x - 15$
9. $6x^2 + 23x + 20$
10. $24x^2 - 76x + 40$
11. $6p^2 - 13pq - 28q^2$
12. $2x^2 - 6x + 3$
13. $x^2 + 49 - 14x$
14. $9x^2 - 64$
15. $36 - t^{10}$
16. $x^2 + 16$
17. $a^4 - 81b^4$
18. $3a^3 + 12a^2 - 63a$
19. $x^3 - 8x^2 + 15x$
20. $x^2 + 6x + 9$
21. $18x^3 - 8x$
22. $3x^2 - 42x + 40$
23. $2x^2 + 4x - 1$
24. $2x^3 + 6x^2 + x + 3$
25. $35ac - 3bd - 7ad + 15bc$
26. $5h^2 - 10hj + h - 2j$

Simplify. Assume that no denominator is equal to 0.

27. $\frac{x^2 + 8x + 15}{x^2 + 4x + 3}$
28. $\frac{x^2 + x - 2}{x^2 - 6x + 5}$
29. $\frac{x^2 - 15x + 56}{x^2 - 4x - 21}$
30. $\frac{x^2 + x - 6}{x^3 + 9x^2 + 27x + 27}$

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Lesson 6-6

(pages 349–355)

Simplify.

1. $\frac{18r^3s^2 + 36r^2s^3}{9r^2s^2}$
2. $\frac{15v^3w^2 - 5v^4w^3}{-5v^4w^3}$
3. $\frac{x^2 - x + 1}{x}$
4. $(5bh + 5ch) \div (b + c)$
5. $(25c^4d + 10c^3d^2 - cd) \div 5cd$
6. $(16f^{18} + 20f^9 - 8f^6) \div 4f^3$
7. $(33m^5 + 55mn^5 - 11m^3)(11m)^{-1}$
8. $(8g^3 + 19g^2 - 12g + 9) \div (g + 3)$
9. $(p^{21} + 3p^{14} + p^7 - 2)(p^7 + 2)^{-1}$
10. $(8k^2 - 56k + 98) \div (2k - 7)$
11. $(2r^2 + 5r - 3) \div (r + 3)$
12. $(n^3 + 125) \div (n + 5)$
13. $(10y^4 + 3y^2 - 7) \div (2y^2 - 1)$
14. $(q^4 + 8q^3 + 3q + 17) \div (q + 8)$
15. $(15v^3 + 8v^2 - 21v + 6) \div (5v - 4)$
16. $(-2x^3 + 15x^2 - 10x + 3) \div (x + 3)$
17. $(5s^3 + s^2 - 7) \div (s + 1)$
18. $(t^4 - 2t^3 + t^2 - 3t + 2) \div (t - 2)$
19. $(z^4 - 3z^3 - z^2 - 11z - 4) \div (z - 4)$
20. $(3r^4 - 6r^3 - 2r^2 + r - 6) \div (r + 1)$
21. $(2b^3 - 11b^2 + 12b + 9) \div (b - 3)$

Lesson 6-7

(pages 356–361)

Use synthetic substitution to find $f(3)$ and $f(-4)$ for each function.

1. $f(x) = x^2 - 6x + 2$
2. $f(x) = x^3 + 5x - 6$
3. $f(x) = x^3 - x^2 - 3x + 1$
4. $f(x) = -3x^3 + 5x^2 + 7x - 3$
5. $f(x) = 3x^5 - 5x^3 + 2x - 8$
6. $f(x) = 10x^3 + 2$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

7. $(x^3 - x^2 + x + 14); (x + 2)$
8. $(5x^3 - 17x^2 + 6x); (x - 3)$
9. $(2x^3 + x^2 - 41x + 20); (x - 4)$
10. $(x^3 - 8); (x - 2)$
11. $(x^2 + 6x + 5); (x + 1)$
12. $(x^4 + x^3 + x^2 + x); (x + 1)$
13. $(x^3 - 8x^2 + x + 42); (x - 7)$
14. $(x^4 + 5x^3 - 27x - 135); (x - 3)$
15. $(2x^3 - 15x^2 - 2x + 120); (2x + 5)$
16. $(6x^3 - 17x^2 + 6x + 8); (3x - 4)$
17. $(10x^3 + x^2 - 46x + 35); (5x - 7)$
18. $(x^3 + 9x^2 + 23x + 15); (x + 1)$

Lesson 6-8

(pages 362–368)

Solve each equation. State the number and type of roots.

1. $-5x - 7 = 0$
2. $3x^2 + 10 = 0$
3. $x^4 - 2x^3 = 23x^2 - 60x$

State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

4. $f(x) = 5x^8 - x^6 + 7x^4 - 8x^2 - 3$
5. $f(x) = 6x^5 - 7x^2 + 5$
6. $f(x) = -2x^6 - 5x^5 + 8x^2 - 3x + 1$
7. $f(x) = 4x^3 + x^2 - 38x + 56$
8. $f(x) = 3x^4 - 5x^3 + 2x^2 - 7x + 5$
9. $f(x) = x^5 - x^4 + 7x^3 - 25x^2 + 8x - 13$

Find all of the zeros of the function.

10. $f(x) = x^3 - 7x^2 + 16x - 10$
11. $f(x) = 10x^3 + 7x^2 - 82x + 56$
12. $f(x) = x^3 - 16x^2 + 79x - 114$
13. $f(x) = -3x^3 + 6x^2 + 5x - 8$
14. $f(x) = 24x^3 + 64x^2 + 6x - 10$
15. $f(x) = 2x^3 + 2x^2 - 34x + 30$

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Lesson 6-9

(pages 369–373)

List all of the possible rational zeros for each function.

1. $f(x) = 3x^5 - 7x^3 - 8x + 6$
2. $f(x) = 4x^3 + 2x^2 - 5x + 8$
3. $f(x) = 6x^9 - 7$

Find all of the rational zeros for each function.

4. $f(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$
5. $f(x) = 6x^4 - 31x^3 - 119x^2 + 214x + 560$
6. $f(x) = 20x^4 - 16x^3 + 11x^2 - 12x - 3$
7. $f(x) = 2x^4 - 30x^3 + 117x^2 - 75x + 280$
8. $f(x) = 3x^4 + 8x^3 + 9x^2 + 32x - 12$
9. $f(x) = x^5 - x^4 + x^3 + 3x^2 - x$

Find all of the zeros of each function.

10. $f(x) = x^4 + 8x^2 - 9$
11. $f(x) = 3x^4 - 9x^2 - 12$
12. $f(x) = 4x^4 + 19x^2 - 63$

Lesson 7-1

(pages 384–390)

Find $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$ for each $f(x)$ and $g(x)$.

1. $f(x) = 3x + 5$, $g(x) = x - 3$
2. $f(x) = \sqrt{x}$, $g(x) = x^2$
3. $f(x) = x^2 - 5$, $g(x) = x^2 + 5$
4. $f(x) = x^2 + 1$, $g(x) = x + 1$

For each set of ordered pairs, find $f \circ g$ and $g \circ f$, if they exist.

5. $f = \{(-1, 1), (2, -1), (-3, 5)\}$, $g = \{(1, -1), (-1, 2), (5, -3)\}$
6. $f = \{(0, 6), (5, -8), (-9, 2)\}$, $g = \{(-8, 3), (6, 4), (2, 1)\}$
7. $f = \{(8, 2), (6, 5), (-3, 4), (1, 0)\}$, $g = \{(2, 8), (5, 6), (4, -3), (0, 1)\}$
8. $f = \{(10, 4), (-1, 2), (5, 6), (-1, 0)\}$, $g = \{(-4, 10), (2, -9), (-7, 5), (-2, -1)\}$

Find $[g \circ h](x)$ and $[h \circ g](x)$.

9. $g(x) = 8 - 2x$, $h(x) = 3x$
10. $g(x) = x^2 - 7$, $h(x) = 3x + 2$
11. $g(x) = 2x + 7$, $h(x) = \frac{x - 7}{2}$
12. $g(x) = 3x + 2$, $h(x) = 5 - 3x$

If $f(x) = x^2 + 1$, $g(x) = 2x$, and $h(x) = x - 1$, find each value.

13. $g[f(1)]$
14. $[f \circ h](3)$
15. $[h \circ f](3)$
16. $[g \circ f](-2)$
17. $g[h(-20)]$
18. $f[h(-3)]$
19. $g[f(a)]$
20. $[f \circ (g \circ f)](c)$

Lesson 7-2

(pages 391–396)

Find the inverse of each relation.

1. $\{(-2, 7), (3, 0), (5, -8)\}$
2. $\{(-3, 9), (-2, 4), (3, 9), (-1, 1)\}$

Find the inverse of each function. Then graph the function and its inverse.

3. $f(x) = x - 7$
4. $y = 2x + 8$
5. $g(x) = 3x - 8$
6. $y = -5x - 6$
7. $y = -2$
8. $g(x) = 5 - 2x$
9. $h(x) = \frac{x}{5} + 1$
10. $h(x) = -\frac{2}{3}x$
11. $y = \frac{x - 5}{3}$
12. $y = \frac{1}{2}x - 1$
13. $f(x) = \frac{3x + 8}{4}$
14. $g(x) = \frac{2x - 1}{3}$

Determine whether each pair of functions are inverse functions.

15. $f(x) = \frac{2x - 3}{5}$, $g(x) = \frac{3x - 5}{3}$
16. $f(x) = 5x - 6$, $g(x) = \frac{x + 6}{5}$
17. $f(x) = 6 - 3x$, $g(x) = 2 - \frac{1}{3}x$
18. $f(x) = 3x - 7$, $g(x) = \frac{1}{3}x + 7$

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Lesson 7-3

(pages 397–401)

Graph each function. State the domain and range of the function.

1. $y = \sqrt{x - 4}$
2. $y = \sqrt{x + 3} - 1$
3. $y = \frac{1}{3}\sqrt{x + 2}$
4. $y = \sqrt{2x + 5}$
5. $y = -\sqrt{4x}$
6. $y = 2\sqrt{x}$
7. $y = -3\sqrt{x}$
8. $y = \sqrt{x} + 5$
9. $y = \sqrt{2x} - 1$
10. $y = 5\sqrt{x} + 1$
11. $y = \sqrt{x + 1} - 2$
12. $y = 6 - \sqrt{x + 3}$

Graph each inequality.

13. $y > \sqrt{2x}$
14. $y \leq \sqrt{-5x}$
15. $y \geq \sqrt{x + 6} + 6$
16. $y < \sqrt{3x + 1} + 2$
17. $y \geq \sqrt{8x - 3} + 1$
18. $y < \sqrt{5x - 1} + 3$

Lesson 7-4

(pages 402–406)

Use a calculator to approximate each value to three decimal places.

1. $\sqrt{289}$
2. $\sqrt{7832}$
3. $\sqrt[4]{0.0625}$
4. $\sqrt[3]{-343}$
5. $\sqrt[10]{32^4}$
6. $\sqrt[3]{49}$
7. $\sqrt[5]{5}$
8. $-\sqrt[4]{25}$

Simplify.

9. $\sqrt{9h^{22}}$
10. $\sqrt[5]{0}$
11. $\sqrt{\frac{16}{9}}$
12. $\sqrt{\left(-\frac{2}{3}\right)^4}$
13. $\sqrt[5]{-32}$
14. $-\sqrt{-144}$
15. $\sqrt[4]{a^{16}b^8}$
16. $\pm\sqrt[4]{81x^4}$
17. $\sqrt[5]{\frac{1}{100{,}000}}$
18. $\sqrt[3]{-d^6}$
19. $\sqrt[5]{p^{25}q^{15}r^5s^{20}}$
20. $\sqrt[4]{(2x^2 - y^8)^8}$
21. $\pm\sqrt{16m^6n^2}$
22. $-\sqrt[3]{(2x - y)^3}$
23. $\sqrt[4]{(r + s)^4}$
24. $\sqrt{9a^2 + 6a + 1}$
25. $\sqrt{4y^2 + 12y + 9}$
26. $-\sqrt{x^2 - 2x + 1}$
27. $\pm\sqrt{x^2 + 2x + 1}$
28. $\sqrt[3]{a^3 + 6a^2 + 12a + 8}$

Lesson 7-5

(pages 408–414)

Simplify.

1. $\sqrt{75}$
2. $7\sqrt{12}$
3. $\sqrt[3]{81}$
4. $\sqrt{5r^5}$
5. $\sqrt[4]{7^8x^5y^6}$
6. $3\sqrt{5} + 6\sqrt{5}$
7. $\sqrt{18} - \sqrt{50}$
8. $4\sqrt[3]{32} + \sqrt[3]{500}$
9. $\sqrt{12}\sqrt{27}$
10. $3\sqrt{12} + 2\sqrt{300}$
11. $\sqrt[3]{54} - \sqrt[3]{24}$
12. $\sqrt{10}(2 - \sqrt{5})$
13. $-\sqrt{3}(2\sqrt{6} - \sqrt{63})$
14. $(5 + \sqrt{2})(3 + \sqrt{3})$
15. $(2 + \sqrt{5})(2 - \sqrt{5})$
16. $(8 + \sqrt{11})^2$
17. $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6})$
18. $(\sqrt{8} + \sqrt{13})^2$
19. $(1 - \sqrt{7})(4 + \sqrt{7})$
20. $(5 - 2\sqrt{7})^2$
21. $\sqrt{\frac{3m^3}{24n^5}}$
22. $\frac{\sqrt{18}}{\sqrt{32}}$
23. $2\sqrt[3]{\frac{r^5}{2s^2t}}$
24. $\sqrt[3]{\frac{4}{7}}$
25. $\sqrt[5]{\frac{32}{a^4}}$
26. $\sqrt{\frac{2}{3}} - \sqrt{\frac{3}{8}}$
27. $\frac{5}{3 - \sqrt{10}}$
28. $\frac{\sqrt{5}}{1 + \sqrt{3}}$
29. $\frac{-2 + \sqrt{7}}{2 + \sqrt{7}}$
30. $\frac{1 - \sqrt{3}}{1 + \sqrt{8}}$
31. $\frac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$
32. $\frac{x + \sqrt{5}}{x - \sqrt{5}}$

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Lesson 7-6

(pages 415–421)

Write each expression in radical form.

1. $10^{\frac{1}{3}}$
2. $8^{\frac{1}{4}}$
3. $a^{\frac{2}{3}}$
4. $(b^2)^{\frac{3}{4}}$

Write each radical using rational exponents.

5. $\sqrt{35}$
6. $\sqrt[4]{32}$
7. $3\sqrt{27a^2x}$
8. $\sqrt[5]{25ab^3c^4}$

Evaluate each expression.

9. $2401^{\frac{1}{4}}$
10. $27^{\frac{4}{3}}$
11. $(-32)^{\frac{2}{5}}$
12. $-81^{\frac{3}{4}}$
13. $(-125)^{-\frac{2}{3}}$
14. $16^{\frac{5}{2}} \cdot 16^{\frac{1}{2}}$
15. $8^{-\frac{2}{3}} \cdot 64^{\frac{1}{6}}$
16. $\left(\frac{48}{1875}\right)^{-\frac{5}{4}}$

Simplify each expression.

17. $7^{\frac{5}{9}} \cdot 7^{\frac{4}{9}}$
18. $32^{\frac{2}{3}} \cdot 32^{\frac{3}{5}}$
19. $\left(k^{\frac{8}{5}}\right)^5$
20. $x^{\frac{2}{5}} \cdot x^{\frac{8}{5}}$
21. $m^{\frac{2}{5}} \cdot m^{\frac{4}{5}}$
22. $\left(p^{\frac{5}{4}} \cdot q^{\frac{7}{2}}\right)^{\frac{8}{3}}$
23. $\left(4^{\frac{9}{2}}c^{\frac{3}{2}}\right)^2$
24. $\frac{7^{\frac{3}{4}}}{7^{\frac{5}{3}}}$
25. $\frac{1}{t^{\frac{9}{5}}}$
26. $a^{-\frac{8}{7}}$
27. $\frac{r}{r^{\frac{7}{5}}}$
28. $\sqrt[4]{36}$
29. $\sqrt[4]{9a^2}$
30. $\sqrt[3]{\sqrt{81}}$
31. $\frac{v^{\frac{11}{7}} - v^{\frac{4}{7}}}{v^{\frac{4}{7}}}$
32. $\frac{1}{5^{\frac{1}{2}} + 3^{\frac{1}{2}}}$

Lesson 7-7

(pages 422–427)

Solve each equation or inequality.

1. $\sqrt{x} = 16$
2. $\sqrt{z+3} = 7$
3. $\sqrt[3]{a+5} = 1$
4. $5\sqrt{s} - 8 = 3$
5. $\sqrt[4]{m+7} + 11 = 9$
6. $d + \sqrt{d^2 - 8} = 4$
7. $g\sqrt{5} + 4 = g + 4$
8. $\sqrt{x-8} = \sqrt{13+x}$
9. $\sqrt{3x+9} > 2$
10. $\sqrt{3n-1} \leq 5$
11. $2 - 4\sqrt{21-6c} < -6$
12. $\sqrt{5y+4} > 8$
13. $\sqrt{2w+3} + 5 \geq 7$
14. $\sqrt{2c+3} - 7 > 0$
15. $\sqrt{3z-5} - 3 = 1$
16. $\sqrt{5y+1} + 6 < 10$
17. $\sqrt{3n+1} - 2 \leq 6$
18. $\sqrt{y-5} - \sqrt{y} \geq 1$
19. $(5n-1)^{\frac{1}{2}} = 0$
20. $(7x-6)^{\frac{1}{3}} + 1 = 3$
21. $(6a-8)^{\frac{1}{4}} + 9 \geq 10$

Lesson 8-1

(pages 442–449)

Simplify each expression.

1. $\frac{25xy^2}{15y}$
2. $\frac{-4a^2b^3}{28ab^4}$
3. $\frac{(-2cd^3)^2}{8c^2d^5}$
4. $\frac{3x^3}{-2} \cdot \frac{-4}{9x}$
5. $\frac{21x^2}{-5} \cdot \frac{10}{7x^3}$
6. $\frac{2u^2}{3} \div \frac{6u^3}{5}$
7. $\frac{15x^3}{14} \div \frac{18x}{7}$
8. $\frac{xy^2}{2} \cdot \frac{x^2}{2y} \cdot \frac{2}{x^2y}$
9. $axy \div \frac{ax}{y}$
10. $\frac{9u^2}{28v} \div \frac{27u^2}{8v^2}$
11. $\frac{x^2-4}{4x^2-1} \cdot \frac{2x-1}{x+2}$
12. $\frac{x^2-1}{2x^2-x-1} \div \frac{x^2-4}{2x^2-3x-2}$
13. $\frac{2x^2+x-1}{2x^2+3x-2} \div \frac{x^2-2x+1}{x^2+x-2}$
14. $\frac{\frac{(ab)^2}{c}}{\frac{xa^3b}{cx^2}}$
15. $\frac{x^4-y^4}{x^3+y^3} \div \frac{x^3-y^3}{x+y}$

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Lesson 8-2

(pages 450–456)

Find the LCM of each set of polynomials.

1. $2a^2b,\ 4ab^2,\ 20a$
2. $x^2 - 4x - 12,\ x^2 + 7x + 10$

Simplify each expression.

3. $\frac{12}{7d} - \frac{3}{14d}$
4. $\frac{x+1}{x} - \frac{x-1}{x^2}$
5. $\frac{2x+1}{4x^2} - \frac{x+3}{6x}$
6. $\frac{7x}{13y^2} + \frac{4y}{6x^2}$
7. $\frac{x}{x-1} + \frac{1}{1-x}$
8. $\frac{1}{3v^2} + \frac{1}{uv} + \frac{3}{4u^2}$
9. $\frac{1}{x^2-x} + \frac{1}{x^2+x}$
10. $\frac{1}{x^2-1} - \frac{1}{(x-1)^2}$
11. $\frac{5}{x} - \frac{3}{x+5}$
12. $y - 1 + \frac{1}{y-1}$
13. $3m + 1 - \frac{2m}{3m+1}$
14. $\frac{3x}{x-y} + \frac{4x}{y-x}$
15. $\frac{4}{a^2-4} - \frac{3}{a^2+4a+4}$
16. $\frac{4}{3-3z^2} - \frac{2}{z^2+5z+4}$
17. $\frac{2c}{c^2-9} - \frac{1}{c^2+6c+9}$
18. $\frac{\frac{1}{x+y}}{\frac{1}{x}+\frac{1}{y}}$
19. $\frac{1-\frac{1}{x+1}}{1+\frac{1}{x-1}}$
20. $\frac{4+\frac{1}{x-2}}{3-\frac{1}{x-2}}$

Lesson 8-3

(pages 457–463)

Determine the equations of any vertical asymptotes and the values of x for any holes in the graph of each rational function.

1. $f(x) = \frac{1}{x+4}$
2. $f(x) = \frac{x-2}{x+3}$
3. $f(x) = \frac{5}{(x+1)(x-8)}$
4. $f(x) = \frac{x}{x+2}$
5. $f(x) = \frac{x^2-4}{x+2}$
6. $f(x) = \frac{x^2+x-6}{x^2+8x+15}$

Graph each rational function.

7. $f(x) = \frac{1}{x-5}$
8. $f(x) = \frac{3x}{x+1}$
9. $f(x) = \frac{x^2-16}{x-4}$
10. $f(x) = \frac{x}{x-6}$
11. $f(x) = \frac{1}{(x-3)^2}$
12. $f(x) = \frac{2}{(x+3)(x-4)}$
13. $f(x) = \frac{x+4}{x^2-1}$
14. $f(x) = \frac{x+2}{x+3}$
15. $f(x) = \frac{x^2+5x-14}{x^2+9x+14}$

Lesson 8-4

(pages 465–471)

State whether each equation represents a *direct*, *joint*, or *inverse* variation. Then name the constant of variation.

1. $xy = 10$
2. $\frac{x}{7} = y$
3. $\frac{x}{y} = -6$
4. $10x = y$
5. $x = \frac{2}{y}$
6. $A = \ell w$
7. $\frac{1}{4}b = -\frac{3}{5}c$
8. $D = rt$
9. If y varies directly as x and $y = 16$ when $x = 4$, find y when $x = 12$.
10. If x varies inversely as y and $x = 12$ when $y = -3$, find x when $y = -18$.
11. If m varies directly as w and $m = -15$ when $w = 2.5$, find m when $w = 12.5$.
12. If y varies jointly as x and z and $y = 10$ when $z = 4$ and $x = 5$, find y when $x = 4$ and $z = 2$.
13. If y varies inversely as x and $y = \frac{1}{4}$ when $x = 24$, find y when $x = \frac{3}{4}$.

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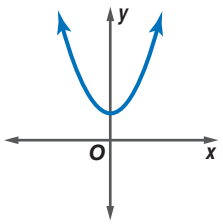
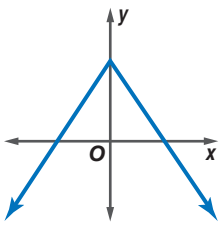
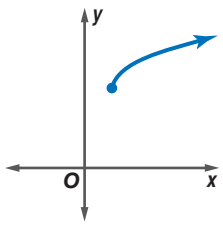
Lesson 8-5

(pages 473–478)

Identify the type of function represented by each graph.

1. 

2. 

3. 

Identify the function represented by each equation. Then graph the equation.

4. $y = \sqrt{5x}$
5. $y = \frac{3}{4}x$
6. $y = |x| + 3$
7. $y = x^2 - 2$
8. $y = \frac{2}{x}$
9. $y = 2[\![x]\!]$
10. $y = -2x^2 + 1$
11. $y = \frac{x^2 + 2x - 3}{x^2 + 7x + 12}$
12. $y = -3$

Lesson 8-6

(pages 479–486)

Solve each equation or inequality. Check your solutions.

1. $\frac{x}{x-3} = \frac{1}{4}$
2. $\frac{5}{x} + \frac{3}{5} = \frac{2}{x}$
3. $\frac{5}{b-2} < 5$
4. $\frac{4}{a+3} > 2$
5. $\frac{x-2}{x} = \frac{x-4}{x-6}$
6. $-6 - \frac{8}{n} < n$
7. $\frac{2}{d} + \frac{1}{d-2} = 1$
8. $\frac{1}{2+3x} + \frac{2}{2-3x} = 0$
9. $\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}$
10. $\frac{p}{p+1} + \frac{3}{p-3} + 1 = 0$
11. $\frac{5z+2}{z^2-4} = \frac{-5z}{2-z} + \frac{2}{z+2}$
12. $\frac{1}{x-3} + \frac{2}{x^2-9} = \frac{5}{x+3}$
13. $\frac{1}{m^2-1} = \frac{2}{m^2+m-2}$
14. $\frac{12}{x^2-16} - \frac{24}{x-4} = 3$
15. $n + \frac{1}{n+3} = \frac{n^2}{n-1}$

Lesson 9-1

(pages 498–506)

Sketch the graph of each function. Then state the function's domain and range.

1. $y = 3(5)^x$
2. $y = 0.5(2)^x$
3. $y = 3\left(\frac{1}{4}\right)^x$
4. $y = 2(1.5)^x$

Determine whether each function represents exponential *growth* or *decay*.

5. $y = 4(3)^x$
6. $y = 10^{-x}$
7. $y = 5\left(\frac{1}{2}\right)^x$
8. $y = 2\left(\frac{5}{4}\right)^x$

Write an exponential function for the graph that passes through the given points.

9. (0, 6) and (2, 54)
10. (0, −4) and (−4, −64)
11. (0, 1.5) and (3, 40.5)

Solve each equation or inequality. Check your solution.

12. $27^{2x-1} = 3$
13. $8^{2+x} \geq 2$
14. $4^{2x+5} < 8^{x+1}$
15. $6^{x+1} = 36^{x-1}$
16. $10^{x-1} > 100^{4-x}$
17. $\left(\frac{1}{5}\right)^{x-3} = 125$
18. $2^{x^2+1} = 32$
19. $36^x = 6^{x^2-3}$

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Lesson 9-2

(pages 509–517)

Write each equation in logarithmic form.

1. $3^5 = 243$
2. $10^3 = 1000$
3. $4^{-3} = \frac{1}{64}$

Write each equation in exponential form.

4. $\log_2 \frac{1}{8} = -3$
5. $\log_{25} 5 = \frac{1}{2}$
6. $\log_7 \frac{1}{7} = -1$

Evaluate each expression.

7. $\log_4 16$
8. $\log_{10} 10{,}000$
9. $\log_3 \frac{1}{9}$
10. $\log_2 1024$
11. $\log_6 6^5$
12. $\log_{\frac{1}{2}} 8$
13. $\log_{11} 121$
14. $5^{\log_5 10}$

Solve each equation or inequality. Check your solutions.

15. $\log_8 b = 2$
16. $\log_4 x < 3$
17. $\log_{\frac{1}{9}} n = -\frac{1}{2}$
18. $\log_x 7 = 1$
19. $\log_{\frac{2}{3}} a < 3$
20. $\log_2 (x^2 - 9) = 4$

Lesson 9-3

(pages 520–526)

Use $\log_3 5 \approx 1.4651$ and $\log_3 7 \approx 1.7712$ to approximate the value of each expression.

1. $\log_3 \frac{7}{5}$
2. $\log_3 245$
3. $\log_3 35$

Solve each equation. Check your solutions.

4. $\log_2 x + \log_2 (x - 2) = \log_2 3$
5. $\log_3 x = 2 \log_3 3 + \log_3 5$
6. $\log_5 (x^2 + 7) = \frac{2}{3} \log_5 64$
7. $\log_2 (x^2 - 9) = 4$
8. $\log_3 (x + 2) + \log_3 6 = 3$
9. $\log_6 x + \log_6 (x - 5) = 2$
10. $\log_5 (x + 3) = \log_5 8 - \log_5 2$
11. $2 \log_3 x - \log_3 (x - 2) = 2$
12. $\log_6 x = \frac{3}{2} \log_6 9 + \log_6 2$
13. $\log_8 (x + 6) + \log_8 (x - 6) = 2$
14. $\log_3 14 + \log_3 x = \log_3 42$
15. $\log_{10} x = \frac{1}{2} \log_{10} 81$

Lesson 9-4

(pages 528–533)

Use a calculator to evaluate each expression to four decimal places.

1. $\log 55$
2. $\log 6.7$
3. $\log 3.3$
4. $\log 0.08$
5. $\log 9.9$
6. $\log 0.6$

Solve each equation or inequality. Round to four decimal places.

7. $2^x = 15$
8. $4^{2a} > 45$
9. $7^{2x} = 35$
10. $11^{x+4} > 57$
11. $1.5^{a-7} = 9.6$
12. $3^{b^2} = 64$
13. $7^{3c} < 35^{2c-1}$
14. $5^{m^2+1} = 30$
15. $7^{3y-1} < 2^{2y+4}$
16. $9^{n-3} = 2^{n+3}$
17. $11^{t+1} \leq 22^{t+3}$
18. $2^{3a-1} = 3^{a+2}$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

19. $\log_3 21$
20. $\log_4 62$
21. $\log_5 28$
22. $\log_2 25$

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Lesson 9-5

(pages 536–542)

Use a calculator to evaluate each expression to four decimal places.

1. e^3
2. $e^{0.75}$
3. e^{-4}
4. $e^{-2.5}$
5. $\ln 5$
6. $\ln 8$
7. $\ln 8.4$
8. $\ln 0.6$

Write an equivalent exponential or logarithmic equation.

9. $e^x = 10$
10. $\ln x \approx 2.3026$
11. $e^3 = 9x$
12. $\ln 0.2 = x$

Solve each equation or inequality.

13. $25e^x = 1000$
14. $e^{0.075x} > 25$
15. $e^x < 3.8$
16. $-2e^x + 5 = 1$
17. $5 + 4e^{2x} = 17$
18. $e^{-3x} \leq 15$
19. $\ln 7x = 10$
20. $\ln 4x = 8$
21. $3 \ln 2x \geq 9$
22. $\ln (x + 2) = 4$
23. $\ln (2x + 3) > 0$
24. $\ln (3x - 1) = 5$

Lesson 9-6

(pages 544–550)

1. **FARMING** Mr. Rogers purchased a combine for $175,000 for his farming operation. It is expected to depreciate at a rate of 18% per year. What will be the value of the combine in 3 years?
2. **REAL ESTATE** The Jacksons bought a house for $65,000 in 1992. Houses in the neighborhood have appreciated at the rate of 4.5% a year. How much is the house worth in 2003?
3. **POPULATION** In 1950, the population of a city was 50,000. Since then, the population has increased by 2.25% per year. If it continues to grow at this rate, what will the population be in 2005?
4. **BEARS** In a particular state, the population of black bears has been decreasing at the rate of 0.75% per year. In 1990, it was estimated that there were 400 black bears in the state. If the population continues to decline at the same rate, what will the population be in 2010?

Lesson 10-1

(pages 562–566)

Find the midpoint of the line segment with endpoints at the given coordinates.

1. (7, −3), (−11, 13)
2. (16, 29), (−7, 2)
3. (43, −18), (−78, −32)
4. (−7.54, 3.42), (4.89, −9.28)
5. $\left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{2}{3}, \frac{3}{5}\right)$
6. $\left(-\frac{1}{4}, \frac{2}{3}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right)$

Find the distance between each pair of points with the given coordinates.

7. (5, 7), (3, 19)
8. (−2, −1), (5, 3)
9. (−3, 15), (7, −8)
10. (6, −3), (−4, −9)
11. (3.89, −0.38), (4.04, −0.18)
12. $(5\sqrt{3}, 2\sqrt{2}), (-11\sqrt{3}, -4\sqrt{2})$
13. $\left(\frac{1}{4}, 0\right), \left(-\frac{2}{3}, \frac{1}{2}\right)$
14. $\left(4, -\frac{5}{6}\right), \left(-2, \frac{1}{6}\right)$
15. A circle has a radius with endpoints at (−3, 1) and (2, −5). Find the circumference and area of the circle. Write the answer in terms of π.
16. Triangle ABC has vertices $A(0, 0)$, $B(-3, 4)$, and $C(2, 6)$. Find the perimeter of the triangle.

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Lesson 10-2

(pages 567–573)

Write each equation in standard form.

1. $y = x^2 - 4x + 7$
2. $y = 2x^2 + 12x + 17$
3. $x = 3y^2 - 6y + 5$

Identify the coordinates of the vertex and focus, the equations of the axis of symmetry and directrix, and the direction of opening of the parabola with the given equation. Then find the length of the latus rectum and graph the parabola.

4. $y + 4 = x^2$
5. $y = 5(x + 2)^2$
6. $4(y + 2) = 3(x - 1)^2$
7. $5x + 3y^2 = 15$
8. $y = 2x^2 - 8x + 7$
9. $x = 2y^2 - 8y + 7$
10. $3(x - 8)^2 = 5(y + 3)$
11. $x = 3(y + 4)^2 + 1$
12. $8y + 5x^2 + 30x + 101 = 0$
13. $x = -\frac{1}{5}y^2 + \frac{8}{5}y - 7$
14. $6x = y^2 - 6y + 39$
15. $-8y = x^2$
16. $y = 4x^2 + 24x + 38$
17. $y = x^2 - 6x + 3$
18. $y = x^2 + 4x + 1$

Write an equation for each parabola described below. Then graph.

19. focus (1, 1), directrix $y = -1$
20. vertex (−1, 2), directrix $y = -4$

Lesson 10-3

(pages 574–579)

Write an equation for the circle that satisfies each set of conditions.

1. center (3, 2), $r = 5$ units
2. center (−5, 8), $r = 3$ units
3. center (1, −6), $r = \frac{2}{3}$ units
4. center (0, 7), tangent to x-axis
5. center (−2, −4), tangent to y-axis
6. endpoints of a diameter at (−9, 0) and (2, −5)
7. endpoints of a diameter at (4, 1) and (−3, 2)
8. center (6, −10), passes through origin
9. center (0.8, 0.5), passes through (2, 2)

Find the center and radius of the circle with the given equation. Then graph.

10. $x^2 + y^2 = 36$
11. $(x - 5)^2 + (y + 4)^2 = 1$
12. $x^2 + 3x + y^2 - 5y = 0.5$
13. $x^2 + y^2 = 14x - 24$
14. $x^2 + y^2 = 2(y - x)$
15. $x^2 + 10x + \left(y - \sqrt{3}\right)^2 = 11$
16. $x^2 + y^2 = 4x + 9$
17. $x^2 + y^2 - 6x + 4y = 156$
18. $x^2 + y^2 - 2x + 7y = 1$

Lesson 10-4

(pages 581–588)

Write an equation for the ellipse that satisfies each set of conditions.

1. endpoints of major axis at (−2, 7) and (4, 7), endpoints of minor axis at (1, 5) and (1, 9)
2. endpoints of minor axis at (1, −4) and (1, 5), endpoints of major axis at (−4, 0.5) and (6, 0.5)
3. major axis 24 units long and parallel to the y-axis, minor axis 4 units long, center at (0, 3)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

4. $\frac{x^2}{36} + \frac{y^2}{81} = 1$
5. $\frac{x^2}{121} + \frac{(y - 5)^2}{16} = 1$
6. $\frac{(x + 2)^2}{12} + \frac{(y + 1)^2}{16} = 1$
7. $8x^2 + 2y^2 = 32$
8. $7x^2 + 3y^2 = 84$
9. $9x^2 + 16y^2 = 144$
10. $169x^2 - 338x + 169 + 25y^2 = 4225$
11. $x^2 + 4y^2 + 8x - 64y = -128$
12. $4x^2 + 5y^2 = 6(6x + 5y) + 658$
13. $9x^2 + 16y^2 - 54x + 64y + 1 = 0$

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Lesson 10-5

(pages 590–597)

Find the coordinates of the vertices and foci and the equations of the asymptotes for the hyperbola with the given equation. Then graph the hyperbola.

1. $\frac{y^2}{25} - \frac{x^2}{9} = 1$
2. $\frac{x^2}{4} - \frac{y^2}{9} = 1$
3. $\frac{x^2}{81} - \frac{y^2}{36} = 1$
4. $\frac{(x-4)^2}{64} - \frac{(y+1)^2}{16} = 1$
5. $\frac{(y-7)^2}{2.25} - \frac{(x-3)^2}{4} = 1$
6. $(x+5)^2 - \frac{(y+3)^2}{48} = 1$
7. $x^2 - 9y^2 = 36$
8. $4x^2 - 9y^2 = 72$
9. $49x^2 - 16y^2 = 784$
10. $576y^2 = 49x^2 + 490x + 29{,}449$
11. $25(y+5)^2 - 20(x-1)^2 = 500$

Write an equation for the hyperbola that satisfies each set of conditions.

12. vertices $(-3, 0)$ and $(3, 0)$; conjugate axis of length 8 units
13. vertices $(0, -7)$ and $(0, 7)$; conjugate axis of length 25 units
14. center $(0, 0)$; horizontal transverse axis of length 12 units and a conjugate axis of length 10 units

Lesson 10-6

(pages 598–602)

Write each equation in standard form. State whether the graph of the equation is a *parabola, circle, ellipse,* or *hyperbola.* Then graph the equation.

1. $9x^2 - 36x + 36 = 4y^2 + 24y + 72$
2. $x^2 + 4x + 2y^2 + 16y + 32 = 0$
3. $x^2 + 6x + y^2 - 6y + 9 = 0$
4. $9y^2 = 25x^2 + 400x + 1825$
5. $2y^2 + 12y - x + 6 = 0$
6. $x^2 + y^2 = 10x + 2y + 23$
7. $3x^2 + y = 12x - 17$
8. $9x^2 - 18x + 16y^2 + 160y = -265$
9. $x^2 + 10x + 5 = 4y^2 + 16$
10. $\frac{(y-5)^2}{4} - (x+1)^2 = 4$
11. $9x^2 + 49y^2 = 441$
12. $4x^2 - y^2 = 4$

Without writing the equation in standard form, state whether the graph of each equation is a *parabola, circle, ellipse,* or *hyperbola.*

13. $(x+3)^2 = 8(y+2)$
14. $x^2 + 4x + y^2 - 8y = 2$
15. $2x^2 - 13y^2 + 5 = 0$
16. $16(x-3)^2 + 81(y+4)^2 = 1296$

Lesson 10-7

(pages 603–608)

Solve each system of inequalities by graphing.

1. $\frac{x^2}{16} - \frac{y^2}{1} \geq 1$
   $x^2 + y^2 \leq 49$
2. $\frac{x^2}{25} + \frac{y^2}{16} \leq 1$
   $y \leq x - 2$
3. $y \geq x + 3$
   $x^2 + y^2 < 25$
4. $4x^2 + (y-3)^2 \leq 16$
   $x + 2y \geq 4$

Find the exact solution(s) of each system of equations.

5. $\frac{x^2}{16} + \frac{y^2}{16} = 1$
   $y = x + 3$
6. $x = y^2$
   $(x+3)^2 + y^2 = 53$
7. $\frac{x^2}{3} - \frac{(y+2)^2}{4} = 1$
   $x^2 = y^2 + 11$
8. $\frac{(x-1)^2}{5} + \frac{y^2}{2} = 1$
   $y = x + 1$
9. $x^2 + y^2 = 13$
   $x^2 - y^2 = -5$
10. $\frac{x^2}{25} - \frac{y^2}{5} = 1$
    $y = x - 4$
11. $x^2 + y = 0$
    $x + y = -2$
12. $x^2 - 9y^2 = 36$
    $x = y$
13. $4x^2 + 6y^2 = 360$
    $y = x$

Extra Practice

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Lesson 11-1 (pages 622–628)

Find the next four terms of each arithmetic sequence.

1. 9, 7, 5, . . .
2. 3, 4.5, 6, . . .
3. 40, 35, 30, . . .
4. 2, 5, 8, . . .

Find the first five terms of each arithmetic sequence described.

5. $a_1 = 1, d = 7$
6. $a_1 = -5, d = 2$
7. $a_1 = 1.2, d = 3.7$
8. $a_1 = -\frac{5}{4}, d = -\frac{1}{2}$

Find the indicated term of each arithmetic sequence.

9. $a_1 = 4, d = 5, n = 10$
10. $a_1 = -30, d = -6, n = 5$
11. $a_1 = -3, d = 32, n = 8$

Write an equation for the *n*th term of each arithmetic sequence.

12. 3, 5, 7, 9, . . .
13. 2, −1, −4, −7, . . .
14. 20, 28, 36, 44, . . .

Find the arithmetic means in each sequence.

15. 2, ?, ?, ?, 34
16. 0, ?, ?, ?, −28
17. −10, ?, ?, ?, 14

Lesson 11-2 (pages 629–635)

Find S_n for each arithmetic series described.

1. $a_1 = 3, a_n = 20, n = 6$
2. $a_1 = 90, a_n = -4, n = 10$
3. $a_1 = 16, a_n = 14, n = 12$
4. $a_1 = -1, d = 10, n = 30$
5. $a_1 = 4, d = -5, n = 11$
6. $a_1 = 5, d = -\frac{1}{2}, n = 17$

Find the sum of each arithmetic series.

7. $\sum_{n=1}^{6}(n + 2)$
8. $\sum_{n=5}^{10}(2n - 5)$
9. $\sum_{k=1}^{5}(40 - 2k)$
10. $\sum_{k=8}^{12}(6 - 3k)$
11. $\sum_{n=1}^{4}(10n + 2)$
12. $\sum_{n=6}^{10}(2 + 3n)$

Find the first three terms of each arithmetic series described.

13. $a_1 = 11, a_n = 38, S_n = 245$
14. $n = 12, a_n = 13, S_n = -42$
15. $n = 11, a_n = 5, S_n = 0$

Lesson 11-3 (pages 636–641)

Find the next two terms of each geometric sequence.

1. 5, 15, 45, . . .
2. 2, 10, 50, . . .
3. 64, 16, 4, . . .
4. −9, 27, −81, . . .
5. 0.5, 0.75, 1.125, . . .
6. $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$

Find the first five terms of each geometric sequence described.

7. $a_1 = -2, r = 6$
8. $a_1 = 4, r = -5$
9. $a_1 = 0.8, r = 2.5$
10. $a_1 = -\frac{1}{3}, r = -\frac{3}{5}$

Find the indicated term of each geometric sequence.

11. $a_1 = 5, r = 7, n = 6$
12. $a_1 = 200, r = -\frac{1}{2}, n = 10$
13. $a_1 = 60, r = -2, n = 4$

Write an equation for the *n*th term of each geometric sequence.

14. 20, 40, 80, . . .
15. $-\frac{1}{2}, -\frac{1}{8}, -\frac{1}{32}, \ldots$

Find the geometric means in each sequence.

16. 1, ?, ?, ?, 81
17. 5, ?, ?, ?, 6480

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Lesson 11-4

(pages 643–649)

Find S_n for each geometric series described.

1. $a_1 = \frac{1}{81}, r = 3, n = 6$
2. $a_1 = 1, r = -2, n = 7$
3. $a_1 = 5, r = 4, n = 5$
4. $a_1 = -27, r = -\frac{1}{3}, n = 6$
5. $a_1 = 1000, r = \frac{1}{2}, n = 7$
6. $a_1 = 125, r = -\frac{2}{5}, n = 5$
7. $a_1 = 10, r = 3, n = 6$
8. $a_1 = 1250, r = -\frac{1}{5}, n = 5$
9. $a_1 = 1215, r = \frac{1}{3}, n = 5$
10. $a_1 = 16, r = \frac{3}{2}, n = 5$
11. $a_1 = 7, r = 2, n = 7$
12. $a_1 = -\frac{3}{2}, r = -\frac{1}{2}, n = 6$

Find the sum of each geometric series.

13. $\sum_{k=1}^{5} 2^k$
14. $\sum_{n=0}^{3} 3^{-n}$
15. $\sum_{n=0}^{3} 2(5^n)$
16. $\sum_{k=2}^{5} -(-3)^{k-1}$

Find the indicated term for each geometric series described.

17. $S_n = 300, a_n = 160, r = 2; a_1$
18. $S_n = -171, n = 9, r = -2; a_5$
19. $S_n = -4372, a_n = -2916, r = 3; a_4$

Lesson 11-5

(pages 650–655)

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 54, r = \frac{1}{3}$
2. $a_1 = 2, r = -1$
3. $a_1 = 1000, r = -0.2$
4. $a_1 = 7, r = \frac{3}{7}$
5. $49 + 14 + 4 + \ldots$
6. $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \ldots$
7. $12 - 4 + \frac{4}{3} - \ldots$
8. $3 - 9 + 27 - \ldots$
9. $3 - 2 + \frac{4}{3} - \ldots$
10. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$
11. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{10}\right)^{n-1}$
12. $\sum_{n=1}^{\infty} -\frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$

Write each repeating decimal as a fraction.

13. $0.\overline{4}$
14. $0.\overline{27}$
15. $0.\overline{123}$
16. $0.\overline{645}$
17. $0.\overline{67}$
18. $0.8\overline{53}$

Lesson 11-6

(pages 658–662)

Find the first five terms of each sequence.

1. $a_1 = 4, a_{n+1} = 2a_{n+1}$
2. $a_1 = 6, a_{n+1} = a_n + 7$
3. $a_1 = 16, a_{n+1} = a_n + (n + 4)$
4. $a_1 = 1, a_{n+1} = \frac{n}{n+2} \cdot a_n$
5. $a_1 = -\frac{1}{2}, a_{n+1} = 2a_n + \frac{1}{4}$
6. $a_1 = \frac{1}{3}, a_2 = \frac{1}{4}, a_{n+1} = a_n + a_{n-1}$

Find the first three iterates of each function for the given initial value.

7. $f(x) = 3x - 1, x_0 = 3$
8. $f(x) = 2x^2 - 8, x_0 = -1$
9. $f(x) = 4x + 5, x_0 = 0$
10. $f(x) = 3x^2 + 1, x_0 = 1$
11. $f(x) = x^2 + 4x + 4, x_0 = 1$
12. $f(x) = x^2 + 9, x_0 = 2$
13. $f(x) = 2x^2 + x + 1, x_0 = -\frac{1}{2}$
14. $f(x) = 3x^2 + 2x - 1, x_0 = \frac{2}{3}$

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Lesson 11-7

(pages 664–669)

Evaluate each expression.

1. $6!$
2. $4!$
3. $\frac{13!}{6!}$
4. $\frac{10!}{3!7!}$
5. $\frac{14!}{4!10!}$
6. $\frac{7!}{2!5!}$
7. $\frac{9!}{8!}$
8. $\frac{10!}{10!0!}$

Expand each power.

9. $(z-3)^5$
10. $(m+1)^4$
11. $(x+6)^4$
12. $(z-y)^2$
13. $(m+n)^5$
14. $(a-b)^4$
15. $(2n+1)^4$
16. $(3n-4)^3$
17. $(2n-m)^0$
18. $(4x-a)^4$
19. $(3r-4s)^5$
20. $\left(\frac{b}{2}-1\right)^4$

Find the indicated term of each expansion.

21. sixth term of $(x+3)^8$
22. fourth term of $(x-2)^7$
23. fifth term of $(a+b)^6$
24. fourth term of $(x-y)^9$
25. sixth term of $(x+4y)^7$
26. fifth term of $(3x+5y)^{10}$

Lesson 11-8

(pages 670–673)

Prove that each statement is true for all positive integers.

1. $2+4+6+\ldots+2n=n^2+n$
2. $1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$
3. $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\ldots+\frac{1}{n(n+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}$
4. $1\cdot3+2\cdot4+3\cdot5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6}$
5. $\frac{5}{1\cdot2}\cdot\frac{1}{3}+\frac{7}{2\cdot3}\cdot\frac{1}{3^2}+\frac{9}{3\cdot4}\cdot\frac{1}{3^3}+\ldots+\frac{2n+3}{n(n+1)}\cdot\frac{1}{3^n}=1-\frac{1}{3^n(n+1)}$

Find a counterexample for each statement.

6. n^2+2n-1 is divisible by 2.
7. 2^n+3^n is prime.
8. $2^{n-1}+n=2^n+2-n$ for all integers $n\geq2$
9. $3^n-2n=3^n-2^n$ for all integers $n\geq1$

Lesson 12-1

(pages 684–689)

For Exercises 1–5, state whether the events are *independent* or *dependent*.

1. tossing a penny and rolling a number cube
2. choosing first and second place in an academic competition
3. choosing from three pairs of shoes if only two pairs are available
4. A comedy video and an action video are selected from the video store.
5. The numbers 1–10 are written on pieces of paper and are placed in a hat. Three of them are selected one after the other without replacement.
6. In how many different ways can a 10-question true-false test be answered?
7. A student council has 6 seniors, 5 juniors, and 1 sophomore as members. In how many ways can a 3-member council committee be formed that includes one member of each class?
8. How many license plates of 5 symbols can be made using a letter for the first symbol and digits for the remaining 4 symbols?

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Lesson 12-2

(pages 690–695)

Evaluate each expression.

1. $P(3, 2)$
2. $P(5, 2)$
3. $P(10, 6)$
4. $P(4, 3)$
5. $P(12, 2)$
6. $P(7, 2)$
7. $C(8, 6)$
8. $C(20, 17)$
9. $C(9, 4) \cdot C(5, 3)$
10. $C(6, 1) \cdot C(4, 1)$
11. $C(10, 5) \cdot C(8, 4)$
12. $C(7, 6) \cdot C(3, 1)$

Determine whether each situation involves a *permutation* or a *combination*. Then find the number of possibilities.

13. choosing a team of 9 players from a group of 20
14. selecting the batting order of 9 players in a baseball game
15. arranging the order of 8 songs on a CD
16. finding the number of 5-card hands that include 4 diamonds and 1 club

Lesson 12-3

(pages 697–702)

A jar contains 3 red, 4 green, and 5 orange marbles. If three marbles are drawn at random and not replaced, find each probability.

1. P(all green)
2. P(1 red, then 2 not red)

Find the odds of an event occurring, given the probability of the event.

3. $\frac{5}{9}$
4. $\frac{4}{8}$
5. $\frac{3}{10}$

Find the probability of an event occurring, given the odds of the event.

6. $\frac{2}{7}$
7. $\frac{6}{13}$
8. $\frac{1}{19}$

The table shows the number of ways to achieve each product when two dice are tossed. Find each probability.

Product	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
Ways	1	2	2	3	2	4	2	1	2	4	2	1	2	2	2	1	2	1

9. $P(6)$
10. $P(12)$
11. P(not 36)
12. P(not 12)

Lesson 12-4

(pages 703–709)

An octahedral die is rolled twice. The sides are numbered 1–8. Find each probability.

1. P(1, then 8)
2. P(two different numbers)
3. P(8, then any number)

Two cards are drawn from a standard deck of cards. Find each probability if no replacement occurs.

4. P(jack, jack)
5. P(heart, club)
6. P(two diamonds)
7. P(2 of hearts, diamond)
8. P(2 red cards)
9. P(2 black aces)

Determine whether the events are *independent* or *dependent*. Then find the probability.

10. According to the weather reports, the probability of rain on a certain day is 70% in Yellow Falls and 50% in Copper Creek. What is the probability that it will rain in both cities?
11. The odds of winning a carnival game are 1 to 5. What is the probability that a player will win the game three consecutive times?

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Lesson 12-5 (pages 710–715)

An octahedral die is rolled. The sides are numbered 1–8. Find each probability.

1. P(7 or 8)
2. P(less than 4)
3. P(greater than 6)
4. P(not prime)
5. P(odd or prime)
6. P(multiple of 5 or odd)

Ten slips of paper are placed in a container. Each is labeled with a number from 1 through 10. Determine whether the events are *mutually exclusive* or *inclusive*. Then find the probability.

7. P(1 or 10)
8. P(3 or odd)
9. P(6 or less than 7)
10. Two letters are chosen at random from the word GEESE and two are chosen at random from the word PLEASE. What is the probability that all four letters are Es or none of the letters is an E?
11. Three dice are rolled. What is the probability they all show the same number?
12. Two marbles are simultaneously drawn at random from a bag containing 3 red, 5 blue, and 6 green marbles. Find each probability.
    a. P(at least one red marble)
    b. P(at least one green marble)
    c. P(two marbles of the same color)
    d. P(two marbles of different colors)

Lesson 12-6 (pages 717–723)

Find the mean, median, mode, and standard deviation of each set of data. Round to the nearest hundredth, if necessary.

1. [4, 1, 2, 1, 1]
2. [86, 71, 74, 65, 45, 42, 76]
3. [16, 20, 15, 14, 24, 23, 25, 10, 19]
4. [25.5, 26.7, 20.9, 23.4, 26.8, 24.0, 25.7]
5. [18, 24, 16, 24, 22, 24, 22, 22, 24, 13, 17, 18, 16, 20, 16, 7, 22, 5, 4, 24]
6. [55, 50, 50, 55, 65, 50, 45, 35, 50, 40, 70, 40, 70, 50, 90, 30, 35, 55, 55, 40, 75, 35, 40, 45, 65, 50, 60]
7. [364, 305, 217, 331, 305, 311, 352, 319, 272, 238, 311, 226, 220, 226, 215, 160, 123, 4, 24, 238, 99]

Lesson 12-7 (pages 724–728)

For Exercises 1–4, use the following information.

The diameters of metal fittings made by a machine are normally distributed. The diameters have a mean of 7.5 centimeters and a standard deviation of 0.5 centimeters.

1. What percent of the fittings have diameters between 7.0 and 8.0 centimeters?
2. What percent of the fittings have diameters between 7.5 and 8.0 centimeters?
3. What percent of the fittings have diameters greater than 6.5 centimeters?
4. Of 100 fittings, how many will have a diameter between 6.0 and 8.5 centimeters?

For Exercises 5–7, use the following information.

A college entrance exam was administered at a state university. The scores were normally distributed with a mean of 510, and a standard deviation of 80.

5. What percent would you expect to score above 510?
6. What percent would you expect to score between 430 and 590?
7. What is the probability that a student chosen at random scored between 350 and 670?

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Lesson 12-8 (pages 729–733)

HORSES **For Exercises 1 and 2, use the following information.**

The average lifespan of a horse is 20 years.

1. What is the probability that a randomly selected horse will live more than 25 years?
2. What is the probability that a randomly selected horse will live less than 10 years?

MINIATURE GOLF **For Exercises 3 and 4, use the following information.**

The probability of reaching in a basket of golf balls at a miniature golf course and picking out a yellow golf ball is 0.25.

3. If 5 golf balls are drawn, what is the probability that at least 2 will be yellow?
4. What is the expected number of yellow golf balls if 8 golf balls are drawn?

Lesson 12-9 (pages 735–739)

Find each probability if a coin is tossed 5 times.

1. P(0 heads)
2. P(exactly 4 heads)
3. P(exactly 3 tails)

Ten percent of a batch of toothpaste is defective. Five tubes of toothpaste are selected at random from this batch. Find each probability.

4. P(0 defective)
5. P(exactly one defective)
6. P(at least three defective)
7. P(less than three defective)

On a 20-question true-false test, you guess at every question. Find each probability.

8. P(all answers correct)
9. P(exactly 10 correct)

Lesson 12-10 (pages 741–744)

Determine whether each situation would produce a random sample. Write *yes* or *no* and explain your answer.

1. finding the most often prescribed pain reliever by asking all of the doctors at a hospital
2. taking a poll of the most popular baby girl names this year by studying birth announcements in newspapers from different cities across the country
3. polling people who are leaving a pizza parlor about their favorite restaurant in the city

For Exercises 4–6, find the margin of sampling error to the nearest percent.

4. $p = 45\%, n = 125$
5. $p = 62\%, n = 240$
6. $p = 24\%, n = 600$
7. A poll conducted on the favorite breakfast choice of students in your school showed that 75% of the 2250 students asked indicated oatmeal as their favorite breakfast.

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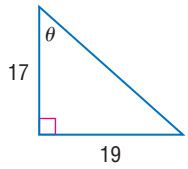
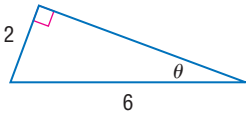
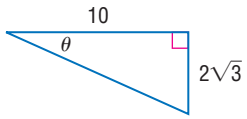
Lesson 13-1

(pages 759–767)

Find the values of the six trigonometric functions for angle θ.

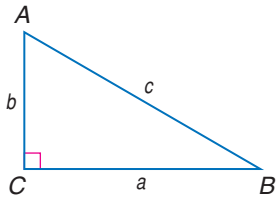
1. 

2. 

3. 

Solve $\triangle ABC$ using the diagram at the right and the given measurements. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

4. $B = 42°, c = 30$
5. $A = 84°, a = 4$
6. $B = 19°, b = 34$
7. $A = 75°, c = 55$
8. $b = 24, c = 36$
9. $a = 51, c = 115$
10. $\cos B = \frac{2}{5}, a = 12$
11. $\tan A = \frac{3}{2}, b = 22$



Lesson 13-2

(pages 768–774)

Draw an angle with the given measure in standard position.

1. $60°$
2. $250°$
3. $315°$
4. $150°$

Rewrite each degree measure in radians and each radian measure in degrees.

5. $-135°$
6. $-315°$
7. $45°$
8. $80°$
9. $24°$
10. $-54°$
11. $-\pi$
12. $\frac{9\pi}{4}$
13. $\frac{3\pi}{2}$
14. $-\frac{7\pi}{2}$
15. $\frac{9\pi}{10}$
16. $\frac{17\pi}{30}$
17. $\frac{7\pi}{12}$
18. 1
19. $-2\frac{1}{3}$

Find one angle with positive measure and one angle with negative measure coterminal with each angle.

20. $50°$
21. $-75°$
22. $125°$
23. $-400°$
24. $550°$
25. 3π
26. -2π
27. $\frac{2\pi}{3}$
28. $\frac{12\pi}{5}$
29. 0

Lesson 13-3

(pages 776–783)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. $P(3, -4)$
2. $P\left(1, \sqrt{3}\right)$
3. $P(0, 24)$
4. $P(-5, -5)$
5. $P\left(\sqrt{2}, -\sqrt{2}\right)$

Find the exact value of each trigonometric function.

6. $\cos 225°$
7. $\sin\left(-\frac{5\pi}{3}\right)$
8. $\tan \frac{7\pi}{6}$
9. $\tan(-300°)$
10. $\cos \frac{7\pi}{4}$

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

11. $\cos \theta = -\frac{1}{3}$; Quadrant III
12. $\sec \theta = 2$; Quadrant IV
13. $\sin \theta = \frac{2}{3}$; Quadrant II
14. $\tan \theta = -4$; Quadrant IV
15. $\csc \theta = -5$; Quadrant III
16. $\cot \theta = -2$; Quadrant II
17. $\tan \theta = \frac{1}{3}$; Quadrant III
18. $\cos \theta = \frac{1}{4}$; Quadrant I
19. $\csc \theta = -\frac{5}{2}$; Quadrant IV

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Lesson 13-4

(pages 785–792)

Find the area of $\triangle ABC$. Round to the nearest tenth.

1. $a = 11$ m, $b = 13$ m, $C = 31°$
2. $a = 15$ ft, $b = 22$ ft, $C = 90°$
3. $a = 12$ cm, $b = 12$ cm, $C = 50°$

Solve each triangle. Round to the nearest tenth.

4. $A = 18°, B = 37°, a = 15$
5. $A = 60°, C = 25°, c = 3$
6. $B = 40°, C = 32°, b = 10$
7. $B = 10°, C = 23°, c = 8$
8. $A = 12°, B = 60°, b = 5$
9. $A = 35°, C = 45°, a = 30$

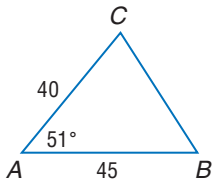
Determine whether each triangle has no solution, one solution, or two solutions. Then solve each triangle. Round to the nearest tenth.

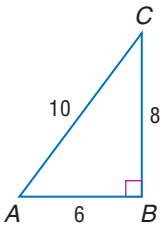
10. $A = 40°, B = 60°, c = 20$
11. $B = 70°, C = 58°, a = 84$
12. $A = 40°, a = 5, b = 12$
13. $A = 58°, a = 26, b = 29$
14. $A = 38°, B = 63°, c = 15$
15. $A = 150°, a = 6, b = 8$
16. $A = 57°, a = 12, b = 19$
17. $A = 25°, a = 125, b = 150$
18. $C = 98°, a = 64, c = 90$
19. $A = 40°, B = 60°, c = 20$
20. $A = 132°, a = 33, b = 50$
21. $A = 5\,45°, a = 83, b = 79$

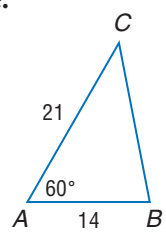
Lesson 13-5

(pages 793–798)

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle.

1. 

2. 

3. 

4. $a = 14, b = 15, c = 16$
5. $B = 41°, C = 52°, c = 27$
6. $a = 19, b = 24.3, c = 21.8$
7. $A = 112°, a = 32, c = 20$
8. $b = 8, c = 7, A = 28°$
9. $a = 5, b = 6, c = 7$
10. $C = 25°, a = 12, b = 9$
11. $a = 8, A = 49°, B = 58°$
12. $A = 42°, b = 120, c = 160$
13. $c = 10, A = 35°, C = 65°$
14. $a = 10, b = 16, c = 19$
15. $B = 45°, a = 40, c = 48$
16. $B = 100°, a = 10, c = 8$
17. $A = 40°, B = 45°, c = 4$
18. $A = 20°, b = 100, c = 84$

Lesson 13-6

(pages 799–805)

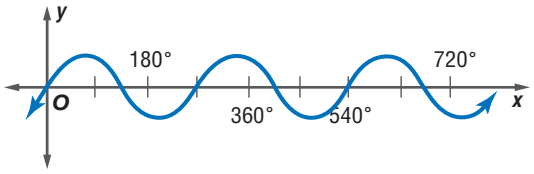
The given point P is located on the unit circle. Find $\sin\theta$ and $\cos\theta$.

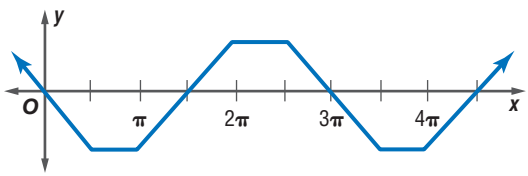
1. $P\left(\frac{4}{5}, \frac{3}{5}\right)$
2. $P\left(\frac{12}{3}, -\frac{5}{13}\right)$
3. $P\left(-\frac{8}{17}, -\frac{15}{17}\right)$
4. $P\left(\frac{3}{7}, \frac{2\sqrt{10}}{7}\right)$
5. $P\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$

Find the exact value of each function.

6. $\sin 210°$
7. $\cos 150°$
8. $\cos(2135°)$
9. $\cos$
10. $\sin 570°$
11. $\sin 390°$
12. $\sin\frac{4\pi}{3}$
13. $\cos -\frac{7\pi}{3}$
14. $\cos 30° + \cos 60°$
15. $5(\sin 45°)(\cos 45°)$
16. $\frac{\sin 210° + \cos 240°}{3}$
17. $\frac{6\cos 120° + 4\sin 150°}{5}$

Determine the period of each function.

18. 

19. 

Extra Practice

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Lesson 13–7

(pages 806–811)

Write each equation in the form of an inverse function.

1. $\text{Sin } m + n$
2. $\text{Tan } 45° = 1$
3. $\text{Cos } x = \frac{1}{2}$
4. $\text{Sin } 65° = a$
5. $\text{Tan } 60° = \sqrt{3}$
6. $\text{Sin } x = \frac{\sqrt{2}}{2}$

Solve each equation.

7. $y = \text{Sin}^{-1} -\frac{\sqrt{2}}{2}$
8. $\text{Tan}^{-1}(1) = x$
9. $a = \text{Arccos}\left(\frac{\sqrt{3}}{2}\right)$
10. $\text{Arcsin}(0) = x$
11. $y = \text{Cos}^{-1}\frac{1}{2}$
12. $y = \text{Sin}^{-1}(1)$

Find each value. Round to the nearest hundredth.

13. $\text{Arccos}\left(-\frac{\sqrt{2}}{2}\right)$
14. $\text{Sin}^{-1}(-1)$
15. $\cos\left[\text{Arcsin}\left(\frac{\sqrt{2}}{2}\right)\right]$
16. $\tan\left[\text{Sin}^{-1}\left(\frac{15}{13}\right)\right]$
17. $\sin\left[\text{Arccos}\frac{1}{2}\right]$
18. $\sin\left[\text{Arccos}\left(\frac{5}{17}\right)\right]$
19. $\sin\left[\text{Tan}^{-1}\left(\frac{5}{12}\right)\right]$
20. $\tan\left[\text{Arccos} -\left(\frac{\sqrt{3}}{2}\right)\right]$
21. $\sin^{-1}[\text{Cos}^{-1}(1) - 1]$
22. $\text{Cos}^{-1}\left[\tan\frac{\pi}{4}\right]$
23. $\cos\left[\text{Sin}^{-1}\frac{1}{2}\right]$
24. $\sin[\text{Cos}^{-1}(0)]$

Lesson 14-1

(pages 822–828)

Find the amplitude, if it exists, and period of each function. Then graph each function.

1. $y = 2\cos\theta$
2. $y = \frac{1}{3}\sin\theta$
3. $y = \sin 3\theta$
4. $y = 3\sec\theta$
5. $y = \sec\frac{1}{3}\theta$
6. $y = 2\csc\theta$
7. $y = 3\tan\theta$
8. $y = 3\sin\frac{2}{3}\theta$
9. $y = 2\sin\frac{1}{5}\theta$
10. $y = 3\sin 2\theta$
11. $y = \frac{1}{2}\cos\frac{3}{4}\theta$
12. $y = 5\csc 3\theta$
13. $y = 2\cot 6\theta$
14. $y = 2\csc 6\theta$
15. $y = 3\tan\frac{1}{3}\theta$

Lesson 14-2

(pages 829–836)

State the phase shift for each function. Then graph the function.

1. $y = \sin(\theta + 60°)$
2. $y = \cos(\theta - 90°)$
3. $y = \tan\left(\theta + \frac{\pi}{2}\right)$
4. $y = \sin\theta + \frac{\pi}{6}$

State the vertical shift and the equation of the midline for each function. Then graph the function.

5. $y = \cos\theta + 3$
6. $y = \sin\theta - 2$
7. $y = \sec\theta + 5$
8. $y = \csc\theta - 6$
9. $y = 2\sin\theta - 4$
10. $y = \frac{1}{3}\sin\theta + 7$

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

11. $y = 3\cos[2(\theta + 30°)] + 4$
12. $y = 2\tan[3(\theta - 60°)] - 2$
13. $y = \frac{1}{2}\sin[4(\theta + 45°)] + 1$
14. $y = \frac{2}{5}\cos[6(\theta + 45°)] - 5$
15. $y = 6 - 2\sin\left[3\left(\theta + \frac{\pi}{2}\right)\right]$
16. $y = 3 + 3\cos\left[2\left(\theta - \frac{\pi}{3}\right)\right]$

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Lesson 14-3

(pages 837–841)

Find the value of each expression.

1. $\sin \theta$, if $\cos \theta = \frac{4}{5}$; $0° \le \theta \le 90°$
2. $\tan \theta$, if $\sin \theta = \frac{1}{2}$; $0° \le \theta \le 90°$
3. $\csc \theta$, if $\sin \theta = \frac{3}{4}$; $90° \le \theta \le 180°$
4. $\cos \theta$, if $\tan \theta = 24$; $90° \le \theta \le 180°$
5. $\sec \theta$, if $\tan \theta = 24$; $90° \le \theta \le 180°$
6. $\sin \theta$, if $\cot \theta = -\frac{1}{4}$; $270° \le \theta \le 360°$
7. $\tan \theta$, if $\sec \theta = 23$; $90° \le \theta \le 180°$
8. $\sin \theta$, if $\cos \theta = \frac{3}{5}$; $270° \le \theta \le 360°$
9. $\cos \theta$, if $\sin \theta = -\frac{1}{2}$; $270° \le \theta \le 360°$
10. $\csc \theta$, if $\cot \theta = -\frac{1}{4}$; $90° \le \theta \le 180°$
11. $\csc \theta$, if $\sec \theta = -\frac{5}{3}$; $180° \le \theta \le 270°$
12. $\cos \theta$, if $\tan \theta = 5$; $180° \le \theta \le 270°$

Simplify each expression.

13. $\csc^2 \theta - \cot^2 \theta$
14. $\sin \theta \tan \theta \csc \theta$
15. $\tan \theta \csc \theta$
16. $\sec \theta \cot \theta \cos \theta$
17. $\cos \theta (1 - \cos^2 \theta)$
18. $\frac{1 - \sin^2 \theta}{\cos^2 \theta}$
19. $\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2}$
20. $\frac{1 + \tan^2 \theta}{1 + \cot^2 \theta}$
21. $\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta}$

Lesson 14-4

(pages 842–846)

Verify that each of the following is an identity.

1. $\sin^2 \theta + \cos^2 \theta + \tan^2 \theta = \sec^2 \theta$
2. $\frac{\tan \theta}{\sin \theta} = \sec \theta$
3. $\frac{\tan \theta}{\cot \theta} = \tan^2 \theta$
4. $\csc^2 \theta (1 - \cos^2 \theta) = 1$
5. $1 - \cot^4 \theta = 2 \csc^2 \theta - \csc^4 \theta$
6. $\sin^4 \theta - \cos^4 \theta = \sin^2 \theta - \cos^2 \theta$
7. $\sin^2 \theta + \cot^2 \theta \sin^2 \theta = 1$
8. $\frac{\cos \theta}{\csc \theta} - \frac{\csc \theta}{\sec \theta} = -\frac{\cos^3 \theta}{\sin \theta}$
9. $\frac{\cos \theta}{\sec \theta} - \frac{1 + \cos \theta}{\sec \theta + 1} = 2 \cot^2 \theta$
10. $\frac{1 + \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 - \cos \theta}$
11. $\sec \theta + \tan \theta = \frac{\cos \theta}{1 - \sin \theta}$
12. $\tan \theta + \cot \theta = \csc \theta \sec \theta$
13. $\frac{\cot^2 \theta}{1 + \cot^2 \theta} = 1 - \sin^2 \theta$
14. $\frac{\tan \theta 2 \sin \theta}{\sec \theta} = \frac{\sin^3 \theta}{1 + \cos \theta}$
15. $\sin^2 \theta (1 - \cos^2 \theta) = \sin 4\theta$
16. $\sin^2 \theta + \sin^2 \theta \tan^2 \theta = \tan^2 \theta$
17. $\frac{\sec \theta - 1}{\sec \theta + 1} + \frac{\cos \theta - 1}{\cos \theta + 1} = 0$
18. $\tan^2 \theta (1 - \sin^2 \theta) = \sin^2 \theta$
19. $\tan \theta + \frac{\cos \theta}{1 + \sin \theta} = \sec \theta$
20. $\frac{\tan \theta}{\sec \theta + 1} = \frac{1 - \cos \theta}{\sin \theta}$
21. $\csc \theta - \frac{\sin \theta}{1 + \cos \theta} = \cot \theta$

Lesson 14-5

(pages 848–852)

Find the exact value of each expression.

1. $\sin 195°$
2. $\cos 285°$
3. $\sin 255°$
4. $\sin 105°$
5. $\cos 15°$
6. $\sin 15°$
7. $\cos 375°$
8. $\sin 165°$
9. $\sin (-225°)$
10. $\cos (-210°)$
11. $\cos (-225°)$
12. $\sin (-30°)$
13. $\sin 120°$
14. $\sin 225°$
15. $\cos (-30°)$

Verify that each of the following is an identity.

16. $\sin (90° + \theta) = \cos \theta$
17. $\cos (180° - \theta) = -\cos \theta$
18. $\sin (p + \theta) = -\sin \theta$
19. $\sin (\theta + 30°) + \sin (\theta + 60°) = \sqrt{3} + \frac{1}{2}(\sin \theta + \cos \theta)$
20. $\cos (30° - \theta) + \cos (30° + \theta) = \sqrt{3} \cos \theta$

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Lesson 14-6

(pages 853–859)

Find the exact value of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

1. $\cos \theta = \frac{7}{25}; 0 < \theta < 90°$
2. $\sin \theta = \frac{2}{7}; 0 < \theta < 90°$
3. $\cos \theta = -\frac{1}{8}; 180° < \theta < 270°$
4. $\sin \theta = -\frac{5}{13}; 270° < \theta < 360°$
5. $\sin \theta = \frac{\sqrt{35}}{6}; 0° < \theta < 90°$
6. $\cos \theta = -\frac{17}{18}; 90° < \theta < 180°$

Find the exact value of each expression by using the half-angle formulas.

7. $\sin 75°$
8. $\cos 75°$
9. $\sin \frac{\pi}{8}$
10. $\cos \frac{13\pi}{12}$
11. $\cos 22.5°$
12. $\cos \frac{\pi}{4}$

Verify that each of the following is an identity.

13. $\frac{\sin 2\theta}{2 \sin^2 \theta} = \cot \theta$
14. $1 + \cos 2\theta = \frac{2}{1 + \tan^2 \theta}$
15. $\csc \theta \sec \theta = 2 \csc 2\theta$
16. $\sin 2\theta (\cot \theta + \tan \theta) = 2$
17. $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta$
18. $\frac{\csc \theta + \sin \theta}{\csc \theta - \sin \theta} = \frac{1 + \sin 2\theta}{\cos 2\theta}$

Lesson 14-7

(pages 861–866)

Find all the solutions for each equation for $0° \leq \theta < 360°$.

1. $\cos \theta = -\frac{\sqrt{3}}{2}$
2. $\sin 2\theta = -\frac{\sqrt{3}}{2}$
3. $\cos 2\theta = 8 - 15 \sin \theta$
4. $\sin \theta + \cos \theta = 1$
5. $2 \sin^2 \theta + \sin \theta = 0$
6. $\sin 2\theta = \cos \theta$

Solve each equation for all values of θ if θ is measured in radians.

7. $\cos 2\theta \sin \theta = 1$
8. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$
9. $\cos 2\theta + 4 \cos \theta = -3$
10. $\sin \frac{\theta}{2} + \cos \theta = 1$
11. $3 \tan^2 \theta - \sqrt{3} \tan \theta = 0$
12. $4 \sin \theta \cos \theta = -\sqrt{3}$

Solve each equation for all values of θ if θ is measured in degrees.

13. $2 \sin^2 \theta - 1 = 0$
14. $\cos \theta - 2 \cos \theta \sin \theta = 0$
15. $\cos 2\theta \sin \theta = 1$
16. $(\tan \theta - 1)(2 \cos \theta + 1) = 0$
17. $2 \cos^2 \theta = 0.5$
18. $\sin \theta \tan \theta - \tan \theta = 0$

Solve each equation for all values of θ.

19. $\tan \theta = 1$
20. $\cos 8\theta = 1$
21. $\sin \theta + 1 = \cos 2\theta$
22. $8 \sin \theta \cos \theta = 2\sqrt{3}$
23. $\cos \theta = 1 + \sin \theta$
24. $2 \cos^2 \theta = \cos \theta$

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Mixed Problem Solving

Chapter 1 Equations and Inequalities

(pp. 4–55)

GEOMETRY **For Exercises 1 and 2, use the following information.**

The formula for the surface area of a sphere is $SA = 4\pi r^2$, and the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. (Lesson 1-1)

1. Find the volume and surface area of a sphere with radius 2 inches. Write your answer in terms of π.
2. Is it possible for a sphere to have the same numerical value for the surface area and volume? If so, find the radius of such a sphere.

3. **CONSTRUCTION** The Birtic family is building a family room on their house. The dimensions of the room are 26 feet by 28 feet. Show how to use the Distributive Property to mentally calculate the area of the room. (Lesson 1-2)

GEOMETRY **For Exercises 4–6, use the following information.**

The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$. (Lesson 1-2)

4. Use the Distributive Property to rewrite the formula by factoring out the greatest common factor of the two terms.
5. Find the surface area for a cylinder with radius 3 centimeters and height 10 centimeters using both formulas. Leave the answer in terms of π.
6. Which formula do you prefer? Explain your reasoning.

POPULATION **For Exercises 7 and 8, use the following information.**

In 2004, the population of Bay City was 19,611. For each of the next five years, the population decreased by an average of 715 people per year. (Lesson 1-3)

7. What was the population in 2009?
8. If the population continues to decline at the same rate as from 2004 to 2009, what would you expect the population to be in 2020?

ASTRONOMY **For Exercises 9 and 10, use the following information.**

The planets in our solar system travel in orbits that are not circular. For example, Pluto's farthest distance from the Sun is 4539 million miles, and its closest distance is 2756 million miles. (Lesson 1-4)

9. What is the average of the two distances ?
10. Write an equation that can be solved to find the minimum and maximum distances from the Sun to Pluto.

HEALTH **For Exercises 11 and 12, use the following information.**

The National Heart Association recommends that less than 30% of a person's total daily Caloric intake come from fat. One gram of fat yields nine Calories. Jason is a healthy 21-year-old male whose average daily Caloric intake is between 2500 and 3300 Calories. (Lesson 1-5)

11. Write an inequality that represents the suggested fat intake for Jason.
12. What is the greatest suggested fat intake for Jason?

TRAVEL **For Exercises 13 and 14, use the following information.**

Bonnie is planning a 5-day trip to a convention. She wants to spend no more than $1000. The plane ticket is $375, and the hotel is $85 per night. (Lesson 1-5)

13. Let f represent the cost of food for one day. Write an inequality to represent this situation.
14. Solve the inequality and interpret the solution.

15. **PAINTING** Phil owns and operates a home remodeling business. He estimates that he will need 12–15 gallons of paint for a particular project. If each gallon of paint costs $18.99, write and solve a compound inequality to determine what the cost c of the paint could be. (Lesson 1-6)

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AGRICULTURE **For Exercises 1–3, use the following information.**

The table shows the average prices received by farmers for a bushel of corn. (Lesson 2-1)

Year	Price	Year	Price
1940	$0.62	1980	$3.11
1950	$1.52	1990	$2.28
1960	$1.00	2000	$1.85
1970	$1.33		

Source: *The World Almanac*

1. Write a relation to represent the data.
2. Graph the relation.
3. Is the relation a function? Explain.

MEASUREMENT **For Exercises 4 and 5, use the following information.**

The equation $y = 0.3937x$ can be used to convert any number of centimeters x to inches y. (Lesson 2-2)

4. Find the number of inches in 100 centimeters.
5. Find the number of centimeters in 12 inches.

POPULATION **For Exercises 6 and 7, use the following information.**

The table shows the growth in the population of Miami, Florida. (Lesson 2-3)

Year	Population	Year	Population
1950	249,276	1990	358,648
1970	334,859	2000	362,437
1980	346,681	2003	376,815

Source: *The World Almanac*

6. Graph the data in the table.
7. Find the average rate of change.

HEALTH **For Exercises 8–10, use the following information.**

In 1985, 39% of people in the United States age 12 and over reported using cigarettes. The percent of people using cigarettes has decreased about 1.7% per year following 1985.

Source: *The World Almanac* (Lesson 2-4)

8. Write an equation that represents how many people use cigarettes in x years.
9. If the percent of people using cigarettes continues to decrease at the same rate, what percent of people would you predict to be using cigarettes in 2005?
10. If the trend continues, when would you predict there to be no people using cigarettes in the U.S.? How accurate is your prediction?

EMPLOYMENT **For Exercises 11–15, use the table that shows unemployment statistics for 1993 to 1999.** (Lesson 2-5)

Year	Number Unemployed	Percent Unemployed
1993	8,940,000	6.9
1994	7,996,000	6.1
1995	7,404,000	5.6
1996	7,236,000	5.4
1997	6,739,000	4.9
1998	6,210,000	4.5
1999	5,880,000	4.2

Source: *The World Almanac*

11. Draw two scatter plots of the data. Let x represent the year.
12. Use two ordered pairs to write an equation for each scatter plot.
13. Compare the two equations.
14. Predict the percent of people that will be unemployed in 2005.
15. In 1999, what was the total number of people in the United States?
16. **EDUCATION** At Madison Elementary, each classroom can have at most 25 students. Draw a graph of a step function that shows the relationship between the number of students x and the number of classrooms y that are needed. (Lesson 2-6)

CRAFTS **For Exercises 17–19, use the following information.**

Priscilla sells stuffed animals at a local craft show. She charges $10 for the small and $15 for the large ones. To cover her expenses, she needs to sell at least $350. (Lesson 2-7)

17. Write an inequality for this situation.
18. Graph the inequality.
19. If she sells 10 small and 15 large animals, will she cover her expenses?

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EXERCISE **For Exercises 1–4, use the following information.**

At Everybody's Gym, you have two options for becoming a member. You can pay \$400 per year or you can pay \$150 per year plus \$5 per visit. (Lesson 3-1)

1. For each option, write an equation that represents the cost of belonging to the gym.
2. Graph the equations. Estimate the break-even point for the gym memberships.
3. Explain what the break-even point means.
4. If you plan to visit the gym at least once per week during the year, which option should you choose?

5. **GEOMETRY** Find the coordinates of the vertices of the parallelogram whose sides are contained in the lines whose equations are $y = 3$, $y = 7$, $y = 2x$, and $y = 2x - 13$. (Lesson 3-2)

EDUCATION **For Exercises 6–9, use the following information.**

Mr. Gunlikson needs to purchase equipment for his physical education classes. His budget for the year is \$4250. He decides to purchase cross-country ski equipment. He is able to find skis for \$75 per pair and boots for \$40 per pair. He knows that he should buy more boots than skis because the skis are adjustable to several sizes of boots. (Lesson 3-3)

6. Let y be the number of pairs of boots and x be the number of pairs of skis. Write a system of inequalities for this situation. (Remember that the number of pairs of boots and skis must be positive.)
7. Graph the region that shows how many pairs of boots and skis he can buy.
8. Give an example of three different purchases that Mr. Gunlikson can make.
9. Suppose Mr. Gunlikson wants to spend all of the money. What combination of skis and boots should he buy? Explain.

MANUFACTURING **For Exercises 10–14, use the following information.**

A shoe manufacturer makes outdoor and indoor soccer shoes. There is a two-step process for both kinds of shoes. Each pair of outdoor shoes requires 2 hours in step one, 1 hour in step two, and produces a profit of \$20. Each pair of indoor shoes requires 1 hour in step one, 3 hours in step two, and produces a profit of \$15. The company has 40 hours of labor per day available for step one and 60 hours available for step two. (Lesson 3-4)

10. Let x represent the number of pairs of outdoor shoes and let y represent the number of indoor shoes that can be produced per day. Write a system of inequalities to represent the number of pairs of outdoor and indoor soccer shoes that can be produced in one day.
11. Draw the graph showing the feasible region.
12. List the coordinates of the vertices of the feasible region.
13. Write a function for the total profit.
14. What is the maximum profit? What is the combination of shoes for this profit?

GEOMETRY **For Exercises 15–17, use the following information.**

An isosceles trapezoid has shorter base of measure a, longer base of measure c, and congruent legs of measure b. The perimeter of the trapezoid is 58 inches. The average of the bases is 19 inches and the longer base is twice the leg plus 7. (Lesson 3-5)

15. Write a system of three equations that represents this situation.
16. Find the lengths of the sides of the trapezoid.
17. Find the area of the trapezoid.

18. **EDUCATION** The three American universities with the greatest endowments in 2000 were Harvard, Yale, and Stanford. Their combined endowments are \$38.1 billion. Harvard had \$0.1 billion more in endowments than Yale and Stanford together. Stanford's endowments trailed Harvard's by \$10.2 billion. What were the endowments of each of these universities? (Lesson 3-5)

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AGRICULTURE For Exercises 1 and 2, use the following information.

In 2003, the United States produced 63,590,000 metric tons of wheat, 9,034,000 metric tons of rice, and 256,905,000 metric tons of corn. In that same year, Russia produced 34,062,000 metric tons of wheat, 450,000 metric tons of rice, and 2,113,000 metric tons of corn.

Source: *The World Almanac* (Lesson 4-1)

1. Organize the data in two matrices.
2. What are the dimensions of the matrices?

LIFE EXPECTANCY For Exercises 3–5, use the life expectancy table. (Lesson 4-2)

Year	1910	1930	1950	1970	1990
Male	48.4	58.1	65.6	67.1	71.8
Female	51.8	61.6	71.1	74.7	78.8

Source: *The World Almanac*

3. Organize all the data in a matrix.
4. Show how to organize the data in two matrices in such a way that you can find the difference between the life expectancies of males and females for the given years. Then find the difference.
5. Does addition of any two of the matrices make sense? Explain.

CRAFTS For Exercises 6 and 7, use the following information.

Mrs. Long is selling crocheted items. She sells large afghans for \$60, baby blankets for \$40, doilies for \$25, and pot holders for \$5. She takes the following number of items to the fair: 12 afghans, 25 baby blankets, 45 doilies, and 50 pot holders. (Lesson 4-3)

6. Write an inventory matrix for the number of each item and a cost matrix for the price of each item.
7. Suppose Mrs. Long sells all of the items. Find her total income as a matrix.

GEOMETRY For Exercises 8–11, use the following information.

A trapezoid has vertices $T(3, 3)$, $R(-1, 3)$, $A(-2, -4)$, and $P(5, -4)$. (Lesson 4-4)

8. Show how to use a reflection matrix to find the vertices of $TRAP$ after a reflection over the x-axis.
9. The area of a trapezoid is found by multiplying one-half the sum of the bases by the height. Find the areas of $TRAP$ and $T'R'A'P'$. How do they compare?
10. Show how to use a matrix and scalar multiplication to find the vertices of $TRAP$ after a dilation that triples its perimeter.
11. Find the areas of $TRAP$ and $T'R'A'P'$ in Exercise 10. How do they compare?

AGRICULTURE For Exercises 12 and 13, use the following information.

A farm has a triangular plot defined by the coordinates $\left(-\frac{1}{2}, -\frac{1}{4}\right)$, $\left(\frac{1}{3}, \frac{1}{2}\right)$, and $\left(\frac{2}{3}, -\frac{1}{2}\right)$, where units are in square miles. (Lesson 4-5)

12. Find the area of the region in square miles.
13. One square mile equals 640 acres. To the nearest acre, how many acres are in the triangular plot?

ART For Exercises 14 and 15, use the following information.

Small beads sell for \$5.80 per pound, and large beads sell for \$4.60 per pound. Bernadette bought a bag of beads for \$33 that contained 3 times as many pounds of the small beads as the large beads. (Lesson 4-6)

14. Write a system of equations using the information given.
15. How many pounds of small and large beads did Bernadette buy?

MATRICES For Exercises 16 and 17, use the following information.

Two 2×2 inverse matrices have a sum of $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$. The value of each entry is no less than -3 and no greater than 2. (Lesson 4-7)

16. Find the two matrices that satisfy the conditions.
17. Explain your method.

18. **CONSTRUCTION** Alan charges \$750 to build a small deck and \$1250 to build a large deck. During the spring and summer, he built 5 more small decks than large decks. If he earned \$11,750, how many of each type of deck did he build? (Lesson 4-8)

Mixed Problem Solving

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PHYSICS **For Exercises 1–3, use the following information.**

A model rocket is shot straight up from the top of a 100-foot building at a velocity of 800 feet per second. (Lesson 5-1)

1. The height $h(t)$ of the model rocket t seconds after firing is given by $h(t) = -16t^2 + at + b$, where a is the initial velocity in feet per second and b is the initial height of the rocket above the ground. Write an equation for the rocket.
2. Find the maximum height of the rocket and the time that the height is reached.
3. Suppose a rocket is fired from the ground (initial height is 0). Find values for a, initial velocity, and t, time, if the rocket reaches a height of 32,000 feet at time t.

RIDES **For Exercises 4 and 5, use the following information.**

An amusement park ride carries riders to the top of a 225-foot tower. The riders then free-fall in their seats until they reach 30 feet above the ground. (Lesson 5-2)

4. Use the formula $h(t) = -16t^2 + h_0$, where the time t is in seconds and the initial height h_0 is in feet, to find how long the riders are in free-fall.
5. Suppose the designer of the ride wants the riders to experience free-fall for 5 seconds before stopping 30 feet above the ground. What should be the height of the tower?

CONSTRUCTION **For Exercises 6 and 7, use the following information.**

Nicole's new house has a small deck that measures 6 feet by 12 feet. She would like to build a larger deck. (Lesson 5-3)

6. By what amount must each dimension be increased to triple the area of the deck?
7. What are the new dimensions of the deck?

NUMBER THEORY **For Exercises 8 and 9, use the following information.**

Two complex conjugate numbers have a sum of 12 and a product of 40. (Lesson 5-4)

8. Find the two numbers.
9. Explain the method you used.

CONSTRUCTION **For Exercises 10 and 11, use the following information.**

A contractor wants to construct a rectangular pool with a length that is twice the width. He plans to build a two-meter-wide walkway around the pool. He wants the area of the walkway to equal the surface area of the pool. (Lesson 5-5)

10. Find the dimensions of the pool to the nearest tenth of a meter.
11. What is the surface area of the pool to the nearest square meter?

PHYSICS **For Exercises 12–14, use the following information.**

A ball is thrown into the air vertically with a velocity of 112 feet per second. The ball was released 6 feet above the ground. The height above the ground t seconds after release is modeled by the equation $h(t) = -16t^2 + 112t + 6$. (Lesson 5-6)

12. When will the ball reach 130 feet?
13. Will the ball ever reach 250 feet? Explain.
14. In how many seconds after its release will the ball hit the ground?

WEATHER **For Exercises 15–17, use the following information.**

The normal monthly high temperatures for Albany, New York, are 21, 24, 34, 46, 58, 67, 72, 70, 61, 50, 40, and 27 degrees Fahrenheit, respectively. **Source:** *The World Almanac* (Lesson 5-7)

15. Suppose January = 1, February = 2, and so on. A graphing calculator gave the following function as a model for the data: $y = -1.5x^2 + 21.2x - 8.5$. Graph the points in the table and the function on the same coordinate plane.
16. Identify the vertex, axis of symmetry, and direction of opening for this function.
17. Discuss how well you think the function models the actual temperature data.
18. **MODELS** John is building a display table for model cars. He wants the perimeter of the table to be 26 feet, but he wants the area of the table to be no more than 30 square feet. What could the width of the table be? (Lesson 5-8)

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1. **EDUCATION** In 2002 in the United States, there were 3,034,065 classroom teachers and 48,201,550 students. An average of $7731 was spent per student. Find the total amount of money spent for students in 2002. Write the answer in both scientific and standard notation. **Source:** *The World Almanac* (Lesson 6-1)

POPULATION For Exercises 2–4, use the following information.

In 2000, the population of Mexico City was 18,131,000, and the population of Bombay was 18,066,000. It is projected that, until the year 2015, the population of Mexico City will increase at the rate of 0.4% per year and the population of Bombay will increase at the rate of 3% per year. **Source:** *The World Almanac* (Lesson 6-2)

2. Let r represent the rate of increase in population for each city. Write a polynomial to represent the population of each city in 2002.
3. Predict the population of each city in 2015.
4. If the projected rates are accurate, in what year will the two cities have approximately the same population?

POPULATION For Exercises 5–8, use the following information.

The table shows the percent of the U.S. population that was foreign-born during various years. The x-values are years since 1900 and the y-values are the percent of the population. **Source:** *The World Almanac* (Lesson 6-3 and 6-4)

U.S. Foreign-Born Population			
x	y	x	y
0	13.6	60	5.4
10	14.7	70	4.7
20	13.2	80	6.2
30	11.6	90	8.0
40	8.8	100	10.4
50	6.9		

5. Graph the function.
6. Describe the turning points of the graph and its end behavior.
7. What do the relative maxima and minima represent?
8. If this graph were modeled by a polynomial equation, what is the least degree the equation could have?

GEOMETRY For Exercises 9 and 10, use the following information.

Hero's formula for the area of a triangle is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the lengths of the sides of the triangle and $s = 0.5(a+b+c)$. (Lesson 6-5)

9. Find the lengths of the sides of the triangle given in this application of Hero's formula: $A = \sqrt{s^4 - 12s^3 + 47s^2 - 60s}$.
10. What type of triangle is this?

GEOMETRY For Exercises 11 and 12, use the following information.

The volume of a rectangular box can be written as $6x^3 + 31x^2 + 53x + 30$, and the height is always $x + 2$. (Lesson 6-6)

11. What are the width and length of the box?
12. Will the ratio of the dimensions of the box always be the same regardless of the value of x? Explain .

SALES For Exercises 13 and 14, use the following information.

The sales of items related to information technology can be modeled by $S(x) = -1.7x^3 + 18x^2 + 26.4x + 678$, where x is the number of years since 1996 and y is billions of dollars. **Source:** *The World Almanac* (Lesson 6-7)

13. Use synthetic substitution to estimate the sales for 2003 and 2006.
14. Do you think this model is useful in estimating future sales? Explain.
15. **MANUFACTURING** A box measures 12 inches by 16 inches by 18 inches. The manufacturer will increase each dimension of the box by the same number of inches and have a new volume of 5985 cubic inches. How much should be added to each dimension? (Lesson 6-8)
16. **CONSTRUCTION** A picnic area has the shape of a trapezoid. The longer base is 8 more than 3 times the length of the shorter base and the height is 1 more than 3 times the shorter base. What are the dimensions if the area is 4104 square feet? (Lesson 6-9)

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EMPLOYMENT **For Exercises 1 and 2, use the following information.**

From 1994 to 1999, the number of employed women and men in the United States, age 16 and over, can be modeled by the following equations, where x is the number of years since 1994 and y is the number of people in thousands. **Source:** *The World Almanac* (Lesson 7-1)

women: $y = 1086.4x + 56{,}610$

men: $y = 999.2x + 66{,}450$

1. Write a function that models the total number of men and women employed in the United States during this time.
2. If f is the function for the number of men, and g is the function for the number of women, what does $(f - g)(x)$ represent?
3. **HEALTH** The average weight of a baby born at a certain hospital is $7\frac{1}{2}$ pounds, and the average length is 19.5 inches. One kilogram is about 2.2 pounds, and 1 centimeter is about 0.3937 inches. Find the average weight in kilograms and the length in centimeters. (Lesson 7-2)

SAFETY **For Exercises 4 and 5, use the following information.**

The table shows the total stopping distance x, in meters, of a vehicle and the speed y, in meters per second. (Lesson 7-3)

Distance	92	68	49	32	18
Speed	29	25	20	16	11

4. Graph the data in the table.
5. Graph the function $y = 2\sqrt{2x}$ on the same coordinate plane. How well do you think this function models the given data? Explain.
6. **PHYSICS** The speed of sound in a liquid is $s = \sqrt{\frac{B}{d}}$, where B is known as the bulk modulus of the liquid and d is the density of the liquid. For water, $B = 2.1 \cdot 10^9$ N/m^2 and $d = 10^3$ kg/m^3. Find the speed of sound in water to the nearest meter per second. (Lesson 7-4)

LAW ENFORCEMENT **For Exercises 7 and 8, use the following information.**

The approximate speed s in miles per hour that a car was traveling if it skidded d feet is given by the formula $s = 5.5\sqrt{kd}$, where k is the coefficient of friction. (Lesson 7-5)

7. For a dry concrete road, $k = 0.8$. If a car skids 110 feet on a dry concrete road, find its speed in miles per hour to the nearest whole number.
8. Another formula using the same variables is $s = 2\sqrt{5kd}$. Compare the results using the two formulas. Explain any variations in the answers.

PHYSICS **For Exercises 9–11, use the following information.**

Kepler's Third Law of planetary motion states that the square of the orbital period of any planet, in Earth years, is equal to the cube of the planet's distance from the Sun in astronomical units (AU). **Source:** *The World Almanac* (Lesson 7-6)

9. The orbital period of Mercury is 87.97 Earth days. What is Mercury's distance from the Sun in AU?
10. Pluto's period of revolution is 247.66 Earth years. What is Pluto's distance from the Sun?
11. What is Earth's distance from the Sun in AU? Explain your result.

PHYSICS **For Exercises 12 and 13, use the following information.**

The time T in seconds that it takes a pendulum to make a complete swing back and forth is given by the formula $T = 2\pi\sqrt{\frac{L}{g}}$, where L is the length of the pendulum in feet and g is the acceleration due to gravity, 32 feet per second squared. (Lesson 7-7)

12. In Tokyo, Japan, a huge pendulum in the Shinjuku building measures 73 feet 9.75 inches. How long does it take for the pendulum to make a complete swing?
Source: *The Guinness Book of Records*
13. A clockmaker wants to build a pendulum that takes 20 seconds to swing back and forth. How long should the pendulum be?

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MANUFACTURING **For Exercises 1–3, use the following information.**

The volume of a shipping container in the shape of a rectangular prism can be represented by the polynomial $6x^3 + 11x^2 + 4x$, where the height is x. (Lesson 8-1)

1. Find the length and width of the container.
2. Find the ratio of the three dimensions of the container when $x = 2$.
3. Will the ratio of the three dimensions be the same for all values of x?

PHOTOGRAPHY **For Exercises 4–6, use the following information.**

The formula $\frac{1}{q} = \frac{1}{f} - \frac{1}{p}$ can be used to determine how far the film should be placed from the lens of a camera. The variable q represents the distance from the lens to the film, f represents the focal length of the lens, and p represents the distance from the object to the lens. (Lesson 8-2)

4. Solve the formula for $\frac{1}{p}$.
5. Write the expression containing f and q as a single rational expression.
6. If a camera has a focal length of 8 centimeters and the lens is 10 centimeters from the film, how far should an object be from the lens so that the picture will be in focus?

PHYSICS **For Exercises 7 and 8, use the following information.**

The Inverse Square Law states that the relationship between two variables is related to the equation $y = \frac{1}{x^2}$. (Lesson 8-3)

7. Graph $y = \frac{1}{x^2}$.
8. Give the equations of any asymptotes.

PHYSICS **For Exercises 9 and 10, use the following information.**

The formula for finding the gravitational force between two objects is $F = G\frac{m_A m_B}{d^2}$, where F is the gravitational force between the objects, G is the universal constant, m_A is the mass of the first object, m_B is the mass of the second object, and d is the distance between the centers of the objects. (Lesson 8-4)

9. If the mass of object A is constant, does Newton's formula represent a *direct* or *inverse* variation between the mass of object B and the distance?
10. The value of G is 6.67×10^{-11} N • m^2/kg^2. If two objects each weighing 5 kilograms are placed so that their centers are 0.5 meter apart, what is the gravitational force between the two objects?

EDUCATION **For Exercises 11–13, use the table that shows the average number of students per computer in United States public schools for various years.** (Lesson 8-5)

Year	Students	Year	Students
1988	32	1996	10
1989	25	1997	7.8
1990	22	1998	6.1
1991	20	1999	5.7
1992	18	2000	5.4
1993	16	2001	5.0
1994	14	2002	4.9
1995	10.5	2003	4.9

Source: *The World Almanac*

11. Let x represent years where 1988 = 1, 1989 = 2, and so on. Let y represent the number of students. Graph the data.
12. What type of function does the graph most closely resemble?
13. Use a graphing calculator to find an equation that models the data.

TRAVEL **For Exercises 14 and 15, use the following information.**

A trip between two towns takes 4 hours under ideal conditions. The first 150 miles of the trip is on an interstate, and the last 130 miles is on a highway with a speed limit that is 10 miles per hour less than on the interstate. (Lesson 8-6)

14. If x represents the speed limit on the interstate, write expressions for the time spent at that speed and for the time spent on the other highway.
15. Write and solve an equation to find the speed limits on the two highways.

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POPULATION **For Exercises 1–4, use the following information.**

In 1950, the world population was about 2.556 billion. By 1980, it had increased to about 4.458 billion. **Source:** *The World Almanac* (Lesson 9-1)

1. Write an exponential function of the form $y = ab^x$ that could be used to model the world population y in billions for 1950 to 1980. Write the equation in terms of x, the number of years since 1950. (Round the value of b to the nearest ten-thousandth.)
2. Suppose the population continued to grow at that rate. Estimate the population in 2000.
3. In 2000, the population of the world was about 6.08 billion. Compare your estimate to the actual population.
4. Use the equation you wrote in Exercise 1 to estimate the world population in the year 2020. How accurate do you think the estimate is? Explain your reasoning.

EARTHQUAKES **For Exercises 5–8, use the following information.**

The table shows the Richter scale that measures earthquake intensity. Column 2 shows the increase in intensity between each number. For example, an earthquake that measures 7 is 10 times more intense than one measuring 6. (Lesson 9-2)

Richter Number x	Increase in Magnitude y
1	1
2	10
3	100
4	1000
5	10,000
6	100,000
7	1,000,000
8	10,000,000

Source: The New York Public Library

5. Graph this function.
6. Write an equation of the form $y = b^{x-c}$ for the function in Exercise 5. (*Hint:* Write the values in the second column as powers of 10 to see a pattern and find the value of c.)
7. Graph the inverse of the function in Exercise 6.
8. Write an equation of the form $y = \log_{10} x + c$ for the function in Exercise 7.

EARTHQUAKES **For Exercises 9 and 10, use the table showing the magnitude of some major earthquakes.** (Lesson 9-3)

Year/Location	Magnitude
1939/Turkey	8.0
1963/Yugoslavia	6.0
1970/Peru	7.8
1988/Armenia	7.0
2004/Morocco	6.4

Source: *The World Almanac*

9. For which two earthquakes was the intensity of one 10 times that of the other? For which two was the intensity of one 100 times that of the other?
10. What would be the magnitude of an earthquake that is 1000 times as intense as the 1963 earthquake in Yugoslavia?
11. Suppose you know that $\log_7 2 \approx 0.3562$ and $\log_7 3 \approx 0.5646$. Describe two different methods that you could use to approximate $\log_7 2.5$. (You can use a calculator, of course.) Then describe how you can check your result. (Lesson 9-4)

WEATHER **For Exercises 12 and 13, use the following information.**

The atmospheric pressure P, in bars, of a given height on Earth is given by using the formula $P = s \cdot e^{-\frac{k}{H}}$. In the formula, s is the surface pressure on Earth, which is approximately 1 bar, h is the altitude for which you want to find the pressure in kilometers, and H is always 7 kilometers. (Lesson 9-5)

12. Find the pressure for 2, 4, and 7 kilometers.
13. What do you notice about the pressure as altitude increases?

AGRICULTURE **For Exercises 14–16, use the following information.**

An equation that models the decline in the number of U.S. farms is $y = 3{,}962{,}520(0.98)^x$, where x is years since 1960 and y is the number of farms. **Source:** *Wall Street Journal* (Lesson 9-6)

14. How can you tell that the number is declining?
15. By what annual rate is the number declining?
16. Predict when the number of farms will be less than 1.5 million.

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GEOMETRY For Exercises 1–4, use the following information.

Triangle ABC has vertices $A(2, 1)$, $B(-6, 5)$, and $C(-2, -3)$. (Lesson 10-1)

1. An isosceles triangle has two sides with equal length. Is $\triangle ABC$ isosceles? Explain.
2. An equilateral triangle has three sides of equal length. Is $\triangle ABC$ equilateral? Explain.
3. Triangle EFG is formed by joining the midpoints of the sides of $\triangle ABC$. What type of triangle is $\triangle EFG$? Explain.
4. Describe any relationship between the lengths of the sides of the two triangles.

ENERGY For Exercises 5–8, use the following information.

A parabolic mirror can be used to collect solar energy. The mirrors reflect the rays from the Sun to the focus of the parabola. The latus rectum of a particular mirror is 40 feet long. (Lesson 10-2)

5. Write an equation for the parabola formed by the mirror if the vertex of the mirror is 9.75 feet below the origin.
6. One foot is exactly 0.3048 meter. Rewrite the equation in terms of meters.
7. Graph one of the equations for the mirror.
8. Which equation did you choose to graph? Explain.

COMMUNICATION For Exercises 9–11, use the following information.

The radio tower for KCGM has a circular radius for broadcasting of 65 miles. The radio tower for KVCK has a circular radius for broadcasting of 85 miles. (Lesson 10-3)

9. Let the radio tower for KCGM be located at the origin. Write an equation for the set of points at the maximum broadcast distance from the tower.
10. The radio tower for KVCK is 50 miles south and 15 miles west of the KCGM tower. Let each mile represent one unit. Write an equation for the set of points at the maximum broadcast distance from the KVCK tower.
11. Graph the two equations and show the area where the radio signals overlap.

ASTRONOMY For Exercises 12–14, use the table that shows the closest and farthest distances of Venus and Jupiter from the Sun in millions of miles. (Lesson 10-4)

Planet	Closest	Farthest
Venus	66.8	67.7
Jupiter	460.1	507.4

Source: *The World Almanac*

12. Write an equation for the orbit of each planet, assuming that the center of the orbit is the origin, the center of the Sun is a focus, and the Sun lies on the x-axis.
13. Find the eccentricity for each planet.
14. Which planet has an orbit that is closer to a circle? Explain your reasoning.
15. A comet follows a path that is one branch of a hyperbola. Suppose Earth is the center of the hyperbolic curve and has coordinates (0, 0). Write an equation for the path of the comet if $c = 5{,}225{,}000$ miles and $a = 2{,}500{,}000$ miles. Let the x-axis be the transverse axis. (Lesson 10-5)

AVIATION For Exercises 16–18, use the following information.

The path of a military jet during an air show can be modeled by a conic section with equation $24x^2 + 1000y - 31{,}680x - 45{,}600 = 0$, where distances are in feet. (Lesson 10-6)

16. Identify the shape of the path of the jet. Write the equation in standard form.
17. If the jet begins its ascent at (0, 0), what is the horizontal distance traveled by the jet from the beginning of the ascent to the end of the descent?
18. What is the maximum height of the jet?

SATELLITES For Exercises 19 and 20, use the following information.

The equations of the orbits of two satellites are $\frac{x^2}{(300)^2} + \frac{y^2}{(900)^2} = 1$ and $\frac{x^2}{(600)^2} + \frac{y^2}{(690)^2} = 1$, where distances are in km and Earth is the center of each curve. (Lesson 10-7)

19. Solve each equation for y.
20. Use a graphing calculator to estimate the intersection points of the two orbits.

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CLUBS **For Exercises 1 and 2, use the following information.**

A quilting club consists of 9 members. Every week, each member must bring one completed quilt square. (Lesson 11-1)

1. Find the first eight terms of the sequence that describes the total number of squares that have been made after each meeting.
2. One particular quilt measures 72 inches by 84 inches and is being designed with 4-inch squares. After how many meetings will the quilt be complete?

ART **For Exercises 3 and 4, use the following information.**

Alberta is making a beadwork design consisting of rows of colored beads. The first row consists of 10 beads, and each consecutive row will have 15 more beads than the previous row. (Lesson 11-2)

3. Write an equation for the number of beads in the nth row.
4. Find the number of beads in the design if it contains 25 rows.

GAMES **For Exercises 5 and 6, use the following information.**

An audition is held for a TV game show. At the end of each round, one half of the prospective contestants are eliminated from the competition. On a particular day, 524 contestants begin the audition. (Lesson 11-3)

5. Write an equation for finding the number of contestants that are left after n rounds.
6. Using this method, will the number of contestants that are to be eliminated always be a whole number? Explain.

SPORTS **For Exercises 7–9, use the following information.**

Caitlin is training for a marathon (about 26 miles). She begins by running 2 miles. Then, when she runs every other day, she runs one and a half times the distance she ran the time before. (Lesson 11-4)

7. Write the first five terms of a sequence describing her training schedule.
8. When will she exceed 26 miles in one run?
9. When will she have run 100 total miles?

GEOMETRY **For Exercises 10–12, use a square of paper at least 8 inches on a side.** (Lesson 11-5)

10. Let the square be one unit. Cut away one half of the square. Call this piece Term 1. Next, cut away one half of the remaining sheet of paper. Call this piece Term 2. Continue cutting the remaining paper in half and labeling the pieces with a term number as long as possible. List the fractions represented by the pieces.
11. If you could cut squares indefinitely, you would have an infinite series. Find the sum of the series.
12. How does the sum of the series relate to the original square of paper?

BIOLOGY **For Exercises 13–15, use the following information.**

In a particular forest, scientists are interested in how the population of wolves will change over the next two years. One model for animal population is the Verhulst population model, $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n, and r is the growth factor. (Lesson 11-6)

13. To find the population of the wolves after one year, evaluate $p_1 = 0.45 + 1.5(0.45)(1 - 0.45)$.
14. Explain what each number in the expression in Exercise 13 represents.
15. The current population of wolves is 165. Find the new population by multiplying 165 by the value in Exercise 13.

16. **PASCAL'S TRIANGLE** Study the first eight rows of Pascal's triangle. Write the sum of the terms in each row as a list. Make a conjecture about the sums of the rows of Pascal's triangle. (Lesson 11-7)

17. **NUMBER THEORY** Two statements that can be proved using mathematical induction are $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$ and $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$.

    Write and prove a conjecture involving $\frac{1}{5}$ that is similar to the statements. (Lesson 11-8)

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NUMBER THEORY **For Exercises 1–3, use the following information.**

According to the Rational Zero Theorem, if $\frac{p}{q}$ is a rational root, then p is a factor of the constant of the polynomial, and q is a factor of the leading coefficient. (Lesson 12-1)

1. What is the maximum number of possible rational roots that you may need to check for the polynomial $3x^4 - 5x^3 + 2x^2 - 7x + 10$? Explain your answer.
2. Why may you not need to check the maximum number of possible roots?
3. Are choosing the numerator and the denominator for a possible rational root independent or dependent events?
4. **GARDENING** A gardener is selecting plants for a special display. There are 15 varieties of pansies from which to choose. The gardener can only use 9 varieties in the display. How many ways can 9 varieties be chosen from the 15 varieties? (Lesson 12-2)

SPEED LIMITS **For Exercises 5 and 6, use the following information.**

Speed Limit	Number of States
60	1
65	20
70	16
75	13

Source: *The World Almanac*

The table shows the number of states having each maximum speed limit for their rural interstates. (Lesson 12-3)

5. If a state is randomly selected, what is the probability that its speed limit is 75? 60?
6. If a state is randomly selected, what is the probability that its speed limit is 60 or greater?
7. **LOTTERIES** A lottery number for a particular state has seven digits, which can be any digit from 0 to 9. It is advertised that the odds of winning the lottery are 1 to 10,000,000. Is this statement about the odds correct? Explain your reasoning. (Lesson 12-4)

For Exercises 8 and 9, use the table that shows the most popular colors for luxury cars in 2003. (Lesson 12-5)

Color	% of cars	Color	% of cars
gray	23.3	red	3.9
silver	18.8	blue	3.8
wh. metallic	17.8	gold	3.6
white	12.6	lt. blue	3.1
black	10.9	other	2.2

Source: *The World Almanac*

8. If a car is randomly selected, what is the probability that it is gray or silver?
9. In a parking lot of 1000 cars sold in 2003, how many cars would you expect to be white or black?

EDUCATION **For Exercises 10–12, use the following information.**

The list shows the average scores for each state for the ACT for 2003–2004. (Lesson 12-6)

20.2, 21.3, 21.5, 20.4, 21.6, 20.3, 22.5, 21.5, 17.8, 20.5, 20.0, 21.7, 21.3, 20.3, 21.6, 22.0, 21.6, 20.3, 19.8, 22.6, 20.8, 22.4, 21.4, 22.2, 18.8, 21.5, 21.7, 21.7, 21.2, 22.5, 21.2, 20.1, 22.3, 20.3, 21.2, 21.4, 20.6, 22.5, 21.8, 21.9, 19.3, 21.5, 20.5, 20.3, 21.5, 22.7, 20.9, 22.5, 22.2, 21.4

10. Compare the mean and median of the data.
11. Find the standard deviation of the data. Round to the nearest hundredth.
12. Suppose the state with an average score of 20.0 incorrectly reported the results. The score for the state is actually 22.5. How are the mean and median of the data affected by this data change?
13. **HEALTH** The heights of students at Madison High School are normally distributed with a mean of 66 inches and a standard deviation of 2 inches. Of the 1080 students in the school, how many would you expect to be less than 62 inches tall? (Lesson 12-7)
14. **SURVEY** A poll of 1750 people shows that 78% enjoy travel. Find the margin of the sampling error for the survey. (Lesson 12-9)

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CABLE CARS **For Exercises 1 and 2, use the following information.**

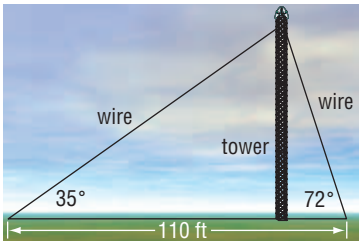
The longest cable car route in the world begins at an altitude of 5379 feet and ends at an altitude of 15,629 feet. The ride is 8-miles long. Source: The Guinness Book of Records (Lesson 13-1)

1. Draw a diagram to represent this situation.
2. To the nearest degree, what is the average angle of elevation of the cable car ride?

RIDES **For Exercises 3 and 4, use the following information.**

In 2000, a gigantic Ferris wheel, the London Eye, opened in England. The wheel has 32 cars evenly spaced around the circumference. (Lesson 13-2)

3. What is the measure, in degrees, of the angle between any two consecutive cars?
4. If a car is located such that the measure in standard position is 260°, what are the measures of one angle with positive measure and one angle with negative measure coterminal with the angle of this car?
5. **BASKETBALL** A person is selected to try to make a shot at a distance of 12 feet from the basket. The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a basketball shot with an initial velocity of V_0 feet per second at an angle of u with the ground. If the basketball was shot with an initial velocity of 24 feet per second at an angle of 75°, how far will the basketball travel? (Lesson 13-3)
6. **COMMUNICATIONS** A telecommunications tower needs to be supported by two wires. The angle between the ground and the tower on one side must be 35° and the angle between the ground and the second tower must be 72°. The distance between the two wires is 110 feet.



To the nearest foot, what should be the lengths of the two wires? (Lesson 13-4)

SURVEYING **For Exercises 7 and 8, use the following information.**

A triangular plot of farm land measures 0.9 by 0.5 by 1.25 miles. (Lesson 13-5)

7. If the plot of land is fenced on the border, what will be the angles at which the fences of the three sides meet? Round to the nearest degree.
8. What is the area of the plot of land? (*Hint:* Use the area formula in Lesson 13-4.)
9. **WEATHER** The monthly normal temperatures, in degrees Fahrenheit, for New York City are given in the table. January is assigned a value of 1, February a value of 2, and so on. (Lesson 13-6)

Month	Temperature	Month	Temperature
1	32	7	77
2	34	8	76
3	42	9	68
4	53	10	58
5	63	11	48
6	72	12	37

A trigonometric model for the temperature T in degrees Fahrenheit of New York City at t months is given by $T = 22.5 \sin \left(\frac{\pi}{6x} - 2.25\right) + 54.3$. A quadratic model for the same situation is $T = -1.34x^2 + 18.84x + 5$. Which model do you think best fits the data? Explain your reasoning.

PHYSICS **For Exercises 10–12, use the following information.**

When light passes from one substance to another, it may be reflected and refracted. Snell's law can be used to find the angle of refraction as a beam of light passes from one substance to another. One form of the formula for Snell's law is $n_1 \sin \theta_1 = n_2 \sin \theta_2$, where n_1 and n_2 are the indices of refraction for the two substances and θ_1 and θ_2 are the angles of the light rays passing through the two substances. (Lesson 13-7)

10. Solve the equation for $\sin \theta_1$.
11. Write an equation in the form of an inverse function that allows you to find θ_1.
12. If a light beam in air with index of refraction of 1.00 hits a diamond with index of 2.42 at an angle of 30°, find the angle of refraction.

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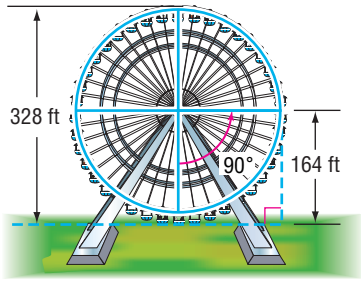
1. **TIDES** The world's record for the hightest tide is held by the Minas Basin in Nova Scotia, Canada, with a tidal range of 54.6 feet. A tide is at equilibrium when it is at its normal level halfway between its highest and lowest points. Write an equation to represent the height h of the tide. Assume that the tide is at equilibrium at $t = 0$, that the high tide is beginning, and that the tide completes one cycle in 12 hours. (Lesson 14-1)

RIDES For Exercises 2 and 3, use the following information.

The Cosmoclock 21 is a huge Ferris wheel in Yokohama City, Japan. The diameter is 328 feet. Suppose that a rider enters the ride at 0 feet and then rotates in 90° increments counterclockwise. The table shows the angle measures of rotation and the height above the ground of the rider. (Lesson 14-2)



Angle	Height	Angle	Height
0°	0	450°	164
90°	164	540°	328
180°	328	630°	164
270°	164	720°	0
360°	0		

2. A function that models the data is $y = 164 \cdot (\sin (x - 90°)) + 164$. Identify the vertical shift, amplitude, period, and phase shift of the graph.

3. Write an equation using the sine that models the position of a rider on the Vienna Giant Ferris Wheel in Vienna, Austria, with a diameter of 200 feet. Check your equation by plotting the points and the equation with a graphing calculator.

4. **TRIGONOMETRY** Using the exact values for the sine and cosine functions, show that the identity $\cos^2 \theta + \sin^2 \theta = 1$ is true for angles of measure 30°, 45°, 60°, 90°, and 180°. (Lesson 14-3)

5. **ROCKETS** In the formula $h = \frac{v^2 \sin^2 \theta}{2g} = h$ is the maximum height reached by a rocket, θ is the angle between the ground and the initial path of the object, v is the rocket's initial velocity, and g is the acceleration due to gravity. Verify the identity $\frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \cos^2 \theta}{2g \cot^2 \theta}$. (Lesson 14-4)

WEATHER For Exercises 6 and 7, use the following information.

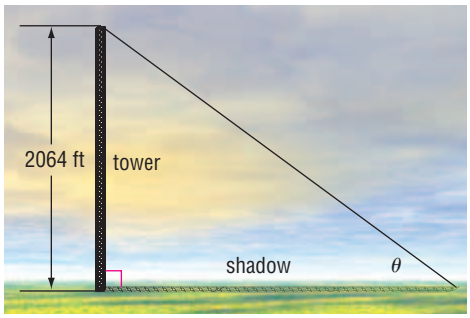
The monthly high temperatures for Minneapolis, Minnesota, can be modeled by the equation $y = 31.65 \sin \left(\frac{\pi}{6x} - 2.09\right) + 52.35$, where the months x are January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by the equation $y = 30.15 \sin \left(\frac{\pi}{6x} - 2.09\right) + 32.95$. (Lesson 14-5)

6. Write a new function by adding the expressions on the right side of each equation and dividing the result by 2.

7. What is the meaning of the function you wrote in Exercise 6?

8. Begin with one of the Pythagorean Identities. Perform equivalent operations on each side to create a new trigonometric identity. Then show that the identity is true. (Lesson 14-6)

9. **TELEVISION** The tallest structure in the world is a television transmitting tower located near Fargo, North Dakota, with a height of 2064 feet.



What is the measure of θ if the length of the shadow is 1 mile? **Source:** *The Guinness Book of Records* (Lesson 14-7)

Mixed Problem Solving

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Preparing for Standardized Tests

Becoming a Better Test-Taker

At some time in your life, you will probably have to take a standardized test. Sometimes this test may determine if you go on to the next grade or course, or even if you will graduate from high school. This section of your textbook is dedicated to making you a better test-taker.

TYPES OF TEST QUESTIONS In the following pages, you will see examples of four types of questions commonly seen on standardized tests. A description of each type of question is shown in the table below.

Type of Question	Description	See Pages
multiple choice	Four or five possible answer choices are given from which you choose the best answer.	942–943
gridded response	You solve the problem. Then you enter the answer in a special grid and shade in the corresponding circles.	944–947
short response	You solve the problem, showing your work and/or explaining your reasoning.	948–951
extended response	You solve a multipart problem, showing your work and/or explaining your reasoning.	952–956

PRACTICE After being introduced to each type of question, you can practice that type of question. Each set of practice questions is divided into five sections that represent the concepts most commonly assessed on standardized tests.

- Number and Operations
- Algebra
- Geometry
- Measurement
- Data Analysis and Probability

USING A CALCULATOR On some tests, you are permitted to use a calculator. You should check with your teacher to determine if calculator use is permitted on the test you will be taking, and if so, what type of calculator can be used.

TEST-TAKING TIPS In addition to Test-Taking Tips like the one shown at the right, here are some additional thoughts that might help you.

- Get a good night's rest before the test. Cramming the night before does not improve your results.
- Budget your time when taking a test. Don't dwell on problems that you cannot solve. Just make sure to leave that question blank on your answer sheet.
- Watch for key words like NOT and EXCEPT. Also look for order words like LEAST, GREATEST, FIRST, and LAST.

TEST-TAKING TIP

If you are allowed to use a calculator, make sure you are familiar with how it works so that you won't waste time trying to figure out the calculator when taking the test.

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Multiple-Choice Questions

Multiple-choice questions are the most common type of questions on standardized tests. These questions are sometimes called *selected-response questions*. You are asked to choose the best answer from four or five possible answers. To record a multiple-choice answer, you may be asked to shade in a bubble that is a circle or an oval or to just write the letter of your choice. Always make sure that your shading is dark enough and completely covers the bubble.

Incomplete shading
(A) (B) (C) (D)
Too light shading
(A) (B) (C) (D)
Correct shading
(A) (B) (C) (D)

The answer to a multiple-choice question is usually not immediately obvious from the choices, but you may be able to eliminate some of the possibilities by using your knowledge of mathematics. Another answer choice might be that the correct answer is not given.

EXAMPLE 1

White chocolate pieces sell for \$3.25 per pound while dark chocolate pieces sell for \$2.50 per pound. How many pounds of white chocolate are needed to produce a 10-pound mixture of both kinds that sells for \$2.80 per pound?

A 2 lb **B** 4 lb **C** 6 lb **D** 10 lb

STRATEGY
Reasonableness Check to see that your answer is reasonable with the given information.

The question asks you to find the number of pounds of the white chocolate. Let w be the number of pounds of white chocolate and let d be the number of pounds of dark chocolate. Write a system of equations.

$w + d = 10$ There is a total of 10 pounds of chocolate.
$3.25w + 2.50d = 2.80(10)$ The price is \$2.80 × 10 for the mixed chocolate.

Use substitution to solve.

$3.25w + 2.50d = 2.80(10)$ Original equation
$3.25w + 2.50(10 - w) = 28$ Solve the first equation for d and substitute.
$3.25w + 25 - 2.5w = 28$ Distributive Property
$0.75w = 3$ Simplify.
$w = 4$ Divide each side by 0.75.

The answer is B.

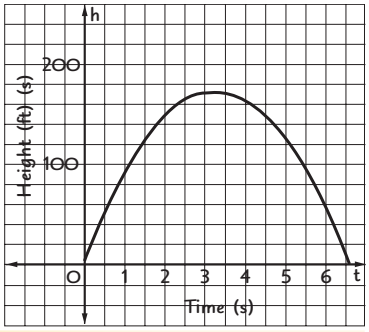
EXAMPLE 2

Josh throws a baseball upward at a velocity of 105 feet per second, releasing the baseball when it is 5 feet above the ground. The height of the baseball t seconds after being thrown is given by the formula $h(t) = -16t^2 + 105t + 5$. Find the time at which the baseball reaches its maximum height. Round to the nearest tenth of a second.

F 1.0 s **G** 3.3 s **H** 6.6 s **J** 177.3 s

STRATEGY
Diagrams Drawing a diagram for a situation may help you to answer the question.

Graph the equation. The graph is a parabola. Make sure to label the horizontal axis as t (time in seconds) and the vertical axis as h for height in feet. The ball is at its maximum height at the vertex of the graph.



The graph indicates that the maximum height is achieved between 3 and 4 seconds after launch.

The answer is G.

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Multiple-Choice Practice

Choose the best answer.

Number and Operations

1. In 2002, 1.8123×10^8 people in the United States and Canada used the Internet, while 5.442×10^8 people worldwide used the Internet. What percentage of users were from the United States and Canada?

A 33% **B** 35% **C** 37% **D** 50%

2. Serena has 6 plants to put in her garden. How many different ways can she arrange the plants?

F 21 **G** 30 **H** 360 **J** 720

Algebra

3. The sum of Kevin's, Anna's, and Tia's ages is 40. Anna is 1 year more than twice as old as Tia. Kevin is 3 years older than Anna. How old is Anna?

A 7 **B** 14 **C** 15 **D** 18

4. Rafael's Theatre Company sells tickets for \$10. At this price, they sell 400 tickets. Rafael estimates that they would sell 40 fewer tickets for each \$2 price increase. What charge would give the most income?

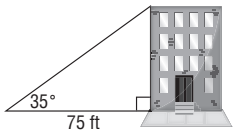
F 10 **G** 13 **H** 15 **J** 20

Geometry

5. Hai stands 75 feet from the base of a building and sights the top at a 35° angle. What is the height of the building to the nearest tenth of a foot?

A 0.0 ft **C** 52.5 ft

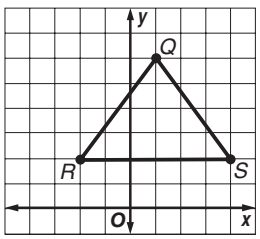
B 43.0 ft **D** 61.4 ft



6. Samone draws $\triangle QRS$ on grid paper to use for a design in her art class. She needs to rotate the triangle 180° counterclockwise. What will be the y-coordinate of the image of S?

F −6 **H** −1

G −2 **J** 2



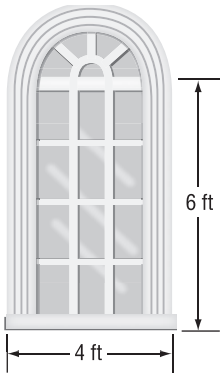
Measurement

7. Lakeisha is teaching a summer art class for children. For one project, she estimates that she will need $\frac{2}{3}$ yard of string for each 3 students. How many yards will she need for 16 students?

A 3 yd **C** $10\frac{2}{3}$ yd

B $3\frac{5}{9}$ yd **D** 16 yd

8. Kari works at night so she needs to make her room as dark as possible during the day to sleep. How much black paper will she need to cover the window in her room, which is shaped as shown. Use 3.14 for π. Round to the nearest tenth of a square foot.



F 24.0 ft^2 **H** 30.3 ft^2

G 24.6 ft^2 **J** 36.6 ft^2

Data Analysis and Probability

9. A card is drawn from a standard deck of 52 cards. If one card is drawn, what is the probability that it is a heart or a 2?

A $\frac{1}{52}$ **B** $\frac{1}{13}$ **C** $\frac{1}{4}$ **D** $\frac{4}{13}$

10. The weight of candy in boxes is normally distributed with a mean of 12 ounces and a standard deviation of 0.5 ounce. About what percent of the time will you get a box that weighs over 12.5 ounces?

F 13.5% **G** 16% **H** 50% **J** 68%

TEST-TAKING TIP

Question 8 Many standardized tests include a reference sheet with common formulas that you may use during the test. If it is available before the test, familiarize yourself with the reference sheet for quick reference during the test.

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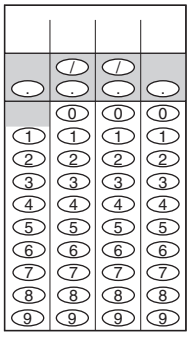
Gridded-Response Questions

Gridded-response questions are another type of question on standardized tests. These questions are sometimes called *student-produced response griddable*, or *grid-in*, because you must create the answer yourself, not just choose from four or five possible answers.



For gridded response, you must mark your answer on a grid printed on an answer sheet. The grid contains a row of four or five boxes at the top, two rows of ovals or circles with decimal and fraction symbols, and four or five columns of ovals, numbered 0–9. At the right is an example of a grid from an answer sheet.

EXAMPLE

1 **Find the x-coordinate for the solution of the given system of equations.**

$$4x - y = 14$$
$$-3x + y = -11$$

What value do you need to find?

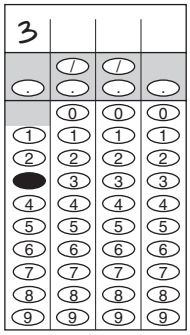
You need to find only the x-coordinate of the point where the graphs of the two equations intersect. You could graph the system, but that takes time. The easiest method is probably the substitution method since the second equation can be solved easily for y.

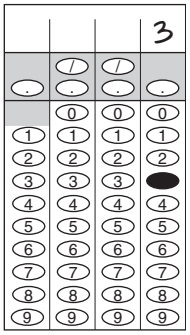
$-3x + y = -11$	Second equation
$y = -11 + 3x$	Solve the second equation for y.
$4x - y = 14$	First equation
$4x - (-11 + 3x) = 14$	Substitute for y.
$4x + 11 - 3x = 14$	Distributive Property
$x = 3$	Simplify.

The answer to be filled in on the grid is 3.

How do you fill in the grid for the answer?

- Write your answer in the answer boxes.
- Write only one digit or symbol in each answer box.
- Do not write any digits or symbols outside the answer boxes.
- You may write your answer with the first digit in the left answer box, or with the last digit in the right answer box. You may leave blank any boxes you do not need on the right or the left side of your answer.
- Fill in only one bubble for every answer box that you have written in. Be sure not to fill in a bubble under a blank answer box.





Many gridded response questions result in an answer that is a fraction or a decimal. These values can also be filled in on the grid.

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EXAMPLE 2

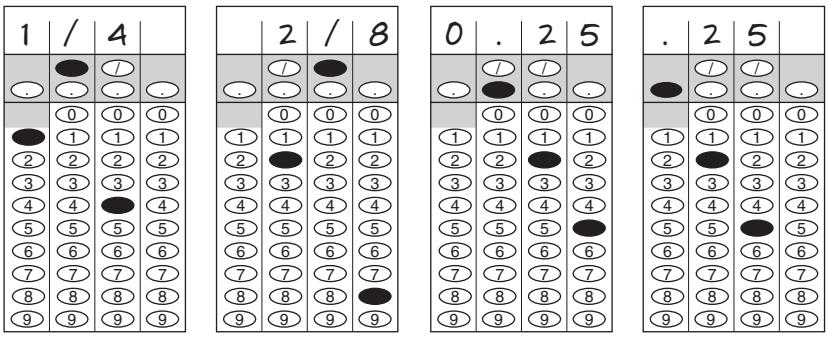
Zuri is solving a problem about the area of a room. The equation she needs to solve is $4x^2 + 11x - 3 = 0$. Since the answer will be a length, she only needs to find the positive root. What is the solution?

Since you can see the equation is not easily factorable, use the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-11 \pm \sqrt{11^2 - 4(4)(-3)}}{2(4)}$$
$$= \frac{-11 + 13}{8}$$ There are two roots, but you only need the positive one.
$$= \frac{2}{8} \text{ or } \frac{1}{4}$$

How do you grid the answer?

You can either grid the fraction $\frac{1}{4}$, or rewrite it as 0.25 and grid the decimal. Be sure to write the decimal point or fraction bar in the answer box. The following are acceptable answer responses that represent $\frac{1}{4}$ and 0.25.

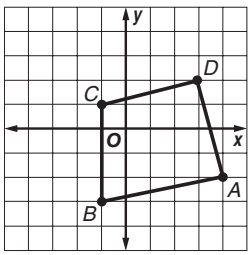


Do not leave a blank answer box in the middle of an answer.

Some problems may result in an answer that is a mixed number. Before filling in the grid, change the mixed number to an equivalent improper fraction or decimal. For example, if the answer is $1\frac{1}{2}$, do not enter 1 1/2, as this will be interpreted as $\frac{11}{2}$. Instead, either enter 3/2 or 1.5.

EXAMPLE 3

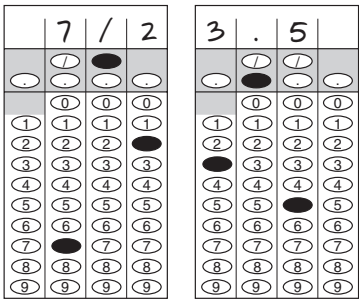
José is using this figure for a computer graphics design. He wants to dilate the figure by a scale factor of $\frac{7}{4}$. What will be the y-coordinate of the image of D?



To find the y-coordinate of the image of D, multiply the x-coordinate by the scale factor of $\frac{7}{4}$.

$$2 \cdot \frac{7}{4} = \frac{7}{2}$$

Grid in 7/2 or 3.5. Do not grid 3 1/2.



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Gridded-Response Practice

Solve each problem. Then copy and complete a grid.

Number and Operations

1. Rewrite 16^3 as a power of 2. What is the value of the exponent for 2?

2. Wolf 359 is the fourth closest star to Earth. It is 45,531,250 million miles from Earth. A light-year, the distance light travels in a year, is 5.88×10^{12} miles. What is the distance from Earth to Wolf 359 in light-years? Round to the nearest tenth.

3. A store received a shipment of coats. The coats were marked up 50% to sell to the customers. At the end of the season, the coats were discounted 60%. Find the ratio of the discounted price of a coat to the original cost of the coat to the store.

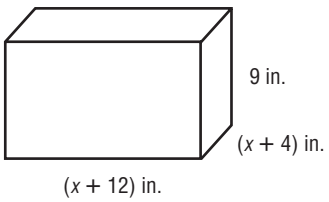
4. Find the value of the determinant $\begin{vmatrix} -1 & 4 \\ -3 & 0 \end{vmatrix}$.

5. Kendra is displaying eight sweaters in a store window. There are four identical red sweaters, three identical brown sweaters, and one white sweater. How many different arrangements of the eight sweaters are possible?

6. An electronics store reduced the price of a DVD player by 10% because it was used as a display model. If the reduced price was $107.10, what was the cost in dollars before it was reduced? Round to the nearest cent if necessary.

Algebra

7. The box shown can be purchased to ship merchandise at the Pack 'n Ship Store. The volume of the box is 945 cubic inches. What is the measure of the greatest dimension of the box in inches?

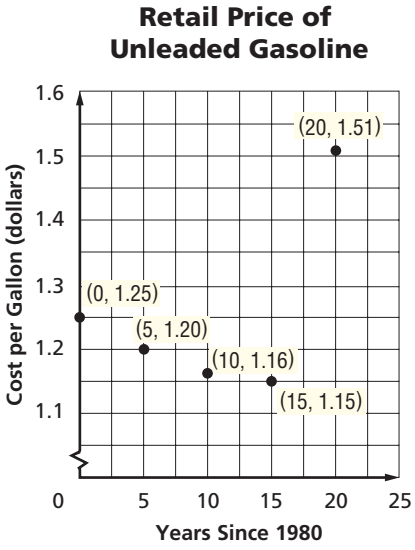


8. If $f(x) = 2x^2 - 3x + 10$, find $f(-1)$.

TEST-TAKING TIP

Question 3 Fractions do not have to be written in lowest terms. Any equivalent fraction that fits the grid is correct.

9. The graph shows the retail price per gallon of unleaded gasoline in the U.S. from 1980 to 2000. What is the slope if a line is drawn through the points for 15 and 20 years since 1980?

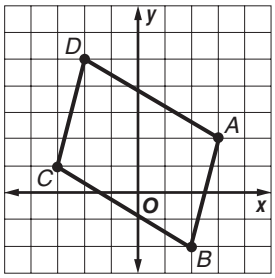


Source: U.S. Dept. of Energy

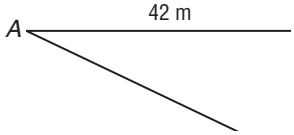
10. Solve $\sqrt{x + 11} - 9 = \sqrt{x} - 8$.

Geometry

11. Polygon $DABC$ is rotated 90° counterclockwise and then reflected over the line $y = x$. What is the x-coordinate of the final image of A?



12. A garden is shaped as shown below. What is the measure of $\angle A$ to the nearest degree?



13. A circle of radius r is circumscribed about a square. What is the ratio of the area of the circle to the area of the square? Express the ratio as a decimal rounded to the nearest hundredth.

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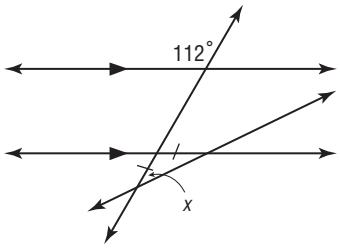
15. Find the value of x



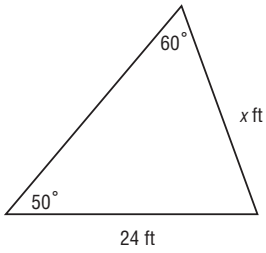
16. Each angle of a regular polygon measures 150°. How many sides does the polygon have?

Measurement

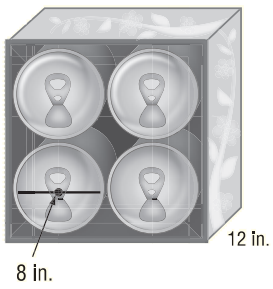
17. The Pascal (P) is a measure of pressure that is equivalent to 1 Newton per square meter. The typical pressure in an automobile tire is 2×105 P while typical blood pressure is 1.6×104 P. How many times greater is the pressure in a tire than typical blood pressure?

18. A circular ride at an amusement park rotated $\frac{7\pi}{4}$ radians while loading riders. What is the degree measure of the rotation?

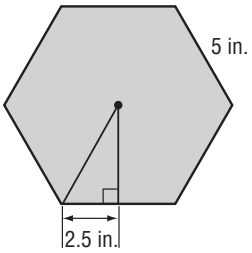
19. Find the value of x in the triangle. Round to the nearest tenth of a foot.



20. Four equal-sized cylindrical juice cans are packed tightly in the box shown. What is the volume of space in the box that is not occupied by the cans in cubic inches? Use 3.14 for π and round to the nearest cubic inch.



21. Caroline is making a quilt. The diagram shows a piece of cloth she will cut for a portion of the pattern. Find the area of the entire hexagonal piece to the nearest tenth of a square inch.



Data Analysis and Probability

22. Often girls on a team, three have blue eyes. If two girls are chosen at random, what is the probability that neither has blue eyes?

23. In order to win a game, Miguel needs to advance his game piece 4 spaces. What is the probability that the sum of the numbers on the two dice he rolls will be 4?

24. The table shows the number of televisions owned per 1000 people in each country. What is the absolute value of the difference between the mean and the median of the data?

Country	Televisions
United States	844
Latvia	741
Japan	719
Canada	715
Australia	706
United Kingdom	652
Norway	648
Finland	643
France	623

Source: International Telecommunication Union

25. The table shows the amount of breakfast cereal eaten per person each year by the ten countries that eat the most. Find the standard deviation of the data set. Round to the nearest tenth of a pound.

Country	Cereal (lb)
Sweden	23
Canada	17
Australia	16
United Kingdom	15
Nauru	14
New Zealand	14
Ireland	12
United States	11
Finland	10
Denmark	7

Source: Euromonitor

26. Two number cubes are rolled. If the two numbers appearing on the faces of the number cubes are different, find the probability that the sum is 6. Round to the nearest hundredth.

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Short-Response Questions

Short-response questions require you to provide a solution to the problem, as well as any method, explanation, and/or justification you used to arrive at the solution. These are-sometimes called *constructed-response, open-response, open-ended, free-response,* or *student-produced questions*. The following is a sample rubric, or scoring guide, for scoring short-response questions.

Criteria	Score
Full credit: The answer is correct and a full explanation is provided that shows each step in arriving at the final answer.	2
Partial credit: There are two different ways to receive partial credit. • The answer is correct, but the explanation provided is incomplete or incorrect. • The answer is incorrect, but the explanation and method of solving the problem is correct.	1
No credit: Either an answer is not provided or the answer does not make sense.	0

On some standardized tests, no credit is given for a correct answer if your work is not shown.

EXAMPLE

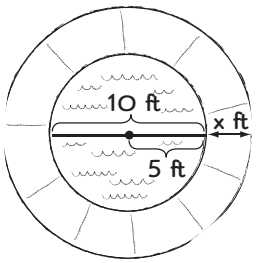
1 **Mr. Youngblood has a fish pond in his backyard. It is circular with a diameter of 10 feet. He wants to build a walkway of equal width around the pond. He wants the total area of the pond and walkway to be about 201 square feet. To the nearest foot, what should be the width of the walkway?**

Full Credit Solution

STRATEGY

Diagrams
Draw a diagram of the pond and the walkway. Label important information.

First draw a diagram to represent the situation.



Since the diameter of the pond is 10 feet, the radius is 5 ft. Let the width of the walkway be x feet.

$A = \pi r^2$

$201 = \pi(x + 5)^2$ $\quad r = x + 5,\ A = 201$

$201 = \pi(x^2 + 10x + 25)$ $\quad$ Multiply.

$\frac{201}{\pi} = \frac{\pi(x^2 + 10x + 25)}{\pi}$ $\quad$ Divide by π, using 3.14 for π.

$64 = x^2 + 10x + 25$ or $x^2 + 10x - 39 = 0$

The steps, calculations, and reasoning are clearly stated.

Use the Quadratic Formula.

Before taking a standardized test, memorize common formulas, like the Quadratic Formula, to save time.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= -10 \pm \frac{\sqrt{(-10)^2 - 4(1)(-39)}}{2(1)} \quad a = 1,\ b = 10,\ c = -39$$

$$= \frac{-10 + \sqrt{256}}{2} \text{ or } 3$$

Since length must be positive, eliminate the negative solution.

The width of the walkway should be 3 feet.

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Partial Credit Solution

In this sample solution, the equation that can be used to solve the problem is correct. However, there is no justification for any of the calculations.

There is no explanation of how the quadratic equation was found.

$x^2 + 10x - 39 = 0$

$$x = \frac{-10 + \sqrt{256}}{2}$$

$= -13$ or 3

The walkway should be 3 feet wide.

Partial Credit Solution

In this sample solution, the answer is incorrect because the wrong root was chosen.

Since the diameter of the pond is 10 feet, the radius is 5 ft. Let the width of the walkway be x feet. Use the formula for the area of a circle.

$A = \pi r^2$

$201 = \pi(x + 5)^2$

$201 = \pi(x^2 + 10x + 25)$

$$\frac{201}{\pi} = \frac{\pi(x^2 + 10x + 25)}{\pi}$$

$64 = x^2 + 10x + 25$ or $x^2 + 10x - 39 = 0$

Use the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$= -13$ or 3

The walkway should be 13 feet wide.

The negative root was chosen as the solution.

No Credit Solution

Use the formula for the area of a circle.

$A = \pi r^2 x$

$201 = \pi(5)^2 x$

$201 = 3.14(25)x$

$201 = 78.5x$

$x = 2.56$

Build the walkway 3 feet wide.

The width of the walkway x is used incorrectly in the area formula for a circle. However, when the student rounds the value for the width of the walkway, the answer is correct. No credit is given for an answer achieved using faulty reasoning.

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Short-Response Practice

Solve each problem. Show all your work.

Number and Operations

1. An earthquake that measures a value of 1 on the Richter scale releases the same amount of energy as 170 grams of TNT, while one that measures 4 on the scale releases the energy of 5 metric tons of TNT. One metric ton is 1000 kilograms and 1000 grams is 1 kilogram. How many times more energy is released by an earthquake measuring 4 than one measuring 1?

2. In 2000, Cook County, Illinois was the second largest county in the U.S. with a population of about 5,377,000. This was about 43.3% of the population of Illinois. What was the approximate population of Illinois in 2000?

3. Show why $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix for multiplication for 2×2 matrices.

4. The total volume of the oceans on Earth is 3.24×10^8 cubic miles. The total surface area of the water of the oceans is 139.8 million square miles. What is the average depth of the oceans?

5. At the Blaine County Fair, there are 12 finalists in the technology project competition. How many ways can 1st, 2nd, 3rd, and 4th place be awarded?

Algebra

6. Factor $3x^2a^2 + 3x^2b^2$. Explain each step.

7. Solve and graph $7 - 2a < \frac{15 - 2a}{6}$.

8. Solve the system of equations.
$x^2 + 9y^2 = 25 \quad y - x = -5$

9. The table shows what Miranda Richards charges for landscaping services for various numbers of hours. Write an equation to find the charge for any amount of time, where y is the total charge in dollars and x is the amount of time in hours. Explain the meaning of the slope and y-intercept of the graph of the equation.

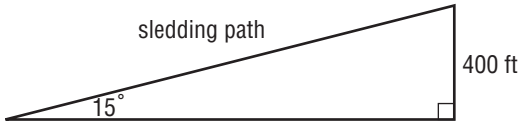
Hours	Charge (dollars)	Hours	Charge (dollars)
0	17.50	3	64.00
1	33.00	4	79.50
2	48.50	5	95.00

10. Write an equation that fits the data in the table.

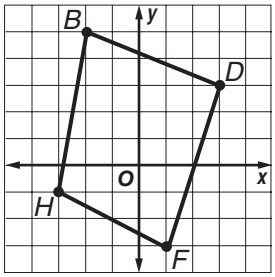
x	y
−3	12
−1	4
0	3
2	7
4	19

Geometry

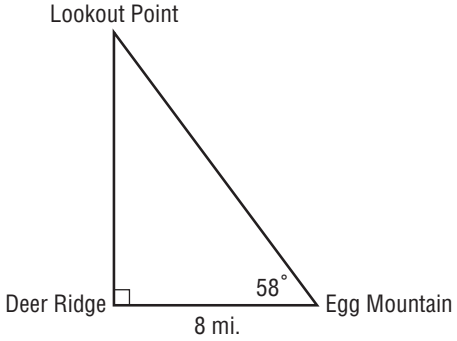
11. A sledding hill at the local park has an angle of elevation of 15°. Its vertical drop is 400 feet. What is the length of the sledding path?



12. Polygon $BDFH$ is transformed using the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Graph $B'D'F'H'$ and identify the type of transformation.



13. The map shows the trails that connect three hiking destinations. If Amparo hikes from Deer Ridge to Egg Mountain to Lookout Point and back to Deer Ridge, what is the distance she will have traveled?



14. Mr Washington is making a cement table for his backyard. The tabletop will be circular with a diameter of 6 feet and a depth of 6 inches. How much cement will Mr. Washington need to make the top of the table? Use 3.14 for π and round to the nearest cubic foot.

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15. A triangular garden is plotted on grid paper, where each unit is 1 meter. Its sides are segments that are parts of the lines with equations $y = -\frac{5}{4}x + 2$, $2y - 5x = 4$, and $y = -3$. Graph the triangle and find its area.

Measurement

16. Dylan is flying a kite. He wants to know how high above the ground it is. He knows that he has let out 75 feet of string and that it is flying directly over a nearby fence post. If he is 50 feet from the fence post, how high is the kite? Round to the nearest tenth of a foot.

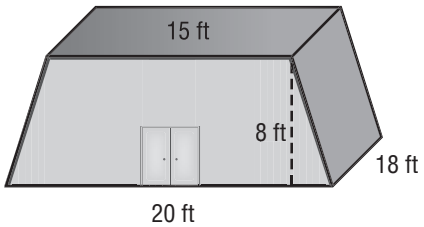
17. The temperature of the Sun can reach 27,000,000°F. The relationship between Fahrenheit F and Celsius C temperatures is given by the equation $F = 1.8C + 32$. Find the temperature of the Sun in degrees Celsius.

18. In 2003, Monaco was the most densely populated country in the world. There were about 32,130 people occupying the country at the rate of 16,477 people per square kilometer. What is the area of Monaco?

19. A box containing laundry soap is a cylinder with a diameter of 10.5 inches and a height of 16 inches. What is the surface area of the box?

20. Light travels at 186,291 miles per second or 299,792 kilometers per second. What is the relationship between miles and kilometers?

21. Home Place Hardware sells storage buildings for your backyard. The front of the building is a trapezoid as shown. The store manager wants to advertise the total volume of the building. Find the volume in cubic feet.



TEST-TAKING TIP

Questions 15, 16, and 22 Be sure to read the instructions of each problem carefully. Some questions ask for more than one solution, specify how to round answers, or require an explanation.

Data Analysis and Probability

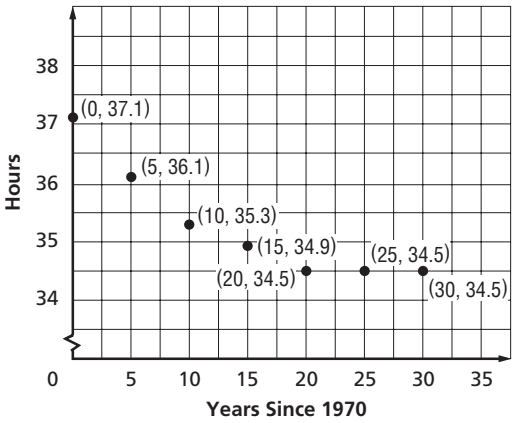
22. The table shows the 2000 populations of the six largest cities in Tennessee. Which measure, mean or median, do you think best represents the data? Explain your answer.

City	Population
Chattanooga	155,404
Clarksville	105,898
Jackson	60,635
Knoxville	173,661
Memphis	648,882
Nashville-Davidson	545,915

Source: International Communication Union

23. The scatter plot shows the number of hours worked per week for U.S. production workers from 1970 through 2000. Let y be the hours worked per week and x be the years since 1970. Write an equation that you think best models the data.

Average Hours Worked per Week for Production Workers



Source: Bureau of Labor Statistics

24. A day camp has 240 participants. Children can sign up for various activities. Suppose 135 children take swimming, 160 take soccer, and 75 take both swimming and soccer. What is the probability that a child selected at random takes swimming or soccer?

25. In how many different ways can seven members of a student government committee sit around a circular table?

26. Illinois residents can choose to buy an environmental license plate to support Illinois parks. Each environmental license plate displays 3 or 4 letters followed by a number 1 thru 99. How many different environmental license plates can be issued?

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Extended-Response Questions

Extended-response questions are often called open-ended or constructed-response questions. Most extended-response questions have multiple parts. You must answer all parts to receive full credit.

Extended-response questions are similar to short-response questions in that you must show all of your work in solving the problem and a rubric is used to determine whether you receive full, partial, or no credit. The following is a sample rubric for scoring extended-response questions.

Criteria	Credit
Full credit: The answer is correct and a full explanation is provided that shows each step in arriving at the final answer.	2
Partial credit: There are two different ways to receive partial credit. • The answer is correct, but the explanation provided is incomplete or incorrect. • The answer is incorrect, but the explanation and method of solving the problem is correct.	1
No credit: Either an answer is not provided or the answer does not make sense.	0

On some standardized tests, no credit is given for a correct answer if your work is not shown.

Make sure that when the problem says to *Show your work,* you show every aspect of your solution including figures, sketches of graphing calculator screens, or reasoning behind computations.

EXAMPLE

1 **Libby throws a ball into the air with a velocity of 64 feet per second. She releases the ball 5 feet above the ground. The height of the ball above the ground t seconds after release is modeled by an equation of the form $h(t) = -16t^2 + v_o t + h_o$ where v_o is the initial velocity in feet per second and h_o is the height at which the ball is released.**

a. Write an equation for the flight of the ball. Sketch the graph of the equation.

b. Find the maximum height that the ball reaches and the time that this height is reached.

c. Change only the speed of the release of the ball such that the ball will reach a maximum height greater than 100 feet. Write an equation for the flight of the ball.

Full Credit Solution

Part a A complete graph includes appropriate scales and labels for the axes, and points that are correctly graphed.

- A complete graph also shows the basic characteristics of the graph. The student should realize that the graph of this equation is a parabola opening downward with a maximum point reached at the vertex.
- The student should choose appropriate points to show the important characteristics of the graph.
- Students should realize that t and x and $h(t)$ and y are interchangeable on a graph on the coordinate plane.

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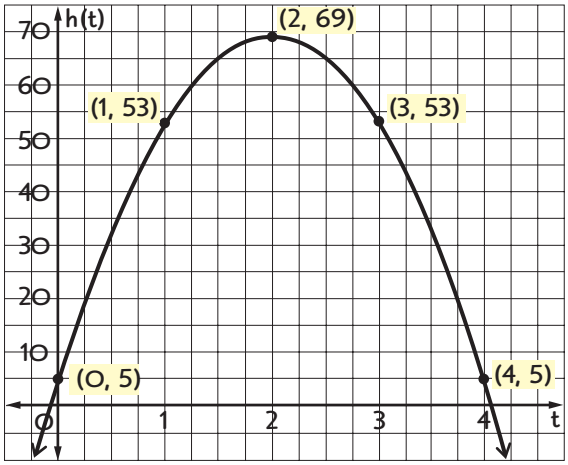
To write the equation for the ball, I substituted $v_o = 64$ and $h_o = 5$ into the equation $h(t) = -16t^2 + v_o t + h_o$, so the equation is $h(t) = -16t^2 + 64t + 5$. To graph the equation, I found the equation of the axis of symmetry and the vertex.

$x = -\frac{b}{2a}$

$= -\frac{64}{2(-16)}$ or 2

The equation of the axis of symmetry is $x = 2$, so the x-coordinate of the vertex is 2. You let $t = x$ and then $h(t) = -16t^2 + 64t + 5 = -16(2)^2 + 64(2) + 5$ or 69. The vertex is (2, 69). I found some other points and sketched the graph. I graphed points $(t, h(t))$ as (x, y).



Part b

The maximum height of the ball is reached at the vertex of the parabola. So, the maximum height is 69 feet and the time it takes to reach the maximum height is 2 seconds.

Part c

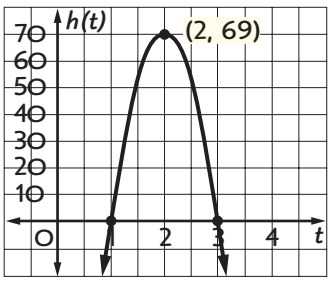
Since I have a graphing calculator, I changed the value of v_o until I found a graph in which the y or $h(t)$ coordinate was greater than 100. The equation I used was $h(t) = -16t^2 + 80t + 5$.

In part c, any equation whose graph has a vertex with y-coordinate greater than 100 would be a correct answer earning full credit.

Partial Credit Solution

Part a This sample answer includes no labels for the graph or the axes and one of the points is not graphed correctly.

$h(t) = -16t^2 + 64t + 5$; (2, 69)



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Part b Full credit is given because the vertex is correct and is interpreted correctly.

> The vertex shows the maximum height of the ball. The time it takes to reach the maximum height of 69 feet is 2 seconds.

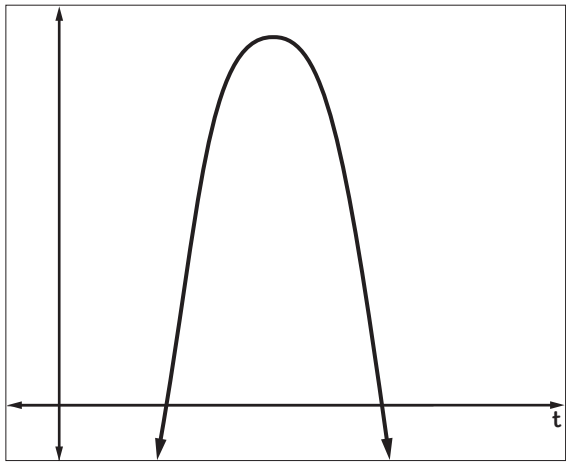
Part c Partial credit is given for part c since no explanation is given for using this equation. The student did not mention that the vertex would have a y-coordinate greater than 100.

> I will write the equation $h(t) = -16t^2 + 100t + 5$.

This sample answer might have received a score of 2 or 1, depending on the judgment of the scorer. Had the student sketched a more accurate graph and given more complete explanations for Parts a and c, the score would probably have been a 3.

No Credit Solution

Part a No credit is given because the equation is incorrect with no explanation and the sketch of the graph has no labels, making it impossible to determine whether the student understands the relationship between the equation for a parabola and the graph.

> $h(t) = -16t^2 + 5t + 64$
>
> 
>

Part b

> It reaches about 10 feet.

Part c

> A good equation for the ball is $h(t) = -16t^2 + 5t + 100$.

In this sample answer, the student does not understand how to substitute the given information into the equation, graph a parabola, or interpret the vertex of a parabola.

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Extended Response Practice

Solve each problem. Show your work.

Number and Operations

1. Mrs. Ebbrect is assigning identification (ID) numbers to freshman students. She plans to use only the digits 2, 3, 5, 6, 7, and 9. The ID numbers will consist of three digits with no repetitions.

a. How many 3-digit ID numbers can be formed?

b. How many more ID numbers can Mrs. Ebbrect make if she allows repetitions?

c. What type of system could Mrs. Ebbrect use to choose the numbers if there are at least 400 students who need ID numbers?

2. Use these four matrices.

$$A = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & 5 \\ 0 & 1 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} -7 & 3 \\ -6 & 2 \end{bmatrix} \qquad D = [-3 \;\; 1]$$

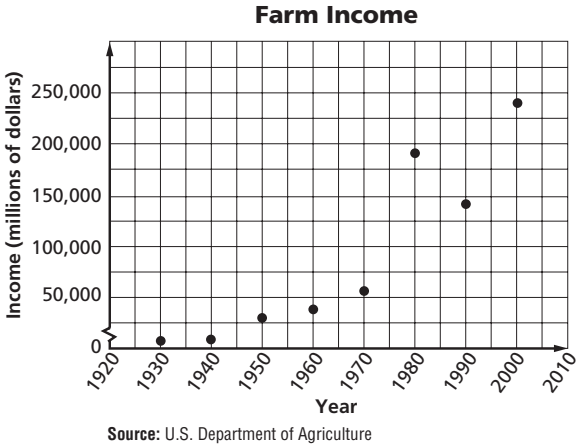
a. Find $A + C$.

b. Compare the dimensions of AB and DB.

c. Compare the matrices BC and CB.

Algebra

3. Roger is using the graph showing the gross cash income for all farms in the U.S. from 1930 through 2000 to make some predictions for the future.



Source: U.S. Department of Agriculture

a. Write an equation in slope-intercept form for the line passing through the point for 1930 and the point for 1970.

b. Write an equation in slope-intercept form for the line passing through the point for 1980 and the point for 2000. Compare the slope of this line to the slope of the line in part a.

c. Which equation, if any, do you think Roger should use to model the data? Explain.

d. Suggest an equation that is not linear for Roger to use.

4. Brad is coaching the bantam age division (8 years old and younger) swim team. On the first day of practice, he has the team swim 4 laps of the 25-meter pool. For each of the next practices, he increases the laps by 3. In other words, the children swim 4 laps the first day, 7 laps the second day, 10 laps the third day, and so on.

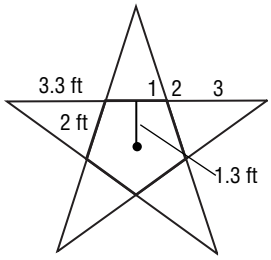
a. Write a formula for the nth term of the sequence of the number of laps each day. Explain how you found the formula.

b. How many laps will the children swim on the 10th day?

c. Brad's goal is to have the children swim at least one mile during practice on the 20th day. If one mile is approximately 1.6 kilometers, will Brad reach his goal?

Geometry

5. Alejandra is planning to use a star shape in a galaxy-themed mural on her wall. The pentagon in the center is regular, and the triangles forming the points are isosceles.



a. Find the measures of $\angle 1$, $\angle 2$, and $\angle 3$. Explain your method.

b. The approximate dimensions of the design are given. The segment of length 1.3 feet is the apothem of the pentagon. Find the approximate area of the design.

c. If Alejandra circumscribes a circle about the star, what is the area of the circle?

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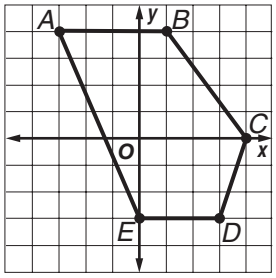
6. Kareem is using polygon $ABCDE$, shown on a coordinate plane, as a basis for a computer graphics design. He plans to perform various transformations on the polygon to produce a variety of interesting designs.



a. First, Kareem creates polygon $A'B'C'D'E'$ by rotating $ABCDE$ counterclockwise about the origin 270°. Graph polygon $A'B'C'D'E'$ and describe the relationship between the coordinates of $ABCDE$ and $A'B'C'D'E'$.

b. Next, Kareem reflects polygon $A'B'C'D'E'$ in the line $y = x$ to produce polygon $A''B''C''D''E''$. Graph $A''B''C''D''E''$ and describe the relationship between the coordinates of $A'B'C'D'E'$ and $A''B''C''D''E''$.

c. Describe how Kareem could transform polygon $ABCDE$ to polygon $A''B''C''D''E''$ with only one transformation.

Measurement

7. The speed of a satellite orbiting Earth can be found using the equation $v = \sqrt{\frac{GmE}{\frac{r}{mE}}}$.

G is the gravitational constant for Earth, mE is the mass of Earth, and r is the radius of the orbit which includes the radius of Earth and the height of the satellite.

a. The radius of Earth is 6.38×10^6 meters. The distance of a particular satellite above Earth is 350 kilometers. What is the value of r? (*Hint:* The center of the orbit is the center of Earth.)

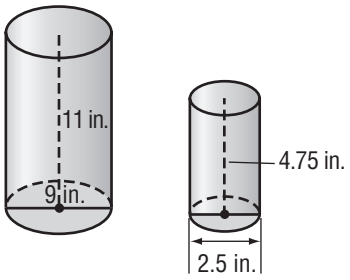
b. The gravitational constant for Earth is 6.67×10^{-11} N • m^2/kg^2. The mass of Earth is 5.97×10^{24} kg. Find the speed of the satellite in part a.

c. As a satellite increases in distance from Earth, what is the effect on the speed of the orbit? Explain your reasoning.

TEST-TAKING TIP

Question 6 When questions require graphing, make sure your graph is accurate to receive full credit for your correct solution.

8. A cylindrical cooler has a diameter of 9 inches and a height of 11 inches. Scott plans to use it for soda cans that have a diameter of 2.5 inches and a height of 4.75 inches.



a. Scott plans to place two layers consisting of 9 cans each into the cooler. What is the volume of the space that will not be filled with cans?

b. Find the ratio of the volume of the cooler to the volume of the cans in part a.

Data Analysis and Probability

9. The table shows the total world population from 1950 through 2000.

Year	Population	Year	Population
1950	2,566,000,053	1980	4,453,831,714
1960	3,039,451,023	1990	5,278,639,789
1970	3,706,618,163	2000	6,082,966,429

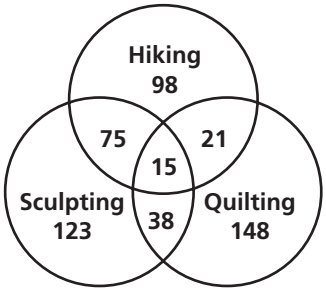
a. Between which two decades was the percent increase in population the greatest?

b. Make a scatter plot of the data.

c. Find a function that models the data.

d. Predict the world population for 2030.

10. Each year, a university sponsors a conference for women. The Venn diagram shows the number of participants in three activities for the 680 women that attended. Suppose women who attended are selected at randomfor a survey.



a. What is the probability that a woman selected participated in hiking or sculpting?

b. Describe a set of women such that the probability of their being selected is about 0.39.

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Glossary/Glossario

MathOnline A mathematics multilingual glossary is available at www.math.glencoe.com/multilingual_glossary. The glossary includes the following languages.

Arabic	Haitian Creole	Portuguese	Tagalog
Bengali	Hmong	Russian	Urdu
Cantonese	Korean	Spanish	Vietnamese
English			

Cómo usar el glosario en español:

1. Busca el término en inglés que desees encontrar.
2. El término en español, junto con la definición, se encuentran en la columna de la derecha.

English

Español

A

absolute value (p. 27) A number's distance from zero on the number line, represented by $|x|$.

valor absoluto Distancia entre un número y cero en una recta numérica; se denota con $|x|$.

absolute value function (p. 96) A function written as $f(x) = |x|$, where $f(x) \geq 0$ for all values of x.

función del valor absoluto Una función que se escribe $f(x) = |x|$, donde $f(x) \geq 0$, para todos los valores de x.

algebraic expression (p. 6) An expression that contains at least one variable.

expresión algebraica Expresión que contiene al menos una variable.

amplitude (p. 823) For functions in the form $y = a \sin b\theta$ or $y = a \cos b\theta$, the amplitude is $|a|$.

amplitud Para funciones de la forma $y = a$ sen $b\theta$ o $y = a \cos b\theta$, la amplitud es $|a|$.

angle of depression (p. 764) The angle between a horizontal line and the line of sight from the observer to an object at a lower level.

ángulo de depresión Ángulo entre una recta horizontal y la línea visual de un observador a una figura en un nivel inferior.

angle of elevation (p. 764) The angle between a horizontal line and the line of sight from the observer to an object at a higher level.

ángulo de elevación Ángulo entre una recta horizontal y la línea visual de un observador a una figura en un nivel superior.

arccosine (p. 807) The inverse of $y = \cos x$, written as $x = \arccos y$.

arcocoseno La inversa de $y = \cos x$, que se escribe como $x = \arccos y$.

arcsine (p. 807) The inverse of $y = \sin x$, written as $x = \arcsin y$.

arcoseno La inversa de $y =$ sen x, que se escribe como $x = \arcsen y$.

arctangent (p. 807) The inverse of $y = \tan x$ written as $x = \arctan y$.

arcotangente La inversa de $y = \tan x$ que se escribe como $x = \arctan y$.

area diagram (p. 703) A model of the probability of two events occurring.

diagrama de área Modelo de la probabilidad de que ocurran dos eventos.

arithmetic mean (p. 624) The terms between any two nonconsecutive terms of an arithmetic sequence.

media aritmética Cualquier término entre dos términos no consecutivos de una sucesión aritmética.

arithmetic sequence (p. 622) A sequence in which each term after the first is found by adding a constant, the common difference d, to the previous term.

sucesión aritmética Sucesión en que cualquier término después del primero puede hallarse sumando una constante, la diferencia común d, al término anterior.

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arithmetic series (p. 629) The indicated sum of the terms of an arithmetic sequence.

serie aritmética Suma específica de los términos de una sucesión aritmética.

asymptote (p. 457, 591) A line that a graph approaches but never crosses.

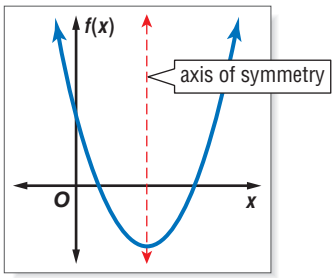
asíntota Recta a la que se aproxima una gráfica, sin jamás cruzarla.

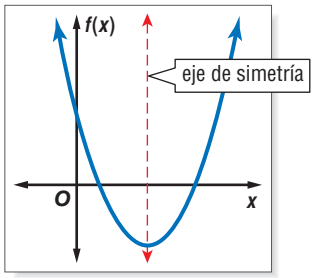
augmented matrix (p. 223) A coefficient matrix with an extra column containing the constant terms.

matriz ampliada Matriz coeficiente con una columna extra que contiene los términos constantes.

axis of symmetry (p. 237) A line about which a figure is symmetric.

eje de simetría Recta respecto a la cual una figura es simétrica.





B

$b^{\frac{1}{n}}$ (p. 415) For any real number b and for any positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except when $b < 0$ and n is even .

$b^{\frac{1}{n}}$ Para cualquier número real b y para cualquier entero positivo n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, excepto cuando $b < 0$ y n es par.

binomial (p. 7) A polynomial that has two unlike terms.

binomio Polinomio con dos términos diferentes.

binomial experiment (p. 730) An experiment in which there are exactly two possible outcomes for each trial, a fixed number of independent trials, and the probabilities for each trial are the same.

experimento binomial Experimento con exactamente dos resultados posibles para cada prueba, un número fijo de pruebas independientes y en el cual cada prueba tiene igual probabilidad.

Binomial Theorem (p. 665) If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n+1)}{1 \cdot 2}a^{n-2}b^2 + \ldots + 1a^0b^n$.

teorema del binomio Si n es un entero no negativo, entonces $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n+1)}{1 \cdot 2}a^{n-2}b^2 + \ldots + 1a^0b^n$.

bivariate data (p. 86) Data with two variables.

datos bivariados Datos con dos variables.

boundary (p. 102) A line or curve that separates the coordinate plane into two regions.

frontera Recta o curva que divide un plano de coordenadas en dos regiones.

bounded (p. 138) A region is bounded when the graph of a system of constraints is a polygonal region.

acotada Una región está acotada cuando la gráfica de un sistema de restricciones es una región poligonal.

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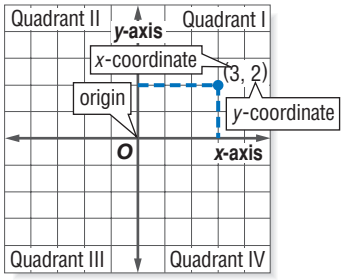
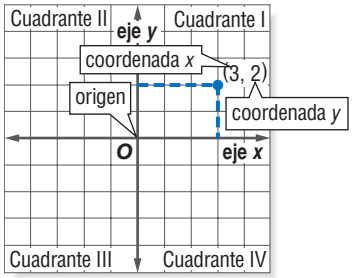
Cartesian coordinate plane (p. 58) A plane divided into four quadrants by the intersection of the x-axis and the y-axis at the origin.

plano de coordenadas cartesiano Plano dividido en cuatro cuadrantes mediante la intersección en el origen de los ejes x y y.





center of a circle (p. 574) The point from which all points on a circle are equidistant.

centro de un círculo El punto desde el cual todos los puntos de un círculo están equidistantes.

center of a hyperbola (p. 591) The midpoint of the segment whose endpoints are the foci.

centro de una hipérbola Punto medio del segmento cuyos extremos son los focos.

center of an ellipse (p. 582) The point at which the major axis and minor axis of an ellipse intersect.

centro de una elipse Punto de intersección de los ejes mayor y menor de una elipse.

Change of Base Formula (p. 530) For all positive numbers a, b, and n, where $a \neq 1$ and $b \neq 1$,

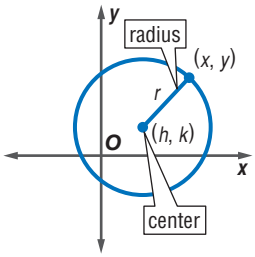
$$\log_a n = \frac{\log_b n}{\log_b a}.$$

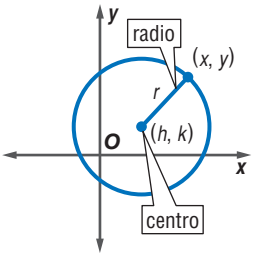
fórmula del cambio de base Para todo número positivo a, b y n, donde $a \neq 1$ y $b \neq 1$,

$$\log_a n = \frac{\log_b n}{\log_b a}.$$

circle (p. 574) The set of all points in a plane that are equidistant from a given point in the plane, called the center.

círculo Conjunto de todos los puntos en un plano que equidistan de un punto dado del plano llamado centro.





circular functions (p. 800) Functions defined using a unit circle.

funciones circulares Funciones definidas en un círculo unitario.

coefficient (p. 7) The numerical factor of a monomial.

coeficiente Factor numérico de u monomio.

column matrix (p. 163) A matrix that has only one column.

matriz columna Matriz que sólo tiene una columna.

combination (p. 692) An arrangement of objects in which order is not important.

combinación Arreglo de elementos en que el orden no es importante.

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common difference (p. 622) The difference between the successive terms of an arithmetic sequence.

diferencia común Diferencia entre términos consecutivos de una sucesión aritmética.

common logarithms (p. 528) Logarithms that use 10 as the base.

logaritmos comunes El logaritmo de base 10.

common ratio (p. 636) The ratio of successive terms of a geometric sequence.

razón común Razón entre términos consecutivos de una sucesión geométrica.

completing the square (p. 269) A process used to make a quadratic expression into a perfect square trinomial.

completar el cuadrado Proceso mediante el cual una expresión cuadrática se transforma en un trinomio cuadrado perfecto.

complex conjugates (p. 263) Two complex numbers of the form $a + bi$ and $a - bi$.

conjugados complejos Dos números complejos de la forma $a + bi$ y $a - bi$.

complex fraction (p. 445) A rational expression whose numerator and/or denominator contains a rational expression.

fracción compleja Expresión racional cuyo numerador o denominador contiene una expresión racional.

complex number (p. 261) Any number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary unit.

número complejo Cualquier número que puede escribirse de la forma $a + bi$, donde a y b son números reales e i es la unidad imaginaria.

composition of functions (p. 385) A function is performed, and then a second function is performed on the result of the first function. The composition of f and g is denoted by $f \cdot g$, and $[f \cdot g](x) = f[g(x)]$.

composición de funciones Se evalúa una función y luego se evalúa una segunda función en el resultado de la primera función. La composición de f y g se define con $f \cdot g$, y $[f \cdot g](x) = f[g(x)]$.

compound event (p. 710) Two or more simple events.

evento compuesto Dos o más eventos simples.

compound inequality (p. 41) Two inequalities joined by the word *and* or *or*.

desigualdad compuesta Dos desigualdades unidas por las palabras *y* u *o*.

conic section (p. 567) Any figure that can be obtained by slicing a double cone.

sección cónica Cualquier figura obtenida mediante el corte de un cono doble.

conjugate axis (p. 591) The segment of length $2b$ units that is perpendicular to the transverse axis at the center.

eje conjugado El segmento de $2b$ unidades de longitud que es perpendicular al eje transversal en el centro.

conjugates (p. 411) Binomials of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers.

conjugados Binomios de la forma $a\sqrt{b} + c\sqrt{d}$ y $a\sqrt{b} - c\sqrt{d}$, donde a, b, c, y d son números racionales.

consistent (p. 118) A system of equations that has at least one solution.

consistente Sistema de ecuaciones que posee por lo menos una solución.

constant (p. 7) Monomials that contain no variables.

constante Monomios que carecen de variables.

constant function (p. 96) A linear function of the form $f(x) = b$.

función constante Función lineal de la forma $f(x) = b$.

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constant of variation (p. 465) The constant k used with direct or inverse variation.

constante de variación La constante k que se usa en variación directa o inversa.

constant term (p. 236) In $f(x) = ax^2 + bx + c$, c is the constant term.

término constante En $f(x) = ax^2 + bx + c$, c es el término constante.

constraints (p. 138) Conditions given to variables, often expressed as linear inequalities.

restricciones Condiciones a que están sujetas las variables, a menudo escritas como desigualdades lineales.

continuity (p. 457) A graph of a function that can be traced with a pencil that never leaves the paper.

continuidad La gráfica de una función que se puede calcar sin levantar nunca el lápiz del papel.

continuous probability distribution (p. 724) The outcome can be any value in an interval of real numbers, represented by curves.

distribución de probabilidad continua El resultado puede ser cualquier valor de un intervalo de números reales, representados por curvas.

continuous relation (p. 59) A relation that can be graphed with a line or smooth curve.

relación continua Relación cuya gráfica puede ser una recta o una curva suave.

cosecant (p. 759) For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\csc \alpha = \frac{r}{y}$.

cosecante Para cualquier ángulo de medida α, un punto P(x, y) en su lado terminal, $r = \sqrt{x^2} + y^2$, $\csc \alpha = \frac{r}{y}$.

cosine (p. 759) For any angle, with measure α , a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\cos \alpha = \frac{x}{r}$.

coseno Para cualquier ángulo de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, $\cos \alpha = \frac{x}{r}$.

cotangent (p. 759) For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\cot \alpha = \frac{x}{y}$.

cotangente Para cualquier ángulo de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, $\cot \alpha = \frac{x}{y}$.

coterminal angles (p. 771) Two angles in standard position that have the same terminal side.

ángulos coterminales Dos ángulos en posición estándar que tienen el mismo lado terminal.

convergent series (p. 651) An infinite series with a sum.

serie convergente Serie infinita con una suma.

counterexample (p. 17) A specific case that shows that a statement is false.

contraejemplo Caso específico que demuestra la falsedad de un enunciado.

Cramer's Rule (p. 201) A method that uses determinants to solve a system of linear equations.

regla de Crámer Método que usa determinantes para resolver un sistema de ecuaciones lineales.

D

degree (p. 7) The sum of the exponents of the variables of a monomial.

grado Suma de los exponentes de las variables de un monomio.

degree of a polynomial (p. 320) The greatest degree of any term in the polynomial.

grado de un polinomio Grado máximo de cualquier término del polinomio.

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dependent events (p. 686) The outcome of one event does affect the outcome of another event.

eventos dependientes El resultado de un evento afecta el resultado de otro evento.

dependent system (p. 118) A consistent system of equations that has an infinite number of solutions.

sistema dependiente Sistema de ecuaciones que posee un número infinito de soluciones.

dependent variable (p. 61) The other variable in a function, usually y, whose values depend on x.

variable dependiente La otra variable de una función, por lo general y, cuyo valor depende de x.

depressed polynomial (p. 357) The quotient when a polynomial is divided by one of its binomial factors.

polinomio reducido El cociente cuando se divide un polinomio entre uno de sus factores binomiales.

determinant (p. 194) A square array of numbers or variables enclosed between two parallel lines.

determinante Arreglo cuadrado de números o variáblesencerrados entre dos rectas paralelas

dilation (p. 187) A transformation in which a geometric figure is enlarged or reduced.

homotecia Transformación en que se amplía o se reduce un figura geométrica.

dimensional analysis (p. 315) Performing operations with units.

anállisis dimensional Realizar operaciones con unidades.

dimensions of a matrix (p. 163) The number of rows, m, and the number of columns, n, of the matrix written as $m \times n$.

tamaño de una matriz El número de filas, m, y columnas, n, de una matriz, lo que se escribe $m \times n$.

directrix (p. 567) See parabola.

directriz Véase parábola.

direct variation (p. 465) y varies directly as x if there is some nonzero constant k such that $y = kx$. k is called the constant of variation.

variación directa y varía directamente con x si hay una constante no nula k tal que $y = kx$. k se llama la constante de variación.

discrete probability distributions (p. 724) Probabilities that have a finite number of possible values.

distribución de probabilidad discreta Probabilidades que tienen un número finito de valores posibles.

discrete relation (p. 59) A relation in which the domain is a set of individual points.

relación discreta Relación en la cual el dominio es un conjunto de puntos individuales.

discriminant (p. 279) In the Quadratic Formula, the expression $b^2 - 4ac$.

discriminante En la fórmula cuadrática, la expresión $b^2 - 4ac$.

dispersion (p. 718) Measures of variation of data.

dispersión Medidas de variación de los datos.

Distance Formula (p. 563) The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

fórmula de la distancia La distancia entre dos puntos (x_1, y_1) and (x_2, y_2) viene dada por $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

domain (p. 58) The set of all x-coordinates of the ordered pairs of a relation.

dominio El conjunto de todas las coordenadas x de los pares ordenados de una relación.

E

e (p. 536) The irrational number 2.71828.... e is the base of the natural logarithms.

e El número irracional 2.71828.... e es la base de los logaritmos naturales.

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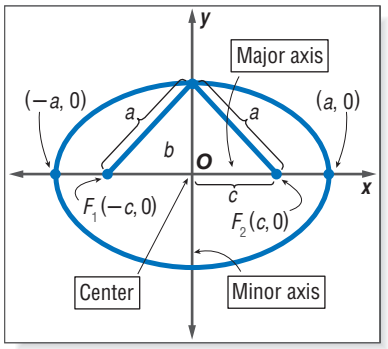
element (p. 163) Each value in a matrix.

elemento Cada valor de una matriz.

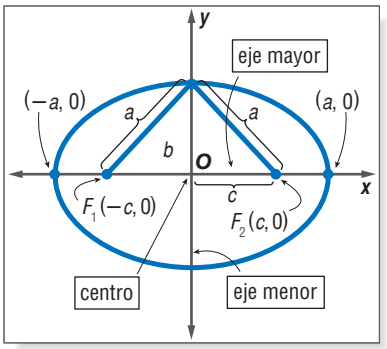
elimination method (p. 125) Eliminate one of the variables in a system of equations by adding or subtracting the equations.

método de eliminación Eliminar una de las variables de un sistema de ecuaciones sumando o restando las ecuaciones.

ellipse (p. 581) The set of all points in a plane such that the sum of the distances from two given points in the plane, called foci, is constant.



elipse Conjunto de todos los puntos de un plano en los que la suma de sus distancias a dos puntos dados del plano, llamados focos, es constante.



empty set (p. 28) The solution set for an equation that has no solution, symbolized by { } or ø.

conjunto vacío Conjunto solución de una ecuación que no tiene solución, denotado por { } o ø.

end behavior (p. 334) The behavior of the graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$).

comportamiento final El comportamiento de una gráfica a medida que x tiende a más infinito ($+\infty$) o menos infinito ($-\infty$).

equal matrices (p. 164) Two matrices that have the same dimensions and each element of one matrix is equal to the corresponding element of the other matrix.

matrices iguales Dos matrices que tienen las mismas dimensiones y en las que cada elemento de una de ellas es igual al elemento correspondiente en la otra matriz.

equation (p. 18) A mathematical sentence stating that two mathematical expressions are equal.

ecuación Enunciado matemático que afirma la igualdad de dos expresiones matemáticas.

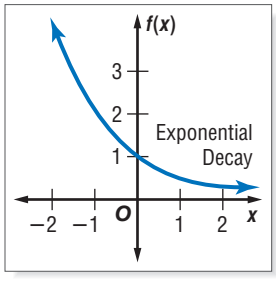
event (p. 684) One or more outcomes of a trial.

evento Uno o más resultados de una prueba.

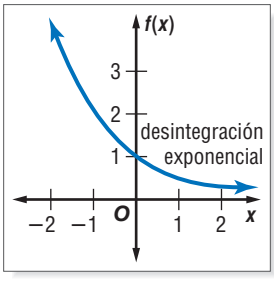
expansion by minors (p. 195) A method of evaluating a third or high order determinant by using determinants of lower order.

expansión por determinantes menores Un método de calcular el determinante de tercer orden o mayor mediante el uso de determinantes de orden más bajo.

exponential decay (p. 500) Exponential decay occurs when a quantity decreases exponentially over time.



desintegración exponencial Ocurre cuando una cantidad disminuye exponencialmente con el tiempo.



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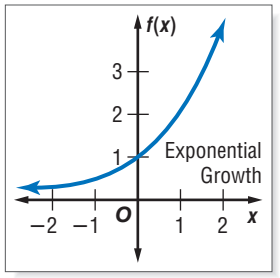
exponential equation (p. 501) An equation in which the variables occur as exponents.

ecuación exponencial Ecuación en que las variables aparecen en los exponentes.

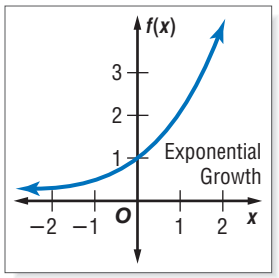
exponential function (p. 499) A function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

función exponencial Una función de la forma $y = ab^x$, donde $a \neq 0$, $b > 0$, y $b \neq 1$.

exponential growth (p. 500) Exponential growth occurs when a quantity increases exponentially over time.



crecimiento exponencial El que ocurre cuando una cantidad aumenta exponencialmente con el tiempo.



exponential inequality (p. 502) An inequality involving exponential functions.

desigualdad exponencial Desigualdad que contiene funciones exponenciales.

extraneous solution (p. 422) A number that does not satisfy the original equation.

solución extraña Número que no satisface la ecuación original.

extrapolation (p. 87) Predicting for an x-value greater than any in the data set.

extrapolación Predicción para un valor de x mayor que cualquiera de los de un conjunto de datos.

F

factorial (p. 666) If n is a positive integer, then $n! = n(n-1)(n-2) \ldots 2 \cdot 1$.

factorial Si n es un entero positivo, entonces $n! = n(n-1)(n-2) \ldots 2 \cdot 1$.

failure (p. 697) Any outcome other than the desired outcome.

fracaso Cualquier resultado distinto del deseado.

family of graphs (p. 73) A group of graphs that displays one or more similar characteristics.

familia de gráficas Grupo de gráficas que presentan una o más características similares.

feasible region (p. 138) The intersection of the graphs in a system of constraints.

región viable Intersección de las gráficas de un sistema de restricciones.

Fibonacci sequence (p. 658) A sequence in which the first two terms are 1 and each of the additional terms is the sum of the two previous terms.

sucesión de Fibonacci Sucesión en que los dos primeros términos son iguales a 1 y cada término que sigue es igual a la suma de los dos anteriores.

focus (pp. 567, 581, 590) See parabola, ellipse, hyperbola.

foco Véase parábola, elipse, hipérbola.

FOIL method (p. 253) The product of two binomials is the sum of the products of **F** the *first* terms, **O** the *outer* terms, **I** the *inner* terms, and **L** the *last* terms.

método FOIL El producto de dos binomios es la suma de los productos de los primeros (***First***) términos, los términos exteriores (***Outer***), los términos interiores (***Inner***) y los últimos (***Last***) términos.

formula (p. 7) A mathematical sentence that expresses the relationship between certain quantities.

fórmula Enunciado matemático que describe la relación entre ciertas cantidades.

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function (p. 59) A relation in which each element of the domain is paired with exactly one element in the range.

función Relación en que a cada elemento del dominio le corresponde un solo elemento del rango.

function notation (p. 61) An equation of y in terms of x can be rewritten so that $y = f(x)$. For example, $y = 2x + 1$ can be written as $f(x) = 2x + 1$.

notación funcional Una ecuación de y en términos de x puede escribirse en la forma $y = f(x)$. Por ejemplo, $y = 2x + 1$ puede escribirse como $f(x) = 2x + 1$.

Fundamental Counting Principle (p. 685) If event M can occur in m ways and is followed by event N that can occur in n ways, then event M followed by event N can occur in $m \cdot n$ ways.

principio fundamental de conteo Si el evento M puede ocurrir de m maneras y es seguido por el evento N que puede ocurrir de n maneras, entonces el evento M seguido por el evento N pueden ocurrir de $m \cdot n$ maneras.

G

geometric mean (p. 638) The terms between any two nonsuccessive terms of a geometric sequence.

media geométrica Cualquier término entre dos términos no consecutivos de una sucesión geométrica.

geometric sequence (p. 636) A sequence in which each term after the first is found by multiplying the previous term by a constant r, called the common ratio.

sucesión geométrica Sucesión en que cualquier término después del primero puede hallarse multiplicando el término anterior por una constante r, llamada razón común .

geometric series (p. 643) The sum of the terms of a geometric sequence.

serie geométrica La suma de los términos de una sucesión geométrica.

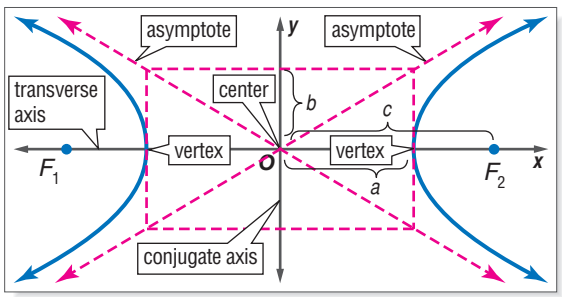
greatest integer function (p. 95) A step function, written as $f(x) = [\![x]\!]$, where $f(x)$ is the greatest integer less than or equal to x.

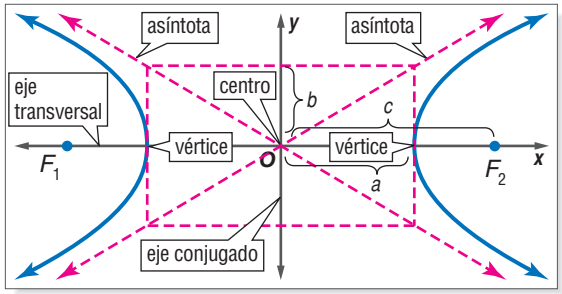
función del máximo entero Una función etapa que se escribe $f(x) = [\![x]\!]$, donde $f(x)$ es el meaximo entero que es menor que o igual a x.

H

hyperbola (p. 590) The set of all points in the plane such that the absolute value of the difference of the distances from two given points in the plane, called foci, is constant.

hipérbola Conjunto de todos los puntos de un plano en los que el valor absoluto de la diferencia de sus distancias a dos puntos dados del plano, llamados focos, es constante.





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I

identity function (p. 96, 393) The function $I(x) = x$.

identity matrix (p. 208) A square matrix that, when multiplied by another matrix, equals that same matrix. If A is any $n \times n$ matrix and I is the $n \times n$ identity matrix, then $A \cdot I = A$ and $I \cdot A = A$.

image (p. 185) The graph of an object after a transformation.

imaginary unit (p. 260) i, or the principal square root of -1.

inclusive (p. 712) Two events whose outcomes may be the same.

inconsistent (p. 118) A system of equations that has no solutions.

independent events (p. 684) Events that do not affect each other.

independent system (p. 118) A system of equations that has exactly one solution.

independent variable (p. 61) In a function, the variable, usually x, whose values make up the domain.

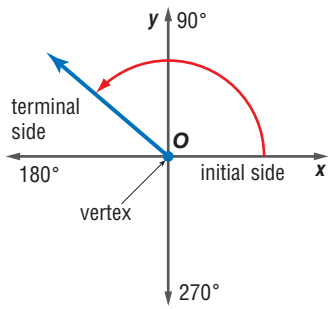
index of summation (p. 631) The variable used with the summation symbol. In the expression below, the index of summation is n.

$$\sum_{n=1}^{3} 4n$$

inductive hypothesis (p. 670) The assumption that a statement is true for some positive integer k, where $k \geq n$.

infinite geometric series (p. 650) A geometric series with an infinite number of terms.

initial side of an angle (p. 768) The fixed ray of an angle.



unción identidad La función $I(x) = x$.

matriz identidad Matriz cuadrada que al multiplicarse por otra matriz, es igual a la misma matriz. Si A es una matriz de $n \times n$ e I es la matriz identidad de $n \times n$, entonces $A \cdot I = A$ y $I \cdot A = A$.

imagen Gráfica de una figura después de una transformación.

unidad imaginaria i, o la raíz cuadrada principal de -1.

inclusivo Dos eventos que pueden tener los mismos resultados.

inconsistente Sistema de ecuaciones que no tiene solución alguna.

eventos independientes Eventos que no se afectan mutuamente.

sistema independiente Sistema de ecuaciones que sólo tiene una solución.

variable independiente En una función, la variable, por lo general x, cuyos valores forman el dominio.

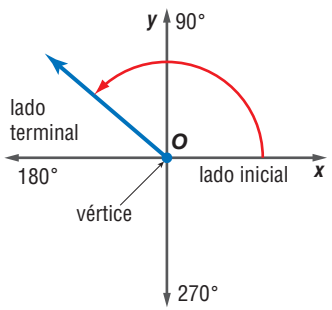
índice de suma Variable que se usa con el símbolo de suma. En la siguiente expresión, el índice de suma es n.

$$\sum_{n=1}^{3} 4n$$

hipótesis inductiva El suponer que un enunciado es verdadero para algún entero positivo k, donde $k \geq n$.

serie geométrica infinita Serie geométrica con un número infinito de términos.

lado inicial de un ángulo El rayo fijo de un ángulo.



Glossary/Glosario

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intercept form (p. 253) A quadratic equation in the form $y = a(x - p)(x - q)$ where p and q represent the x-intercept of the graph.

forma intercepción Ecuación cuadrática de la forma $y = a(x - p)(x - q)$ donde p y q representan la intersección x de la gráfica.

interpolation (p. 87) Predicting for an x-value between the least and greatest values of the set.

interpolación Predecir un valor de x entre los valores máximo y mínimo del conjunto de datos.

intersection (p. 41) The graph of a compound inequality containing *and*.

intersección Gráfica de una desigualdad compuesta que contiene la palabra *y*.

inverse (p. 209) Two $n \times n$ matrices are inverses of each other if their product is the identity matrix.

inversa Dos matrices de $n \times n$ son inversas mutuas si su producto es la matriz identidad.

inverse function (p. 392) Two functions f and g are inverse functions if and only if both of their compositions are the identity function.

función inversa Dos funciones f y g son inversas mutuas si y sólo si las composiciones de ambas son la función identidad.

inverse of a trigonometric function (p. 806) The arccosine, arcsine, and arctangent relations.

inversa de una función trigonométrica Las relaciones arcocoseno, arcoseno y arcotangente.

inverse relations (p. 391) Two relations are inverse relations if and only if whenever one relation contains the element (a, b) the other relation contains the element (b, a).

relaciones inversas Dos relaciones son relaciones inversas mutuas si y sólo si cada vez que una de las relaciones contiene el elemento (a, b), la otra contiene el elemento (b, a).

inverse variation (p. 467) y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

variación inversa y varía inversamente con x si hay una constante no nula k tal que $xy = k$ o $y = \frac{k}{x}$, donde $x \neq 0$ y $y \neq 0$.

irrational number (p. 11) A real number that is not rational. The decimal form neither terminates nor repeats.

número irracional Número que no es racional. Su expansión decimal no es ni terminal ni periódica.

iteration (p. 660) The process of composing a function with itself repeatedly.

iteración Proceso de componer una función consigo misma repetidamente.

J

joint variation (p. 466) y varies jointly as x and z if there is some nonzero constant k such that $y = kxz$.

variación conjunta y varía conjuntamente con x y z si hay una constante no nula k tal que $y = kxz$.

L

latus rectum (p. 569) The line segment through the focus of a parabola and perpendicular to the axis of symmetry.

latus rectum El segmento de recta que pasa por el foco de una parábola y que es perpendicular a su eje de simetría.

Law of Cosines (pp. 793–794) Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides, and opposite angles with measures A, B, and C, respectively. Then the following equations are true.

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Ley de los cosenos Sea $\triangle ABC$ un triángulo cualquiera, con a, b y c las longitudes de los lados y con ángulos opuestos de medidas A, B y C, respectivamente. Entonces se cumplen las siguientes ecuaciones.

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

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Law of Sines (p. 786) Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C, respectively. Then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Ley de los senos Sea $\triangle ABC$ cualquier triángulo con a, b y c las longitudes de los lados y con ángulos opuestos de medidas A, B y C, respectivamente.
Entonces $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

leading coefficient (p. 331) The coefficient of the term with the highest degree.

coeficiente líder Coeficiente del término de mayor grado.

like radical expressions (p. 411) Two radical expressions in which both the radicands and indices are alike.

expresiones radicales semejantes Dos expresiones radicales en que tanto los radicandos como los índices son semejantes.

like terms (p. 7) Monomials that can be combined.

términos semejantes Monomios que pueden combinarse.

limit (p. 642) The value that the terms of a sequence approach.

límite El valor al que tienden los términos de una sucesión.

linear correlation coefficient (p. 92) A value that shows how close data points are to a line.

coeficiente de correlación lineal Valor que muestra la cercanía de los datos a una recta.

linear equation (p. 66) An equation that has no operations other than addition, subtraction, and multiplication of a variable by a constant.

ecuación lineal Ecuación sin otras operaciones que las de adición, sustracción y multiplicación de una variable por una constante.

linear function (p. 66) A function whose ordered pairs satisfy a linear equation.

función lineal Función cuyos pares ordenados satisfacen una ecuación lineal.

linear permutation (p. 690) The arrangement of objects or people in a line.

permutación lineal Arreglo de personas o figuras en una línea.

linear programming (p. 140) The process of finding the maximum or minimum values of a function for a region defined by inequalities.

programación lineal Proceso de hallar los valores máximo o mínimo de una función lineal en una región definida por las desigualdades.

linear term (p. 236) In the equation $f(x) = ax^2 + bx + c$, bx is the linear term.

término lineal En la ecuación $f(x) = ax^2 + bx + c$, el término lineal es bx.

line of best fit (p. 92) A line that best matches a set of data.

recta de óptimo ajuste Recta que mejor encaja un conjunto de datos.

line of fit (p. 86) A line that closely approximates a set of data.

recta de ajuste Recta que se aproxima estrechamente a un conjunto de datos.

Location Principle (p. 340) Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

principio de ubicación Sea $y = f(x)$ una función polinómica con a y b dos números tales que $f(a) < 0$ y $f(b) > 0$. Entonces la función tiene por lo menos un resultado real entre a y b.

logarithm (p. 510) In the function $x = b^y$, y is called the logarithm, base b, of x. Usually written as $y = \log_b x$ and is read "y equals log base b of x."

logaritmo En la función $x = b^y$, y es el logaritmo en base b, de x. Generalmente escrito como $y = \log_b x$ y se lee "y es igual al logaritmo en base b de x."

logarithmic equation (p. 512) An equation that contains one or more logarithms.

ecuación logarítmica Ecuación que contiene uno o más logaritmos.

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logarithmic function (p. 511) The function $y = \log_b x$, where $b > 0$ and $b \neq 1$, which is the inverse of the exponential function $y = bx$.

función logarítmica La función $y = \log_b x$, donde $b > 0$ y $b \neq 1$, inversa de la función exponencial $y = bx$.

logarithmic inequality (p. 512) An inequality that contains one or more logarithms.

desigualdad logarítmica Desigualdad que contiene uno o más logaritmos.

M

major axis (p. 582) The longer of the two line segments that form the axes of symmetry of an ellipse.

eje mayor El más largo de dos segmentos de recta que forman los ejes de simetría de una elipse.

mapping (p. 58) How each member of the domain is paired with each member of the range.

transformaciones La correspondencia entre cada miembro del dominio con cada miembro del rango.

margin of sampling error (ME) (p. 735) The limit on the difference between how a sample responds and how the total population would respond.

margen de error muestral (EM) Límite en la diferencia entre las respuestas obtenidas con una muestra y cómo pudiera responder la población entera.

mathematical induction (p. 670) A method of proof used to prove statements about positive integers.

inducción matemática Método de demostrar enunciados sobre los enteros positivos.

matrix (p. 162) Any rectangular array of variables or constants in horizontal rows and vertical columns.

matriz Arreglo rectangular de variables o constantes en filas horizontales y columnas verticales.

matrix equation (p. 216) A matrix form used to represent a system of equations.

ecuación matriz Forma de matriz que se usa para representar un sistema de ecuaciones.

maximum value (p. 238) The y-coordinate of the vertex of the quadratic function $f(x) = ax^2 + bx + c$, where $a < 0$.

valor máximo La coordenada y del vértice de la función cuadrática $f(x) = ax^2 + bx + c$, where $a < 0$.

measure of central tendency (p. 717) A number that represents the center or middle of a set of data.

medida de tendencia central Número que representa el centro o medio de un conjunto de datos.

measure of variation (p. 718) A representation of how spread out or scattered a set of data is.

medida de variación Número que representa la dispersión de un conjunto de datos.

midline (p. 831) A horizontal axis used as the reference line about which the graph of a periodic function oscillates.

recta central Eje horizontal que se usa como recta de referencia alrededor de la cual oscila la gráfica de una función periódica.

minimum value (p. 238) The y-coordinate of the vertex of the quadratic function $f(x) = ax^2 + bx + c$, where $a > 0$.

valor mínimo La coordenada y del vértice de la función cuadrática $f(x) = ax^2 + bx + c$, donde $a > 0$.

minor (p. 195) The determinant formed when the row and column containing that element are deleted.

determinante menor El que se forma cuando se descartan la fila y columna que contienen dicho elemento.

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minor axis (p. 582) The shorter of the two line segments that form the axes of symmetry of an ellipse.

eje menor El más corto de los dos segmentos de recta de los ejes de simetría de una elipse.

monomial (p. 6) An expression that is a number, a variable, or the product of a number and one or more variables.

monomio Expresión que es un número, una variable o el producto de un número por una o más variables.

mutually exclusive (p. 710) Two events that cannot occur at the same time.

mutuamente exclusivos Dos eventos que no pueden ocurrir simultáneamente.

N

***nth* root** (p. 402) For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an *nth* root of b.

raíz *enésima* Para cualquier número real a y b y cualquier entero positivo n, si $a^n = b$, entonces a se llama una raíz *enésima* de b.

natural base, *e* (p. 536) An irrational number approximately equal to 2.71828… .

base natural, *e* Número irracional aproximadamente igual a 2.71828…

natural base exponential function (p. 536) An exponential function with base e, $y = e^x$.

función exponencial natural La función exponencial de base e, $y = e^x$.

natural logarithm (p. 537) Logarithms with base e, written $\ln x$.

logaritmo natural Logaritmo de base e, el que se escribe $\ln x$.

natural logarithmic function (p. 537) $y = \ln x$, the inverse of the natural base exponential function $y = e^x$.

función logarítmica natural $y = \ln x$, la inversa de la función exponencial natural $y = e^x$.

negative exponent (p. 312) For any real number $a \neq 0$ and any integer n, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.

exponente negativo Para cualquier número real $a \neq 0$ cualquier entero positivo n, $a^{-n} = \frac{1}{a^n}$ y $\frac{1}{a^{-n}} = a^n$.

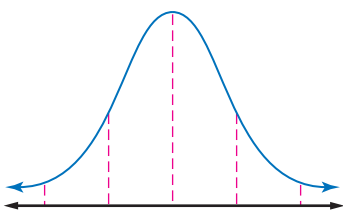
nonrectangular hyperbola (p. 596) A hyperbola with asymptotes that are not perpendicular.

hipérbola no rectangular Hipérbola con asíntotas que no son perpendiculares.

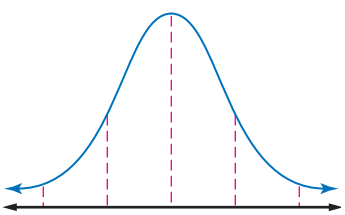
normal distribution (p. 724) A frequency distribution that often occurs when there is a large number of values in a set of data: about 68% of the values are within one standard deviation of the mean, 95% of the values are within two standard deviations from the mean, and 99% of the values are within three standard deviations.

distribución normal Distribución de frecuencia que aparece a menudo cuando hay un número grande de datos: cerca del 68% de los datos están dentro de una desviación estándar de la media, 95% están dentro de dos desviaciones estándar de la media y 99% están dentro de tres desviaciones estándar de la media.

Normal Distribution



Distribución normal



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O

one-to-one function (p. 394) **1.** A function where each element of the range is paired with exactly one element of the domain **2.** A function whose inverse is a function.

función biunívoca **1.** Función en la que a cada elemento del rango le corresponde sólo un elemento del dominio. **2.** Función cuya inversa es una función.

open sentence (p. 18) A mathematical sentence containing one or more variables.

enunciado abierto Enunciado matemático que contiene una o más variables.

ordered pair (p. 58) A pair of coordinates, written in the form (x, y), used to locate any point on a coordinate plane.

par ordenado Un par de números, escrito en la forma (x, y), que se usa para ubicar cualquier punto en un plano de coordenadas.

ordered triple (p. 146) **1.** The coordinates of a point in space **2.** The solution of a system of equations in three variables x, y, and z.

triple ordenado **1.** Las coordenadas de un punto en el espacio **2.** Solución de un sistema de ecuaciones en tres variables x, y y z.

Order of Operations (p. 6)
Step 1 Evaluate expressions inside grouping symbols.
Step 2 Evaluate all powers.
Step 3 Do all multiplications and/or divisions from left to right.
Step 4 Do all additions and subtractions from left to right.

orden de las operaciones
Paso 1 Evalúa las expresiones dentro de símbolos de agrupamiento.
Paso 2 Evalúa todas las potencias.
Paso 3 Ejecuta todas las multiplicaciones y divisiones de izquierda a derecha.
Paso 4 Ejecuta todas las adiciones y sustracciones de izquierda a derecha.

outcomes (p. 684) The results of a probability experiment or an event.

resultados Lo que produce un experimento o evento probabilístico.

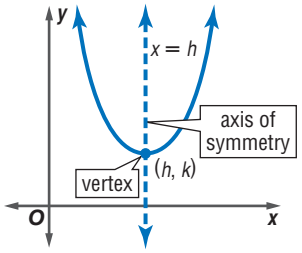
outlier (p. 87) A data point that does not appear to belong to the rest of the set.

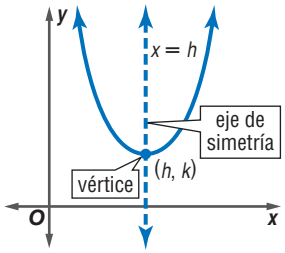
valor atípico Dato que no parece pertenecer al resto el conjunto.

P

parabola (p. 236, 567) The set of all points in a plane that are the same distance from a given point, called the focus, and a given line, called the directrix.

parábola Conjunto de todos los puntos de un plano que están a la misma distancia de un punto dado, llamado foco, y de una recta dada, llamada directriz.





parallel lines (p. 73) Nonvertical coplanar lines with the same slope.

rectas paralelas Rectas coplanares no verticales con la misma pendiente.

parent graph (p. 73) The simplest of graphs in a family.

gráfica madre La gráfica más sencilla en una familia de gráficas.

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partial sum (p. 650) The sum of the first n terms of a series.

suma parcial La suma de los primeros n términos de una serie.

Pascal's triangle (p. 664) A triangular array of numbers such that the $(n + 1)^{th}$ row is the coefficient of the terms of the expansion $(x + y)^n$ for $n = 0, 1, 2 \ldots$

triángulo de Pascal Arreglo triangular de números en el que la fila $(n + 1)^n$ proporciona los coeficientes de los términos de la expansión de $(x + y)^n$ para $n = 0, 1, 2 \ldots$

period (p. 801) The least possible value of a for which $f(x) = f(x + a)$.

período El menor valor positivo posible para a, para el cual $f(x) = f(x + a)$.

periodic function (p. 801) A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function.

función periódica Función para la cual hay un número a tal que $f(x) = f(x + a)$ para todo x en el dominio de la función .

permutation (p. 690) An arrangement of objects in which order is important.

permutación Arreglo de elementos en que el orden es importante.

perpendicular lines (p. 74) In a plane, any two oblique lines the product of whose slopes is 21.

rectas perpendiculares En un plano, dos rectas oblicuas cualesquiera cuyas pendientes tienen un producto igual a 21.

phase shift (p. 829) A horizontal translation of a trigonometric function.

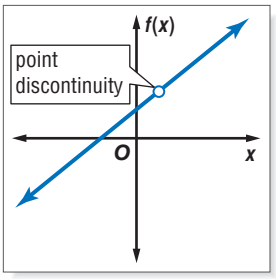
desvío de fase Traslación horizontal de una función trigonométrica.

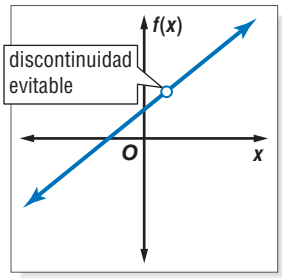
piecewise function (p. 97) A function that is written using two or more expressions.

función a intervalos Función que se escribe usando dos o más expresiones.

point discontinuity (p. 457) If the original function is undefined for $x = a$ but the related rational expression of the function in simplest form is defined for $x = a$, then there is a hole in the graph at $x = a$.

discontinuidad evitable Si la función original no está definida en $x = a$ pero la expresión racional reducida correspondiente de la función está definida en $x = a$, entonces la gráfica tiene una ruptura o corte en $x = a$.





point-slope form (p. 80) An equation in the form $y - y_1 = m(x - x_1)$ where (x_1, y_1) are the coordinates of a point on the line and m is the slope of the line.

forma punto-pendiente Ecuación de la forma $y - y_1 = m(x - x_1)$ donde (x_1, y_1) es un punto en la recta y m es la pendiente de la recta.

polynomial (p. 7) A monomial or a sum of monomials.

polinomio Monomio o suma de monomios.

polynomial function (p. 332) A function that is represented by a polynomial equation.

función polinomial Función representada por una ecuación polinomial.

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polynomial in one variable (p. 331) $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients a_n, a_{n-1}, ..., a_0 represent real numbers, and a_n is not zero and n is a nonnegative integer.

polinomio de una variable $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, donde los coeficientes a_n, a_{n-1}, ..., a_0 son números reales, a_n no es nulo y n es un entero no negativo.

power (p. 7) An expression of the form x^n.

potencia Expresión de la forma x^n.

power function (p. 762) An equation in the form $f(x) = ax^b$, where a and b are real numbers.

función potencia Ecuación de la forma $f(x) = ax^b$, donde a y b son números reales.

prediction equation (p. 86) An equation suggested by the points of a scatter plot that is used to predict other points.

ecuación de predicción Ecuación sugerida por los puntos de una gráfica de dispersión y que se usa para predecir otros puntos.

preimage (p. 185) The graph of an object before a transformation.

preimagen Gráfica de una figura antes de una transformación.

principal root (p. 402) The nonnegative root.

raíz principal La raíz no negativa.

principal values (p. 806) The values in the restricted domains of trigonometric functions.

valores principales Valores en los dominios restringidos de las funciones trigonométricas.

probability (p. 697) A ratio that measures the chances of an event occurring.

probabilidad Razón que mide la posibilidad de que ocurra un evento.

probability distribution (p. 699) A function that maps the sample space to the probabilities of the outcomes in the sample space for a particular random variable.

distribución de probabilidad Función que aplica el espacio muestral a las probabilidades de los resultados en el espacio muestral obtenidos para una variable aleatoria particular.

pure imaginary number (p. 260) The square roots of negative real numbers. For any positive (real number b, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$, or $b\boldsymbol{i}$.

número imaginario puro Raíz cuadrada de un número real negativo. Para cualquier número (real positivo b, $\sqrt{-b^2} = \sqrt{b^2} \cdot \sqrt{-1}$, ó $b\boldsymbol{i}$.

quadrantal angle (p. 778) An angle in standard position whose terminal side coincides with one of the axes.

ángulo de cuadrante Ángulo en posición estándar cuyo lado terminal coincide con uno de los ejes.

quadrants (p. 58) The four areas of a Cartesian coordinate plane.

cuadrantes Las cuatro regiones de un plano de coordenadas Cartesiano.

quadratic equation (p. 246) A quadratic function set equal to a value, in the form $ax^2 + bx + c$, where $a \neq 0$.

ecuación cuadrática Función cuadrática igual a un valor, de la forma $ax^2 + bx + c$, donde $a \neq 0$.

quadratic form (p. 351) For any numbers a, b, and c, except for $a = 0$, an equation that can be written in the form $a[f(x)^2] + b[f(x)] + c = 0$, where $f(x)$ is some expression in x.

forma de ecuación cuadrática Para cualquier número a, b, y c, excepto $a = 0$, una ecuación que puede escribirse de la forma $[f(x)^2] + b[f(x)] + c = 0$, donde $f(x)$ es una expresión en x.

Quadratic Formula (p. 276) The solutions of a quadratic equation of the form $ax^2 + bx + c$, where $a \neq 0$, are given by the Quadratic Formula, which is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

fórmula cuadrática Las soluciones de una ecuación cuadrática de la forma $ax^2 + bx + c$, donde $a \neq 0$, se dan por la fórmula cuadrática, que es $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

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quadratic function (p. 236) A function described by the equation $f(x) = ax^2 + bx + c$, where $a \neq 0$.

quadratic inequality (p. 294) A quadratic equation in the form $y > ax^2 + bx + c$, $y \geq ax^2 + bx + c$, $y < ax^2 + bx + c$, or $y \leq ax^2 + bx + c$.

quadratic term (p. 236) In the equation $f(x) = ax^2 + bx + c$, ax^2 is the quadratic term.

función cuadrática Función descrita por la ecuación $f(x) = ax^2 + bx + c$, donde $a \neq 0$.

desigualdad cuadrática Ecuación cuadrática de la forma $y > ax^2 + bx + c$, $y \geq ax^2 + bx + c$, $y < ax^2 + bx + c$, $y \leq ax^2 + bx + c$.

término cuadrático En la ecuación $f(x) = ax^2 + bx + c$, el término cuadrático es ax^2.

R

radian (p. 770) The measure of an angle θ in standard position whose rays intercept an arc of length 1 unit on the unit circle.

radical equation (p. 422) An equation with radicals that have variables in the radicands.

radical inequality (p. 424) An inequality that has a variable in the radicand.

random (p. 697) All outcomes have an equally likely chance of happening.

random variable (p. 699) The outcome of a random process that has a numerical value.

range (p. 58) The set of all y-coordinates of a relation.

rate of change (p. 71) How much a quantity changes on average, relative to the change in another quantity, often time.

rate of decay (p. 544) The percent decrease r in the equation $y = a(1 - r)^t$.

rate of growth (p. 546) The percent increase r in the equation $y = a(1 + r)^t$.

rational equation (p. 479) Any equation that contains one or more rational expressions.

rational exponent (p. 416) For any nonzero real number b, and any integers m and n, with $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$, except when $b < 0$ and n is even.

rational expression (p. 457) A ratio of two polynomial expressions.

rational function (p. 472) An equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions, and $q(x) \neq 0$.

radián Medida de un ángulo θ en posición normal cuyos rayos intersecan un arco de 1 unidad de longitud en el círculo unitario.

ecuación radical Ecuación con radicales que tienen variables en el radicando.

desigualdad radical Desigualdad que tiene una variable en el radicando.

aleatorio Todos los resultados son equiprobables.

variable aleatoria El resultado de un proceso aleatorio que tiene un valor numérico.

rango Conjunto de todas las coordenadas y de una relación.

tasa de cambio Lo que cambia una cantidad en promedio, respecto al cambio en otra cantidad, por lo general el tiempo.

tasa de desintegración Disminución porcentual r en la ecuación $y = a(1 - r)^t$.

tasa de crecimiento Aumento porcentual r en la ecuación $y = a(1 + r)^t$.

ecuación racional Cualquier ecuación que contiene una o más expresiones racionales.

exponent racional Para cualquier número real no nulo b y cualquier entero m y n, con $n > 1$, $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$, excepto cuando $b < 0$ y n es par.

expresión racional Razón de dos expresiones polinomiales.

función racional Ecuación de la forma $f(x) = \frac{p(x)}{q(x)}$, donde $p(x)$ y $q(x)$ son funciones polinomiales y $q(x) \neq 0$.

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rational inequality (p. 483) Any inequality that contains one or more rational expressions.

desigualdad racional Cualquier desigualdad que contiene una o más expresiones racionales.

rationalizing the denominator (p. 409) To eliminate radicals from a denominator or fractions from a radicand.

racionalizar el denominador La eliminación de radicales de un denominador o de fracciones de un radicando.

rational number (p. 11) Any number $\frac{m}{n}$, where m and n are integers and n is not zero. The decimal form is either a terminating or repeating decimal.

número racional Cualquier número $\frac{m}{n}$, donde m y n son enteros y n no es cero. Su expansión decimal es o terminal o periódica.

real numbers (p. 11) All numbers used in everyday life; the set of all rational and irrational numbers.

números reales Todos los números que se usan en la vida cotidiana; el conjunto de los todos los números racionales e irracionales.

rectangular hyperbola (p. 596) A hyperbola with perpendicular asymptotes.

hipérbola rectangular Hipérbola con asíntotas perpendiculares.

recursive formula (p. 658) Each term is formulated from one or more previous terms.

fórmula recursiva Cada término proviene de uno o más términos anteriores.

reference angle (p. 778) The acute angle formed by the terminal side of an angle in standard position and the x-axis.

ángulo de referencia El ángulo agudo formado por el lado terminal de un ángulo en posición estándar y el eje x.

reflection (p. 188) A transformation in which every point of a figure is mapped to a corresponding image across a line of symmetry.

reflexión Transformación en que cada punto de una figura se aplica a través de una recta de simetría a su imagen correspondiente.

reflection matrix (p. 188) A matrix used to reflect an object over a line or plane.

matriz de reflexión Matriz que se usa para reflejar una figura sobre una recta o plano.

regression line (p. 92) A line of best fit.

reca de regresión Una recta de óptimo ajuste.

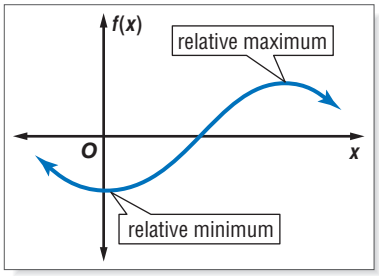
relation (p. 58) A set of ordered pairs.

relación Conjunto de pares ordenados.

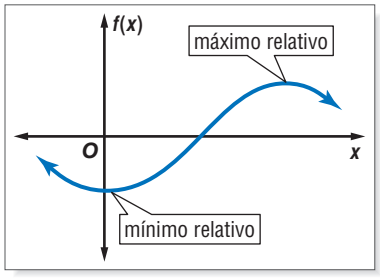
relative frequency histogram (p. 699) A table of probabilities or a graph to help visualize a probability distribution.

histograma de frecuencia relativa Tabla de probabilidades o gráfica para asistir en la visualización de una distribución de probabilidad.

relative maximum (p. 340) A point on the graph of a function where no other nearby points have a greater y-coordinate.



máximo relativo Punto en la gráfica de una función en donde ningún otro punto cercano tiene una coordenada y mayor.



relative minimum (p. 340) A point on the graph of a function where no other nearby points have a lesser y-coordinate.

mínimo relativo Punto en la gráfica de una función en donde ningún otro punto cercano tiene una coordenada y menor.

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root (p. 246) The solutions of a quadratic equation.

raíz Las soluciones de una ecuación cuadrática.

rotation (p. 188) A transformation in which an object is moved around a center point, usually the origin.

rotación Transformación en que una figura se hace girar alrededor de un punto central, generalmente el origen.

rotation matrix (p. 188) A matrix used to rotate an object.

matriz de rotación Matriz que se usa para hacer girar un objeto.

row matrix (p. 163) A matrix that has only one row.

matriz fila Matriz que sólo tiene una fila.

S

sample space (p. 684) The set of all possible outcomes of an experiment.

espacio muestral Conjunto de todos los resultados posibles de un experimento probabilístico.

scalar (p. 171) A constant.

escalar Una constante.

scalar multiplication (p. 171) Multiplying any matrix by a constant called a scalar; the product of a scalar k and an $m \times n$ matrix.

multiplicación por escalares Multiplicación de una matriz por una constante llamada escalar; producto de un escalar k y una matriz de $m \times n$.

scatter plot (p. 86) A set of data graphed as ordered pairs in a coordinate plane.

gráfica de dispersión Conjuntos de datos graficados como pares ordenados en un plano de coordenadas.

scientific notation (p. 315) The expression of a number in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

notación científica Escritura de un número en la forma a x 10^n, donde $1 \leq a < 10$ y n es un entero.

secant (p. 759) For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\sec \alpha = \frac{r}{x}$.

secante Para cualquier ángulo de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, $\sec \alpha = \frac{r}{x}$.

second-order determinant (p. 194) The determinant of a 2×2 matrix.

determinante de segundo orden El determinante de una matriz de 2×2.

sequence (p. 622) A list of numbers in a particular order.

sucesión Lista de números en un orden particular.

series (p. 629) The sum of the terms of a sequence.

serie Suma específica de los términos de una sucesión.

set-builder notation (p. 35) The expression of the solution set of an inequality, for example $\{x|x > 9\}$.

notación de construcción de conjuntos Escritura del conjunto solucion de una desigualdad, por ejemplo, $\{x|x > 9\}$.

sigma notation (p. 631) For any sequence $a_1, a_2, a_3, \ldots$, the sum of the first k terms may be written $\sum_{n=1}^{k} a_n$, which is read "the summation from $n = 1$ to k of a_n." Thus, $\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k$, where k is an integer value.

notación de suma Para cualquier sucesión $a_1, a_2, a_3, \ldots$, la suma de los k primeros términos puede escribirse $\sum_{n=1}^{k} a_n$, lo que se lee "la suma de $n = 1$ a k de los a_n." Así, $\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \ldots + a_k$, donde k es un valor entero.

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simple event (p. 710) One event.

evento simple Un solo evento.

simplify (p. 312) To rewrite an expression without parentheses or negative exponents.

reducir Escribir una expresión sin paréntesis o exponentes negativos.

simulation (p. 734) The use of a probability experiment to mimic a real-life situation.

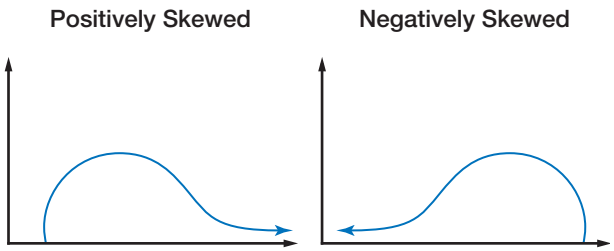
simulación Uso de un experimento probabilístico para imitar una situación de la vida real.

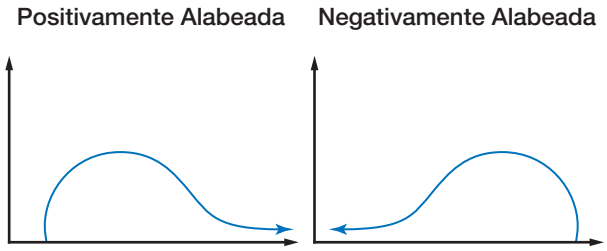
sine (p. 759) For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\sin \alpha = \frac{y}{r}$.

seno Para cualquier ángulo de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, $\sin \alpha = \frac{y}{r}$.

skewed distribution (p. 724) A curve or histogram that is not symmetric.

distribución asimétrica Curva o histograma que no es simétrico.





slope (p. 71) The ratio of the change in y-coordinates to the change in x-coordinates.

pendiente La razón del cambio en coordenadas y al cambio en coordenadas x.

slope-intercept form (p. 79) The equation of a line in the form $y = mx + b$, where m is the slope and b is the y-intercept.

forma pendiente-intersección Ecuación de una recta de la forma $y = mx + b$, donde m es la pendiente y b la intersección.

solution (p. 19) A replacement for the variable in an open sentence that results in a true sentence.

solución Sustitución de la variable de un enunciado abierto que resulta en un enunciado verdadero.

solving a right triangle (p. 762) The process of finding the measures of all of the sides and angles of a right triangle.

resolver un triángulo rectángulo Proceso de hallar las medidas de todos los lados y ángulos de un triángulo rectángulo.

square matrix (p. 163) A matrix with the same number of rows and columns.

matriz cuadrada Matriz con el mismo número de filas y columnas.

square root (p. 259) For any real numbers a and b, if $a^2 = b$, then a is a square root of b.

raíz cuadrada Para cualquier número real a y b, si $a^2 = b$, entonces a es una raíz cuadrada de b.

square root function (p. 397) A function that contains a square root of a variable.

función radical Función que contiene la raíz cuadrada de una variable.

square root inequality (p. 399) An inequality involving square roots.

desigualdad radical Desigualdad que presenta raíces cuadradas.

Square Root Property (p. 260) For any real number n, if $x^2 = n$, then $x = \pm\sqrt{n}$.

Propiedad de la raíz cuadrada Para cualquier número real n, si $x^2 = n$, entonces $x = \pm\sqrt{n}$.

standard deviation (p. 718) The square root of the variance, represented by a.

desviación estándar La raíz cuadrada de la varianza, la que se escribe a.

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standard form (p. 67, 246) **1.** A linear equation written in the form $Ax + By = C$, where A, B, and C are integers whose greatest common factor is 1, $A \geq 0$, and A and B are not both zero. **2.** A quadratic equation written in the form $ax^2 + bx + c = 0$, where a, b, and c are integers, and $a \neq 0$.

forma estándar **1.** Ecuación lineal escrita de la forma $Ax + By = C$, donde A, B, y C son enteros cuyo máximo común divisores 1, $A \geq 0$, y A y B no son cero simultáneamente. **2.** Una ecuación cuadrática escrita en la forma $ax^2 + bx + c = 0$, donde a, b, and c are integers, and $a \neq 0$.

standard notation (p. 315) Typical form for written numbers.

notación estándar Forma típica de escribir números.

standard position (p. 767) An angle positioned so that its vertex is at the origin and its initial side is along the positive x-axis.

posición estándar Ángulo en posición tal que su vértice está en el origen y su lado inicial está a lo largo del eje x positivo.

step function (p. 95) A function whose graph is a series of line segments.

fución etapa Función cuya gráfica es una serie de segmentos de recta.

substitution method (p. 123) A method of solving a system of equations in which one equation is solved for one variable in terms of the other.

método de sustitución Método para resolver un sistema de ecuaciones en que una de las ecuaciones se resuelve en una de las variables en términos de la otra.

success (p. 697) The desired outcome of an event.

éxito El resultado deseado de un evento.

synthetic division (p. 327) A method used to divide a polynomial by a binomial.

división sintética Método que se usa para dividir un polinomio entre un binomio.

synthetic substitution (p. 356) The use of synthetic division to evaluate a function.

sustitución sintética Uso de la división sintética para evaluar una función polinomial.

system of equations (p. 116) A set of equations with the same variables.

sistema de ecuaciones Conjunto de ecuaciones con las mismas variables.

system of inequalities (p. 130) A set of inequalities with the same variables.

sistema de desigualdades Conjunto de desigualdades con las mismas variables.

T

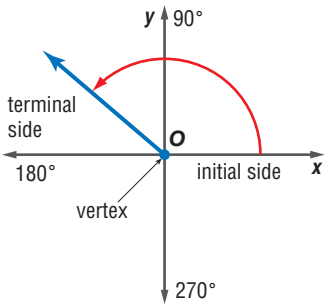
tangent (pp. 427, 759) **1.** A line that intersects a circle at exactly one point. **2.** For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, $\tan \alpha = \frac{y}{x}$.

tangente **1.** Recta que interseca un círculo en un solo punto. **2.** Para cualquier ángulo, de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, $\tan \alpha = \frac{y}{x}$.

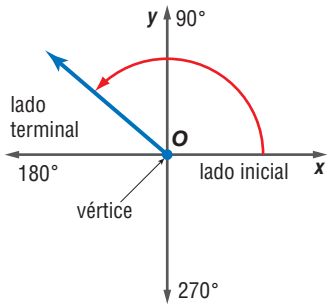
term (p. 7, 622) **1.** The monomials that make up a polynomial. **2.** Each number in a sequence or series.

término **1.** Los monomios que constituyen un polinomio. **2.** Cada número de una sucesión o serie.

terminal side of an angle (p. 767) A ray of an angle that rotates about the center.



lado terminal de un ángulo Rayo de un ángulo que gira alrededor de un centro.



determinante de tercer orden Determinante de

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third-order determinant (p. 195) Determinant of a 3×3 matrix.

transformation (p. 185) Functions that map points of a pre-image onto its image.

translation (p. 185) A figure is moved from one location to another on the coordinate plane without changing its size, shape, or orientation.

translation matrix (p. 185) A matrix that represents a translated figure.

transverse axis (p. 591) The segment of length $2a$ whose endpoints are the vertices of a hyperbola.

trigonometric equation (p. 861) An equation containing at least one trigonometric function that is true for some but not all values of the variable.

trigonometric functions (pp. 759, 775) For any angle, with measure α, a point $P(x, y)$ on its terminal side, $r = \sqrt{x^2 + y^2}$, the trigonometric functions of a are as follows.

$\sin \alpha = \frac{y}{r}$ $\cos \alpha = \frac{x}{r}$ $\tan \alpha = \frac{y}{x}$

$\csc \alpha = \frac{r}{y}$ $\sec \alpha = \frac{r}{x}$ $\cot \alpha = \frac{x}{y}$

trigonometric identity (p. 837) An equation involving a trigonometric function that is true for all values of the variable.

trigonometry (p. 759) The study of the relationships between the angles and sides of a right triangle.

trinomial (p. 7) A polynomial with three unlike terms.

transformación Funciones que aplican puntos de una preimagen en su imagen.

traslación Se mueve una figura de un lugar a otro en un plano de coordenadas sin cambiar su tamaño, forma u orientación.

matriz de traslación Matriz que representa una figura trasladada.

eje transversal El segmento de longitud $2a$ cuyos extremos son los vértices de una hipérbola.

ecuación trigonométrica Ecuación que contiene por lo menos una función trigonométrica y que sólo se cumple para algunos valores de la variable.

funciones trigonométricas Para cualquier ángulo, de medida α, un punto $P(x, y)$ en su lado terminal, $r = \sqrt{x^2 + y^2}$, las funciones trigonométricas de a son las siguientes.

$\text{sen } \alpha = \frac{y}{r}$ $\cos \alpha = \frac{x}{r}$ $\tan \alpha = \frac{y}{x}$

$\csc \alpha = \frac{r}{y}$ $\sec \alpha = \frac{r}{x}$ $\cot \alpha = \frac{x}{y}$

identidad trigonométrica Ecuación que involucra una o más funciones trigonométricas y que se cumple para todos los valores de la variable.

trigonometría Estudio de las relaciones entre los lados y ángulos de un triángulo rectángulo.

trinomio Polinomio con tres términos diferentes.

U

unbiased sample (p. 735) A sample in which every possible sample has an equal chance of being selected.

unbounded (p. 139) A system of inequalities that forms a region that is open.

uniform distribution (p. 699) A distribution where all of the probabilities are the same.

union (p. 42) The graph of a compound inequality containing *or*.

muestra no sesgada Muestra en que cualquier muestra posible tiene la misma posibilidad de seleccionarse.

no acotado Sistema de desigualdades que forma una región abierta.

distribución uniforme Distribución donde todas las probabilidades son equiprobables.

unión Gráfica de una desigualdad compuesta que contiene la palabra o.

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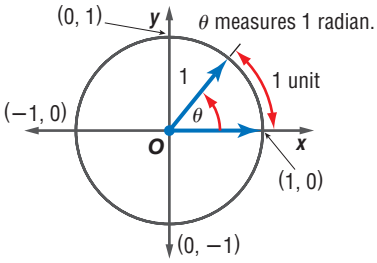
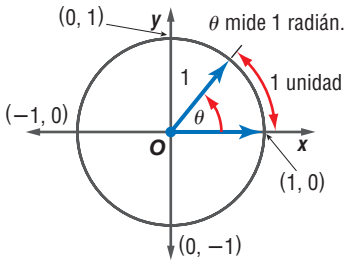
unit circle (p. 768) A circle of radius 1 unit whose center is at the origin of a coordinate system.

círculo unitario Círculo de radio 1 cuyo centro es el origen de un sistema de coordenadas.





univariate date (p. 717) Data with one variable.

datos univariados Datos con una variable.

V

variable (p. 6) Symbols, usually letters, used to represent unknown quantities.

variables Símbolos, por lo general letras, que se usan para representar cantidades desconocidas.

variance (p. 718) The mean of the squares of the deviations from the arithmetic mean.

varianza Media de los cuadrados de las desviaciones de la media aritmética.

vertex (p. 138, 237, 591) **1.** Any of the points of intersection of the graphs of the constraints that determine a feasible region. **2.** The point at which the axis of symmetry intersects a parabola. **3.** The point on each branch nearest the center of a hyperbola.

vértice **1.** Cualqeiera de los puntos de intersección de las gráficas que los contienen y que determinan una región viable. **2.** Punto en el que el eje de simetría interseca una parábola. **3.** El punto en cada rama más cercano al centro de una hipérbola.

vertex form (p. 286) A quadratic function in the form $y = a(x - h)^2 + k$, where (h, k) is the vertex of the parabola and $x = h$ is its axis of symmetry.

forma de vértice Función cuadrática de la forma $y = a(x - h)^2 + k$, donde (h, k) es el vértice de la parábola y $x = h$ es su eje de simetría.

vertex matrix (p. 185) A matrix used to represent the coordinates of the vertices of a polygon.

matriz de vértice Matriz que se usa para escribir las coordenadas de los vértices de un polígono.

vertical asymptote (p. 457) If the related rational expression of a function is written in simplest form and is undefined for $x = a$, then $x = a$ is a vertical asymptote.

asíntota vertical Si la expresión racional que corresponde a una función racional se reduce y está no definida en $x = a$, entonces $x = a$ es una asíntota vertical.

vertical line test (p. 59) If no vertical line intersects a graph in more than one point, then the graph represents a function.

prudba de la recta vertical Si ninguna recta vertical interseca una gráfica en más de un punto, entonces la gráfica representa una función.

X

***x*-intercept** (p. 68) The x-coordinate of the point at which a graph crosses the x-axis.

intersección *x* La coordenada x del punto o puntos en que una gráfica interseca o cruza el eje x.

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Y

y-intercept (p. 68) The y-coordinate of the point at which a graph crosses the y-axis.

intersección y La coordenada y del punto o puntos en que una gráfica interseca o cruza el eje y.

Z

zeros (p. 246) The x-intercepts of the graph of a quadratic equation; the points for which $f(x) = 0$.

ceros Las intersecciones x de la gráfica de una ecuación cuadrática; los puntos x para los que $f(x) = 0$.

zero matrix (p. 163) A matrix in which every element is zero.

matriz nula matriz cuyos elementos son todos igual a cero.

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Selected Answers

Chapter 1 Equations and Inequalities

Page 5 Chapter 1 Get Ready

1. 19.84 **3.** $-\frac{5}{12}$ **5.** $-2\frac{1}{6}$ **7.** 0.48 **9.** $-2\frac{2}{3}$ **11.** $8\frac{4}{5}$ **13.** \$7.31 **15.** 125 **17.** −1 **19.** −1.44 **21.** $\frac{25}{81}$ **23.** 2^5 or 32 **25.** true **27.** true **29.** false **31.** false

Pages 8–10 Lesson 1-1

1. −2.5 **3.** 10.5 **5.** 24 **7.** \$432 **9.** 3.4 **11.** 45 **13.** 5.3 **15.** 40 **17.** −1 **19.** $\frac{1}{4}$ **21.** 31.25 drops per min **23.** $\pi\left(\frac{y+5}{2}\right)^2$ **25.** 75 **27.** −4 **29.** 36.01 **31.** −16 **33.** \$15,954.39 **35.** 98.6 **37.** Sample answer: $4-4+4\div4=1$; $4\div4+4\div4=2$; $(4+4+4)\div4=3$; $4\times(4-4)+4=4$; $(4\times4+4)\div4=5$; $(4+4)\div4+4=6$; $44\div4-4=7$; $(4+4)\times(4\div4)=8$; $4+4+4\div4=9$; $(44-4)\div4=10$ **39.** A table of IV flow rates is limited to those situations listed, while a formula can be used to find any IV flow rate. If a formula used in a nursing setting is applied incorrectly, a patient could die. **41.** H **43.** 4 **45.** 13 **47.** −5 **49.** $\frac{6}{7}$

Pages 15–17 Lesson 1-2

1. Z, Q, R **3.** Q, R **5.** Assoc. (+) **7.** 8, $-\frac{1}{8}$ **9.** −1.5, $\frac{2}{3}$ **11.** \$175.50 **13.** $-17a-1$ **15.** Q, R **17.** I, R **19.** N, W, Z, Q, R **21.** Z, Q, R **23.** Add. Iden. **25.** Comm. (+) **27.** Distributive **29.** −2.5; 0.4 **31.** $\frac{5}{8}$; $-\frac{8}{5}$ **33.** $4\frac{3}{5}$; $-\frac{5}{23}$ **35.** $3\left(2\frac{1}{4}\right)+2\left(1\frac{1}{8}\right)$

$=3\left(2+\frac{1}{4}\right)+2\left(1+\frac{1}{8}\right)$ Def. of a mixed number

$=3(2)+3\left(\frac{1}{4}\right)+2(1)+2\left(\frac{1}{8}\right)$ Distributive Prop.

$=6+\frac{3}{4}+2+\frac{1}{4}$ Multiply.

$=6+2+\frac{3}{4}+\frac{1}{4}$ Commutative Property

$=8+\frac{3}{4}+\frac{1}{4}$ Add.

$=8+\left(\frac{3}{4}+\frac{1}{4}\right)$ Associative Property

$=8+1$ or 9 Add.

37. $10x+2y$ **39.** $11m+10a$ **41.** $32c-46d$ **43.** $4.4p-2.9q$ **45.** 3.6; \$327.60 **47.** $-m$; Add. Inv. **49.** 1 **51.** $\sqrt{2}$ units **53.** W, Z, Q, R **55.** I, R **57.** Sample answer: −2 **59.** true **61.** false; 6 **63.** Yes; $\frac{6+8}{2}=\frac{6}{2}+\frac{8}{2}=7$; dividing by a number is the same as multiplying by its reciprocal. **65.** B **67.** 9 **69.** −5 **71.** 358 in^2 **73.** $\frac{7}{10}$ **75.** 36

Pages 22–26 Lesson 1-3

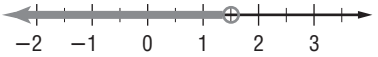
1. $5+4n$ **3.** 9 times a number decreased by 3 is 6. **5.** Reflexive (=) **7.** −21 **9.** −4 **11.** 1.5 **13.** $y=\frac{9+2n}{4}$ **15.** D **17.** $5+3n$ **19.** n^2-4 **21.** $5(9+n)$ **23.** Sample answer: 5 less than a number is 12. **25.** Sample answer: A number squared is equal to 4 times the number. **27.** Substitution (=) **29.** Trans. (=) **31.** 7 **33.** 3.2 **35.** −8 **37.** $\frac{d}{t}=r$ **39.** $\frac{3V}{\pi r^2}=h$ **41.** $\frac{1}{3}$ **43.** s = length of a side; $8s=124$; 15.5 in. **45.** $(n-7)^3$ **47.** $2\pi r(h+r)$ **49.** Sample answer: 7 minus half a number is equal to 3 divided by the square of the number. **51.** $\frac{4x}{1-x}=y$ **53.** −7 **55.** 1 **57.** $\frac{10}{17}$ **59.** n = number of students that can attend each meeting; $2n+3=83$; 40 students **61.** c = cost per student; $50(30-c)+\frac{50}{5}(45)=1800$; \$3 **63.** h = height of can A; $\pi(1.2^2)h=\pi(2^2)3$; $8\frac{1}{3}$ units **65.** Central: 690 mi; Union: 1085 mi **67.** \$295 **69.** Sample answer: $2x-5=-19$ **71.** The Symmetric Property of Equality allows the two sides of an equation to be switched; the order is changed. The Commutative Property of Addition allows the order of terms in an expression on one side of an equation to be changed; the order of terms is changed, but not necessarily on both sides of an equation. **73.** D **75.** $-6x+8y+4z$ **77.** 6.6 **79.** 105 cm^2 **81.** $-\frac{1}{4}$ **83.** $-5+6y$

Pages 29–31 Lesson 1-4

1. 8 **3.** −10.8 **5.** least: 158°F; greatest: 162°F **7.** {−21, 13} **9.** {−11, 29} **11.** ∅ **13.** {8} **15.** 15 **17.** 0 **19.** 3 **21.** −4 **23.** {8, 42} **25.** {−45, 21} **27.** {−2, 16} **29.** $\left\{\frac{3}{2}\right\}$ **31.** ∅ **33.** $|x-200|=5$; maximum: 205°F; minimum: 195°F **35.** $\left\{2, \frac{9}{2}\right\}$ **37.** {−5, 11} **39.** $\left\{-\frac{11}{3}, -3\right\}$ **41.** {8} **43.** 5 **45.** −22 **47.** $|x-13|=5$; maximum: 18 km, minimum: 8 km **49.** Sometimes; it is true only if $a\geq0$ and $b\geq0$ or if $a\leq0$ and $b\leq0$. **51.** Always; since the opposite of 0 is still 0, this equation has only one case, $ax+b=0$. The solution is $-\frac{b}{a}$. **53.** B **55.** $\frac{16}{3}$ **57.** 14 **59.** Distributive **61.** 364 ft^2 **63.** 8 **65.** $\frac{2}{3}$

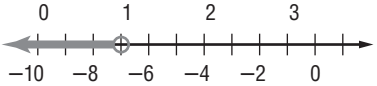
Pages 37–39 Lesson 1-5

1. $\{a|\ a<1.5\}$



3. $\left\{x\middle|\ x\leq\frac{5}{3}\right\}$

5. $\{w|\ w<-7\}$



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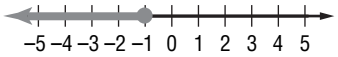
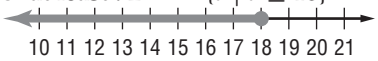
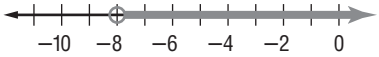
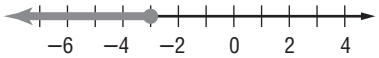
7. $\{n|\ n \le -1\}$ 

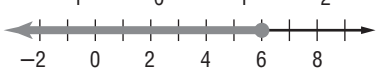
9. at least 92 **11.** $\{b \mid b \le 18\}$ 

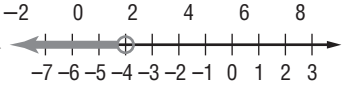
13. $\{d|\ d > -8\}$ 

15. $\{p|\ p \le -3\}$ 

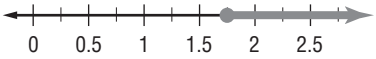
17. $\{y|\ y < 5\}$

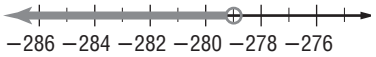
19. $\left\{b \middle|\ b \ge \frac{2}{3}\right\}$

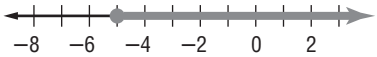
21. $\{r|\ r \le 6\}$ 

23. $\{k|\ k < -4\}$ 

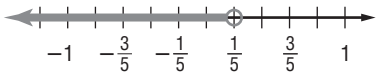
25. at least 25 h **27.** $12n > 36$; $n > 3$ **29.** $n + 8 > 2$; $n > -6$ **31.** $\frac{1}{2}n - 7 \ge 5$; $n \ge 24$

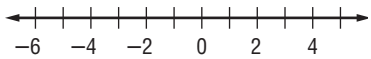
33. $\{n|\ n \ge 1.75\}$ 

35. $\{x|\ x < -279\}$ 

37. $\{d|\ d \ge -5\}$ 

39. $\{g|\ g < 2\}$

41. $\left\{y \middle|\ y < \frac{1}{5}\right\}$ 

43. ∅ 

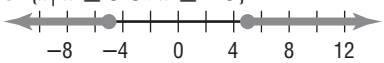
45. $34{,}000 + 0.015(30{,}500n) \ge 50{,}000$ **47.** $2(n + 5) \le 3n + 11$; $n \ge -1$ **49.** $2(7m) \ge 17$; $m \ge \frac{17}{14}$; at least 2 child-care staff members **51.** $s \ge 91$; Flavio must score at least 91 on his next test to have an A test average. **53.** $x \ge -2$ **55.** Sample answer: $x + 2 < x + 1$ **57a.** It holds only for ≤ or ≥; $2 < 2$. **57b.** $1 < 2$ but $2 \not< 1$ **57c.** For all real numbers a, b, and c, if $a < b$ and $b < c$ then $a < c$. **59.** Let n equal the number of minutes used. Write an expression representing the cost of Plan 1 and for Plan 2 for n minutes. The cost for Plan 1 would include a monthly access fee of \$35 plus 40¢ for each minute over 400 minutes or $35 + 0.4(n - 400)$. The cost for Plan 2 for 650 minutes or less would be \$55. To find where Plan 2 would cost less than Plan 1, solve $55 < 35 + 0.4(n - 400)$ for n. The solution set is $\{n \mid n > 450\}$, which means that for more than 450 minutes of calls, Plan 2 is cheaper. **61.** J

63. $\left\{-\frac{5}{4}, \frac{11}{4}\right\}$ **65.** b = billions of dollars spent online each year; $4b + 28.3 = 69.2$; about \$10.2 billion each year **67.** Q, R **69.** $4.25(5.5 + 8)$; $4.25(5.5) + 4.25(8)$ **71.** $\{13, -23\}$ **73.** $\{11, 25\}$ **75.** $\{-18, 10\}$

Pages 45–48 ***Lesson 1-6***

1. $\{d|\ -2 < d < 3\}$

3. $\{y|\ y > 4 \text{ or } y < -1\}$

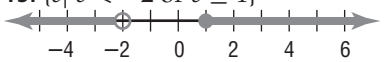
5. $\{a|\ a \ge 5 \text{ or } a \le -5\}$ 

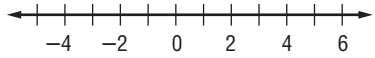
7. $\{h|\ -3 < h < 3\}$

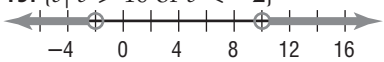
9. $\{k|\ -3 < k < 7\}$

11. $54.45 \le c \le 358.8$ between \$54.45 and \$358.80

13. $\{x|\ -2 < x < 4\}$

15. $\{c|\ c < -2 \text{ or } c \ge 1\}$ 

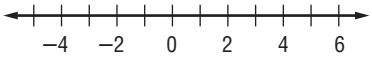
17. ∅ 

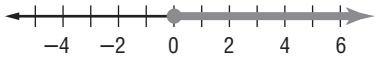
19. $\{b|\ b > 10 \text{ or } b < -2\}$ 

21. $\{r|\ -3 < r < 4\}$

23. $45 \le s \le 55$

25. all real numbers

27. ∅ 

29. $\{n|\ -9\ n- \ge 0\}$ 

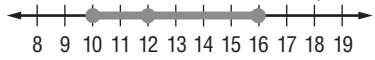
31. $\{n|\ -3 < n < 1\}$

33. $|n| > 1$ **35.** $|n| \ge 1.5$ **37.** $|n + 1| > 1$

39. $|b - 98.6| \ge 8$; $\{b \mid b > 106.6 \text{ or } b < 90.6\}$

41. 84 in. $< L \le 106$ in. **43.** red: $24.35 \le x \le 24.49$; blue: $24.17 \le x \le 24.67$; green: $23.92 \le x \le 24.92$

45. $a - b < c < a + b$ **47.** $2 < x < 3$ **49.** $\text{abs}(2x - 6) > 10$; $\{x \mid x < -2 \text{ or } x > 8\}$ **51.** Sabrina; an absolute value inequality of the form $|a| > b$ should be rewritten as an *or* compound inequality, $a > b$ or $a < -b$.

53. Compound inequalities can be used to describe the acceptable time frame for the fasting state before a glucose tolerance test is administered to a patient suspected of having diabetes. $10 \le h \le 16$; 12 hours would be an acceptable fasting time for this test since it is part of the solution set of $10 \le h \le 16$, as indicated on the graph. 

55. G

57. $\{x|\ x < 4\}$

59. $|x - 587| = 5$; highest: 592 keys, lowest: 582 keys **61.** $\{-11, 4\}$ **63.** Addition (=) **65.** Transitive (=) **67.** 616.69 **69.** $-2m - 7n - 18$ **71.** 92

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Pages 49–52 ***Chapter 1*** ***Study Guide and Review***

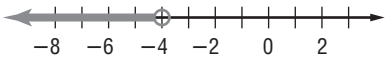
1. empty set **3.** rational numbers **5.** absolute value **7.** coefficient **9.** equation **11.** 22 **13.** 14 **15.** 18 **17.** 7 **19.** 260 mi **21.** Q, R **23.** $-4m + 2n$ **25.** $7x - 16y$ **27.** \$75 **29** −21 **31.** 3 **33.** −4

35. $x = \frac{C - By}{A}$ **37.** $p = \frac{A}{1 + rt}$ **39.** about 1.5 in.

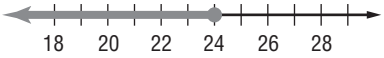
41. {6, −18}

43. {6} **45.** $\left\{-\frac{3}{2}, -1\right\}$

47. $\{w|\ w < -4\}$



49. $\{n|\ n \leq 24\}$

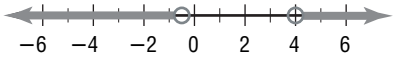


51. $\{z|\ z \geq 6\}$

53. $6(9 + 1.25x) \leq 75$, $x \leq 2.8$; 2 or fewer toppings

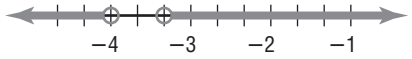
55. $\{a|\ -1 < a < 4\}$

57. $\left\{y \middle|\ y > 4 \text{ or } y < -\frac{1}{3}\right\}$



59. $\{y|\ -9 \leq y \leq 18\}$

61. $\left\{b \middle|\ b > \frac{-10}{3} \text{ or } b < -4\right\}$



Chapter 2 Linear Relations and Functions

Page 57 ***Chapter 2*** ***Get Ready***

1. (−3, 3) **3.** (−3, −1) **5.** (0, − 4)

7.

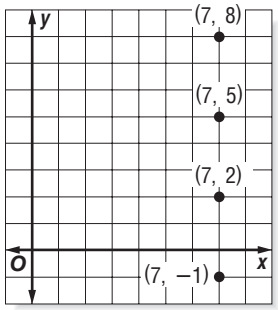
x	y	(x, y)
1	9	(1, 9)
2	18	(2, 18)
3	27	(3, 27)
4	36	(4, 36)

9. −2 **11.** 9 **13.** 2 **15.** $x + 1$ **17.** $2x + 6$
19. $(120x + 165)$ mi

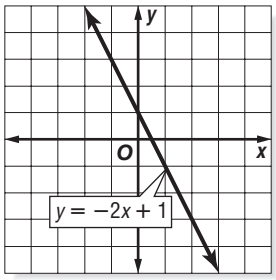
Pages 62–64 ***Lesson 2-1***

1. D = {−6, 2, 3}, R = {1, 5}; yes **3.** D = {−1, 2, 3}, R = {1, 2, 3, 4}; no **5.** {(97, 134), (78, 117), (86, 109), (98, 119)}

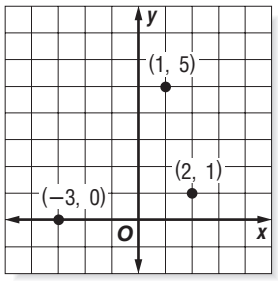
7. D = {7}, R = {−1, 2, 5, 8}; no; discrete



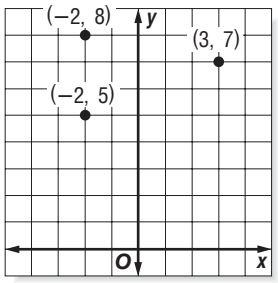
9. D = all reals, R = all reals; yes; continuous



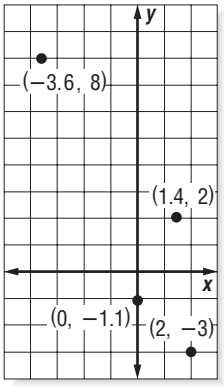
11. 10 **13.** D = {10, 20, 30}, R = {1, 2, 3}; yes
15. D = {0.5, 2}, R = {−3, 0.8, 8}; no
17. D = all reals, R = all reals; no **19.** discrete
21. discrete **23.** D = {−3, 1, 2}, R = {0, 1, 5}; yes; discrete



25. D = {−2, 3}, R = {5, 7, 8}; no; discrete



27. D = {−3.6, 0, 1.4, 2}, R = {−3, −1.1, 2, 8}; yes; discrete



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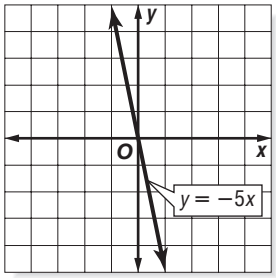
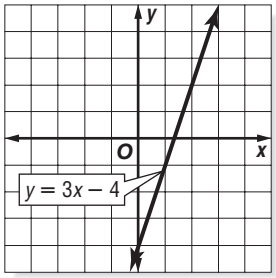
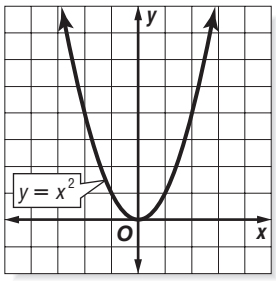
29. D = all reals, R = all reals; yes; continuous



31. D = all reals, R = all reals; yes; continuous

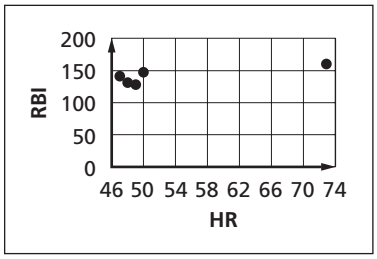


33. D = all reals, R = $\{y \mid y \geq 0\}$; yes; continuous



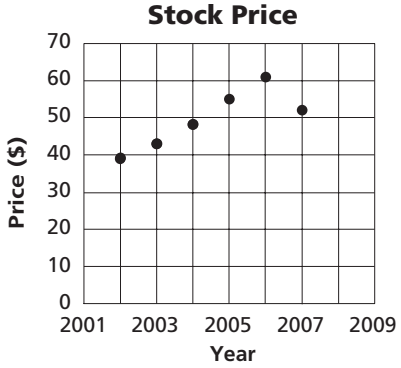
35. -14 **37.** $-\frac{2}{9}$ **39.** $3a - 5$ **41.** -4

43.



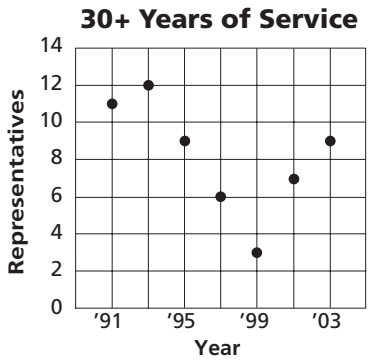
45. Yes; each domain value is paired with only one range value so the relation is a function.

47.



49. Yes; each domain value is paired with only one range value.

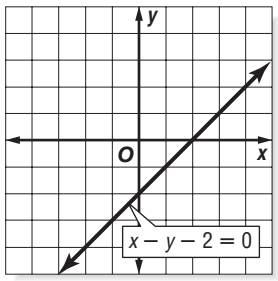
51.



53. Yes; each domain value is paired with only one range value so the relation is a function. **55.** Sample answer: {(−4, 3), (−2, 3), (1, 5), (−2, 1)}. For $x = -2$, there are two different y-values. **57.** Sample answer: $f(x) = 4x - 3$ **59.** C **61.** $\{y \mid -8 < y < 6\}$ **63.** $\{x \mid x < 5.1\}$ **65.** 362 **67.** -1 **69.** 6

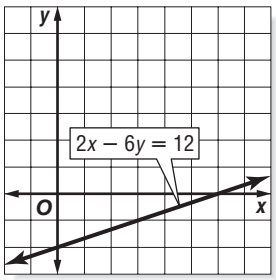
Pages 68–70 ***Lesson 2-2***

1. No; the variables have an exponent other than 1. **3.** \$177.62 **5.** $3x - y = 5$; 3, −1, 5 **7.** $2x - 3y = -3$; 2, −3, −3 **9.** 2, −2

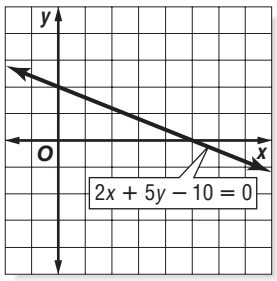


11. yes **13.** No; x has exponents other than 1. **15.** No; x appears in a denominator. **17.** No; x is inside a square root. **19.** Sound travels only 1715 m in 5 seconds in air, so it travels faster in water. **21.** 35,000 ft **23.** $12x - y = 0$; 12, −1, 0 **25.** $x - 7y = 2$; 1, −7, 2 **27.** $x - 2y = -3$; 1, −2, −3

29. 6, −2



31. 5, 2



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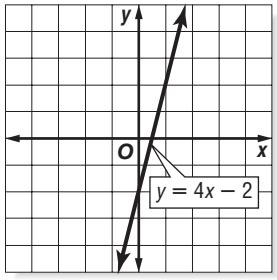
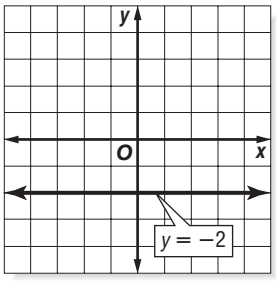
33. $\frac{1}{2}$, -2

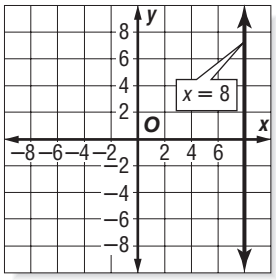


35. $x + y = 12$; 1, 1, 12 **37.** $x = 6$; 1, 0, 6
39. $25x + 2y = 9$; 25, 2, 9

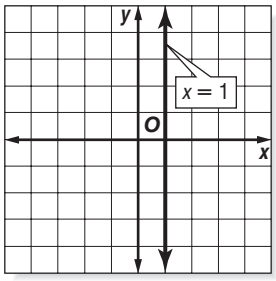
41. none, -2



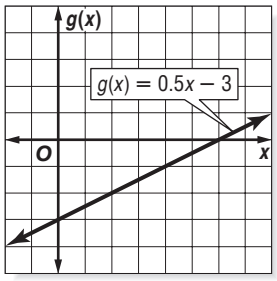
43. 8, none



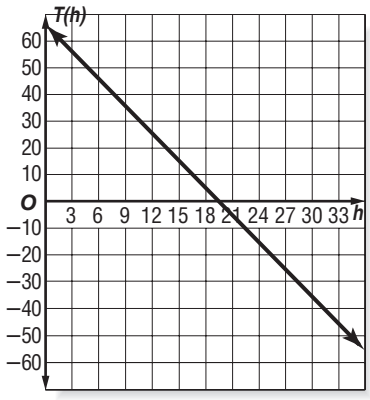
45. 1, none



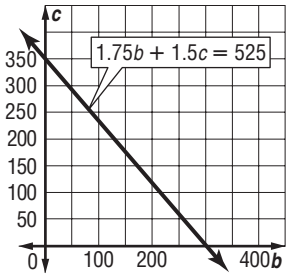
47. 6, -3



49.



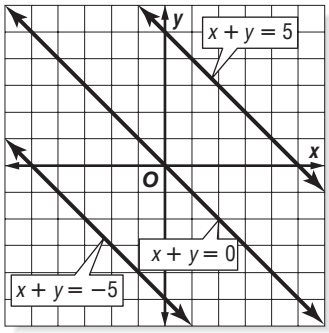
51.



Yes; the graph passes the vertical line test.

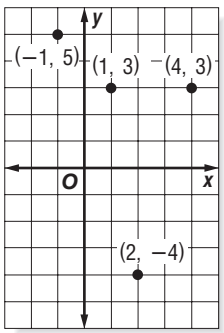
53. Sample answer: $x + y = 2$

55.



The lines are parallel but have different y-intercepts.

57. Regardless of whether 0 is substituted in for x or y, the value of the other variable is also 0. So the only intercept is (0, 0). **59.** B **61.** D = {−1, 1, 2, 4}, R = {−4, 3, 5}; yes



63. $\{x \mid -1 < x < 2\}$ **65.** \$7.95 **67.** 2 **69.** −0.8

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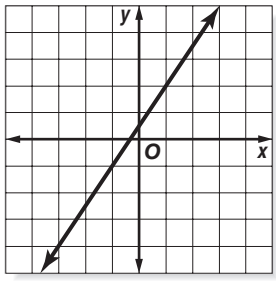
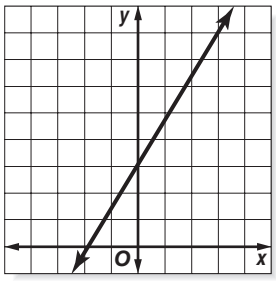
Pages 74–77 **Lesson 2-3**

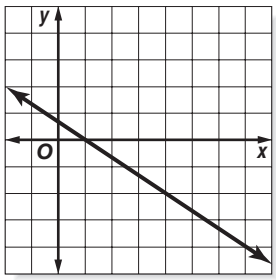
1. $-\frac{1}{2}$ **3.** 1 **5.**



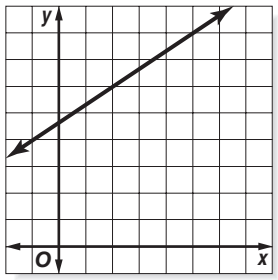
7. 1.25°/h **9.**



11.



13. $-\frac{5}{2}$ **15.** 13 **17.** 0 **19.**



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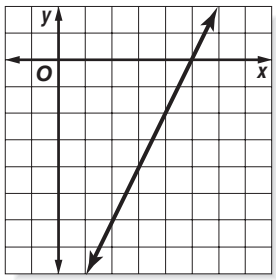
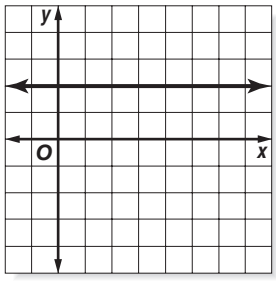
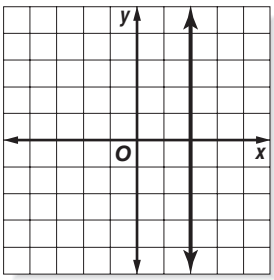


23.



25. about 11 million per year **27.** 55 mph **29.** speed or velocity

31.



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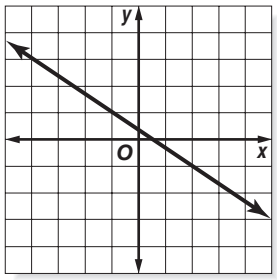

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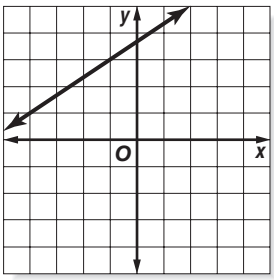
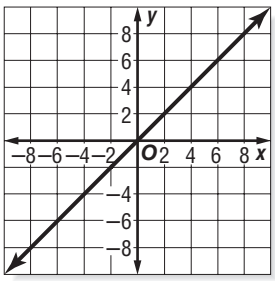
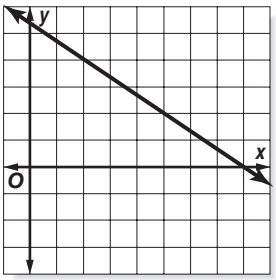


37.



39. $-\frac{5}{4}$ **41.** 0 **43.** 9

45.



47. Yes; slopes show that adjacent sides are perpendicular. **49.** The graphs have the same y-intercept. As the slopes become more negative, the lines get steeper. **51.** −1 **53.** Sometimes; the slope of a vertical line is undefined. **55.** D

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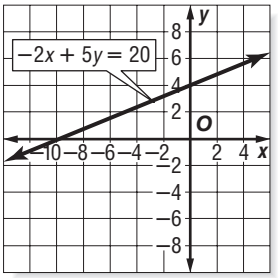
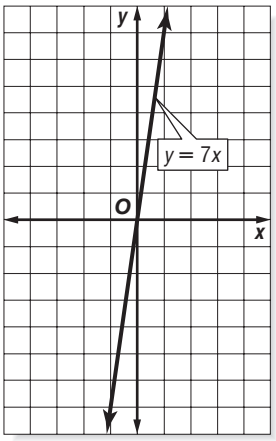
57. $-10, 4$



59. 0, 0



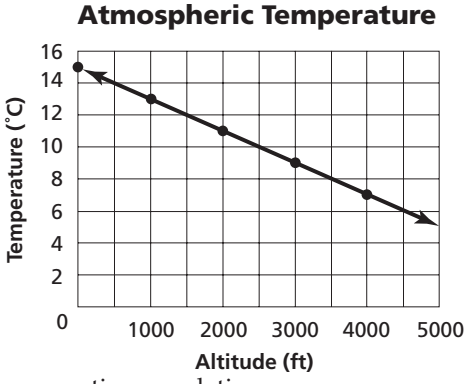
61. 5 **63.** $3a - 4$ **65.** $\{z|\ z \geq 735\}$ **67.** $17a - b$
69. $y = 9 - x$ **71.** $y = -3x + 7$ **73.** $y = \frac{3}{5}x + \frac{4}{5}$

Pages 82–84 *Lesson 2-4*

1. $y = 0.5x + 1$ **3.** $y = 3x - 6$ **5.** $y = -\frac{5}{2}x + 16$
7. $y = 0.8x$ **9.** B **11.** $y = -\frac{4}{3}x + \frac{8}{3}$ **13.** $y = 3x - 6$
15. $y = -\frac{1}{2}x + \frac{7}{2}$ **17.** $y = -\frac{4}{5}x + \frac{17}{5}$ **19.** $y = \frac{2}{3}x + \frac{10}{3}$
21. $y = -4$ **23.** $y = 75x + 6000$ **25.** $d = 180c - 360$
27. 540° **29.** 68°F **31.** $y = -0.5x - 2$ **33.** $y = x + 4$
35. $y = -\frac{1}{15}x - \frac{23}{5}$ **37.** Sample answer: $y = 3x + 2$
39. $y = 2x + 4$ **41.** A **43.** -2 **45.** 0 **47.** ∅
49. $\{r|\ r \geq 6\}$ **51.** 6.5 **53.** 5.85

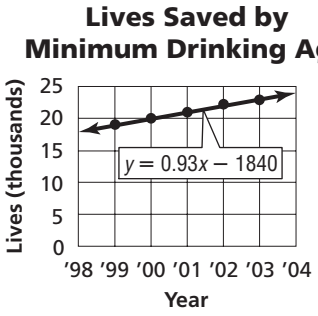
Pages 88–91 *Lesson 2-5*

1a.

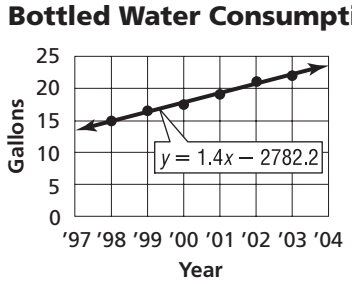


strong negative correlation

1b. Sample answer using (2000, 11.0) and (3000, 9.1): $y = -0.0019x + 14.8$ **1c.** Sample answer: 5.3°C
3a. strong positive correlation



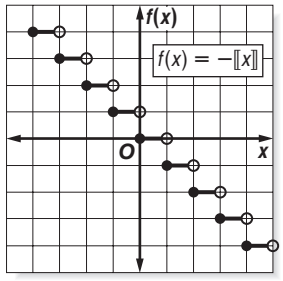
3b. Sample answer using (1999, 19.1) and (2003, 22.8): $y = 0.93x - 1830$ **3c.** Sample answer: 33,900
5a. strong positive correlation



5b. Sample answer using (1998, 15) and (2003, 22): $y = 1.4x - 2782.2$ **5c.** Sample answer: 38.8 gal
7. Sample answer using (2000, 1309.9) and (2003, 1678.9): $y = 123x - 244{,}690.1$ **9.** The value predicted by the equation is significantly lower than the one given in the graph. **11.** No. Past performance is no guarantee of the future performance of a stock. Other factors that should be considered include the companies' earnings data and how much debt they have. **13.** Sample answer using (483.8, −166) and (3647.2, −375): $y = -0.07x - 134.04$ **15.** Sample answer: The predicted value differs from the actual value by only 2°F, less than 1%. **17.** Sample answer using (1980, 66.5) and (1995, 81.7): 102% **19.** Sample answer using (4, 152.5) and (8, 187.6): $y = 8.78x + 117.4$ **21.** D **23.** $y = 4x + 6$ **25.** $y = 0.35x + 1.25$
27. $-\frac{1}{2}$ **29.** undefined **31.** 3 **33.** 0 **35.** 1.5

Pages 99–101 *Lesson 2-6*

1. D = all reals, R = all integers



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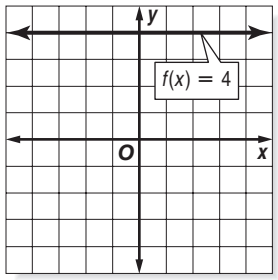
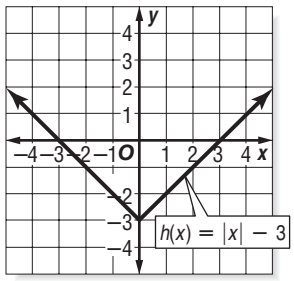
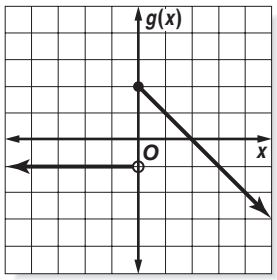
3. D = all reals, R = {4}



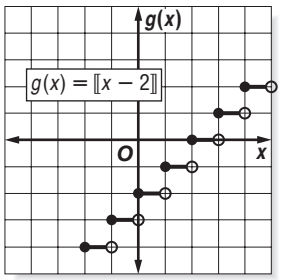
5. D = all reals, R = $\{y|y \geq -3\}$



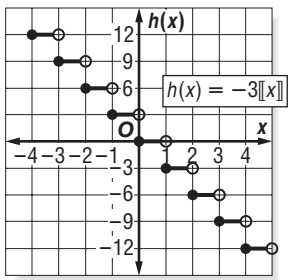
7. D = all reals, R = $\{y|y \leq 2\}$



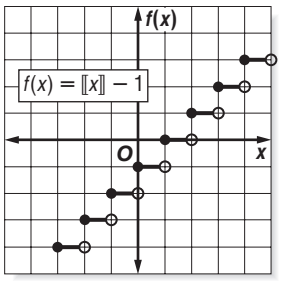
9. A **11.** step function **13.** \$6 **15.** D = all reals, R = all integers



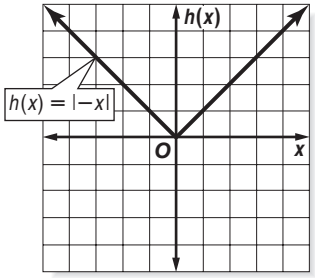
17. D = all reals, R = $\{3a|\ a \text{ is an integer}\}$



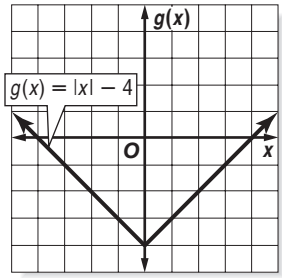
19. D = all reals, R = all integers



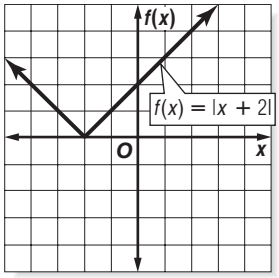
21. D = all reals, R = all nonnegative reals



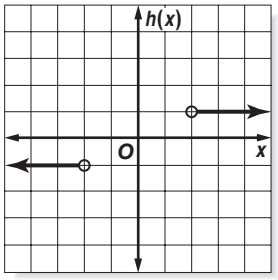
23. D = all reals, R = $\{y|y \geq -4\}$



25. D = all reals, R = all nonnegative reals



27. D = $\{x|x < -2 \text{ or } x > 2\}$, R = {−1, 1}



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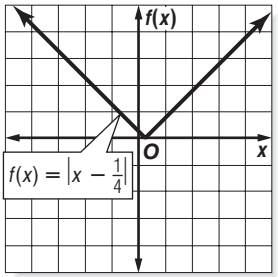
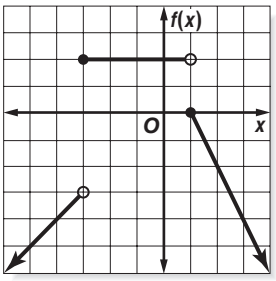
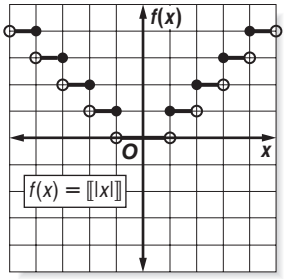
29. A **31.** S **33.** P **35.** D = all reals, R = all nonnegative reals



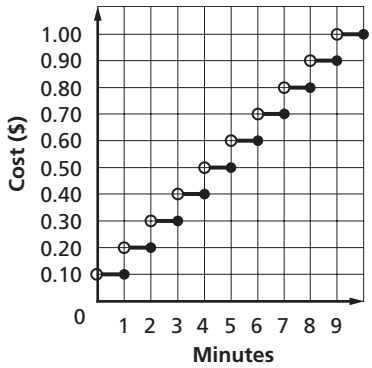
37. D = all reals, R = $\{y|y \leq 0 \text{ or } y = 2\}$



39. D = all reals, R = all whole numbers

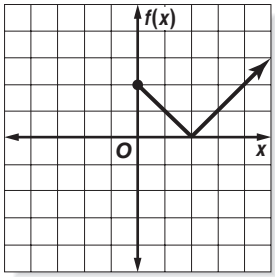


41.



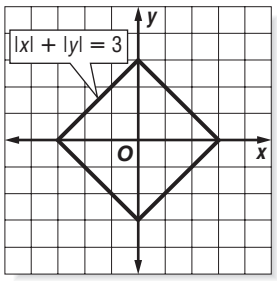
43. $f(x) = |x - 2|$

45.



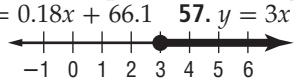
47. Sample answer: $f(x) = |x - 1|$

49.



51. B **53.** B **55.** Sample answer using (10, 69.7) and (47, 76.5): $y = 0.18x + 66.1$ **57.** $y = 3x + 10$

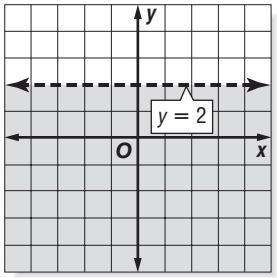
59. $\{x|x \geq 3\}$



61. yes **63.** no **65.** no

Pages 104–105 Lesson 2-7

1.



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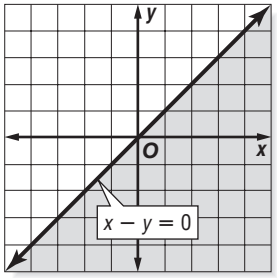

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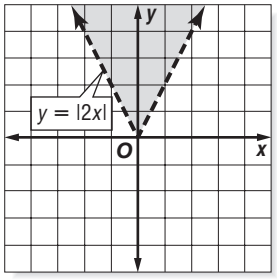
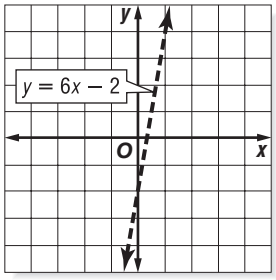


7. $10c + 13d \leq 40$ **9.** No; (2, 3) is not in the shaded region.

11.



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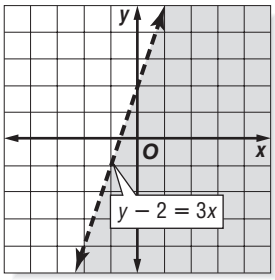

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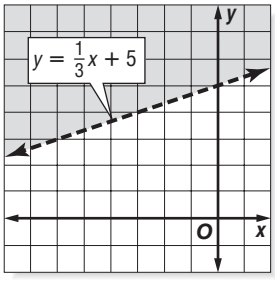

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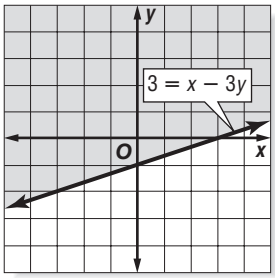

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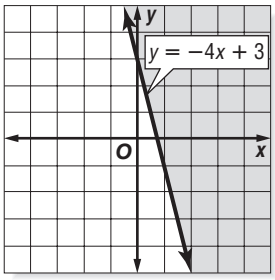

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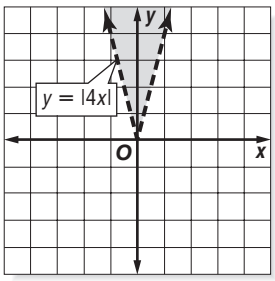
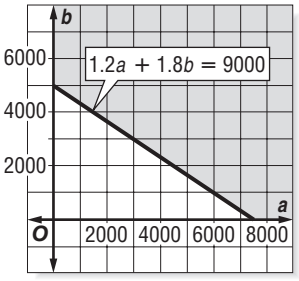
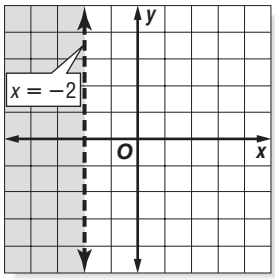


23. yes

25.



27. $x < -2$



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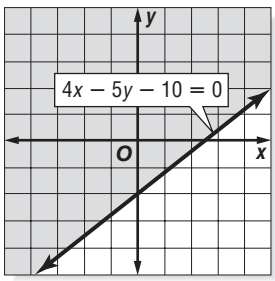

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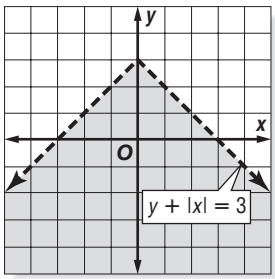

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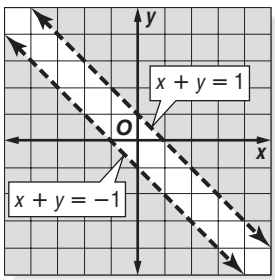

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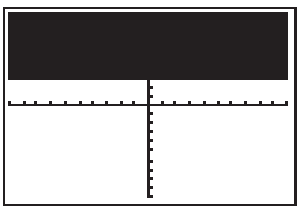

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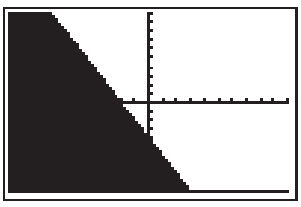
37.



[−10, 10] scl: 1 by [−10, 10] scl: 1

39. Substitute the coordinates of a point not on the boundary into the inequality. If the inequality is satisfied, shade the region containing the point. If the inequality is not satisfied, shade the region that does not contain the point.

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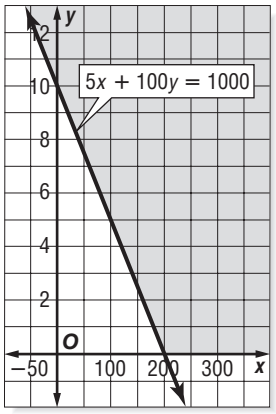
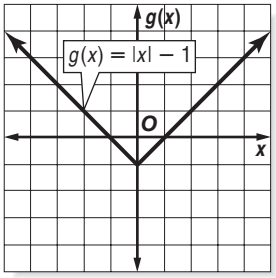
41. Linear inequalities can be used to track the performance of players in fantasy football leagues. Let x be the number of passing yards and let y be the number of touchdowns. The number of points Dana gets from passing yards is $5x$ and the number of points he gets from touchdowns is $100y$. His total number of points is $5x + 100y$. He wants at least 1000 points, so the inequality $5x + 100y \geq 1000$ represents the situation.

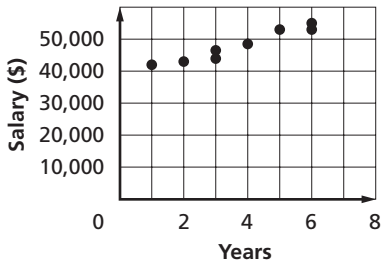


43. J **45.**



D = all reals, R = $\{y \mid y \geq -1\}$

47.



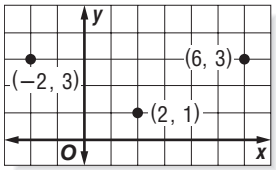
There is a strong positive correlation between salary and experience.

49. Sample answer: $64,000 **51.** 3

Pages 106–110 ***Chapter 2*** ***Study Guide and Review***

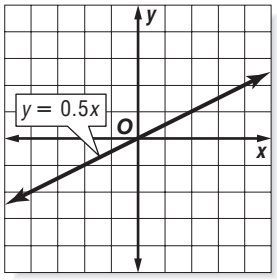
1. identity **3.** slope-intercept **5.** vertical line test

7.



D = {−2, 2, 6}, R = {1, 3}; function; discrete

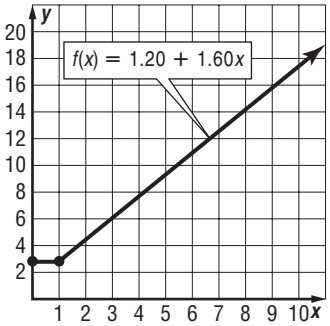
9.



D = all reals, R = all reals; function; continuous

11. 21 **13.** $5y - 9$

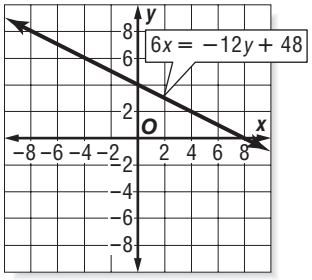
15.



D = all reals, $\{x \mid x \geq 0\}$
R = $\{y \mid y \geq 2.80\}$
function; continuous

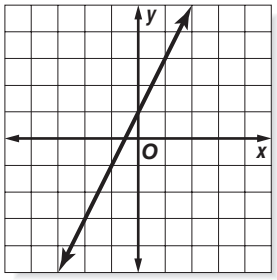
17. No; this function is not linear because the x is under a square root. **19.** $5x + 2y = -4$; 5, 2, −4

21. 8, 4



23. $\frac{5}{6}$

25.



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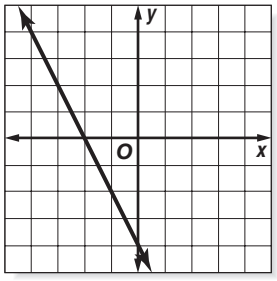

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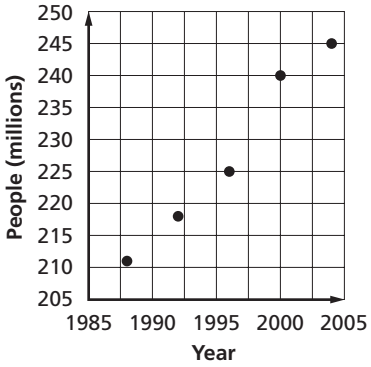
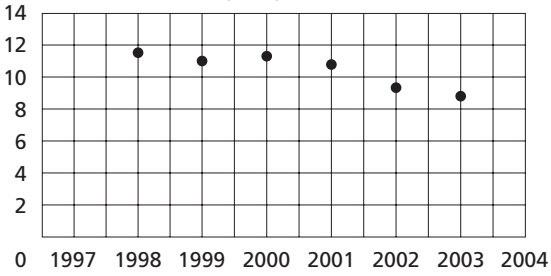
29. $\frac{3}{5}$ **31.** $y = \frac{1}{3}x + \frac{7}{3}$ **33.** $y = -\frac{3}{4}x + \frac{17}{4}$

35.



There is a strong positive correlation.

37. There is a strong negative correlation.



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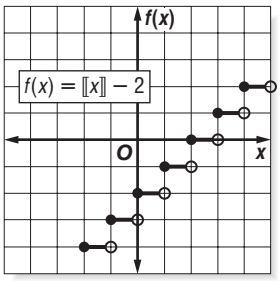
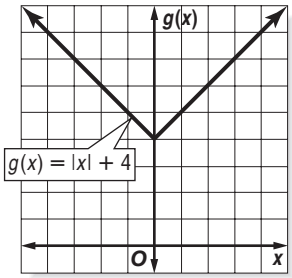


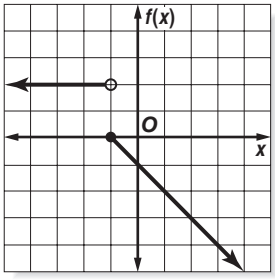
D = all reals,
R = all integers

41.



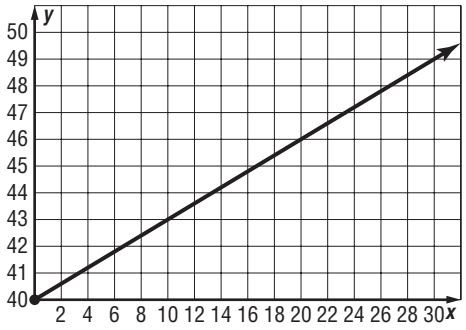
D = all reals,
R = {y| $y \geq 4$}

43.



D = all reals,
R = {y| $y \leq 0$ or $y = 2$}

45.



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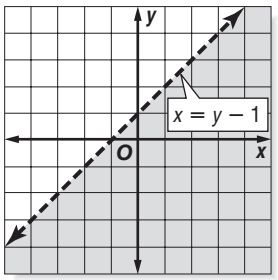

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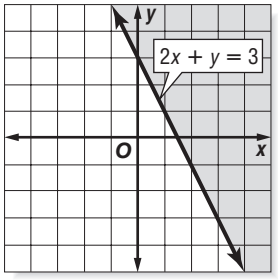

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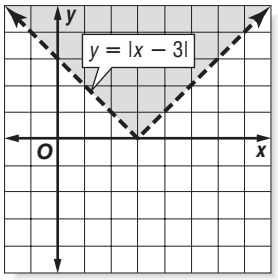

Chapter 3 Systems of Equations and Inequalities

Page 115 ***Chapter 3*** ***Get Ready***

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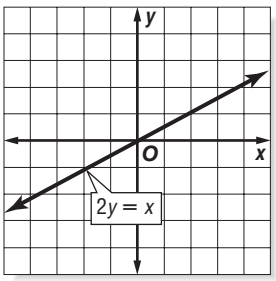

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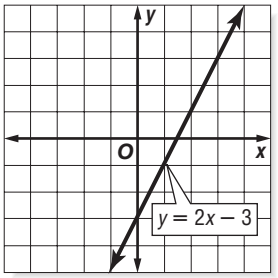

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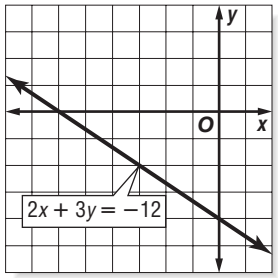
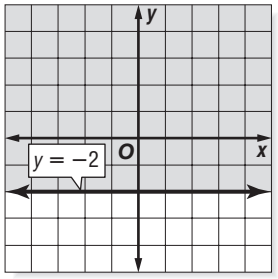


7. $1.75b + 1.5c = 525$ **9.** Yes; the graph passes the vertical line test.

11.



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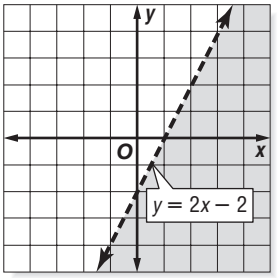

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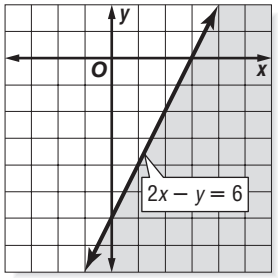

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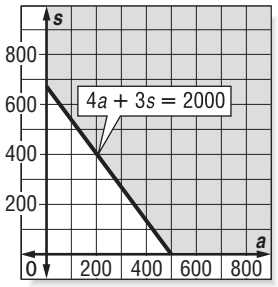
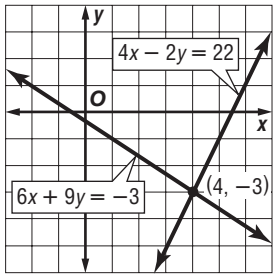


Pages 120–122 ***Lesson 3-1***

1. (−2, 5)

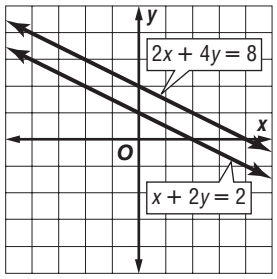
x	y_1	y_2
0	9	3
−1	7	4
−2	5	5
−3	3	6

3. (4, −3)



5. $y = 0.15x + 2.70$, $y = 0.25x$ **7.** You should use Ez Online photos if you are printing more than 27 digital photos and the local pharmacy if you are printing fewer than 27 digital photos.

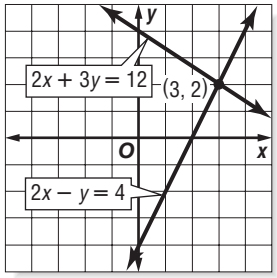
9. inconsistent



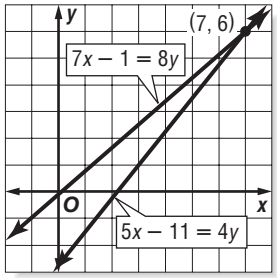
11. (0, −8)

x	y_1	y_2
−1	−11	−9
0	−8	−8
1	−5	−7
2	−2	−6

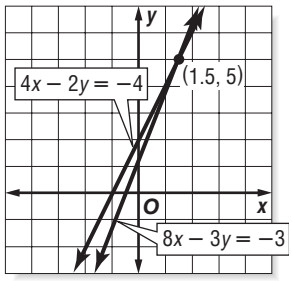
13. (3, 2)



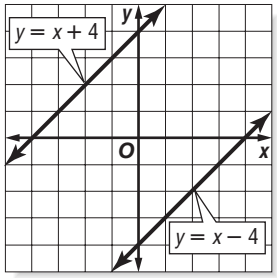
(7, 6)



17. (1.5, 5)



19. inconsistent



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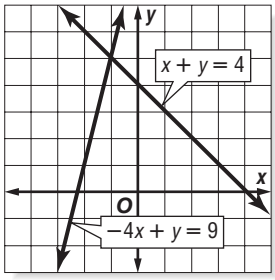
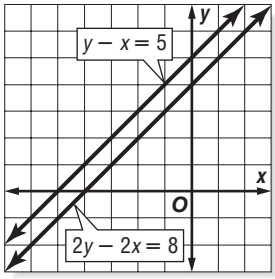
21. 

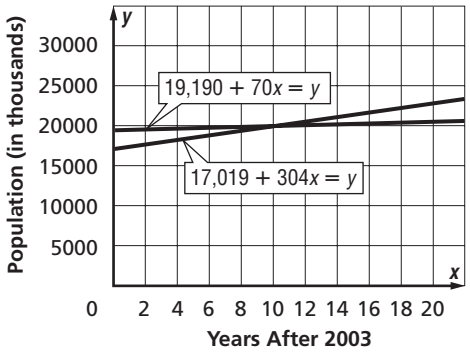
consistent and independent

23. 

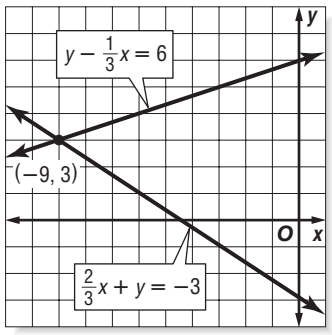
inconsistent

25. (−3, 1)

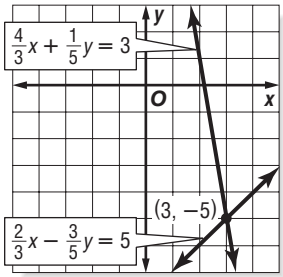
27. Supply, 200,000; demand, 300,000; prices will tend to rise. **29.** 250,000; $10

31. 

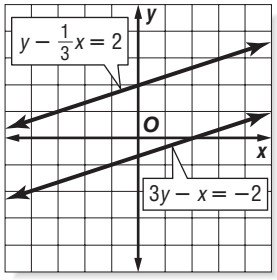
2012

33. 

(−9, 3)

35. 

(3, −5)

37. 

inconsistent

39. (3.40, −2.58) **41.** (4, 3.42) **43.** Two lines cannot intersect in exactly two points. **45.** You can use a system of equations to track sales and make predictions about future growth based on past performance and trends in the graphs. The coordinates (6, 54.2) represent that 6 years after 1999, both the in-store sales and online sales will be $54,200. It would not be very reasonable. The unpredictability of the market, companies, and consumers makes models such as this one accurate for only a short period of time. **47.** H **49.** A **51.** P **53.** $9y + 1$ **55.** $12x + 18y - 6$ **57.** $x + 4y$

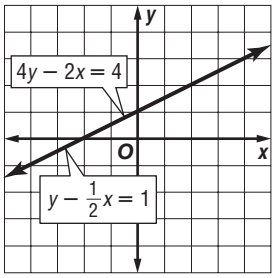
Pages 127–129 ***Lesson 3-2***

1. (4, 8) **3.** (9, 7) **5.** C **7.** (6, −20) **9.** (3, 2) **11.** infinitely many solutions **13.** (2, 7) **15.** (−6, 8) **17.** (1, 1) **19.** (2, −7) **21.** (3, −1) **23.** no solution **25.** 18 members rented skis and 10 members rented snowboards. **27.** 18 printers, 12 monitors **29.** (6, 5)

31. (7, −1) **33.** (−5, 8) **35.** $\left(\frac{1}{3}, 2\right)$ **37.** (2, 4)

39. (12, −3) **41.** 10 true/false, 20 multiple-choice **43.** $a + s = 40$, $11a + 4s = 335$ **45.** one equation should have a variable with a coefficient of 1. **47.** Jamal; Juanita subtracted the two equations incorrectly; $-y - y = -2y$, not 0. **49.** You can use a system of equations to find the monthly fee and rate per minute charged during the months of January and February. The coordinates of the point of intersection are (0.08, 3.5). Currently, Yolanda is paying a monthly fee of $3.50 and an additional 8¢ per minute. **51.** J

53. consistent and dependent



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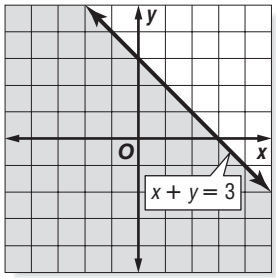
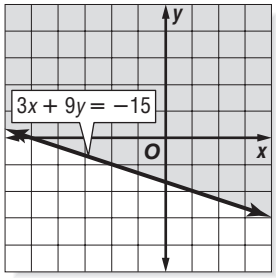
55.



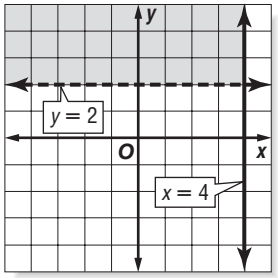
57.



59. $x - y = 0$; 1, −1, 0 **61.** $2x - y = -3$; 2,−1,−3
63. $3x + 2y = 21$; 3, 2, 21 **65.** yes **67.** no

Pages 132–135 ***Lesson 3-3***

1.



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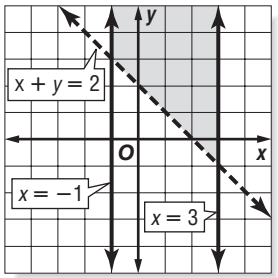

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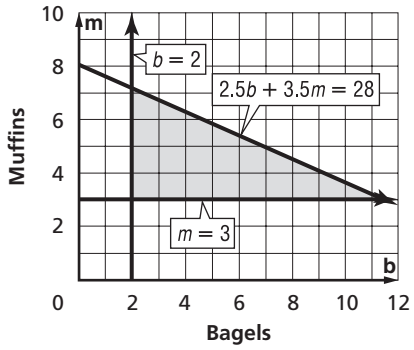
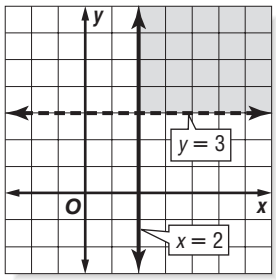


7. (−3, −3), (2, 2), (5, −3)

9.



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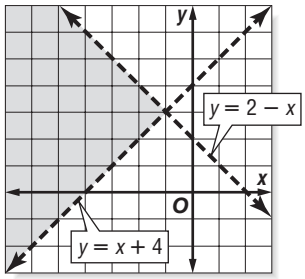

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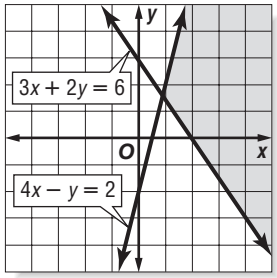
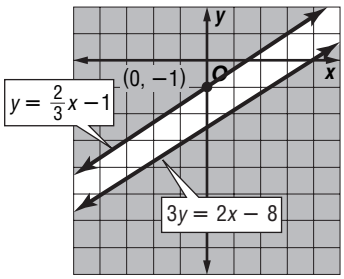
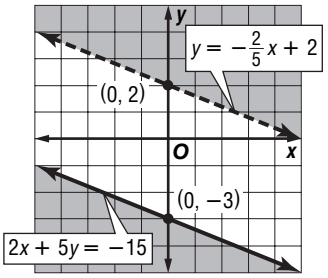


15. no solution



17. no solution



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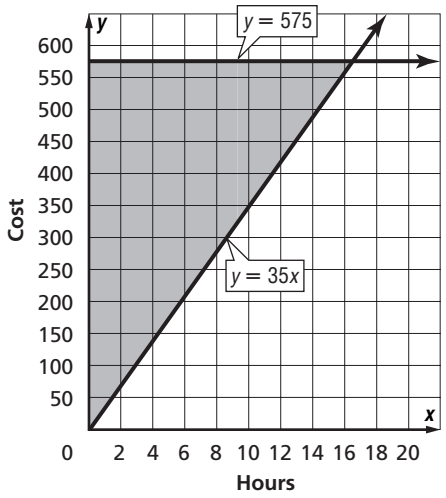
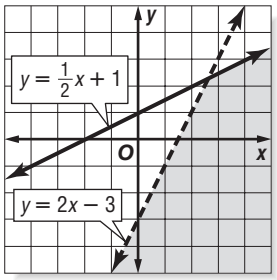
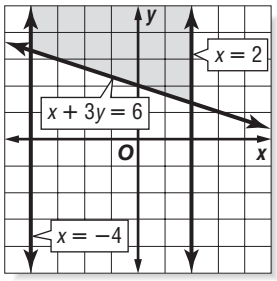


21. $(-3, -4)$, $(5, -4)$, $(1, 4)$ **23.** $(-6, -9)$, $(2, 7)$, $(10, -1)$ **25.** 64 units^2 **27.** category 4; 13–18 ft
29. Sample answer: 2 pumpkin, 8 soda; 4 pumpkin, 6 soda; 8 pumpkin, 4 soda

31.



33.



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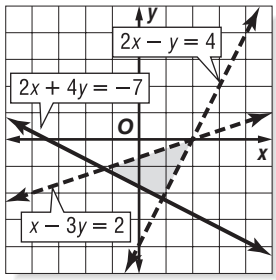
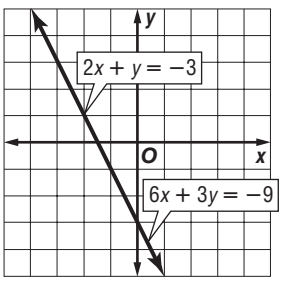


37. Sample answer: $y > x + 3$, $y < x - 2$ **39.** 42 units^2
41. B **43.** $(-3, 8)$ **45.** $(8, -5)$

47.

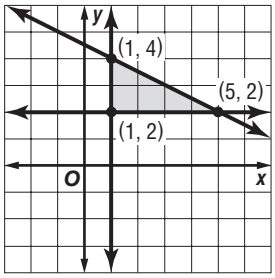


infinitely many

49. $75x + 200$ **51.** -12 **53.** -8.25

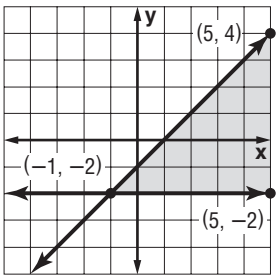
Pages 141–144 ***Lesson 3-4***

1.



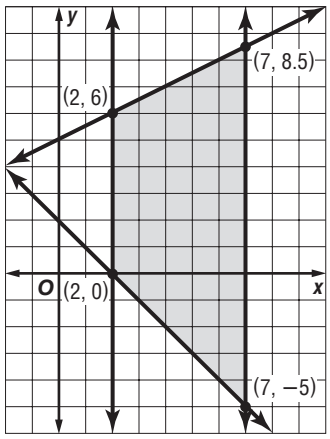
vertices: (1, 2), (1, 4), (5, 2); max: $f(5, 2) = 4$; min: $f(1, 4) = -10$

3.



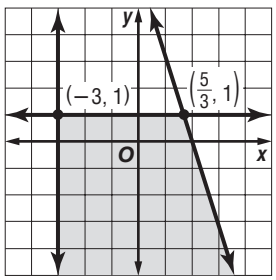
vertices: $(-1, -2)$, $(5, -2)$, $(5, 4)$; max: $f(5, -2) = 9$; min: $f(5, 4) = -3$

5.



vertices: (2, 0), (2, 6), (7, 8.5), (7, −5); max: $f(7, 8.5) = 81.5$; min: $f(2, 0) = 16$

7.



vertices: $(-3, 1)$, $\left(\frac{5}{3}, 1\right)$; no maximum; min: $f(-3, 1) = -17$

9. $c \geq 0$, $\ell \geq 0$, $c + 3\ell \leq 56$, $4c + 2\ell \leq 104$ **11.** (0, 0), (26, 0), (20, 12), $\left(0, 18\frac{2}{3}\right)$ **13.** Make 20 canvas tote bags and 12 leather tote bags.

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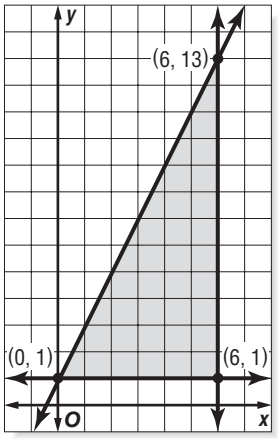
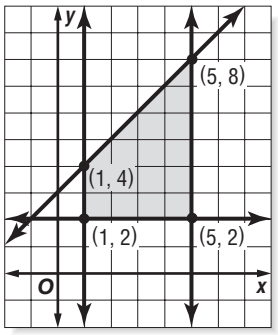
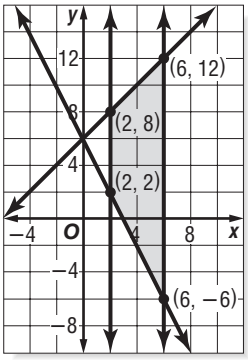
15. 

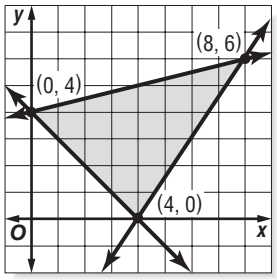
vertices: (0, 1), (6, 1), (6, 13); max $f(6, 13) = 19$; min; $f(0, 1) = 1$

17. 

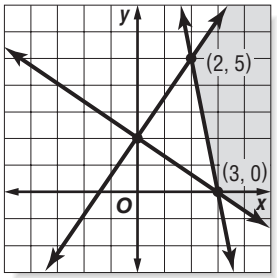
vertices: (1, 4), (5, 8), (5, 2), (1, 2); max $f(5, 2) = 11$; min; $f(1, 4) = -5$

19. 

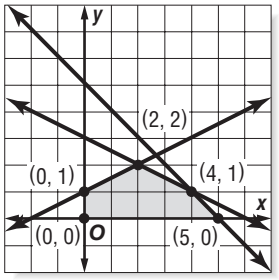
vertices: (2, 2), (2, 8), (6, 12), (6, −6); max $f(6, 12) = 30$; min; $f(6, -6) = -24$

21. 

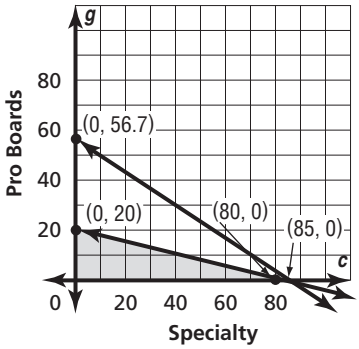
vertices: (0, 4), (4, 0), (8, 6); max $f(4, 0) = 4$; min; $f(0, 4) = -8$

23. 

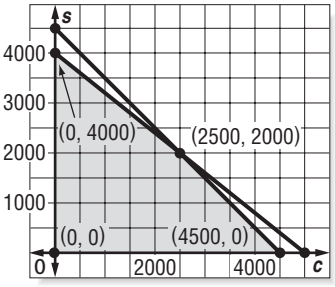
vertices: (2, 5), (3, 0); no maximum; no minimum

25. 

vertices: (0, 0), (0, 1), (2, 2), (4, 1), (5, 0); max $f(5, 0) = 19$; min; $f(0, 1) = -5$

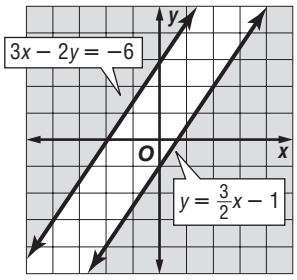
29. 

31. $f(c, g) = 65c + 50g$ **33.** $5200

35. 

(0, 0), (0, 4000), (2500, 2000), (4500, 0)

37. 4500 acres corn, 0 acres soybeans; $130,500
39. Sample answer: $y \geq -x$, $y \geq x - 5$, $y \leq 0$
41. (−2,6); the other coordinates are solutions of the system of inequalities. **43.** There are many variables in scheduling tasks. Linear programming can help make sure that all the requirements are met. Let x = the number of buoy replacements and let y = the number of buoy repairs. Then, $x \geq 0$, $y \geq 0$, $x \leq 8$ and $2.5x + y \leq 24$. The captain would want to maximize the number of buoys that a crew could repair and replace so $f(x, y) = x + y$. **45.** J

47. 

no solution

49. (2, 3) **51.** $y = 2x + 50$ **53.** 5 **55.** −3 **57.** −4

Pages 149–152 ***Lesson 3-5***

1. (6, 3, −4) **3.** infinitely many **5.** no solution
7. $6c + 3s + r = 42$, $c + s + r = 13\frac{1}{2}$, $r = 2s$

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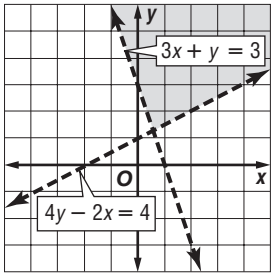
9. (3, 4, 7) **11.** (2, −3, 6) **13.** no solution
15. (1, 2, −1) **17.** infinitely many **19.** 8, 1, 3
21. 52 3-point goals, 168 2-point goals, 142 1-point free throws **23.** \$7.80 **25.** $\left(\frac{1}{3}, -\frac{1}{2}, \frac{1}{4}\right)$ **27.** (−5, 9, 4)

29. You can use elimination or substitution to eliminate one of the variables. Then you can solve two equations in two variables. **31.** $a = \frac{4}{3}$, $b = \frac{1}{3}$, $c = 3$; $y = \frac{4}{3}x^2 + \frac{1}{3}x + 3$ **33.** A **35.** 179 gallons of skim and 21 of whole milk

37.

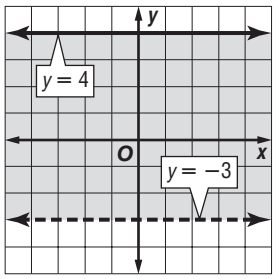


39. Sample answer using (0, 8) and (35, 39): $y = 0.89x + 8$
41. 3830 feet; the y-intercept
43. 5 **45.** 30

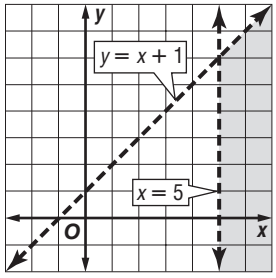
Pages 153–156 ***Chapter 3*** ***Study Guide and Review***

1. constraints **3.** feasible region **5.** consistent system
7. elimination method **9.** system of equations
11. (4, 0) **13.** (−8, −8) **15.** 1 hr **17.** (−3, −5)
19. (6.25, −2.25) **21.** (−1, 2)

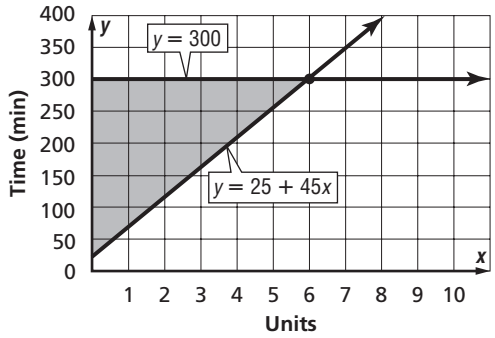
23.



25.



27.



29. (1, 2, 3) **31.** (3, −1, 5)

Chapter 4 Matrices

Page 161 ***Chapter 4*** ***Get Ready***

1. −3; $\frac{1}{3}$ **3.** −8; $\frac{1}{8}$ **5.** −1.25; 0.8 **7.** $\frac{8}{3}$; $-\frac{3}{8}$ **9.** 4
11. (3, 4) **13.** (9, 2)

Pages 165–167 ***Lesson 4-1***

1.

	Fri	Sat	Sun	Mon	Tue
High	88	88	90	86	85
Low	54	54	56	53	52

3. 1 × 5 **5.** (5, 6)

7.
$$\begin{bmatrix} 60{,}060{,}700 & 13{,}215 \\ 29{,}637{,}900 & 12{,}880 \\ 26{,}469{,}500 & 13{,}002 \\ 7{,}848{,}300 & 16{,}400 \\ 5{,}427{,}000 & 3953 \end{bmatrix}$$

9. 2 × 3 **11.** 4 × 3 **13.** 2 × 5 **15.** $\left(3, -\frac{1}{3}\right)$

17. (5, 3) **19.** (4, −3) **21.**
$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 3 & 3 \\ 2 & 4 & 4 & 3 \end{bmatrix}$$

23.

	Evening	Matinee	Twilight
Adult	7.50	5.50	3.75
Child	4.50	4.50	3.75
Senior	5.50	5.50	3.75

25.

	Weekday	Weekend
Single	60	79
Double	70	89
Suite	75	95

29.
$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 & 21 \\ 2 & 5 & 9 & 14 & 20 & 27 \\ 4 & 8 & 13 & 19 & 26 & 34 \\ 7 & 12 & 18 & 25 & 33 & 42 \\ 11 & 17 & 24 & 32 & 41 & 51 \\ 16 & 23 & 31 & 40 & 50 & 61 \\ 22 & 30 & 39 & 49 & 60 & 72 \end{bmatrix}$$

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31. Matrices are used to organize information so it can be read and compared more easily. For example, Sabrina can see that the hybrid SUV has the best price and fuel economy; the standard SUV has the most horsepower, exterior length, and cargo space; the mid-size SUV has a lower price than the standard but high horsepower and cargo space; and the compact SUV has the a low price and good fuel economy. **33.** J **35.** (7, 3, −9) **37.** 0 dresses, 120 skirts **39.** $2x - 3y = -3$ **41.** 2 **43.** 20 **45.** −18 **47.** 75

Pages 173–176 Lesson 4-2

1. impossible **3.** $\begin{bmatrix} 1 & 10 \\ -7 & 5 \end{bmatrix}$

5. Males $\begin{bmatrix} 17{,}389 & 544{,}811 \\ 15{,}221 & 504{,}801 \\ 14{,}984 & 457{,}146 \\ 10{,}219 & 349{,}785 \\ 5758 & 96{,}562 \end{bmatrix}$ Females $\begin{bmatrix} 17{,}061 & 457{,}986 \\ 15{,}089 & 418{,}322 \\ 14{,}181 & 362{,}468 \\ 9490 & 309{,}032 \\ 6176 & 144{,}565 \end{bmatrix}$

7. No; many schools offer the same sport for males and females, so those schools would be counted twice.

9. $\begin{bmatrix} -10 & 20 \\ 30 & -15 \\ 45 & 5 \end{bmatrix}$ **11.** $\begin{bmatrix} -21 & 29 \\ 12 & -22 \end{bmatrix}$ **13.** impossible

15. $\begin{bmatrix} -13 & -1 \\ 2 & 3 \end{bmatrix}$ **17.** $\begin{bmatrix} -7 & 7 & -2 \\ -6 & 5 & 8 \\ -5 & -16 & 8 \end{bmatrix}$ **19.** $\begin{bmatrix} 15 & 0 & 4 \\ 0 & 13 & -5 \end{bmatrix}$

21. $\begin{bmatrix} -8 & -1 & -13 \\ 15 & 13 & -2 \\ -9 & -7 & 19 \end{bmatrix}$ **23.** $\begin{bmatrix} 631 & 377 & 292 & 533 \\ 552 & 431 & 163 & 317 \\ 363 & 367 & 199 & 258 \end{bmatrix}$

25. $\begin{bmatrix} -4 & 8 & -2 \\ 6 & -10 & -16 \\ -14 & -12 & 4 \end{bmatrix}$ **27.** $[15 \quad -29 \quad 65 \quad -2]$

29. $\begin{bmatrix} 13 & 10 \\ 4 & 7 \\ 7 & -5 \end{bmatrix}$ **31.** $\begin{bmatrix} 0 & 16 \\ -8 & 20 \\ 28 & -4 \end{bmatrix}$ **33.** $\begin{bmatrix} -12 & -13 \\ 3 & -8 \\ 13 & 37 \end{bmatrix}$

35. $\begin{bmatrix} 1.8 & 9.08 \\ 3.18 & 31.04 \\ 10.41 & 56.56 \end{bmatrix}$ **37.** $\begin{bmatrix} -4 & -15 \\ \frac{3}{2} & -2 \end{bmatrix}$ **39.** $\begin{bmatrix} 0.5 \\ 0.47 \\ 0.87 \\ 0 \end{bmatrix}$

41. 50m and 100m **43.** $\begin{bmatrix} 1.00 & 1.00 \\ 1.50 & 1.50 \end{bmatrix}$

45. Sample answer: [−3, 1], [3, −1] **47.** You can use matrices to track dietary requirements and add them to find the total each day or each week.

Breakfast	Lunch	Dinner
$\begin{bmatrix} 566 & 18 & 7 \\ 482 & 12 & 17 \\ 530 & 10 & 11 \end{bmatrix}$	$\begin{bmatrix} 785 & 22 & 19 \\ 622 & 23 & 20 \\ 710 & 26 & 12 \end{bmatrix}$	$\begin{bmatrix} 1257 & 40 & 26 \\ 987 & 32 & 45 \\ 1380 & 29 & 38 \end{bmatrix}$

add the three matrices: $\begin{bmatrix} 2608 & 80 & 52 \\ 2091 & 67 & 82 \\ 2620 & 65 & 61 \end{bmatrix}$.

49. J **51.** 1 × 4 **53.** 3 × 3 **55.** 4 × 3 **57.** (5, 3, 7) **59.** (2, 5) **61.** (6, −1) **63.** No, it would cost $6.30. **65.** Assoc. (+) **67.** Comm. (×)

Pages 182–184 Lesson 4-3

1. 3 × 2 **3.** 3 × 22 **5.** $\begin{bmatrix} 0 & 44 \\ 8 & -34 \end{bmatrix}$

7. $\begin{bmatrix} 15 & -5 & 20 \\ 24 & -8 & 32 \end{bmatrix}$ **9.** $\begin{bmatrix} 24 \\ 41 \end{bmatrix}$ **11.** $74,525

13. yes; $A(BC) = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \left(\begin{bmatrix} -4 & 1 \\ 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \right)$

$= \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -13 & -6 \\ 24 & 16 \end{bmatrix} = \begin{bmatrix} -50 & -28 \\ 81 & 62 \end{bmatrix}$

$(AB)C = \left(\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 8 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -16 & 2 \\ 28 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -50 & -28 \\ 81 & 62 \end{bmatrix}$

15. 2 × 2 **17.** 1 × 5 **19.** 3 × 5 **21.** $\begin{bmatrix} 12 & -42 \\ -6 & 21 \end{bmatrix}$

23. $\begin{bmatrix} -6 & 3 \\ 44 & -19 \end{bmatrix}$ **25.** $\begin{bmatrix} -39 \\ 18 \end{bmatrix}$ **27.** $\begin{bmatrix} 12 & 4 \\ -24 & -8 \end{bmatrix}$ **29.** $\begin{bmatrix} 14{,}285 \\ 13{,}270 \\ 4295 \end{bmatrix}$

31. $c(AB) = 3\left(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \right)$

$= 3 \begin{bmatrix} -13 & -4 \\ -8 & 17 \end{bmatrix} = \begin{bmatrix} -39 & -12 \\ -24 & 51 \end{bmatrix}$

$A(cB) = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \left(3 \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \right)$

$= \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -16 & 6 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} -39 & -12 \\ -24 & 51 \end{bmatrix}$

The equation is true.

33. $AC + BC = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 9 \\ 26 & -8 \end{bmatrix} + \begin{bmatrix} -21 & -13 \\ 26 & -8 \end{bmatrix}$

$= \begin{bmatrix} -20 & -4 \\ 52 & -16 \end{bmatrix}$

$(A + B)C = \left(\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} -5 & 2 \\ 4 & 3 \end{bmatrix} \right) \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} -4 & 0 \\ 8 & 8 \end{bmatrix} \cdot \begin{bmatrix} 5 & 1 \\ 2 & -4 \end{bmatrix}$

$= \begin{bmatrix} -20 & -4 \\ 52 & -16 \end{bmatrix}$ The equation is true.

35. $\begin{bmatrix} 72 & 49 \\ 68 & 63 \\ 90 & 56 \\ 86 & 62 \end{bmatrix} \cdot \begin{bmatrix} 1.00 \\ 0.50 \end{bmatrix} = \begin{bmatrix} 96.50 \\ 99.50 \\ 118 \\ 117 \end{bmatrix}$; juniors

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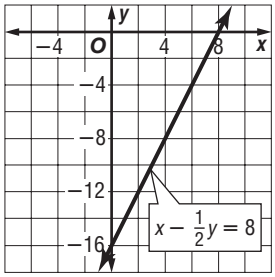
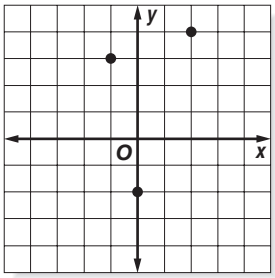
37. $24,900 **39.** $1460 **41.** Never; the inner dimensions will never be equal. **43.** $a = 1, b = 0, c = 0, d = 1$; the original matrix **45.** C **47.** $\begin{bmatrix} 12 & -6 \\ -3 & 21 \end{bmatrix}$ **49.** $\begin{bmatrix} -20 & 2 \\ -28 & 12 \end{bmatrix}$

51. $x = 5, y = -9$ **53.** $2.50; $1.50

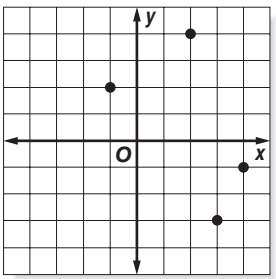
55. 8, −16



57.

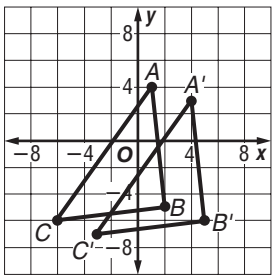


59.



Pages 189–192 ***Lesson 4-4***

1. $\begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}$ **3.**

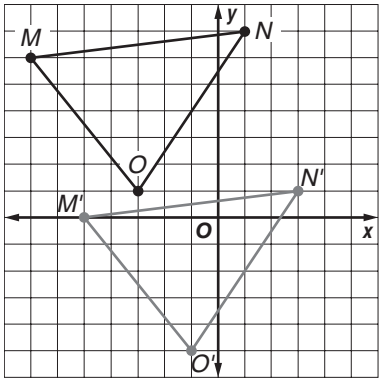


5. $\begin{bmatrix} 0 & 5 & 5 & 0 \\ 4 & 4 & 0 & 0 \end{bmatrix}$ **7.** $A'(0, 2), B'\left(\frac{5}{2}, 2\right), C'\left(\frac{5}{2}, 0\right), D'(0, 0)$

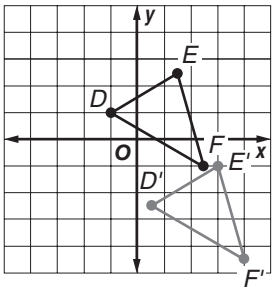
9. $A'(0, 4), B'(-5, 4), C'(-5, 0), D'(0, 0)$

11. $A'(4, 0), B'(4, -5), C'(0, -5), D'(0, 0)$

13. $\begin{bmatrix} 2 & 2 & 2 \\ -6 & -6 & -6 \end{bmatrix}$; $M'(-5, 0), N'(3, 1), O'(-1, -5)$;

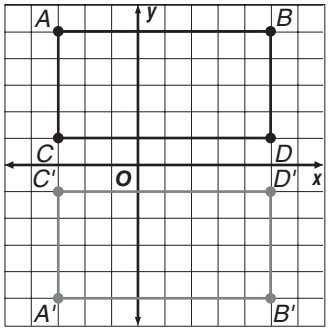


15. $\begin{bmatrix} 3 & 3 & 3 \\ -7 & -7 & -7 \end{bmatrix}$, $E'(6, -2), F'(8, -9)$



17. $\begin{bmatrix} -6 & 4 & 2 \\ 2 & 8 & -6 \end{bmatrix}$, $X'(-3, 1), Y'(2. 4), Z'(1, -3)$;

19. $\begin{bmatrix} -3 & 5 & 5 & -3 \\ 5 & 5 & -1 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $A'(-3, -5), B'(5, -5)$, $D'(5,1), C'(-3,1)$;



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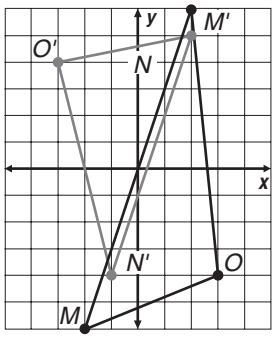
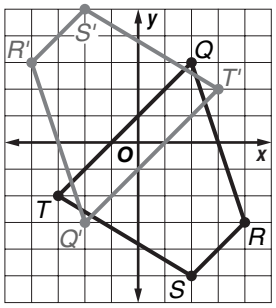
21. $\begin{bmatrix} -2 & 1 & 3 \\ -6 & 4 & -4 \end{bmatrix}$; $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; $M'(2, 6)$, $N'(-1, -4)$, $O'(-3, 4)$;



23.



25. $J(-5, 3)$, $K(7, 2)$, $L(4, -1)$ **27.** $P(2, 2)$, $Q(-4, 1)$, $R(1, -5)$, $S(3, -4)$ **29.** $\begin{bmatrix} 4 & -4 & -4 & 4 \\ -4 & -4 & 4 & 4 \end{bmatrix}$

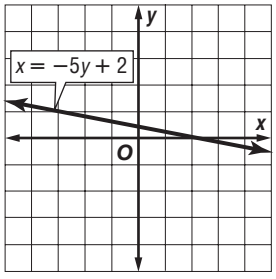
31. The figures in Exercise 28 and Exercise 29 have the same coordinates, but the figure in Exercise 30 has different coordinates. **33.** (6.5, 6.25) **35.** (−3.75, −2.625) **37.** The object is reflected over the x-axis, then translated 6 units to the right. **39.** No; since the translation does not change the y-coordinate, it does not matter whether or not you do the translation or the reflection first. However, if the translation did change the y-coordinate, the order would be important.

41. $\begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -2 \end{bmatrix}$ **43.** Sample answer: $\begin{bmatrix} -4 & -4 & -4 \\ 1 & 1 & 1 \end{bmatrix}$

45. Sometimes; the image of a dilation is congruent to its preimage if and only if the scale factor is 1 or −1.

47. A. 3 **49.** 2 × 2 **51.** 2 × 5 **53.** $\begin{bmatrix} 20 & 10 & -24 \\ 31 & -46 & -9 \\ -10 & 3 & 7 \end{bmatrix}$

55. D = {all real numbers}; R = {all real numbers}; yes



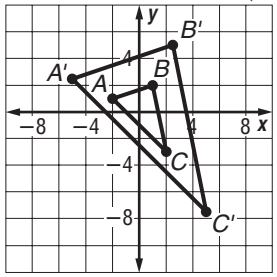
57. $|x| \geq 4$ **59.** $513\frac{2}{3}$ mi **61.** 5 **63.** $\frac{9}{4}$ **65.** $\frac{5}{3}$

Pages 198–200 Lesson 4-5

1. −38 **3.** −28 **5.** 0 **7.** 26 units^2 **9.** 20 **11.** −22 **13.** −14 **15.** 32 **17.** −58 **19.** 62 **21.** 172 **23.** −22 **25.** −5 **27.** 20 ft^2 **29.** 14.5 units^2 **31.** $\frac{5}{3}$, −1 **33.** 6 or 4

35. 0 **37.** Sample answer: $\begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix}$

39. Sample answer: $\begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix}$, $\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$ **41.** If you know the coordinates of the vertices of a triangle, you can use a determinant to find the area. This is convenient since you don't need to know any additional information such as the measure of the angles. You could place a coordinate grid over a map of the Bermuda Triangle with one vertex at the origin. By using the scale of the map, you could determine coordinates to represent the other two vertices and use a determinant to estimate the area. The determinant method is advantageous because you don't need to physically measure the lengths of each side or the measure of the angles at the vertices. **43.** H **45.** $A'(-5, 2.5)$, $B'(2.5, 5)$, $C'(5, -7.5)$;



47. undefined **49.** 138,435 ft **51.** $y = -\frac{4}{3}x$ **53.** $y = \frac{1}{2}x + 5$ **55.** (1, 9)

Pages 205-207 Lesson 4-6

1. (5, 1) **3.** $s + d = 5000$, $0.035s + 0.05d = 227.50$ **5.** no solution **7.** (2, −1) **9.** (3, 5) **11.** (−4, −1.75) **13.** (−1.5, 2) **15.** $6g + 15r = 93$; $7g + 12r = 81$ **17.** \$1.99, \$2.49 **19.** (2, −1, 3) **21.** $\left(\frac{141}{29}, -\frac{102}{29}, \frac{244}{29}\right)$ **23.** $\left(-\frac{155}{28}, \frac{143}{70}, \frac{673}{140}\right)$ **25.** (−8.5625, −19.0625)

27. $\left(\frac{2}{3}, \frac{5}{6}\right)$ **29.** $p + r + c = 5$, $2r - c = 0$, $3.2p + 2.4r + 4c = 16.8$; peanuts, 2 lb; raisins, 1 lb; pretzels, 2 lb

31. $3x + 5y = -6$, $4x - 2y = 30$ **33.** Cramer's Rule is a formula for the variables x and y where (x, y) is a solution for a system of equations. Cramer's Rule uses determinants composed of the coefficients and constant terms in a system of linear equations to solve the system. Cramer's rule is convenient when coefficients are large or involve fractions or decimals. Finding the value of the determinant is sometimes easier than trying to find a greatest common factor if

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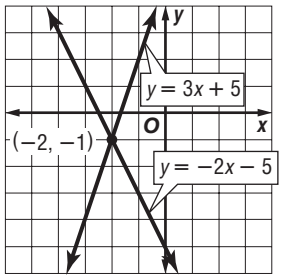
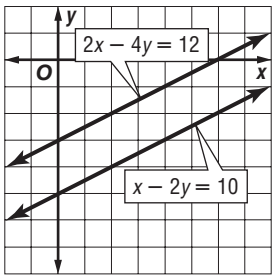
you are solving by using elimination or substituting complicated numbers. **35.** J **37.** 40 **39.** $\begin{bmatrix}1 & 1 & 1\\3 & 3 & 3\end{bmatrix}$

41. $(-2, -1)$



43. no solution



45. $[-4 \quad 32]$ **47.** $\begin{bmatrix}21\\43\end{bmatrix}$

Pages 212–215 ***Lesson 4-7***

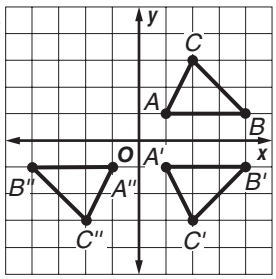
1. no **3.** yes **5.** $\begin{bmatrix}2 & 5\\3 & 8\end{bmatrix}$ **7.** $-\frac{1}{27}\begin{bmatrix}4 & -1\\-7 & -5\end{bmatrix}$ **9.** yes

11. no **13.** $\frac{1}{5}\begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$ **15.** No inverse exists. **17.** $\frac{1}{7}\begin{bmatrix}1 & -1\\4 & 3\end{bmatrix}$

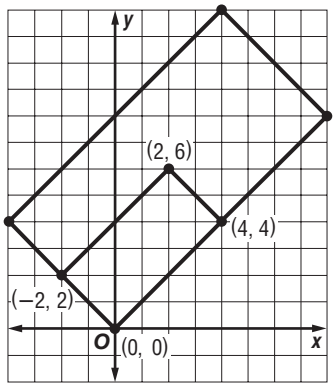
19. $\frac{1}{34}\begin{bmatrix}7 & 3\\-2 & 4\end{bmatrix}$ **21.** No inverse exists.

23. AT_SIX_THIRTY **27.** true **29.** false **31.** yes

33. $4\begin{bmatrix}\frac{1}{4} & \frac{3}{4}\\-\frac{1}{6} & \frac{1}{2}\end{bmatrix}$ **35a.** no **35b.** yes



37. $\begin{bmatrix}0 & -2 & 2 & 4\\0 & 2 & 6 & 4\end{bmatrix}$ **39.** dilation by a scale factor of 2



41. $B^{-1} = \begin{bmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{bmatrix}$; the graph of the inverse transformation is the original figure. **43.** No inverse exists. **45.** Exchange the values for a and d in the first diagonal in the matrix. Multiply the values for b and c by -1 in the second diagonal in the matrix. Find the determinant of the original matrix. Multiply the negative reciprocal of the determinant by the matrix with the above mentioned changes. **47.** $a = \pm 1$, $d = \pm 1$, $b = c = 0$ **49.** B **51.** $(2, -4)$ **53.** $(-5, 4, 1)$ **55.** -14 **57.** $[-4]$ **59.** $[14 \quad -8]$ **61.** $(2, 5)$ **63.** 1 **65.** -5 **67.** $\frac{5}{2}$ **69.** 7.82 tons/in^2 **71.** -2 **73.** 4 **75.** -34

Pages 219–222 ***Lesson 4-8***

1. $\begin{bmatrix}1 & -1\\1 & 3\end{bmatrix} \cdot \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-3\\5\end{bmatrix}$ **3.** $h = 1$, $c = 12$, $o = 16$

5. $(1, 1.75)$ **7.** no solution **9.** $\begin{bmatrix}4 & -7\\3 & 5\end{bmatrix} \cdot \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\9\end{bmatrix}$

11. $\begin{bmatrix}3 & -7\\6 & 5\end{bmatrix} \cdot \begin{bmatrix}m\\n\end{bmatrix} = \begin{bmatrix}-43\\-10\end{bmatrix}$ **13.** 27 h of flight instruction and 23 h in the simulator **15.** no solution **17.** $(2, -3)$

19. no solution **21.** $(3, 1.5)$ **23.** $\left(\frac{3}{2}, \frac{1}{3}\right)$ **25.** carbon = 12; hydrogen = 1 **27.** 2010 **29.** $(-6, 2, 5)$ **31.** $(0, -1, 3)$ **33.** Sample answer: $x + 3y = 8$ and $2x + 6y = 16$ **35.** The solution set is the empty set or infinitely many solutions. **37.** C **39.** D **41.** $\begin{bmatrix}4 & -5\\-7 & 9\end{bmatrix}$ **43.** $(4, -2)$ **45.** $(-6, -8)$

Pages 224–228 ***Chapter 4*** ***Study Guide and Review***

1. identity matrix **3.** rotation **5.** matrix equation **7.** matrix **9.** inverses **11.** dilation **13.** $(-5, -1)$ **15.** $(-1, 0)$ **17.** $\begin{bmatrix}17 & 20 & 23\\12 & 19 & 22\\6 & 7 & 11\end{bmatrix}$ row 3, column 1

19. $[-1.8 \quad -0.4 \quad -3]$ **21.** $\begin{bmatrix}5 & -6 & -13\\10 & -3 & -2\end{bmatrix}$ **23.** $[-18]$

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25. No product exists. **27.** $A'(1, 0)$, $B'(8, -2)$, $C'(3, -7)$ **29.** $A'(3, 5)$, $B'(-4, 3)$, $C'(1, -2)$ **31.** $A'(1, 1)$, $B'(3, 1)$, $C'(3, 3)$, $D'(1, 3)$ **33.** 53 **35.** -36 **37.** -35 **39.** $\left(\frac{2}{3}, 5\right)$ **41.** $(2, -2)$ **43.** $(1, 2, -1)$ **45.** ($5.25, $4.75) **47.** $\frac{1}{2}\begin{bmatrix} 7 & -6 \\ -9 & 8 \end{bmatrix}$ **49.** No inverse exists. **51.** (4, 2) **53.** $(-3, 1)$ **55.** 720 mL of the 50% solution and 780 mL of the 75% solution

Chapter 5 Quadratic Functions and Inequalities

Page 235 ***Chapter 5*** ***Get Ready***

1. -4 **3.** -6 **5.** -4 **7.** 0 **9.** $f(x) = 9x$ **11.** $(x + 6)(x + 5)$ **13.** $(x - 8)(x + 7)$ **15.** prime **17.** $(x - 11)^2$ **19.** $(x + 7)$ feet

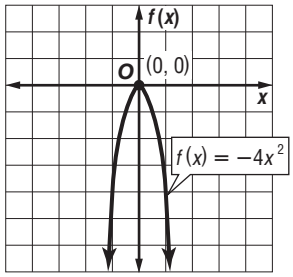
Pages 241–244 ***Lesson 5–1***

1a. 0; $x = 0$; 0

1b.

x	f(x)
−1	−4
0	0
1	−4

1c.

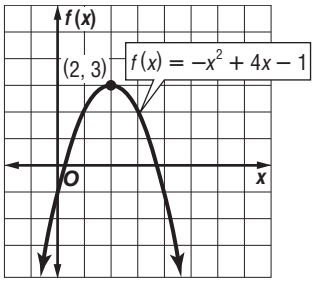


3a. -1; $x = 2$; 2

3b.

x	f(x)
0	−1
1	2
2	3
3	2
4	−1

3c.

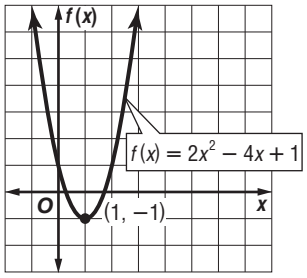


5a. 1; $x = 1$; 1

5b.

x	f(x)
−1	7
0	1
1	−1
2	1
3	7

5c.



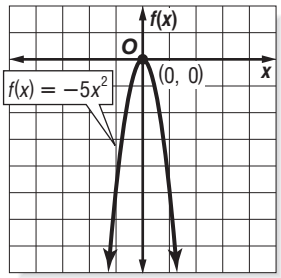
7. max.; 7; D = all reals; R = $\{y \mid y \leq 7\}$
9. min.; 0; D = all reals; R = $\{y \mid y \geq 0\}$
11. $8.75

13a. 0; $x = 0$; 0

13b.

x	f(x)
−2	−20
−1	−5
0	0
1	−5
2	−20

13c.

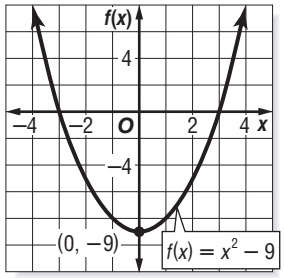


15a. -9; $x = 0$; 0

15b.

x	f(x)
−2	−5
−1	−8
0	−9
1	−8
2	−5

15c.

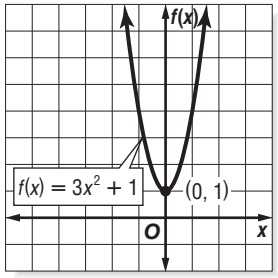


17a. 1; $x = 0$; 0

17b.

x	f(x)
−2	13
−1	4
0	1
1	4
2	13

17c.

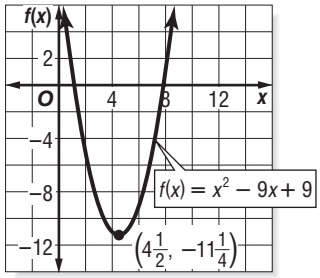


19a. 9; $x = 4.5$; 4.5

19b.

x	f(x)
3	−9
4	−11
4.5	−11.25
5	−11
6	−9

19c.

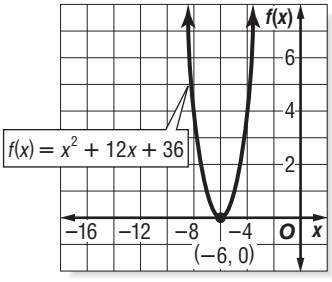


21a. 36; $x = -6$; -6

21b.

x	f(x)
−8	4
−7	1
−6	0
−5	1
−4	4

21c.



23. max.; -9; D = all reals, R = $\{y \mid y \leq -9\}$
25. min.; -11; D = all reals, R = $\{y \mid y \geq -11\}$
27. max.; 12; D = all reals, R = $\{y \mid y \leq 12\}$
29. min.; -1; D = all reals, R = $\{y \mid y \geq -1\}$
31. max.; -60; D = all reals, R = $\{y \mid y \leq -60\}$
33. 40 m **35.** 300 ft, 2.5 s

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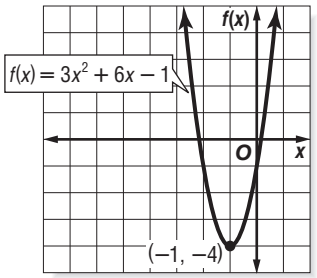
37a. -1; $x = -1$; -1

37c.

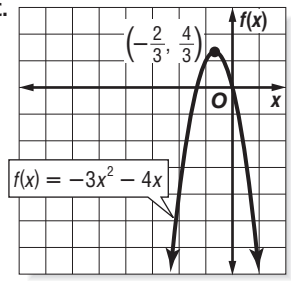


37b.

x	f(x)
−3	8
−2	−1
−1	−4
0	−1
1	8

39a. 0; $x = -\frac{2}{3}; -\frac{2}{3}$

39c.

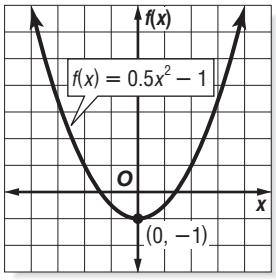


39b.

x	f(x)
−2	−4
−1	1
$-\frac{2}{3}$	$\frac{4}{3}$
0	0
1	−7

41a. -1; $x = 0$; 0

41c.

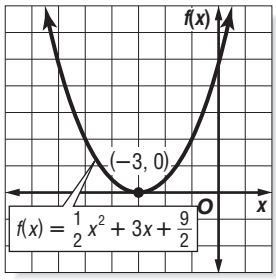


41b.

x	f(x)
−2	1
−1	$-\frac{1}{2}$
0	−1
1	$-\frac{1}{2}$
2	1

43a. $\frac{9}{2}$; $x = -3$; -3

43c.



43b.

x	f(x)
−5	2
−4	0.5
−3	0
−2	0.5
−1	2

45. min.; $\frac{9}{2}$; D = all reals, R = $\left\{y \mid y \geq \frac{9}{2}\right\}$

47. max.; 5; D = all reals, R = $\{y \mid y \leq 5\}$

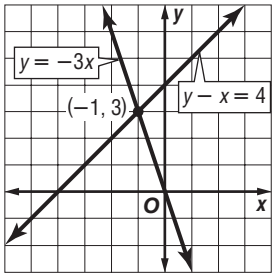
49. max.; 5; D = all reals, R = $\{y \mid y \leq 5\}$

51. $120 - 2x$ **53.** 60 ft by 30 ft **55.** 5 in. by 4 in. **57.** \$2645 **59.** 3.20 **61.** 3.38 **63.** 1.56 **65.** c; the x-coordinate of the vertex of $y = ax^2 + c$ is $-\frac{0}{2a}$ or 0, so the y-coordinate of the vertex, the minimum of the function, is $a(0)^2 + c$ or c; -12.5. **67.** C **69.** (1, 2)

71. $\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$ **73.** $\begin{bmatrix} -7 & 0 \\ 5 & 20 \end{bmatrix}$ **75.** $[10 \quad -4 \quad 5]$

77. $\begin{bmatrix} -28 & 20 & -44 \\ 8 & -16 & 36 \end{bmatrix}$ **79.** \$20, \$35 **81.** (3, −5)

83.



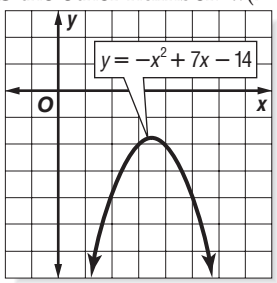
(−1, 3); consistent and independent

85. -1 **87.** -5 **89.** $-\frac{9}{5}$ **91.** $\{-5, 1\}$

93. 256 in^2 **95.** 8 **97.** -1

Pages 249–251 ***Lesson 5–2***

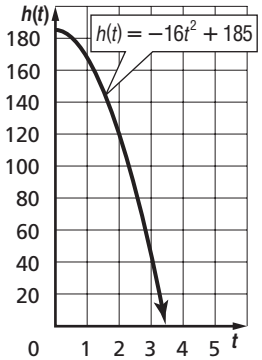
1. $-4, 1$ **3.** -4 **5.** $-4, 6$ **7.** 7 **9.** no real solutions **11.** between -1 and 0; between 1 and 2 **13.** 4 s **15.** 3 **17.** 0 **19.** no real solutions **21.** 0, 4 **23.** 3, 6 **25.** 6 **27.** no real solutions **29.** between -1 and 0; between 2 and 3 **31.** about 12 s **33.** $-\frac{1}{2}, 2\frac{1}{2}$ **35.** $-2\frac{1}{2}, 3$ **37.** between 0 and 1; between 3 and 4 **39.** between -3 and -2; between 2 and 3 **41.** Let x be the first number. Then, $7 - x$ is the other number. $x(7 - x) = 14$; $-x^2 + 7x - 14 = 0$;



Since the graph of the related function does not intersect the x-axis, this equation has no real solutions. Therefore no such numbers exist.

43. $-2, 14$ **45.** about 8 s **47.** The x-intercepts of the related function are the solutions to the equation. You can estimate the solutions by stating the consecutive integers between which the x-intercepts are located.

49.



Locate the positive x-intercept at about 3.4. This represents the time when the height of the ride is 0. Thus, if the ride were allowed to fall to the ground, it would take about 3.4 seconds. **51.** F

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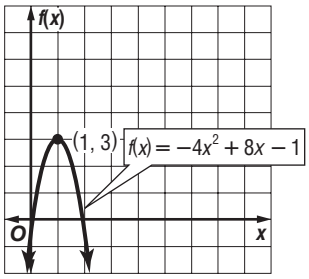
53. -1; $x = 1$; 1;



55. (1, 4) **57.** -8 **59.** \$500 **61.** $(x - 10)(x + 10)$ **63.** $(x - 9)^2$ **65.** $2(3x + 2)(x - 3)$

Pages 256–258 Lesson 5–3

1. $x^2 - 3x - 28 = 0$ **3.** $15x^2 + 14x + 3 = 0$ **5.** $4x(y + 2)(y - 2)$ **7.** $\{0, 11\}$ **9.** $\left\{-\frac{3}{4}, 4\right\}$ **11.** $\{3\}$ **13.** $x^2 - 9x + 20 = 0$ **15.** $x^2 + x - 20 = 0$ **17.** $(x - 6)(x - 1)$ **19.** $3(x + 7)(x - 3)$ **21.** $\{-8, 3\}$ **23.** $\{-5, 5\}$ **25.** $\{-6, 3\}$ **27.** $\{2, 4\}$ **29.** $\{6\}$ **31.** 14, 16 or -14, -16 **33.** $\left\{0, \frac{5}{3}\right\}$ **35.** $\left\{-2, \frac{1}{4}\right\}$ **37.** $\left\{-\frac{1}{2}, -\frac{3}{2}\right\}$ **39.** $\left\{-\frac{8}{3}, -\frac{2}{3}\right\}$ **41.** 0, -6, 5 **43.** $2x^2 - 7x + 3 = 0$ **45.** $12x^2 - x - 6 = 0$ **47.** $\frac{1}{4}$ s **49.** 4; The logs must have a diameter greater than 4 in. for the rule to produce positive board feet values. **51.** Sample answer: Roots 6 and -5; $x^2 - x - 30 = 0$; the sign of the linear term changes, but the others stay the same.

53. To use the Zero Product Property, the equation must be written as a product of factors equal to zero. Move all the terms to one side and factor (if possible). Then set each factor equal to zero and solve for the variable. To use the Zero Product Property, one side of the equation must equal zero, so the equation cannot be solved by setting each factor on the left side equal to 24. **55.** G **57.** $-\frac{1}{2}$ **59.** min.; -19 **61.** $y = -2x - 2$ **63.** Comm. (+) **65.** Assoc. (+)

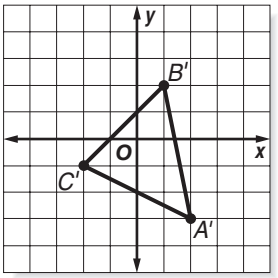
Pages 264–266 Lesson 5–4

1. $2\sqrt{14}$ **3.** $\frac{4\sqrt{3}}{7}$ **5.** $6i$ **7.** 12 **9.** i **11.** $\pm 3i$ **13.** 3, -3 **15.** $10 + 3j$ amps **17.** $6 + 3i$ **19.** $-9 + 2i$ **21.** $\frac{7}{17} - \frac{11}{17}i$ **23.** $7\sqrt{3}$ **25.** $\frac{5\sqrt{14}}{9}$ **27.** $9i$ **29.** $10a^2|b|i$ **31.** $-75i$ **33.** 1 **35.** -3 **37.** 2 **39.** $6 - 7i$ **41.** $\frac{10}{17} - \frac{6}{17}i$ **43.** $\pm 4i$ **45.** $\pm 2i\sqrt{3}$ **47.** 4, -3 **49.** $\frac{5}{3}$, 4 **51.** $4 + 2j$ amps **53.** $(5 - 2i)x^2 + (-1 + i)x + 7 + i$ **55.** $4i$ **57.** 6 **59.** $-8 + 4i$ **61.** $\frac{2}{5} + \frac{1}{5}i$ **63.** $20 + 15i$ **65.** $-\frac{1}{3} - \frac{2\sqrt{2}}{3}i$ **67.** $\pm 2i\sqrt{10}$ **69.** $\pm\frac{\sqrt{5}}{2}i$ **71.** $\frac{67}{11}, \frac{19}{11}$

73. Sample answer: $1 + 3i$ and $1 - 3i$ **75.** $(2i)(3i)(4i)$; The other three expressions represent real numbers, but $(2i)(3i)(4i) = -24i$, which is an imaginary number.

77. Some polynomial equations have complex solutions and cannot be solved using only the real numbers. a and c must have the same sign. The solutions are $\pm i$. **79.** H **81.** $12x^2 + 13x + 3 = 0$ **83.** -4, $-1\frac{1}{2}$ **85.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

87.



89. \$206.25 $< x <$ \$275.00 **91.** yes **93.** no **95.** no

Pages 272–275 Lesson 5–5

1. $\{-10, -4\}$ **3.** $\{-8 \pm \sqrt{7}\}$ **5.** Jupiter

7. Yes; the acceleration due to gravity is significantly greater on Jupiter, so the time to reach the ground should be much less. **9.** $\frac{9}{4}$; $\left(x - \frac{3}{2}\right)^2$ **11.** $\{4 \pm \sqrt{5}\}$ **13.** $\{-2 \pm \sqrt{10}\}$ **15.** $\{3 \pm i\sqrt{3}\}$ **17.** $\{-2, 12\}$ **19.** $\left\{-\frac{11}{2}, -\frac{3}{2}\right\}$ **21.** $\{3 \pm 2\sqrt{2}\}$ **23.** $\left\{\frac{3}{2} \pm \sqrt{6}\right\}$ **25.** 81; $(x - 9)^2$ **27.** $\frac{49}{4}$; $\left(x + \frac{7}{2}\right)^2$ **29.** $\{-12, 10\}$ **31.** $\{2 \pm \sqrt{3}\}$ **33.** $\left\{\frac{1}{2}, 1\right\}$ **35.** $\left\{\frac{1 \pm \sqrt{5}}{3}\right\}$ **37.** $\{-3 \pm 2i\}$ **39.** $\{-4 \pm i\sqrt{2}\}$ **41.** $5\frac{1}{2}$ in. by $5\frac{1}{2}$ in. **43.** $\{-1.6, 0.2\}$ **45.** $\left\{\frac{-5 \pm \sqrt{11}}{3}\right\}$ **47.** 1.44; $(x - 1.2)^2$ **49.** $\frac{25}{16}$; $\left(x + \frac{5}{4}\right)^2$ **51.** $\{0.7, 4\}$ **53.** $\left\{\frac{3}{4} \pm \sqrt{2}\right\}$ **55.** $\left\{\frac{7 \pm i\sqrt{47}}{4}\right\}$ **57.** $\frac{x}{1}, \frac{1}{x - 1}$

59. Sample answers: The golden ratio is found in much of ancient Greek architecture, such as the Parthenon, as well as in modern architecture, such as in the windows of the United Nations building. Many songs have their climax at a point occurring 61.8% of the way through the piece, with 0.618 being about the reciprocal of the golden ratio. The reciprocal of the golden ratio is also used in the design of some violins. **61.** Sample answer: $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{1}{4}$; $\left\{\frac{5}{6}, -\frac{1}{6}\right\}$ **63.** Never; the value of c that makes $ax^2 + bx + c$ a perfect square trinomial is the square of $\frac{b}{2}$ and the square of a number can never be negative.

65. To find the distance traveled by the accelerating racecar in the given situation, you must solve the equation $t^2 + 22t + 121 = 246$ or $t^2 + 22t - 125 = 0$. Since the expression $t^2 + 22t - 125$ is prime, the solutions of $t^2 + 22t + 121 = 246$ cannot be obtained by factoring. Rewrite $t^2 + 22t + 121$ as $(t + 11)^2$. Solve $(t + 11)^2 = 246$ by applying the Square Root Property. Using a calculator, the two solutions are about 4.7 or -26.7. Since time cannot be negative, the driver takes about 4.7 seconds to reach the finish line. **67.** J **69.** $-1 + 3i$ **71.** -2, 0

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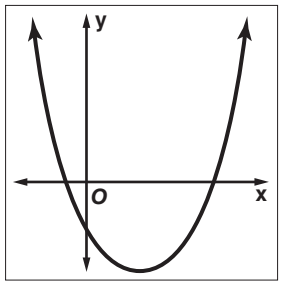
73. $\frac{2}{3}$, 5 **75.** $\left(\frac{43}{21}, -\frac{6}{7}\right)$ **77.** greatest: −255°C; least: −259°C **79.** −16 **81.** 0

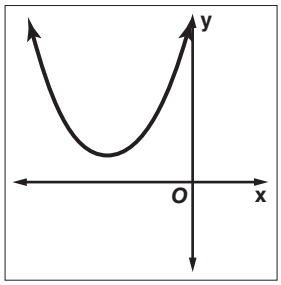
Pages 281–283 ***Lesson 5-6***

1. $\frac{1}{4}, -\frac{5}{2}$ **3.** $-\frac{1}{2}$ **5.** $\frac{2 \pm \sqrt{2}}{2}$ **7.** $\frac{-3 \pm i\sqrt{3}}{2}$
9. at about 0.7 s and again at about 4.6 s **11a.** 484
11b. 2 rational; yes, there were 2 rational roots
13a. 8 **13b.** 2 irrational; yes, there were 2 irrational roots
15a. 121 **15b.** 2 rational **15c.** $-\frac{1}{4}, \frac{2}{3}$ **17a.** 0 **17b.** 1 rational
17c. $\frac{1}{3}$ **19a.** 21 **19b.** 2 irrational **19c.** $\frac{-3 \pm \sqrt{21}}{2}$
21a. 20 **21b.** 2 irrational **21c.** $-2 \pm \sqrt{5}$ **23a.** −16
23b. 2 imaginary **23c.** $1 \pm 2i$ **25.** −2, 32 **27.** $2 \pm i\sqrt{3}$
29. $\pm\sqrt{2}$ **31.** D: $0 \le t \le$ (current year − 1975), R: $73.7 \le A(t) \le 2.3$(current year − 1975)^2 − 12.4(current year − 1975) + 73.7 **33.** No; the fastest the car could have been traveling is about 67.2 mph, which is less than the Texas speed limit. **35a.** −31 **35b.** 2 complex
35c. $\frac{9 \pm i\sqrt{31}}{8}$ **37a.** $\frac{28}{9}$ **37b.** 2 irrational **37c.** $\frac{2 \pm 4\sqrt{7}}{9}$
39a. −0.55 **39b.** 2 complex **39c.** $\frac{-0.1 \pm i\sqrt{0.55}}{0.4}$
41. $\frac{5 \pm \sqrt{46}}{3}$ **43.** $0, -\frac{3}{10}$ **45.** −2, 6 **47.** This means that the cables do not touch the floor of the bridge, since the graph does not intersect the x-axis and the roots are imaginary.

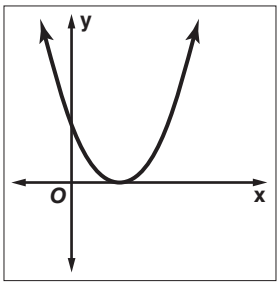
49a.



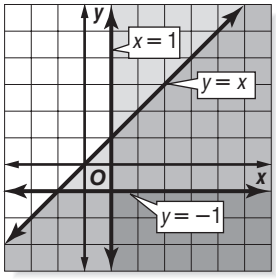
49b.



49c.



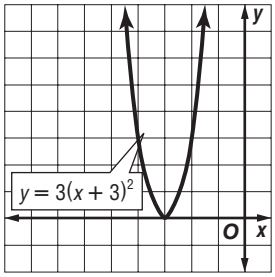
51. $-1, \frac{5}{2}$
53. The diver's height above the pool at any time t can be determined by substituting the value of t in the equation and evaluating. When the diver hits the water, her height above the pool is 0. Substitute 0 for h and use the quadratic formula to find the positive value of t which is a solution to the equation. This is the number of seconds that it will take for the diver to hit the water. **55.** G **57.** $4 \pm \sqrt{7}$ **59.** $\frac{1}{5} + \frac{3}{5}i$ **61.** $\frac{1}{13} + \frac{5}{13}i$
63.



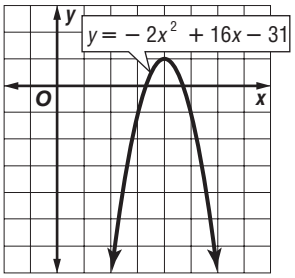
65. $y = -\frac{2}{3}x + 3$ **67.** no **69.** yes; $(2x + 3)^2$ **71.** no

Pages 289–292 ***Lesson 5-7***

1.

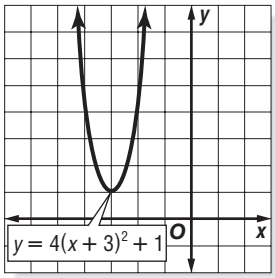


3.



5. (−3, −1); $x = -3$; up **7.** $y = -3(x + 3)^2 + 38$; (−3, 38); $x = -3$; down **9.** $y = -(x + 3)^2 + 6$ **11.** $h(d) = -2d^2 + 4d + 6$ The graph opens downward and is narrower than the parent, and the vertex is at (1, 8).

13.



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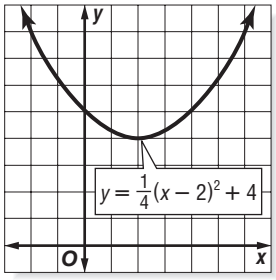
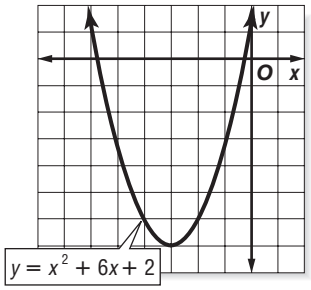
15.

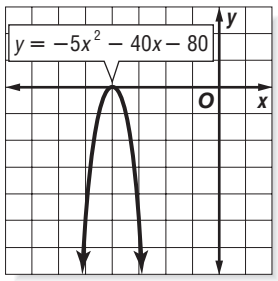


17.

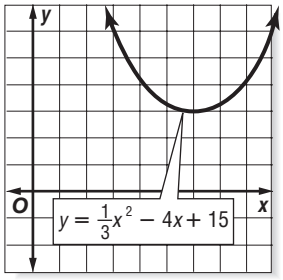


19. The graph is congruent to the original graph, and the vertex moves 7 units down the y-axis. **21.** $(-3, 0)$; $x = -3$; down **23.** $y = -(x+2)^2 + 12$; $(-2, 12)$; $x = -2$; down **25.** $(0, -6)$; $x = 0$; up **27.** $y = 9(x-6)^2 + 1$ **29.** $y = -\frac{2}{3}(x-3)^2$ **31.** $y = \frac{1}{3}x^2 + 5$ **33.** Angle A; the graph of the equation for angle A is higher than the other two since 3.27 is greater than 2.39 or 1.53. **35.** Angle C, Angle A

37.



39.



41. $y = 4(x+3)^2 - 36$; $(-3, -36)$; $x = -3$; up
43. $y = -2(x-5)^2 + 15$; $(5, 15)$; $x = 5$; down
45. $y = 4\left(x - \frac{3}{2}\right)^2 - 20$; $\left(\frac{3}{2}, -20\right)$; $x = \frac{3}{2}$; up
47. $y = \frac{4}{3}(x+3)^2 - 4$ **49.** Sample answer: The graphs have the same shape, but the graph of $y = 2(x-4)^2 - 1$ is 1 unit to the left and 5 units above the graph of $y = 2(x-5)^2 + 4$. **51.** about 1.6 s **53.** about 2.0 s

55. $y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$y = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right] + c - a\left(\frac{b}{2a}\right)^2$$

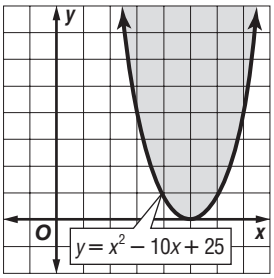
$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

The axis of symmetry is $x = h$ or $-\frac{b}{2a}$.

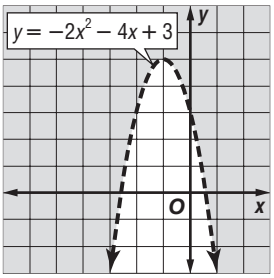
57. The equation of a parabola can be written in the form $y = ax^2 + bx + c$ with $a \neq 0$. For each of the three points, substitute the value of the x-coordinate for x in the equation and substitute the value of the y-coordinate for y in the equation. This will produce three equation in the three variables a, b, and c. Solve the system of equations to find the values of a, b, and c. These values determine the quadratic equation. **59.** D **61.** 12; 2 irrational **63.** -23; 2 complex **65.** $\{3 \pm 3i\}$ **67.** yes **69.** yes

Pages 298–301 Lesson 5-8

1.

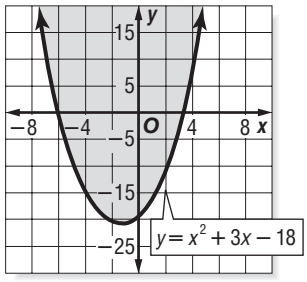


3.



5. $x < 1$ or $x > 5$ **7.** $\{x | x < -3 \text{ or } x > 4\}$
9. $\{x | -\sqrt{3} \leq x \leq \sqrt{3}\}$

11.



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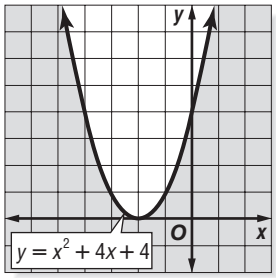
13.



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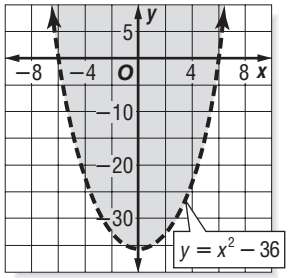
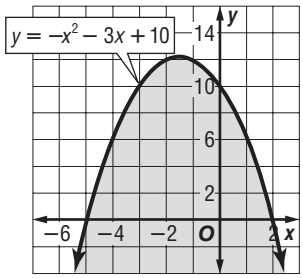


17. 5 **19.** $x < -3$ or $x > 3$ **21.** $\{x \mid x < -3 \text{ or } x > 6\}$ **23.** $\{x \mid -1 \le x \le 5\}$ **25.** $\{x \mid -4 \le x \le 3\}$ **27.** 0 to 10 ft or 24 to 34 ft

29.



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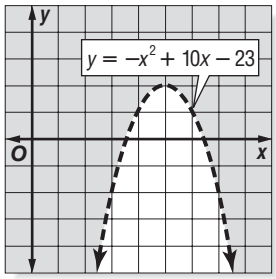
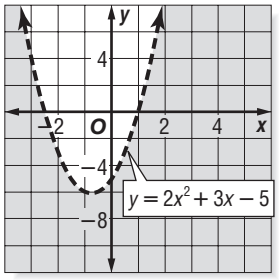


33.



35. $\{x \mid x = \frac{1}{3}\}$ **37.** $\varnothing$ **39.** all reals **41.** $\{x \mid -4 < x < 1 \text{ or } x > 3\}$ **43.** $P(n) = n[15 + 1.5(60 - n)] - 525 = -1.5n^2 + 105n - 525$ **45.** \$1312.50; 35 passengers **47.** Sample answer: −4, 0, and 6

49. $-16t^2 + 42t + 3.75 > 10$; One method of solving this inequality is to graph the related quadratic function $h(t) = -16t^2 + 42t + 3.75 - 10$. The interval in which the graph is above the t-axis represents the times when the trampolinist is above 10 feet. A second method of solving this inequality would be find the roots of the related quadratic equation $-16t^2 + 42t + 3.75 - 10 = 0$ and then test points in the three intervals determined by these roots to see if they satisfy the inequality. The interval in which the inequality is satisfied represent the times when the trampolinist is above 10 feet. **51.** H **53.** $y = -2(x - 4)^2$; (4, 0), $x = 4$; down **55.** −4, −8 **57.** $\frac{-3 \pm 2\sqrt{6}}{3}$ **59.** (−3, −2) **61.** (1, 3) **63.** [−54 6] **65.** C

67a. Sample answer using (2000, 143,590) and (2003, 174,629): $y = 10{,}346x - 20{,}548{,}410$ **67b.** Sample answer: 247,050

Pages 302–306 Chapter 5 Study Guide and Review

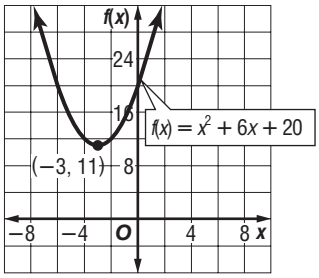
1. parabola **3.** axis of symmetry **5.** roots **7.** discriminant **9.** completing the square

11a. 20; $x = -3$; −3

11b.

x	f(x)
−5	15
−4	12
−3	11
−2	12
−1	15

11c.

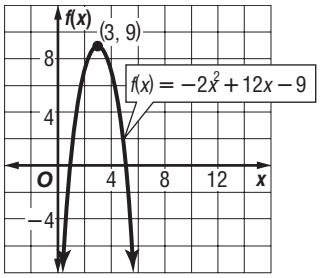


13a. −9; $x = 3$; 3

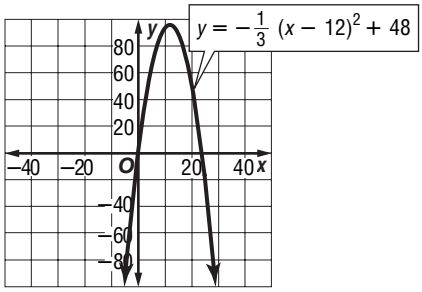
13b.

x	f(x)
1	1
2	7
3	9
4	7
5	1

13c.



15. {−6, 6} **17.** between −2 and −1, between −1 and 0 **19.** 6.25 s **21.** $x^2 - 3x - 70 = 0$ **23.** {−4, 8} **25.** {−4, 4} **27.** $\left\{\frac{1}{3}, -\frac{3}{2}\right\}$ **29.** base = 4 cm; height = 6 cm **31.** $8|n|\sqrt{m}$ **33.** $10 - 10i$ **35.** 7 **37.** $-\frac{2}{5} - \frac{11}{5}i$ **39.** 289; $(x + 17)^2$ **41.** $\left\{-\frac{3}{2}, 5\right\}$ **43.** 8 ft by 16 ft **45a.** 104 **45b.** 2 irrational **45c.** $3 \pm \frac{\sqrt{26}}{2}$ **47.** about 204.88 ft **49.** $y = -\frac{1}{3}(x - 12)^2 + 48$; (12, 48); $x = 12$; down;



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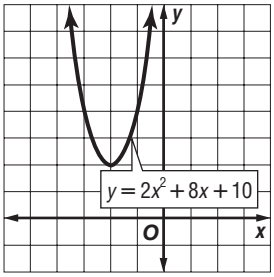
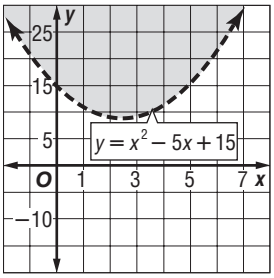
51. $y = 2(x + 2)^2 + 2$; $(-2, 2)$; $x = -2$; up;



53.



55. $\left\{x \mid x < -\frac{4}{3} \text{ or } x > \frac{1}{2}\right\}$

57. $\left\{x \middle| \frac{-1 - \sqrt{10}}{2} \le x \le \frac{-1 + \sqrt{10}}{2}\right\}$

59. $22.087 \le s \le 67.91$ mph

Chapter 6 Polynomial Functions

Page 311 ***Chapter 6*** ***Get Ready***

1. $2 + (-7)$ **3.** $x + (-y)$ **5.** $2xy + (-6yz)$ **7.** $\$4 + (-\$0.50x)$ **9.** $-x - 2$ **11.** $-6x^4 + 15x^2 + 6$ **13.** $-\frac{4}{3} - 4z$ **15.** \$62.15 **17.** $-\frac{3}{2}, -\frac{1}{7}$ **19.** $1, \frac{2}{3}$

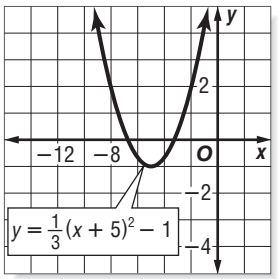
Pages 316–318 ***Lesson 6-1***

1. $-15x^7y^9$ **3.** $-\frac{ab^4}{9}$ **5.** $\frac{1}{w^{12}z^6}$ **7.** 1 **9.** $\frac{1}{4x^6}$ **11.** $\frac{2}{3}a^{10}b^4$ **13.** $\frac{28x^4}{y^2}$ **15.** an **17.** $-\frac{1}{4y^4}$ **19.** n^{16} **21.** $16x^4$ **23.** ab **25.** $\frac{cd^4}{5}$ **27.** 2×10^{-7}; $\pi \times 10^{-14}$ m² **29.** $24x^4y^4$ **31.** $\frac{a^4b^2}{2}$ **33.** $\frac{1}{x^2y^2}$ **35.** $\frac{a^4}{16b^4}$ **37.** $\frac{x^8}{16y^{14}}$ **39.** 7

41. about 330,000 times **43.** Alejandra; when Kyle used the Power of a Product property in his first step, he forgot to use the exponent -2 for both -2 and a. **45.** $100^{10} = (10^2)^{10}$ or 10^{20}, and $10^{100} > 10^{20}$, so $10^{100} > 100^{10}$.

47. D **49.** $\{x \mid 2 < x < 6\}$ **51.** Ø

53.

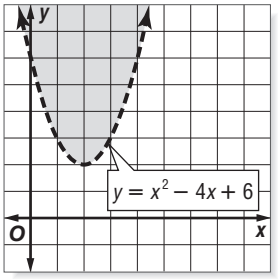


55. -6 **57.** $(2, 3, -1)$ **59.** A **61.** S **63.** Sample answer using (0, 5.4) and (25, 8.3): $y = 0.116x + 5.4$ **65.** 7 **67.** $2x + 2y$ **69.** $4x + 8$ **71.** $-5x + 10y$

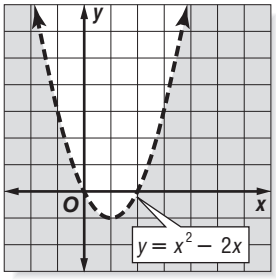
Pages 322–324 ***Lesson 6-2***

1. yes, 1 **3.** no **5.** $-3x^2 - 7x + 8$ **7.** $10p^3q^2 - 6p^5q^3 + 8p^3q^5$ **9.** $x^2 + 9x + 18$ **11.** $4m^2 - 12mn + 9n^2$ **13.** $2x^3 - 9x^2 + 12x - 4$ **15.** yes, 2 **17.** no **19.** yes, 6 **21.** $4x^2 + 3x - 7$ **23.** $r^2 - r + 6$ **25.** $4b^2c - 4bdz$ **27.** $15a^3b^3 - 30a^4b^3 + 15a^5b^6$ **29.** $p^2 + 2p - 24$ **31.** $b^2 - 25$ **33.** $6x^2 + 34x + 48$ **35.** $27b^3 - 27b^2c + 9bc^2 - c^3$ **37.** $29.75 - 0.018x$ **39.** \$5327.50 **41.** $-3x^3 - 16x^2 + 27x - 10$ **43.** $7x^2 - 8xy + 4y^2$ **45.** $2a^4 - 3a^3b + 4a^4b^4$ **47.** $xy^3 + y + \frac{1}{x}$ **49.** $2m^4 - 7m^2 - 15$ **51.** $1 + 8c + 16c^2$ **53.** Sample answer: $x^5 + x^4 + x^3$ **55.** 14; Sample answer: $(x^8 + 1)(x^6 + 1) = x^{14} + x^8 + x^6 + 1$ **57.** D **59.** $-64d^6$ **61.** $\frac{xz^2}{y^2}$

63.



65.



67. max.; 32 **69.** $\begin{bmatrix} -3 & 2 \\ 3 & -4 \\ -2 & 9 \end{bmatrix}$ **71.** $\begin{bmatrix} 29 & -8 \\ 8 & 9 \\ 16 & -16 \end{bmatrix}$

73. $y = \frac{2}{3}x - \frac{4}{3}$ **75.** x^2 **77.** xy^2

Pages 328–330 ***Lesson 6-3***

1. $6y - 3 + 2x$ **3.** $-w + 16 + \frac{1000}{w}$ **5.** $3a^3 - 9a^2 + 7a - 6$ **7.** $x^2 - xy + y^2$ **9.** $b^3 + b - 1 + \frac{2}{b - 2}$ **11.** $2y + 5$ **13.** $3ab - 6b^2$ **15.** $2c^2 - 3d + 4d^2$ **17.** x^2 **19.** $b^2 + 10b$ **21.** $y^2 - y - 1$ **23.** $t^4 + 2t^3 + 4t^2 + 5t + 10$ **25.** $2c^2 + c + 5 + \frac{6}{c - 2}$ **27.** $x^4 - 3x^3 + 2x^2 - 6x + 19 - \frac{56}{x + 3}$ **29.** $x^2 + x - 1 + \frac{2}{4x + 1}$ **31.** $3t^2 - 2t + 3$ **33.** $x^3 - x - \frac{6}{2x + 3}$ **35.** 5; Let x be the number.

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Multiplying by 4 results in $4x$. The sum of the number, 15, and the result of the multiplication is $x + 15 + 4x$ or $5x + 15$.

Dividing by the sum of the number and 3 gives $\frac{5x + 15}{x + 3}$ or 5. The end result is always 5.

37. about 2,423 subscriptions **39.** $x - 2s$

41. Sample answer: $(x^2 + x + 5) \div (x + 1)$ **43.** Jorge; Shelly is subtracting in the columns instead of adding. **45.** Division of polynomials can be used to solve for unknown quantities in geometric formulas that apply to manufacturing situations. $10x$ in. by $14x + 2f$ in. The area of a rectangle is equal to the length times the width. That is, $A = \ell w$. Substitute $140x^2 + 60x$ for A, $10x$ for ℓ, and $14x + 2f$ for w. Solving for f involves dividing $140x^2 + 60x$ by $10x$.

$$\begin{aligned} A &= \ell w \\ 140x^2 + 60x &= 10x(14x + 2f) \\ \frac{140x^2 + 60x}{10x} &= 14x + 2f \\ 14x + 6 &= 14x + 2f \\ 6 &= 2f \\ 3 &= f \end{aligned}$$

The flaps are each 3 inches wide.

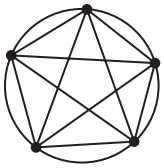
47. H **49.** $y^4z^4 - y^3z^3 + 3y^2z$ **51.** $a^2 - 2ab + b^2$ **53.** 20 **55.** $4a^2 - 10a + 6$

Pages 335–338 Lesson 6-4

1. 6; 5 **3.** −21; 3 **5.** 109 lumens **7.** $100a^2 + 20$ **9. a.** $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; **b.** odd; **c.** 3 **11. a.** $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; **b.** odd; **c.** 1 **13.** 3; 1 **15.** No, this is not a polynomial because the term $\frac{1}{c}$ cannot be written in the form c^n, where n is a nonnegative integer.

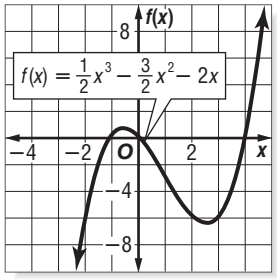
17. 3; −5 **19.** 12; 18 **21.** 1008; −36 **23.** $12a^2 - 8a + 20$ **25.** $12a^6 - 4a^3 + 5$ **27.** $3x^4 + 16x^2 + 26$ **29a.** $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to +\infty$ as $x \to -\infty$; **29b.** even; **29c.** 4 **31a.** $f(x) \to +\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; **31b.** odd; **31c.** 5 **33a.** $f(x) \to -\infty$ as $x \to +\infty$, $f(x) \to -\infty$ as $x \to -\infty$; **33b.** even; **33c.** 2 **35.** 10,345.5 joules **37.** 86; 56 **39.** 7; 4 **41.** $-x^6 + x^3 + 2x^2 + 4x + 2$ **43.** odd **45.** Sample answer: Decrease; the graph appears to be turning at $x = 19$, indicating a maximum at that point. So attendance will decrease after 2005.

47. 16 regions;



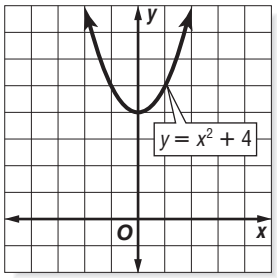
49. $4 = 4x^0$; $x = x^1$ **51.** Sometimes; a polynomial function with 4 real roots may be a sixth-degree polynomial function with 2 imaginary roots. A polynomial function that has 4 real roots is *at least* a fourth-degree polynomial. **53.** −1, 0, 4

55.

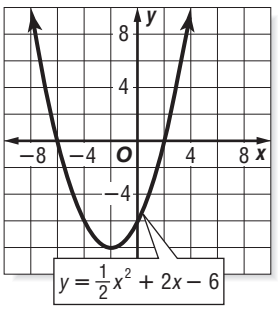


57. C **59.** $t^2 - 2t + 1$ **61.** $x^2 + 2$ **63.** $23{,}450(1 + p)$; $23{,}450(1 + p)^3$ **65.** $\left\{-\frac{7}{6}, \frac{5}{6}\right\}$ **67.** $|x| > 2$ **69.** $|x + 1| < 3$ **71.** Distributive

73.



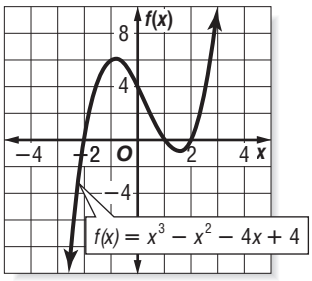
75.



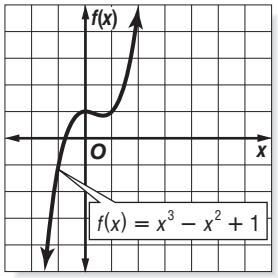
Pages 343–345 Lesson 6-5

1.

x	f(x)
−3	−20
−2	0
−1	6
0	4
1	0
2	0
3	10



3. between −1 and 0



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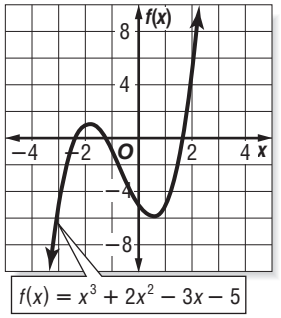
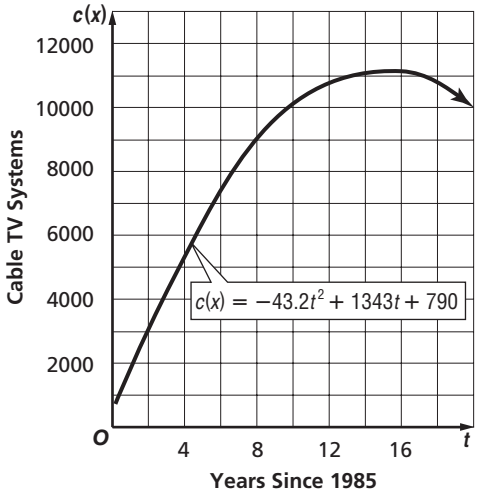
5.



Sample answer: rel. max. at $x \approx -2$, rel. min. at $x \approx 0.5$; domain: all real numbers, range: all real numbers

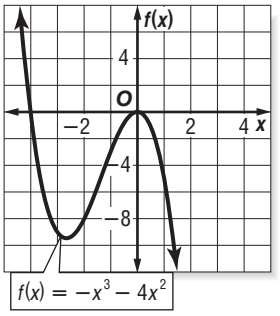
7.



9. The domain is all real numbers. The range is all real numbers less than or equal to approximately 11,225.

11a.

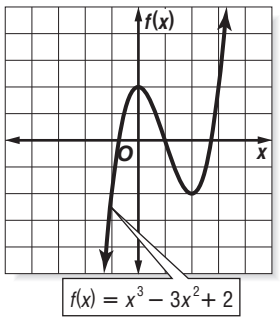
x	f(x)
−5	25
−4	0
−3	-9
−2	-8
−1	−3
0	0
1	−5
2	−24



11b. at $x = -4$ and $x = 0$ **11c.** Sample answer: rel. max. at $x \approx 0$, rel. min. at $x \approx -3$

13a.

x	f(x)
−2	−18
−1	−2
0	2
1	0
2	−2
3	2
4	18

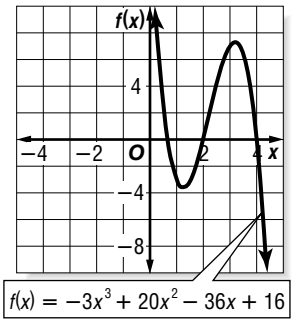


13b. at $x = 1$, between -1 and 0, and between 2 and 3

13c. Sample answer: rel. max. at $x \approx 0$, rel. min. at $x \approx 2$

15a.

x	f(x)
−1	75
0	16
1	−3
2	0
3	7
4	0
5	−39

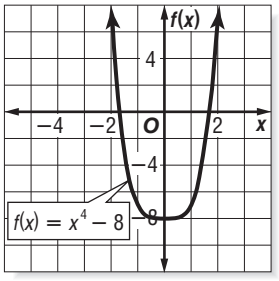


15b. between 0 and 1, at $x = 2$, and at $x = 4$

15c. Sample answer: rel. max. at $x \approx 3$, rel. min. at $x \approx 1$

17a.

x	f(x)
−3	72
−2	8
−1	−7
0	−8
1	−7
2	8
3	73



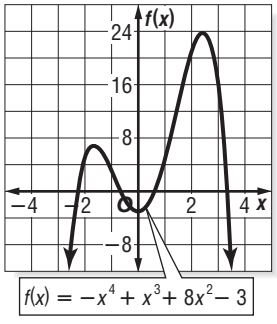
17b. between -2 and -1, and between 1 and 2

17c. Sample answer: no rel. max., rel. min. at $x = 0$

19. highest: 1982; lowest: 2000 **21.** 7 **23.** 0 s and about 5.3 s **25.** about 3.4 s

27a.

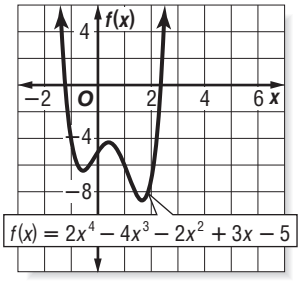
x	f(x)
−3	−39
−2	5
−1	3
0	−3
1	5
2	21
3	15
4	−67



27b. between -3 and -2, between -1 and 0, between 0 and 1, and between 3 and 4 **27c.** Sample answer: rel. max. at $x \approx -1.5$ and at $x \approx 2.5$, rel. min. at $x \approx 0$

29a.

x	f(x)
−2	45
−1	−4
0	−5
1	−6
2	−7
3	40



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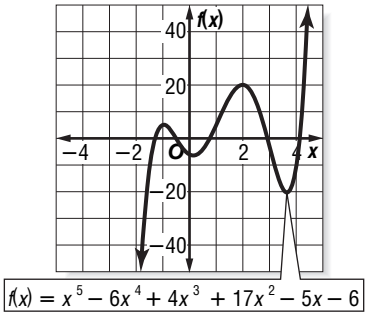
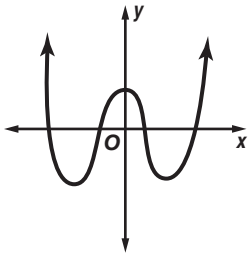
29b. between −2 and −1, and between 2 and 3
29c. Sample answer: rel. max. at $x \approx 0.5$, rel. min. at $x \approx -0.5$ and at $x \approx 1.5$

31a.

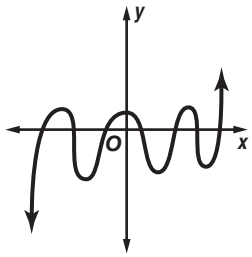
x	f(x)
−2	88
−1	5
0	−6
1	5
2	20
3	−3
4	−10
5	269



b. between −2 and −1, between −1 and 0, between 0 and 1, between 2 and 3, and between 4 and 5
c. Sample answer: rel. max. at $x \approx -1$ and at $x \approx 2$, rel. min. at $x \approx 0$ and at $x \approx 3.5$ **33.** The growth rate for both boys and girls increases steadily until age 18 and then begins to level off, with boys averaging a height of 71 in. and girls a height of 60 in. **35.** 3.41; 0.59
37. 0.52; −0.39, 1.62
39. Sample answer:



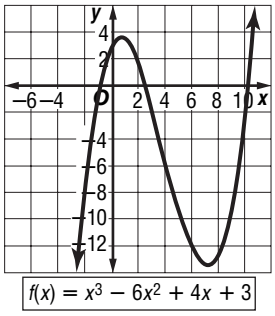
41. Sample answer:



43. The turning points of a polynomial function that models a set of data can indicate fluctuations that may repeat. Polynomial functions best model data that contain turning points. To determine when the percentage of foreign-born citizens was at its highest, look for the rel. max. of the graph which is at about $t \approx 5$. The lowest percentage is found at $t \approx 74$, the rel. min. of the graph. **45.** H **47.** $10c^2 - 25c + 20$ **49.** $3x^3 - 10x^2 + 11x - 6$ **51.** $4x^4 - 9x^3 + 28x^2 - 33x + 20$ **53.** $x^3 + 9x^2 + 41x + 210 + \frac{1050}{x - 5}$ **55.** $14x^2 + 26x - 4$ **57.** (−3, −2) **59.** (1, 3) **61.** 9 **63.** 4 **65.** 6

Pages 353–355 Lesson 6-6

1. $-6x(2x + 1)$ **3.** $(x + 7)(3 - y)$ **5.** $(z - 6)(z + 2)$
7. $(4w + 13)(4w - 13)$ **9.** not possible **11.** −7, −1, 1, 7
13. 8 ft **15.** $6ab^2(a + 3b)$ **17.** prime **19.** $(3a + 1)(x - 5)$ **21.** $(2b - 1)(b + 7)$ **23.** $(t - 2)(t^2 + 2t + 4)$
25. not possible **27.** $b[7(b^2)^2 - 4(b^2) + 2)]$
29. $6\left(x^{\frac{1}{5}}\right)^2 - 4\left(x^{\frac{1}{5}}\right) - 16$ **31.** $-4, 4, -i, i$ **33.** $-4, 2 + 2i\sqrt{3}, 2 - 2i\sqrt{3}$ **35.** $\frac{3}{2}, \frac{-3 + 3i\sqrt{3}}{4}, \frac{-3 - 3i\sqrt{3}}{4}$
37. 3 in. × 3 in. **39.** $w = 4$ cm, $\ell = 8$ cm, $h = 2$ cm
41. $(2y + 1)(y + 4)$ **43.** $(y^2 + z)(y^2 - z)$ **45.** $3(x + 3y)(x - 3y)$ **47.** $(a + 3b)(3a + 5)(a - 1)$ **49.** The height increased by 3, the width increased by 2, and the length increased by 4. **51.** yes **53.** no; $(2x + 1)(x - 3)$
55. Sample answer: $16x^4 - 12x^2 = 0$; $4[4(x^2)^2 - 3x^2] = 0$
57. Sample answer: If $a = 1$ and $b = 1$, then $a^2 + b^2 = 2$ but $(a + b)^2 = 4$. **59.** Solve the cubic equation $4x^3 + (-164x^2) + 1600x = 3600$ in order to determine the dimensions of the cut square if the desired volume is 3600 in^3. Solutions are 10 and $\frac{31 - \sqrt{601}}{2}$ in. There can be more than one square cut to produce the same volume because the height of the box is not specified and 3600 has many factors. **61.** G
63. Sample answer: rel. max. at $x \approx 0.5$, rel. min. at $x \approx 3.5$



65. 17; 27 **67.** $\frac{1715}{3}$; 135 **69.** yes **71.** $x^2 + 5x - 4$
73. $x^3 + 3x^2 - 2$

Pages 359–361 Lesson 6-7

1. 7, −91 **3.** $3.236 billion **5.** Sample answer: Direct substitution, because it can be done quickly with a calculator. **7.** $x - 1, x + 2$ **9.** $x - 2, x^2 + 2x + 4$
11. 37, −19 **13.** 55, 272 **15.** 267, 680 **17.** 422, 3110
19. $x - 4, x + 2$ **21.** $x - 3, x - 1$ **23.** $x - 1, 3x + 4$
25. $x - 1, x + 6$ **27.** $2x - 3, 2x + 3, 4x^2 + 9$
29. $x - 2, x + 2, x^2 + 1$ **31.** $f(6) = 132.96$ ft/s. This means the boat is traveling at 132.96 ft/s when it passes the second buoy. **33.** Yes; 2-ft lengths; the binomial $x - 2$ is a factor of the polynomial since $f(2) = 0$. **35.** 8 **37.** −3 **39.** $16.70 **41.** No, he will still owe $4.40. **43.** dividend: $x^3 + 6x + 32$; divisor: $x + 2$; quotient: $x^2 - 2x + 10$; remainder: 12 **45.** Using the Remainder Theorem you can evaluate a polynomial for a value of a by dividing the polynomial by $x - a$ using synthetic division. It is easier to use the Remainder Theorem when you have polynomials of degree 2 and lower or when you have access to a calculator. The estimated number of international travelers to the U.S. in 2006 is 65.9 million. **47.** G

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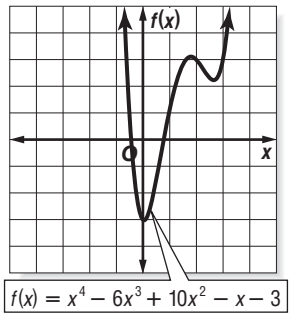
49. $(a + 3)(b - 5)$ **51.** $(c - 6)(c^2 + 6c + 36)$

53.

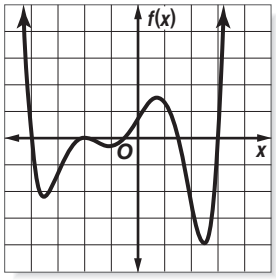
x	$f(x)$
-1	15
0	-3
1	1
2	3
3	3
4	25



55. $\frac{-7 \pm \sqrt{17}}{2}$ **57.** $\frac{-3 \pm i\sqrt{7}}{4}$

Pages 366–368 Lesson 6-8

1. $2i$, $-2i$; 2 imaginary **3.** 2 or 0; 1; 2 or 0 **5.** -4, $1 + 2i$, $1 - 2i$ **7.** $2i$, $-2i$, 3 **9.** $f(x) = x^3 - 2x^2 + 16x - 32$ **11.** $-\frac{8}{3}$; 1 real **13.** 0, $3i$, $-3i$; 1 real, 2 imaginary **15.** 2, -2, $2i$, and $-2i$; 2 real, 2 imaginary **17.** 2 or 0; 1; 2 or 0 **19.** 3 or 1; 0; 2 or 0 **21.** 4, 2, or 0; 1; 4, 2, or 0 **23.** -2, $-2 + 3i$, $-2 - 3i$ **25.** $4 - i$, $4 + i$, -3 **27.** $-\frac{3}{2}$, $1 + 4i$, $1 - 4i$ **29.** $2i$, $-2i$, $\frac{i}{2}$, $\frac{-i}{2}$ **31.** $3 - 2i$, $3 + 2i$, -1, 1 **33.** $f(x) = x^3 - 2x^2 - 19x + 20$ **35.** $f(x) = x^4 + 7x^2 - 144$ **37.** $f(x) = x^3 - 11x^2 + 23x - 45$ **39.** 2 or 0; 1; 2 or 0 **41.** The company needs to produce no fewer than 4 and no more than 24 computers per day. **43.** radius = 4 m, height = 21 m **45.** 1 ft **47.** One root is a double root. Sample graph:



49. Sample answer: $f(x) = x^3 - 6x^2 + 5x + 12$ and $g(x) = 2x^3 - 12x^2 + 10x + 24$; each has zeros at $x = 4$, $x = -1$, and $x = 3$. **51.** C **53.** -127, 41 **55.** $5ab^2(3a - c^2)$ **57.** $4y(y + 3)^2$ **59.** $\pm\frac{1}{2}$, ±1, $\pm\frac{5}{2}$, ±5 **61.** $\pm\frac{1}{9}$, $\pm\frac{1}{3}$, ±1, ±3

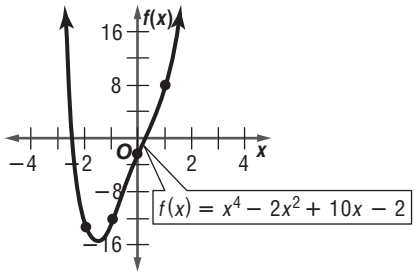
Pages 371–373 Lesson 6-9

1. ±1, ±2, ±5, ±10 **3.** -4, 2, 7 **5.** 2, -2, 3, -3 **7.** 10 cm × 11 cm × 13 cm **9.** 2, -1, i, $-i$ **11.** ±1, ±2, ±3, ±6 **13.** ±1, ±2, ±3, ±6, ±9, ±18 **15.** ±1, $\pm\frac{1}{3}$, $\pm\frac{1}{9}$, ±3, ±9, ±27 **17.** -1, -1, 2 **19.** 0, 2, -2 **21.** -2, -4 **23.** $\frac{4}{5}$, 0, $\frac{5 \pm i\sqrt{3}}{2}$ **25.** -7, 1, 3 **27.** $-\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$ **29.** 3, $\frac{2}{3}$, $-\frac{2}{3}$, $\frac{-3 \pm \sqrt{13}}{2}$ **31.** No; the dimensions of the space are $\ell = 36$ in., $w = 48$ in., $h = 32$ in., so the package is too tall to fit. **33.** 4, $-5 \pm i\sqrt{15}$; 4 **35.** $V = \frac{1}{3}\ell^3 - 3\ell^2$ **37.** $\ell = 30$ in., $w = 30$ in., $h = 21$ in. **39.** g **41.** Sample answer: $f(x) = 2x^2 - 8x + 3$ **43.** The polynomial equation that represents the volume of the compartment is $V = h^3 + 3h^2 - 40h$. Measures of the width of the compartment are, in inches, 1, 2, 3, 4, 6, 7, 9, 11, 12, 14, 18, 21, 22, 28, 33, 36, 42, 44, 63, 66, 77, and 84. The solution shows that $h = 14$ in., $\ell = 22$ in., and $w = 9$ in. **45.** G **47.** -4, $2 + i$, $2 - i$ **49.** -7, $5 + 2i$, $5 - 2i$ **51.** $x - 4$, $3x^2 + 2$

Pages 374–378 Chapter 6 Study Guide and Review

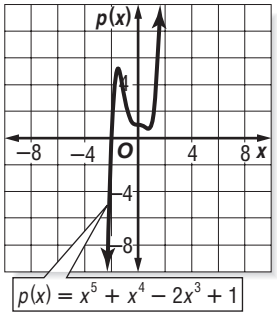
1. relative minimum **3.** quadratic form **5.** scientific notation **7.** depressed polynomial **9.** end behavior **11.** $\frac{1}{f^3}$ **13.** $8xy^4$ **15.** 10.48 times **17.** $4x^2 + 22x - 34$ **19.** $4a^4 + 24a^2 + 36$ **21.** $2x^3 + x - \frac{3}{x - 3}$ **23.** $16x^3 + 4x^2 - 12x + 8$ **25.** 8; $-x - h + 4$ **27.** 21; $x^2 + 2xh + h^2 + 5$ **29.** -129; $2x^3 + 6x^2h + 6xh^2 + 2h^3 - 1$

31a.



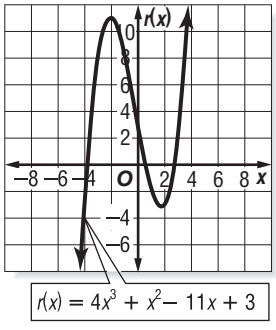
31b. at $x = 3$ **31c.** Sample answer: rel.max. at $x \approx -1.4$, rel. min. at $x \approx 1.4$

33a.



33b. between -2 and -3 **33c.** Sample answer: rel. max. at $x \approx -1.6$, rel. min. at $x \approx 0.8$

35a.



35b. between -2 and -1, between 0 and 1, and between 1 and 2 **35c.** Sample answer: rel. max. at $x \approx -1$, rel. min. at $x \approx 0.9$ **37.** 3: 2 rel. max. and 1 rel.

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min. **39.** $(5w^2 + 3)(w - 4)$ **41.** prime **43.** $-8, 0, 5$ **45.** $4, -2 \pm 2i\sqrt{3}$ **47.** $4, -1$ **49.** $20, -20$ **51.** $x^2 + 2x + 3$ **53.** 3 or 1; 1; 2 or 0 **55.** 3 or 1; 1; 0 or 2 **57.** $(8 - x)(5 - x)(6 - x) = 72$ **59.** $-\frac{1}{2}, 3, 4$ **61.** $1, 2, 4, -3$ **63.** $\frac{1}{2}, 2$

Chapter 7 Radical Equations and Inequalities

Page 383 ***Chapter 7*** ***Get Ready***

1. between 0 and 1, between 4 and 5 **3.** $3x + 4$ **5.** $170 - \frac{170}{t^2 + 1}$

Pages 388–390 ***Lesson 7-1***

1. $4x + 9$; $2x - 1$; $3x^2 + 19x + 20$; $\frac{3x + 4}{x + 5}$, $x \neq -5$ **3.** $\{(-5, 7), (4, 9)\}$; $\{(4, 12)\}$ **5.** $6x - 8$; $6x - 4$ **7.** 30 **9.** 1 **11.** $\frac{3x}{4} - 5$; price of CD when 25% discount is taken and then the coupon is subtracted **13.** Discount first, then coupon; $c[p(49.99)]$ gives a sale price of \$32.49, but $p[c(49.99)]$ gives \$33.74. **15.** $6x + 6$; $-2x - 12$; $8x^2 + 6x - 27$; $\frac{2x - 3}{4x + 9}$, $x \neq -\frac{9}{4}$ **17.** $x^2 + 8x + 15$; $x^2 + 4x + 3$; $2x^3 + 18x^2 + 54x + 54$; $\frac{x + 3}{2}$, $x \neq -3$ **19.** $\frac{x^3 + x^2 - 7x - 15}{x + 2}$, $x \neq -2$; $\frac{x^3 + x^2 - 9x - 9}{x + 2}$, $x \neq -2$; $x^2 - 6x + 9$, $x \neq -2$; $x^2 + 4x + 4$, $x \neq -2, 3$ **21.** $(C - W)(x) = 2x^2 + 7x - 11$ **23.** $\{(2, 4), (4, 4)\}$; $\{(1, 5), (3, 3), (5, 3)\}$ **25.** $\{(4, 5), (2, 5), (6, 12), (8, 12)\}$; does not exist **27.** $\{(2, 3), (2, 2)\}$; $\{(-5, 6), (8, 6), (-9, -5)\}$ **29.** $15x - 5$; $15x + 1$ **31.** $3x^2 - 4$; $3x^2 - 24x + 48$ **33.** $2x^2 - 5x + 9$; $2x^2 - x + 5$ **35.** 50 **37.** 68 **39.** -48 **41.** 1.5 **43.** 104 **45.** 36 **47.** 1,085,000 **49.** $s[p(x)]$; The 30% would be taken off first, and then the sales tax would be calculated on this price. **51.** \$700, \$661.20, \$621.78, \$581.73, \$541.04 **53.** Danette is correct because $[g \circ f](x) = g[f(x)]$ which means you evaluate the f function first and then the g function. Marquan evaluated the functions in the wrong order. **55.** Using the revenue and cost functions, a new function that represents the profit is $p(x) = r(c(x))$. The benefit of combining two functions into one function is that there are fewer steps to compute and it is less confusing to the general population of people reading the formulas.

57. G **59.** $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$

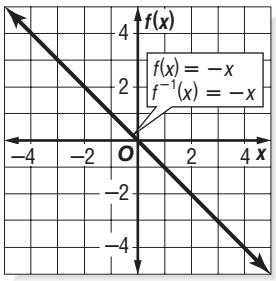
61. 2 or 0; 2 or 0; 4, 2, or 0 **63.** about 1830 times

65. $y = \frac{1 - 4x^2}{-5x}$ **67.** $t = \frac{I}{pr}$ **69.** $m = \frac{Fr^2}{GM}$

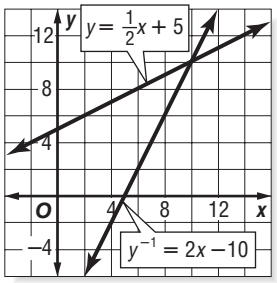
Pages 394–396 ***Lesson 7-2***

1. $\{(4, 2), (1, -3), (8, 2)\}$

3. $f^{-1}(x) = -x$

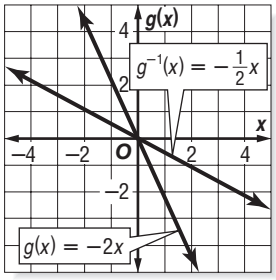


5. $y = 2x - 10$

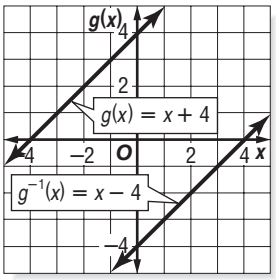


7. 15.24 m/s^2 **9.** no **11.** $\{(8, 3), (-2, 4), (-3, 5)\}$ **13.** $\{(-2, -1), (-2, -3), (-4, -1), (6, 0)\}$ **15.** $\{(8, 2), (5, -6), (2, 8), (-6, 5)\}$

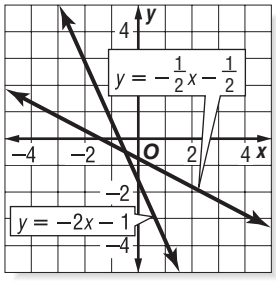
17. $g^{-1}(x) = -\frac{1}{2}x$



19. $g^{-1}(x) = x - 4$



21. $y = -\frac{1}{2}x - \frac{1}{2}$



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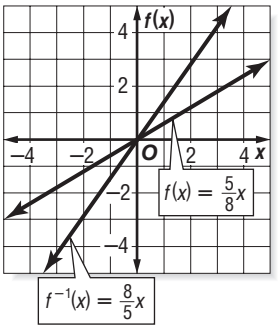
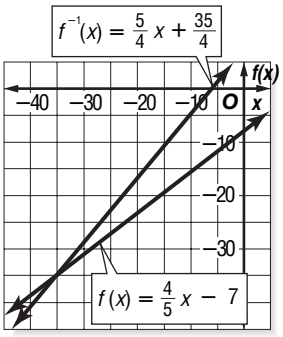
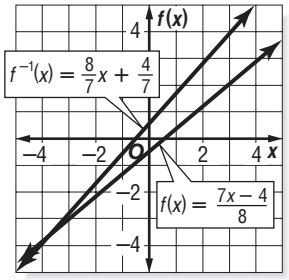
23. $f^{-1}(x) = \frac{8}{5}x$



25. $f^{-1}(x) = \frac{5}{4}x + \frac{35}{4}$



27. $f^{-1}(x) = \frac{8}{7}x + \frac{4}{7}$



29. ≈ 3.39 cm **31.** no **33.** yes **35.** yes

37. $y = 2x + 7$ **39.** $F^{-1}(x) = \frac{5}{9}(x - 32)$; $F[F^{-1}(x)] = F^{-1}[F(x)] = x$. **41.** n is an odd whole number.

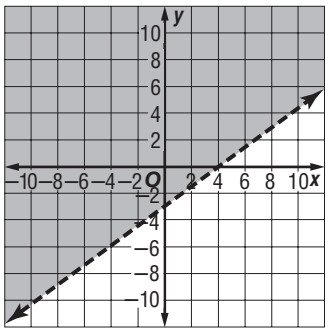
43. Sample answer: $f(x) = x$ and $f^{-1}(x) = x$ or $f(x) = -x$ and $f^{-1}(x) = -x$. **45.** A **47.** 6 **49.** 4

51. $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm1, \pm\frac{5}{4}, \pm2, \pm\frac{5}{2}, \pm4, \pm5, \pm10, \pm20$

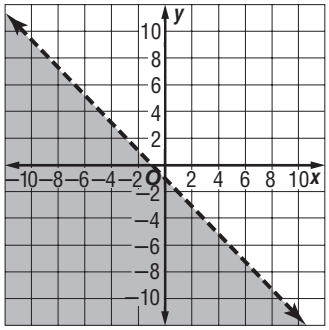
53. $\begin{bmatrix} 1 & 4 \\ -5 & -4 \end{bmatrix}$ **55.** consistent and independent

57. −5 **59.** −2, 4 **61.** $\{x \mid x > 6\}$

63.

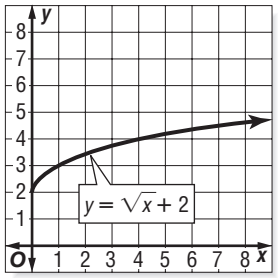


65.

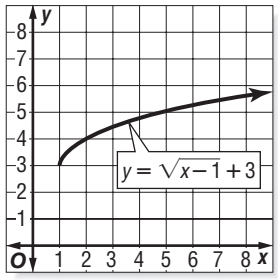


Pages 399–401 ***Lesson 7-3***

1. D: $x \geq 0$, R: $y \geq 2$

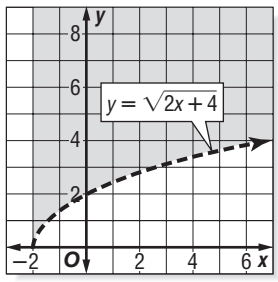


3. D: $x \geq 1$; R: $y \geq 3$

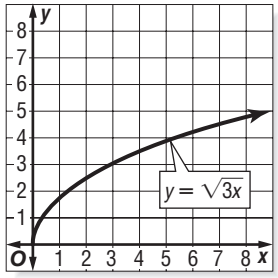


5. Yes; sample answer: the advertised pump will reach a maximum height of 87.9 ft.

7.



9. D: $x \geq 0$, R: $y \geq 0$



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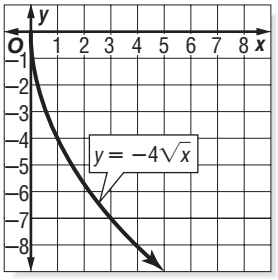
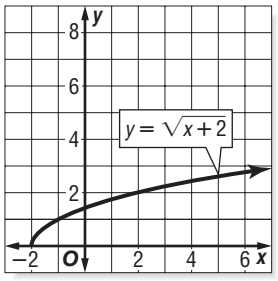
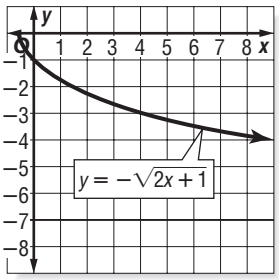
11. D: $x \geq 0$, R: $y \leq 0$



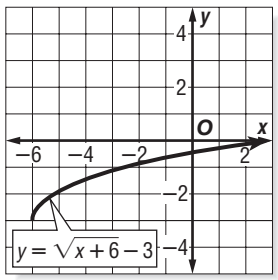
13. D: $x \geq -2$, R: $y \geq 0$



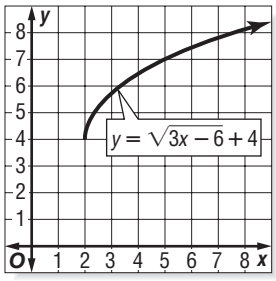
15. D: $x \geq -0.5$, R: $y \leq 0$



17. D: $x \geq -6$, R: $y \geq -3$

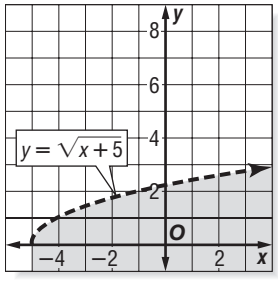


19. D: $x \geq 2$, R: $y \geq 4$

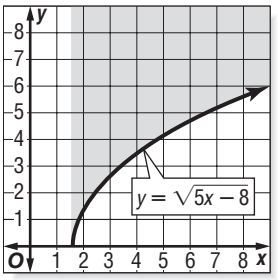


21. $h > 125$ ft **23.** ≈133.25 lb

25.



27.



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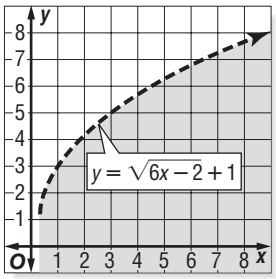
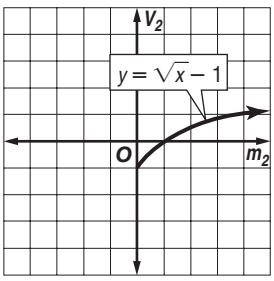


31. If a is negative, the graph is reflected over the x-axis. The larger the value of a, the less steep the graph. If h is positive, the graph is translated to the right, and if h is negative, the graph is translated to the left. When k is positive, the graph is translated up, and when k is negative, the graph is translated down. **33.** Square root functions are used in bridge design because the engineers must determine what diameter of steel cable needs to be used to support a bridge based on its weight. Sample answer: when the weight to be supported is less than 8 tons; 13,608 tons **35.** G **37.** no **39.** $(f + g)(x) = 2x + 2$; $(f - g)(x) = 2$; $(f \cdot g)(x) = x^2 + 2x - 15$; $(f \div g)(x) = \frac{x + 5}{x - 3}$ **41.** $(f + g)(x) = \frac{8x^3 + 12x^2 - 18x - 26}{2x + 3}$; $(f - g)(x) = \frac{8x^3 + 12x^2 - 18x - 28}{2x + 3}$; $(f \cdot g)(x) = 2x - 3$; $(f \div g)(x) = 8x^3 + 12x^2 - 18x - 27$ **43.** rational **45.** rational **47.** irrational

Pages 405–406 Lesson 7-4

1. 4 **3.** −3 **5.** x **7.** $6|a|b^2$ **9.** 8.775 **11.** 2.632 **13.** 15 **15.** not a real number **17.** −3 **19.** $\frac{1}{4}$ **21.** 0.5 **23.** z^2 **25.** $7|m^3|$ **27.** $3r$ **29.** $25g^2$ **31.** $5x^2|y^3|$ **33.** $13x^4y^2$ **35.** $2ab$ **37.** 11.358 **39.** 0.933 **41.** 3.893 **43.** 4.953 **45.** 4.004 **47.** 26.889 **49.** about 4088×10^8 m **51.** Sample answer: 64 **53.** $x = 0$ and $y \geq 0$, or $y = 0$ and $x \geq 0$ **55.** The radius and volume of a sphere can be related by an expression containing a cube root. As the value of V increases, the value of r increases. **57.** G

59.



D = {$x \mid x \geq 0$},
R = {$y \mid y \geq -1$}

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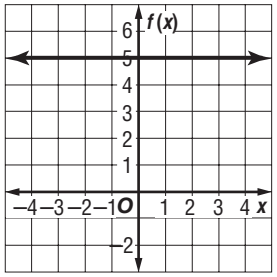
61. no **63.** $29 - 28i$ **65.** $(3, -1)$ **67.** $(-4, 6)$

69.



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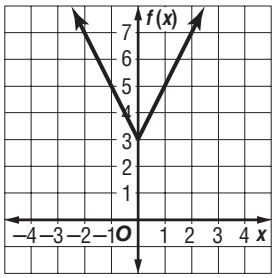


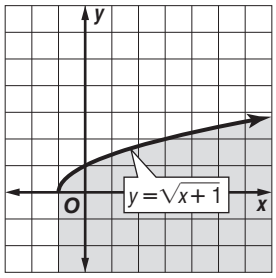
73. $y^2 + 3y - 10$ **75.** $a^2 + 3ab + 2b^2$
77. $6w^2 - 7wz - 5z^2$

Pages 412–414 Lesson 7-5

1. $15\sqrt{7}$ **3.** $5|xy^3|\sqrt{3x}$ **5.** $\frac{|a^3|\sqrt{ab}}{|b^5|}$ **7.** $s = 2\sqrt{5\ell}$
9. $-24\sqrt{35}$ **11.** $\sqrt[3]{25}$ **13.** $22\sqrt[3]{2}$ **15.** 2 **17.** $\frac{19 - 7\sqrt{7}}{2}$
19. $6\sqrt{2}$ **21.** $2\sqrt[4]{6}$ **23.** $2y\sqrt[3]{2}$ **25.** $2|a|b^2\sqrt{10a}$
27. $4mn\sqrt[3]{3mn^2}$ **29.** $\frac{1}{2}wz\sqrt[5]{wz^2}$ **31.** $\frac{\sqrt[4]{54}}{3}$ **33.** $\frac{2r^4Rt}{|t^5|}$
35. $-60\sqrt{30}$ **37.** $40\sqrt{3}$ feet **39.** $5\sqrt{2}$ **41.** $4\sqrt{5} + 23\sqrt{6}$ **43.** $6 + 3\sqrt{6} + 2\sqrt{7} + \sqrt{42}$ **45.** $8 - 2\sqrt{15}$
47. $\frac{5\sqrt{6} - 3\sqrt{2}}{22}$ **49.** $\frac{12 + 7\sqrt{2}}{23}$ **51.** $\sqrt{x} + 1$ **53.** $\frac{\sqrt{10}}{5}$
55. 0 ft/s **57.** about 18.18 m **59.** Sample answer: $2\sqrt{2} + \sqrt{3} + 3\sqrt{27}$; Simplify the term $3\sqrt{27}$ to $9\sqrt{3}$. Then combine $\sqrt{3}$ and $9\sqrt{3}$. The simplified expression is $2\sqrt{2} + 10\sqrt{3}$.

61. The ratio of the lengths of the sides of the rectangle around the face is $\frac{2}{\sqrt{5} - 1}$. You can simplify this expression by multiplying the numerator and denominator by the conjugate of the denominator. The new expression is $\frac{\sqrt{5} + 1}{2}$. **63.** G **65.** $6ab^3$

67.



69. $-\frac{1}{2}\begin{bmatrix} 5 & -6 \\ -7 & 8 \end{bmatrix}$ **71.** does not exist **73.** $\frac{1}{2}$ **75.** $\frac{13}{12}$
77. $\frac{19}{30}$ **79.** $-\frac{5}{12}$

Pages 419–421 Lesson 7-6

1. $\sqrt[3]{7}$ **3.** $26^{\frac{1}{4}}$ **5.** 5 **7.** 9 **9.** \$5.11 **11.** $x^{\frac{2}{3}}$ **13.** $\sqrt{3x}$
15. $\frac{x + 2x^{\frac{1}{2}} + 1}{x - 1}$ **17.** $\sqrt[5]{6}$ **19.** $\sqrt[5]{c^2}$ or $(\sqrt[5]{c})^2$ **21.** $23^{\frac{1}{2}}$
23. $2z^{\frac{1}{2}}$ **25.** 2 **27.** $\frac{1}{5}$ **29.** 81 **31.** $\frac{4}{3}$ **33.** about 4.62 in.
35. y^4 **37.** $b^{\frac{1}{5}}$ **39.** $\frac{w^{\frac{1}{5}}}{w}$ **41.** $\frac{a^{\frac{5}{12}}}{6a}$ **43.** $\sqrt{5}$ **45.** $17\sqrt[6]{17}$
47. $\frac{xy\sqrt{z}}{z}$ **49.** $2\sqrt{6} - 5$ **51.** $6r^{\frac{3}{4}}s^{\frac{3}{4}}$ **53.** $2^{\frac{3}{2}} + 3^{\frac{1}{2}}$

55. about 336 **57.** In radical form, the expression would be $\sqrt{-16}$, which is not a real number because the index is even and the radicand is negative.
59. Always; in exponential form $\sqrt[n]{b^m}$ equals $(b^m)^{\frac{1}{n}}$. By the Power of a Power Property, $(b^m)^{\frac{1}{n}} = b^{\frac{m}{n}}$. But, $b^{\frac{m}{n}}$ is also equal to $\left(\frac{1}{b^n}\right)^m$ by the Power of a Power Property. This last expression is equal to $\left(\sqrt[n]{b}\right)^m$. Thus, $\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$. **61.** B **63.** $2|xy|\sqrt{x}$ **65.** $2\sqrt{2}$
67. $\sqrt{x - 5}$ **69.** $[K \circ C](F) = \frac{5}{9}(F - 32) + 273$
71. 2.5 s **73.** $2x - 3$ **75.** $4x - 12\sqrt{x} + 9$

Pages 425–427 Lesson 7-7

1. 2 **3.** no solution **5.** 18 **7.** 9 **9.** $0 \le b < 4$ **11.** 16
13. no solution **15.** -1 **17.** no solution
19. no solution **21.** 3 **23.** 9 **25.** -20 **27.** $\frac{1}{3}$
29. about 1.82 ft **31.** $x > 1$ **33.** $x \le -11$
35. $0 \le x \le 2$ **37.** $b \ge 5$ **39.** 34 ft

41.
$$\frac{\sqrt{(x^2)^2}}{-x} = x$$
$$\frac{\sqrt{(x^2)^2}}{-x} = x$$
$$\frac{x^2}{-x} = x$$
$$x^2 = (x)(-x)$$

$x^2 \neq -x^2$, Never.

43. Sample answer: $\sqrt{x} + \sqrt{x + 3} = 3$

45. If a company's cost and number of units manufactured are related by an equation involving radicals or rational exponents, then the production level associated with given cost can be found by solving a radical equation.

$$C = 10\sqrt[3]{n^2} + 1500$$
$$10{,}000 = 10n^{\frac{2}{3}} + 1500 \quad C = 10{,}000$$
$$8500 = 10n^{\frac{2}{3}} \quad \text{Subtract 1500 from each side.}$$
$$850 = n^{\frac{2}{3}} \quad \text{Divide each side by 10.}$$
$$850^{\frac{3}{2}} = n \quad \text{Raise each side to the } \frac{3}{2} \text{ power.}$$
$$24{,}781.55 \approx n \quad \text{Use a calculator.}$$

Round down so that the cost does not exceed \$10,000. The company can make at most 24,781 chips. **47.** G

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49. $5^{\frac{3}{7}}$ **51.** $(x^2+1)^{\frac{2}{3}}$ **53.** $\frac{\sqrt[3]{100}}{10}$ **55.** $I(m) = 320 + 0.04m$; \$4500 **57.** $(f+g)(x) = x^2 + x - 2$; $(f-g)(x) = x^2 - x - 6$; $(f \cdot g)(x) = x^3 + 2x^2 - 4x - 8$; $(f \div g)(x) = x - 2$; $x \neq -2$

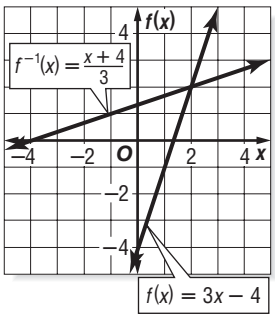
59. 4; If x is your number, you can write the expression $\frac{3x + x + 8}{x + 2}$, which equals 4 after dividing the numerator and denominator by the GCF, $x + 2$.

61. $6p^2 - 2p - 20$ **63.** Sample answer: $y = 0.79x + 4.93$

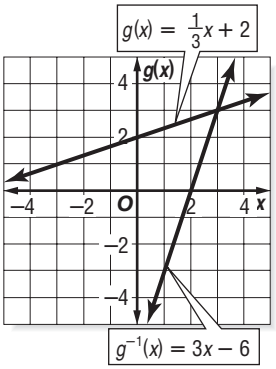
Pages 430–434 Chapter 7 Study Guide and Review

1. radical equation **3.** like radical expressions **5.** inverse functions **7.** one-to-one **9.** inverse relations **11.** $x^2 - 1$; $x^2 - 6x + 11$ **13.** $-15x - 5$; $-15x + 25$ **15.** $|x + 4|$; $|x| + 4$

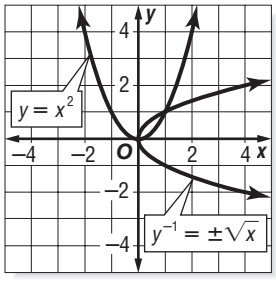
17. $f^{-1}(x) = \frac{x+4}{3}$



19. $g^{-1}(x) = 3x - 6$

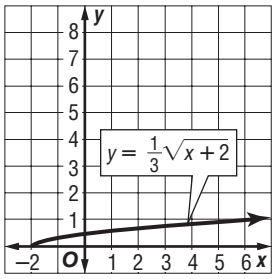


21. $y^{-1}(x) = \pm\sqrt{x}$

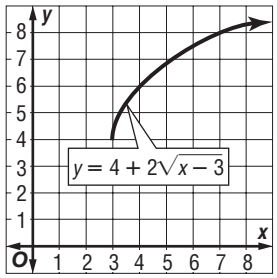


23. $I(m) = 400 + 0.1m$; \$6000

25. D: $x \geq -2$, R: $y \geq 0$



27. D: $x \geq 3$, R: $y \geq 4$



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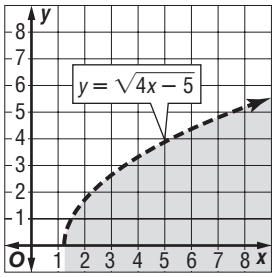

31. ± 16 **33.** 8 **35.** $|x^4 - 3|$ **37.** $2m^2$ **39.** 10 meters per second **41.** $-5\sqrt{3}$ **43.** $20 + 8\sqrt{6}$ **45.** 9

47. $\frac{2\sqrt{10} - \sqrt{5}}{7}$ **49.** $\frac{1}{9}$ **51.** $\frac{9}{4}$ **53.** $\frac{xyz^{\frac{2}{3}}}{z}$ **55.** 6.3 amps

57. 343 **59.** 4 **61.** 5 **63.** 8 **65.** $x > \frac{11}{5}$ **67.** $x \geq 21$

69. $d > -\frac{3}{4}$ **71.** 1 m

Chapter 8 Rational Expressions and Equations

Page 441 Chapter 8 Get Ready

1. $\frac{1}{6}$ **3.** $\frac{5}{8}$ **5.** 16 **7.** $2\frac{1}{2}$ **9.** \$17.50 **11.** 12 **13.** 15 **15.** 15 **17.** 6 **19.** $7\frac{1}{2}$ **21.** \$5250

Pages 446–449 Lesson 8-1

1. $\frac{9m}{4n^4}$ **3.** $x + 3$ **5.** D **7.** $\frac{-b^2 - ab - a^2}{a + b}$ **9.** $\frac{6}{5}$ **11.** $\frac{8y}{9x^2}$ **13.** $\frac{3x + 9}{4x + 24}$ **15.** $\frac{2y(y-2)}{3(y+2)}$ **17.** $-\frac{n}{7m}$ **19.** $\frac{1}{2}$ **21.** $-\frac{t+3}{t+4}$ **23.** $-\frac{4bc}{27a}$ **25.** $-2p^2$ **27.** $\frac{4}{3}$ **29.** $\frac{p-7}{p+7}$ **31.** $-2p$ **33.** $\frac{2x + y}{2x - y}$ **35.** $d = -2, -1$, or 2

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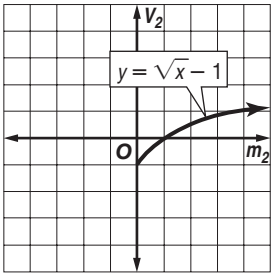
37. $\frac{1}{2}(8x^2 + 18x - 5)$ m^2 **39.** $\frac{s}{3}$ **41.** $\frac{y+2}{3y-1}$
43. $\frac{3x+4}{3(x+2)}$ **45.** $-\frac{b-3}{b+3}$ **47.** $\frac{5by}{3ax}$ **49.** $\frac{xz}{8y}$
51. $\frac{3(r+4)}{r+3}$ **53.** $a = -b$ or b **55.** $\frac{5422+m}{12{,}138+a}$

57. $\frac{25}{27}$; Sample answer: the second airplane travels a bit further than the first airplane. **59.** $-3x + 2$; The expression defines the function $g(x)$. **61.** The tables are the same except for $f(x)$ the value $f(0)$ is undefined.

63. Sample answer: $\frac{x-4}{2}, \frac{3x-12}{6}$ **65.** $\frac{x+1}{\sqrt{x+3}}$ does not belong with the other three. The other three expressions are rational expressions. Since the denominator of $\frac{x+1}{\sqrt{x+3}}$ is not a polynomial, $\frac{x+1}{\sqrt{x+3}}$ is not a rational expression. **67.** A rational expression can be used to express the fraction of a nut mixture that is peanuts. The expression $\frac{8+x}{13+x+y}$ could be used to represent the fraction that is peanuts if x pounds of peanuts and y pounds of cashews were added to the original mixture. **69.** F

71.



D = $\{x \mid x \geq 0\}$, R = $\{y \mid y \geq -1\}$

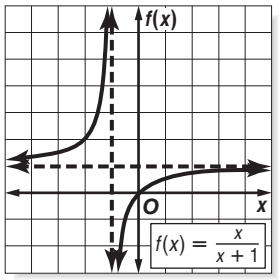
73. no **75.** odd; 3 **77.** about 4.99×10^2 s or about 8 min 19 s
79. $\left\{-\frac{1}{6}, \frac{1}{3}\right\}$ **81.** $\varnothing$ **83.** $-1\frac{1}{9}$ **85.** $1\frac{4}{15}$ **87.** $-\frac{11}{18}$

Pages 453–456 Lesson 8-2

1. $12x^2y^2$ **3.** $x(x-2)(x+2)$ **5.** $\frac{2-x^3}{x^2y}$ **7.** $\frac{37}{42m}$
9. $\frac{5d+16}{(d+2)^2}$ **11.** $\frac{x^2-2x+1}{(x+2)(x-2)}$ **13.** $\frac{8}{5}$ **15.** $\frac{2}{x+2}$
17. $4x + 8$ **19.** $180x^2yz$ **21.** $x^2(x-y)(x+y)$
23. $\frac{31}{12v}$ **25.** $\frac{25-7ab}{5a^2b}$ **27.** $\frac{a+3}{a-4}$ **29.** $\frac{y(y-9)}{(y+3)(y-3)}$
31. $\frac{-8d+20}{(d-4)(d+4)(d-2)}$ **33.** -1 **35.** -1 **37.** $36p^3q^4$
39. $(n-4)(n-3)(n+2)$ **41.** $\frac{2x+15y}{3y}$ **43.** $\frac{110w-423}{90w}$
45. $\frac{x^2-6}{(x+2)^2(x+3)}$ **47.** $\frac{2y^2+y-4}{(y-1)(y-2)}$ **49.** $\frac{a+7}{a+2}$
51. $\frac{3x-4}{2x(x-2)}$ **53.** $\frac{24}{x}$ h **55.** $\frac{48(x-2)}{x(x-4)}$ h **57.** Sample answer: $d^2 - d, d + 1$ **59.** Sample answer: $\frac{1}{x+1}, \frac{1}{x-2}$
61. Subtraction of rational expressions can be used to determine the distance between the lens and the film if the focal length of the lens and the distance between the lens and the object are known. $\frac{1}{q} = \frac{1}{50} - \frac{1}{1000}$ could be used to determine the distance between the lens and the film if the focal length of the lens is 50 mm and the distance between the lens and the object is 1000 mm. **63.** F **65.** $\frac{a(a+2)}{a+1}$ **67.** $\pm i, \pm 3$
69. 5.0 ft **71.** $(x+1)(x+2)$ **73.** $(x+12)(x-1)$
75. $3(x-5)(x+5)$

Pages 460–463 Lesson 8-3

1. asymptote: $x = 2$ **3.**



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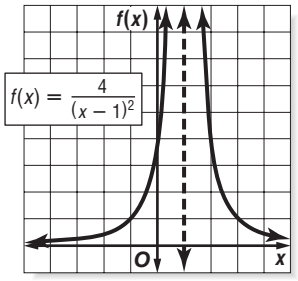

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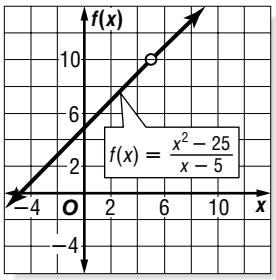

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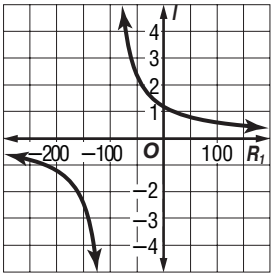
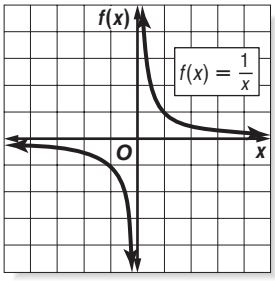


11. 0.5 amperes
13. asymptotes: $x = 2$, $x = 3$ **15.** asymptote: $x = -4$; hole: $x = -3$

17.



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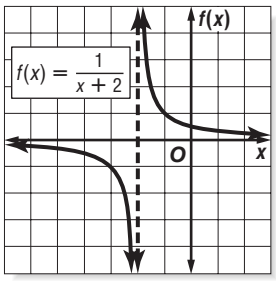

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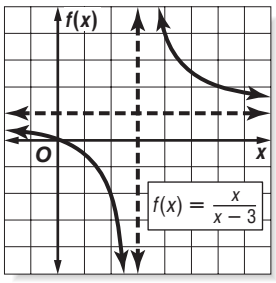

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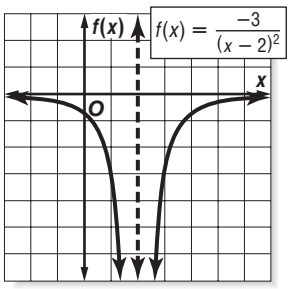

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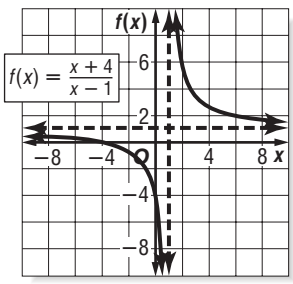

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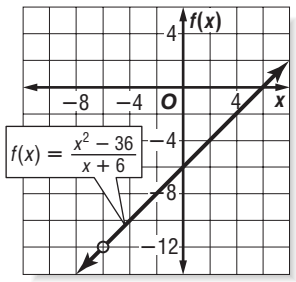

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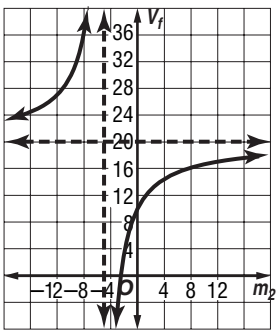
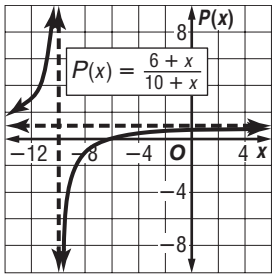


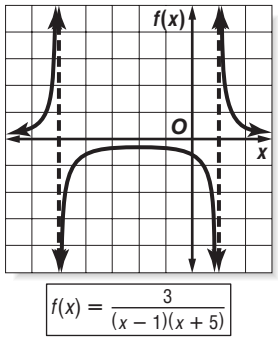
31. $m_2 = -5$, $V_f = 20$; -2.5; 10

33.

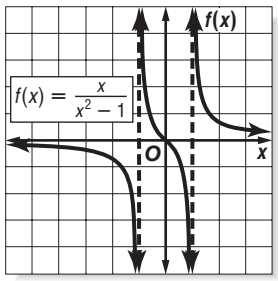


35. It represents her original free-throw percentage of 60%. **37.** hole: $x = 4$

39.



41.



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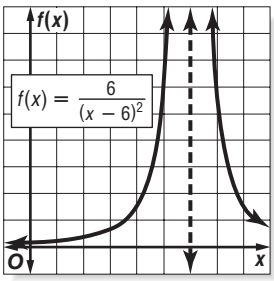

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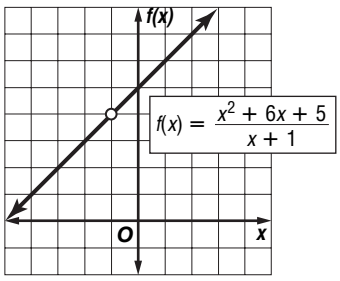

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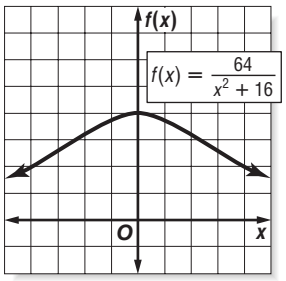
47.



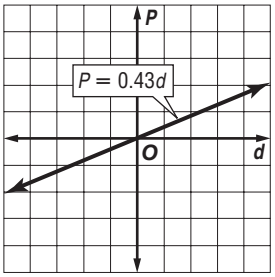
49. Since $\frac{-64}{x^2 + 16} = -\left(\frac{64}{x^2 + 16}\right)$, the graph of $f(x) = \frac{-64}{x^2 + 16}$ is the reflection image of the graph of $f(x) = \frac{64}{x^2 + 16}$ over the x-axis. **51.** Each of the graphs is a straight line passing through $(-5, 0)$ and $(0, 5)$. However, the graph of $f(x) = \frac{(x - 1)(x + 5)}{x - 1}$ has a hole at $(1, 6)$, and the graph of $g(x) = x + 5$ does not have a hole. **53.** Sample answers: $f(x) = \frac{x + 2}{(x + 2)(x - 3)}$, $f(x) = \frac{2(x + 2)}{(x + 2)(x - 3)}$, $f(x) = \frac{5(x + 2)}{(x + 2)(x - 3)}$ **55.** A **57.** $\frac{3m + 4}{m + n}$ **59.** $\frac{5(w - 2)}{(w + 3)^2}$ **61.** $-\frac{1}{2}$, 2, 3 **63.** $\{-4 \pm 2i\}$ **65.** $\left\{\frac{-7 \pm 3\sqrt{13}}{2}\right\}$ **67.** 4.5 **69.** 20

Pages 468–471 Lesson 8-4

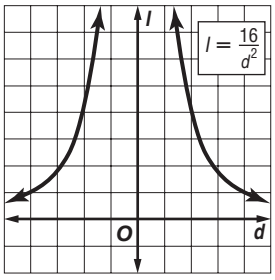
1. 24 **3.** −8 **5.** 25.8 psi

7.

Depth (ft)	Pressure (psi)
0	0
1	0.43
2	0.86
3	1.29
4	1.72



9. 20 **11.** 64 **13.** 4 **15.** about 359.6 mi **17.** direct; 3 **19.** joint; $\frac{1}{3}$ **21.** inverse; 2.5 **23.** direct; −7 **25.** −12.6 **27.** $2\frac{1}{4}$ **29.** 1.25 **31.** $V = \frac{k}{P}$ **33.** $\ell = 15md$ **37.** joint **39.** 30 mph **43.**

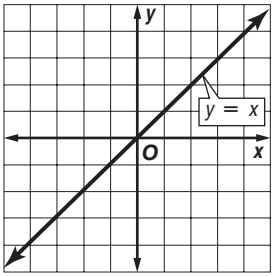


45. about 2×10^{20} Newtons **47.** 6.67×10^{-3} Newtons **49.** Sample answer: If the average student spends $2.50 for lunch in the school cafeteria, write an equation to represent the amount s students will spend for lunch in d days. How much will 30 students spend in a week? $a = 2.50sd$; $375 **51.** C **53.** asymptote: $x = 1$; hole: $x = -1$ **55.** hole: $x = -3$ **57.** $\frac{t^2 - 2t - 2}{(t + 2)(t - 2)}$ **59.** 1×10^{14} **61.** 3; 7 **63.** C **65.** S **67.** A

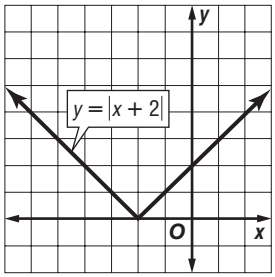
Pages 476–478 Lesson 8-5

1. greatest integer **3.** constant **5.** b

7. direct variation

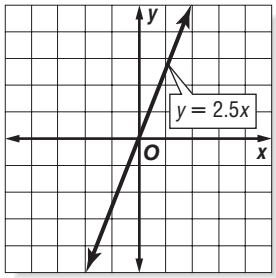


9. absolute value

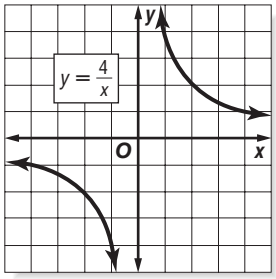


11. square root **13.** direct variation **15.** constant

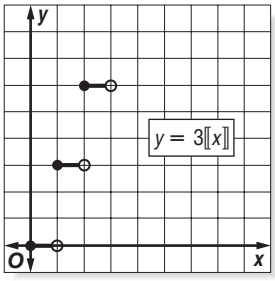
17. direct variation



19. inverse variation or rational



21. greatest integer



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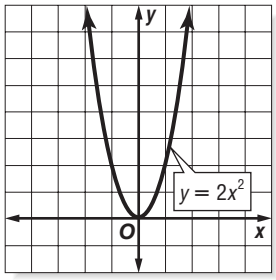
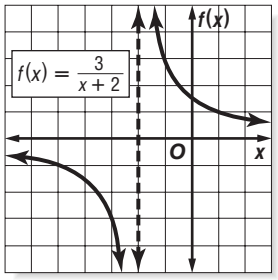
23. quadratic

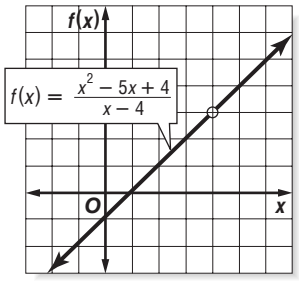


25. e **27.** a **29.** direct variation **31.** parabola **33.** The graph is similar to the graph of the greatest integer function because both graphs look like a series of steps. In the graph of the postage rates, the solid dots are on the right and the circles are on the left. However, in the greatest integer function, the circles are on the right and the solid dots are on the left. **35a.** absolute value **35b.** quadratic **35c.** greatest integer **35d.** square root **37.** There are several types of functions. Each type of function has features which distinguish it from other types. Knowing which features are characteristic of each type of graph can help you determine which type of function best describes the relationship between two quantities. **39.** G

41.



43.



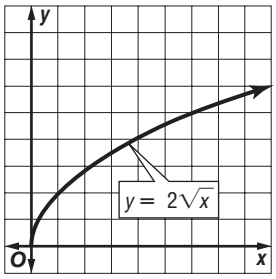
45. $-7, \frac{3}{2}$ **47.** 120 m **49.** $45x^3y^3$ **51.** $3(x - y)(x + y)$ **53.** $(t - 5)(t + 6)(2t + 1)$

Pages 484–486 Lesson 8-6

1. 3 **3.** $\frac{2}{3}$ **5.** 3 **7.** $2\frac{2}{9}$ h; The answer is reasonable. The time to complete the job when working together must be less than the time it would take either person working alone. **9.** $v < 0$ or $v > 1\frac{1}{6}$ **11.** $-\frac{4}{3}$ **13.** $-3, 2$ **15.** 2 **17.** $\varnothing$ **19.** $-1 < m < 1$ **21.** $0 < b < 1$

23. 2 or 4 **25.** $\frac{7}{3}$ **27.** $\frac{1 \pm \sqrt{145}}{4}$ **29.** $p < 0$ or $p > 2\frac{1}{2}$ **31.** 4.8 cm/g **33.** 15 km/h; With the wind, Alfonso's speed would be 18 km/h, and his 36-km trip would take 2 hours. Against the wind, his speed would be 12 km/h, and his 24-km trip would take 2 hours. The answer makes sense. **35.** $\{x \mid x < -2 \text{ or } x > 1\}$ **37.** $\frac{80}{13}$ **39.** Jeff; when Dustin multiplied by $3a$, he forgot to multiply the 2 by $3a$. **41.** If something has a general fee and cost per unit, rational equations can be used to determine how many units a person must buy in order for the actual unit price to be a given number. Since the cost is \$1.00 per download plus \$15.00 per month, the actual cost per download could never be \$1.00 or less. **43.** J

45. square root

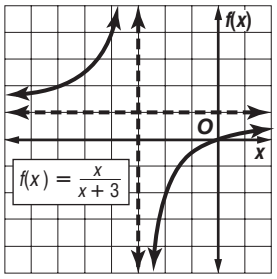


47. 36 **49.** $\{x \mid 0 \le x \le 4\}$ **51.** $\begin{bmatrix} -25 & 23 & -54 \\ 66 & -26 & 57 \end{bmatrix}$ **53.** 196 beats per min **55.** 12

Pages 489–192 Chapter 8 Study Guide and Review

1. false; point discontinuity **3.** false; rational **5.** true **7.** true **9.** $-\frac{4bc}{33a}$ **11.** $\frac{x + 2}{x - 3}$ **13.** $(y + 3)(y - 6)$ **15.** $4(x + 4)$ m **17.** $\frac{7}{5(x + 1)}$ **19.** $\frac{18}{y - 2}$ **21.** $\frac{3(3m^2 - 14m + 27)}{(m + 3)(m - 3)^2}$ **23.** ≈5.8

25.



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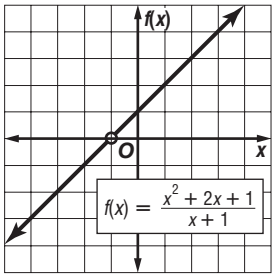

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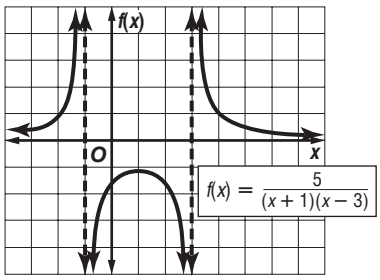
29.



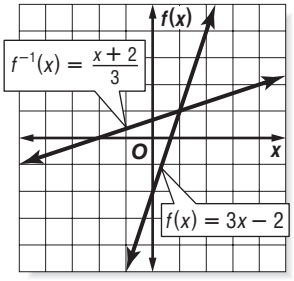
31. $-1\frac{2}{3}$ **33.** $\frac{32}{121}$ **35.** 17.6 **37.** square root **39.** $1\frac{1}{9}$
41. 0 **43.** $-2\frac{1}{2} < b < 0$

Chapter 9 Exponential and Logarithmic Relations

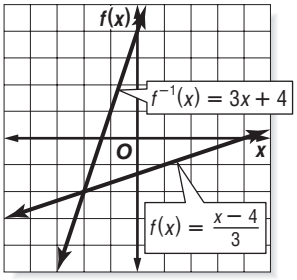
Page 497 ***Chapter 9*** ***Get Ready***

1. x^{12} **3.** $-\frac{12x^3}{7y^5z}$ **5.** $\approx 1.9 \times 10^4$ kg/m^3

7. $f^{-1}(x) = \frac{x+2}{3}$



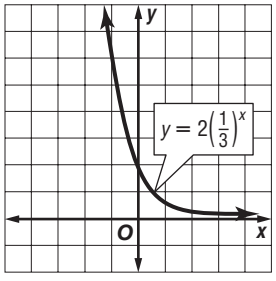
9. $f^{-1}(x) = 3x + 4$



11. $275.77

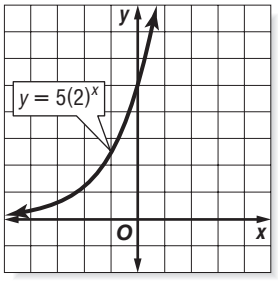
Pages 503–506 ***Lesson 9-1***

1. c **3.** b **5.** D = all real numbers, R = $\{y|y > 0\}$

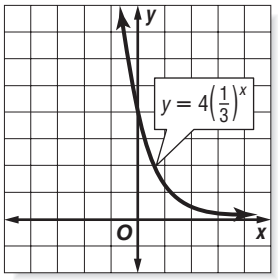


7. growth **9.** $y = -18(3)^x$ **11.** about $2,578,760; Yes, the money is continuing to grow at a faster rate each year. In the first 10 years it grew by $678,000, and in the next ten years it grew about $900,000. **13.** 2 **15.** $x \leq 0$
17. $a \leq -3$

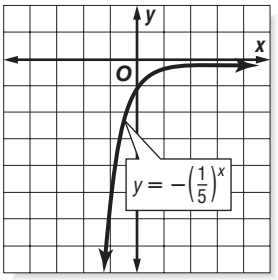
19. D = all real numbers, R = $\{y|y > 0\}$



21. D = all real numbers, R = $\{y|y > 0\}$

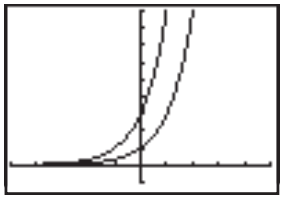


23. growth **25.** growth **27.** decay **29.** $y = 3(5)^x$
31. $y = -5\left(\frac{1}{3}\right)^x$ **33.** $y = -0.3(2)^x$ **35.** about 1,008,290
37. $A(t) = 1000(1.01)^{4t}$ **39.** $\frac{2}{3}$ **41.** $-\frac{8}{3}$ **43.** $\frac{5}{3}$ **45.** $n > 5$
47. $n < 3$ **49.** D = all real numbers. R = $\{y|y > 0\}$



51. $s \cdot 4^x$ **53.** $y = 3.93(1.35)^x$ **55.** 2144.87 million; 281.42 million; No, the growth rate has slowed considerably. The population in 2000 was much smaller than the equation predicts it would be.

57.



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The graphs have the same shape. The graph of $y = 3^{x+1}$ is the graph of $y = 3^x$ translated one unit to the left. The asymptote for the graphs $y = 3^x$ and for $y = 3^{x+1}$ is the line $y = 0$. The graphs have the same domain, all real numbers, and range, $y > 0$. The y-intercept of the graph of $y = 3^x$ is 1 and of the graph of $y = 3^{x+1}$ is 3.

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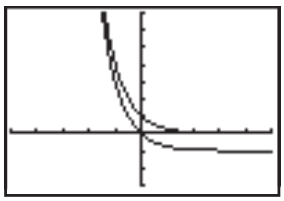
59.



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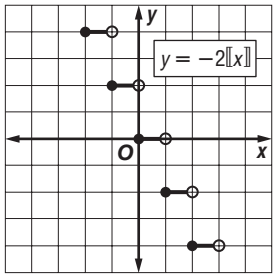
The graphs have the same shape. The graph of $y = \left(\frac{1}{4}\right)^x - 1$ is the graph of $y = \left(\frac{1}{4}\right)^x$ translated one unit down. The asymptote for the graph of $y = \left(\frac{1}{4}\right)^x$ is the line $y = 0$ and for the graph of $y = \left(\frac{1}{4}\right)^x - 1$ is the line $y = -1$. The graphs have the same domain, all real numbers, but the range of $y = \left(\frac{1}{4}\right)^x$ is $y > 0$ and of $y = \left(\frac{1}{4}\right)^x - 1$ is $y > -1$. The y-intercept of the graph of $y = \left(\frac{1}{4}\right)^x$ is 1 and for the graph of $y = \left(\frac{1}{4}\right)^x - 1$ is 0. **61.** Sample answer: 0.8 **63.** Sometimes; true when $b > 1$, but false when $b < 1$. **65.** A **67.** 1, 15 **69.** $-\frac{13}{3}$, 3 **71.** greatest integer



73. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **75.** $\frac{1}{51}\begin{bmatrix} 3 & -6 \\ 11 & -5 \end{bmatrix}$ **77.** $g[h(x)] = 2x - 6$; $h[g(x)] = 2x - 11$ **79.** $g[h(x)] = -2x - 2$; $h[g(x)] = -2x + 11$

Pages 514–517 Lesson 9-2

1. $\log_5 625 = 4$ **3.** $\log_3 243 = 5$ **5.** $36^{\frac{1}{2}} = 6$ **7.** 4 **9.** 3 **11.** 1000 **13.** 10^{13} **15.** $10^{5.5}$ or about 316,228 times **17.** $\{x|\frac{1}{2} < x \leq 5\}$ **19.** $\frac{1}{2}$, 1 **21.** $x > 6$ **23.** $5^3 = 125$ **25.** $4^{-1} = \frac{1}{4}$ **27.** $8^{\frac{2}{3}} = 4$ **29.** $\log_8 512 = 3$ **31.** $\log_5 \frac{1}{125} = -3$ **33.** $\log_{100} 10 = \frac{1}{2}$ **35.** 4 **37.** $\frac{1}{2}$ **39.** −5 **41.** −3 **43.** $3x$ **45.** 125 **47.** ±3 **49.** 11 **51.** $10^{10.67}$ **53.** $0 < y \leq 8$ **55.** $x \geq 24$ **57.** ±8 **59.** $a > 3$

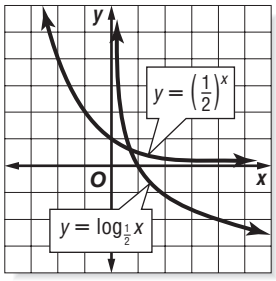
61. $\log_{16} 2 \cdot \log_2 16 \stackrel{?}{=} 1$ Original equation

$\log_{16} 16^{\frac{1}{4}} \cdot \log_2 2^4 \stackrel{?}{=} 1$ $2 = 16^{\frac{1}{4}}$ and $16 = 2^4$

$\frac{1}{4}(4) \stackrel{?}{=} 1$ Inverse Property of Exponents and Logarithms

$1 = 1$ ✓

63.



The graphs are reflections of each other over the line $y = x$. **65.** 10^3 or about 1000 times as great **67.** $10^{1.7}$ or about 50 times **69.** D = $\{x \mid x > 0\}$, D = $\{x \mid x > 0\}$, D = $\{x \mid x > 1\}$, D = $\{x \mid x > -2\}$, respectively; R = {all reals} **71.** $\log_2 16 = 4$; all other choices are equal to 2 **73.** All powers of 1 are 1, so the inverse of $y = 1^x$ is not a function. **75.** B **77.** $x^{2\sqrt{6}}$ **79.** ∅ **81.** $\pm\frac{7}{3}$ **83.** \$4000, CD; \$4000, savings **85.** $8a^6b^3$ **87.** 1

Pages 524–526 Lesson 9-3

1. 2.6309 **3.** 1.1403 **5.** Mt. Everest: 26,855.44 pascals; Mt. Trisuli: 34,963.34 pascals; Mt. Bonete: 36,028.42 pascals; Mt. McKinley: 39,846.22 pascals; Mt. Logan: 41,261.82 pascals **7.** 2.5840 **9.** 2 **11.** 4 **13.** 2.1133 **15.** −0.2519 **17.** 0.1788 **19.** 1.2921 **21.** 2 **23.** 4 **25.** 2 **27.** 4 **29.** 14 **31.** $\frac{x^3}{4}$ **33.** 2 **35.** $\frac{3}{2}$ **37.** 10 **39.** 3 **41.** About 95 decibels; $L = 10 \log_{10} R$, where L is the loudness of the sound in decibels and R is the relative intensity of the sound. Since the crowd increased by a factor 3, we assume that the intensity also increases by a factor of 3. Thus, we need to find the loudness of $3R$. $L = 10 \log_{10} 3R$; $L = 10 (\log_{10} 3 + \log_{10} R)$; $L = 10 \log_{10} 3 + 10 \log_{10} R$; $L \approx 10(0.4771) + 90$; $L \approx 4.771 + 90$ or about 95 **43.** 7.5

45.

$m^p = m^p$	Reflexive property
$(b^{\log_b m})^p = b^{\log_b (m^p)}$	$m = b\log_b m$ and $m^p = b\log_b (m^p)$.
$b^{\log_b mp} = b^{\log_b (m^p)}$	Use Property of Powers on the left hand side of the equation. Power of a Power: $a^{mn} = (a^m)^n$
$\log_b mp = \log_b (m^p)$	Exponents must be equal by the Property of Equality for Exponential Functions.
$p \log_b m = \log_b (m^p)$	Reverse the order of multiplication on the left hand side.

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47. $\log_{\sqrt{a}}(a^2) = \dfrac{\log_x a^2}{\log_x \sqrt{a}}$

$= \log_x a^2 - \log_x \sqrt{a}$

$= \log_x \left(a^{\frac{1}{2}}\right)^4 - \log_x \left(a^{\frac{1}{2}}\right)$

$= 4 \log_x \left(a^{\frac{1}{2}}\right) - \log_x \left(a^{\frac{1}{2}}\right)$

$= \dfrac{4 \log_x \left(a^{\frac{1}{2}}\right)}{\log_x \left(a^{\frac{1}{2}}\right)} = 4$

49. False; $\log_2 (2^2 + 2^3) = \log_2 12$, $\log_2 2^2 + \log_2 2^3 = 2 + 3$, or 5, and $\log_2 12 \neq 5$, since $2^5 \neq 12$.

51. Let $b^x = m$ and $b^y = n$. Then $\log_b m = x$ and $\log_b n = y$.

$\dfrac{b^y}{b^x} = \dfrac{m}{n}$

$b^{x-y} = \dfrac{m}{n}$ Quotient Property

$\log_b b^{x-y} = \log_b \dfrac{m}{n}$ Property of Equality for Logarithmic Equations

$x - y = \log_b \dfrac{m}{n}$ Inverse Property of Exponents and Logarithms

$\log_b m - \log_b n = \log_b \dfrac{m}{n}$ Replace x with $\log_b m$ and y with $\log_b n$.

53. A **55.** 4 **57.** $2x$ **59.** -8 **61.** ≈3.06 s **63.** 5

65. $-\frac{3}{4} < x < 2$

Pages 531–533 Lesson 9-4

1. 0.6021 **3.** −0.3010 **5.** 1.7325 **7.** ±1.1615

9. $n > 0.4907$ **11.** $\dfrac{\log 5}{\log 7}$; 0.8271 **13.** $\dfrac{\log 9}{\log 2}$; 3.1699

15. 1.0792 **17.** 0.3617 **19.** −1.5229 **21.** 8 **23.** 0.5537

25. 4.8362 **27.** 8.0086 **29.** $\{a|a < 1.1590\}$

31. $\{n|n < -1.0178\}$ **33.** $\{y|y \leq 0.4275\}$

35. $\dfrac{\log 20}{\log 5} \approx 1.8614$ **37.** $\dfrac{\log 8}{\log 5} \approx 1.8928$

39. $\dfrac{0.5 \log 5}{\log 6} \approx 0.4491$ **41.** 2.2 **43.** 3.5 **45.** ±2.6281

47. 3.7162 **49.** 4.7095 **51.** 2.7674 **53.** 113.03 cents

55. about 11.19 years

57.

$\log_{\sqrt{a}} 3 = \log_a x$ Original equation

$\dfrac{\log_a 3}{\log_a \sqrt{a}} = \log_a x$ Change of Base Formula

$\log_a 3 = \log_a (a)^{\frac{1}{2}} = \log_a x$ Quotient Property of Logarithms

$\log_a \left(\dfrac{3}{\sqrt{a}}\right) = \log_a x$ Quotient Property of Logarithms

$x = \dfrac{3}{\sqrt{a}}$ Property of Equality for Logarithmic Functions

$x = \dfrac{3\sqrt{a}}{a}$ Rationalize the denominator

59a. $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$

59b. $\log_9 27 = \frac{3}{2}$ and $\log_{27} 9 = \frac{2}{3}$

59c. Conjecture: $\log_a b = \dfrac{1}{\log_b a}$;

Proof:

$\log_a b \stackrel{?}{=} \dfrac{1}{\log_b a}$ Original statement

$\dfrac{\log_b b}{\log_b a} \stackrel{?}{=} \dfrac{1}{\log_b a}$ Change of Base Formula

$\dfrac{1}{\log_b a} = \dfrac{1}{\log_b a}$ ✓ Inverse Property of Exponents and Logarithms

61. C **63.** 1.4248 **65.** 1.8416 **67.** $\left\{z \middle| 0 < z \leq \frac{1}{64}\right\}$ **69.** −22

71. $2^x = 3$ **73.** $5^3 = 125$

Pages 540–542 Lesson 9-5

1. 403.4288 **3.** 1.4191 **5.** −2.3026 **7.** $x = \ln 4$ **9.** 1.0986

11. $h = -26{,}200 \ln \dfrac{P}{101.3}$ **13.** $\{x|x > 3.4012\}$ **15.** 2.4630

17. 54.5982 **19.** 0.3012 **21.** 1.0986 **23.** 1.6901 **25.** $-x = \ln 5$

27. $e^1 = e$ **29.** $x + 1 = \ln 9$ **31.** $e^{2x} = \frac{7}{3}$ **33.** 0.2877

35. 0.2747 **37.** 0 **39.** 0.3662 **41.** about 7.94 billion

43. about 19.8 yr **45.** $100 \ln 2 \approx 70$ **47.** 27.2991 **49.** 1.7183

51. $x < 1.5041$ **53.** $x \geq 0.6438$ **55.** about 6065 people

57. 232.9197 **59.** 2, 6 **61.** Sample answer: $e^x = 8$

63. Always;

$\dfrac{\log x}{\log y} \stackrel{?}{=} \dfrac{\ln x}{\ln y}$ Original statement

$\dfrac{\log x}{\log y} \stackrel{?}{=} \dfrac{\frac{\log x}{\log e}}{\frac{\log y}{\log e}}$ Change of Base Formula

$\dfrac{\log x}{\log y} \stackrel{?}{=} \dfrac{\log x}{\log e} \cdot \dfrac{\log e}{\log y}$ Multiply $\dfrac{\log x}{\log e}$ by the reciprical of $\dfrac{\log y}{\log e}$.

$\dfrac{\log x}{\log y} = \dfrac{\log x}{\log y}$ Simplify.

65. B **67.** $\dfrac{\log 68}{\log 4} \approx 3.0437$ **69.** $\dfrac{\log 23}{\log 50} \approx 0.8015$ **71.** 4

73. joint, 1 **75.** 25 free throws and 17 field goals

77. 1.54 **79.** 33.77 **81.** 9.32

Pages 548–550 Lesson 9-6

1. about 5 h **3.** about 33.5 watts **5.** C **7.** about 284,618 people **9.** about 4.27 hr **11.** more than 44,000 years ago

13. $14,559 billion **15.** about 0.0347 **17.** after the year 2182

19. $t = \frac{20}{3} n^{0.585}$ **21.** Take the common logarithm of each side, use the Power Property to write $\log (1 + r)^t$ as $t \log(1 + r)$, and then divide each side by the quantity $\log(1 + r)$. **23.** Never; theoretically, the amount left will always be half of the amount that existed 1620 years before. **25.** D **27.** $\ln y = 3$ **29.** $4x^2 = e^8$

31. $p > 3.3219$ **33.** $\dfrac{0.5(0.08p)}{6} + \dfrac{0.5(0.08p)}{4}$ **35.** $\dfrac{0.5(0.08p)}{12}$

37. 5.0×10^7

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Pages 552–556 Chapter 9 Study Guide and Review

1. true **3.** false, common logarithm **5.** true **7.** false, logarithmic function **9.** false; Property of Inequality for Logarithms **11.** growth **13.** $y = 7\left(\frac{1}{5}\right)^x$ **15.** -1 **17.** $x \leq -\sqrt{6}$ or $x \geq \sqrt{6}$ **19.** $\log_7 343 = 4$ **21.** $4^3 = 64$ **23.** 9 **25.** $\frac{1}{4}$ **27.** 2 **29.** $-4, 3$ **31.** 1000 **33.** 1.7712 **35.** 1.8856 **37.** 6 **39.** 10 decibels **41.** ± 2.2452 **43.** -0.6309 **45.** 8.0086 **47.** $\frac{\log 15}{\log 2}$; 3.9069 **49.** $\ln 6 = x$ **51.** 0.9163 **53.** 0.3466 **55.** 11.6487 **57.** 23.37 yr **59.** 5.05 days **61.** about 3.6%

Chapter 10 Conic Selections

Page 561 Chapter 10 Get Ready

1. $\{-4, -6\}$ **3.** $\left(\frac{3}{2}, -4\right)$ **5.** 9 in. by 6 in.

7a. $\begin{bmatrix} -2 & 4 & -1 \\ 2 & 0 & -2 \end{bmatrix}$ **7b.** $\begin{bmatrix} 5 & 5 & 5 \\ -3 & -3 & -3 \end{bmatrix}$

7c. $\begin{bmatrix} 3 & 9 & 4 \\ -1 & -3 & -5 \end{bmatrix}$

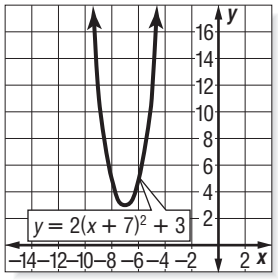
Pages 564–566 Lesson 10–1

1. $\left(-2, \frac{13}{2}\right)$ **3.** (11.5, 5.3) **5.** 10 units **7.** $\sqrt{2.61}$ units **9.** $\sqrt{1885} \approx 43.4$ units **11.** $(-4, -2)$ **13.** $\left(\frac{17}{2}, \frac{27}{2}\right)$ **15.** around 8th St. and 10th Ave. **17.** 25 units **19.** $3\sqrt{17}$ units **21.** $\sqrt{70.25}$ units **23.** $\sqrt{130}$ units **25.** $\left(\frac{3}{10}, -\frac{1}{5}\right)$; 1 unit **27.** $\left(0, \frac{3\sqrt{5}}{8}\right)$; $\frac{\sqrt{813}}{12}$ units **29.** $6\sqrt{10}\pi$ units, 90π units^2 **31.** Sample answer: Draw several line segments across the U.S. One should go from the northeast corner to the southwest corner; another should go from the southeast corner to the northwest corner; another should go across the middle of the U.S. from east to west, and so on. Find the midpoints of these segments. Locate a point to represent all of these midpoints. **33.** about 85 mi **35.** 14 in. **37.** all of the points on the perpendicular bisector of the segment **39.** Most maps have a superimposed grid. Think of the grid as a coordinate system and assign approximate coordinates to the two cities. Then use the Distance Formula to find the distance between the points with those coordinates. **41.** G **43.** -0.4055 **45.** 146.4132 **47.** $y = 2(x + 5)^2$

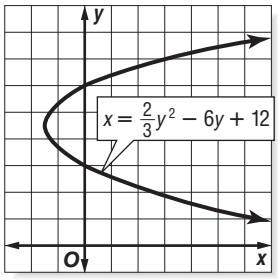
Pages 571–573 Lesson 10–2

1. $y = 2(x - 3)^2 - 12$; vertex $= (3, -12)$; axis of symmetry: $x = 3$; opens upward

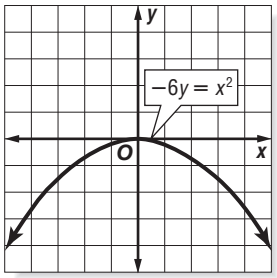
3.



5.



7. $y = (x - 3)^2 + 2$; vertex $= (3, 2)$; axis of symmetry: $x = 3$; opens upward **9.** $y = \frac{1}{2}(x + 12)^2 - 80$; vertex $= (-12, -80)$; axis of symmetry: $x = -12$; opens upward **11.**



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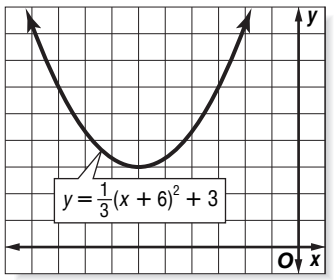
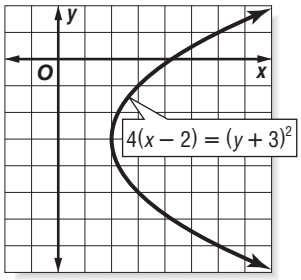


15.



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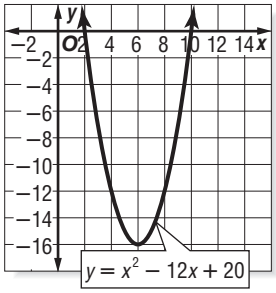
17.



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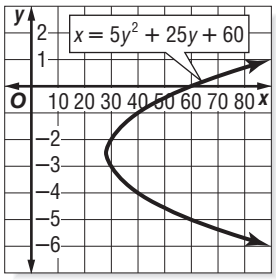
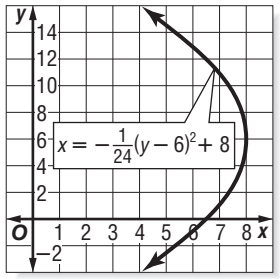


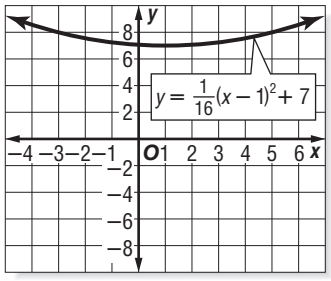
21. 0.75 cm **23.** $y = -\frac{1}{100}(x - 50)^2 + 25$ **25.** $y = -\frac{2}{3}$

27. The graph's vertex is shifted to the left $\frac{1}{3}$ unit and down $\frac{2}{3}$ unit.

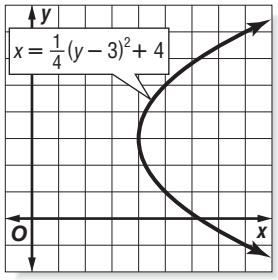
29. $x = -\frac{1}{24}(y-6)^2 + 8$



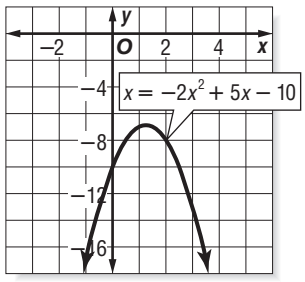
31. $y = \frac{1}{16}(x - 1)^2 + 7$



33. $x = \frac{1}{4}(y - 3)^2 + 4$

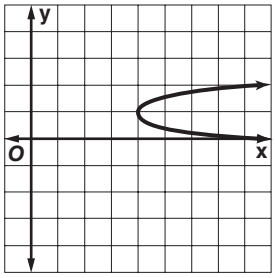


35.

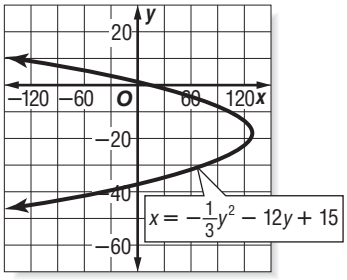


$(\frac{5}{4}, -\frac{55}{8})$, $(\frac{5}{4}, -7)$
$x = \frac{5}{4}$, $y = -\frac{27}{4}$
downward, $\frac{1}{2}$ unit

37. (4,1), $\left(\frac{81}{20}, 1\right)$, $y = 1$, $x = \frac{79}{20}$, right, $\frac{1}{5}$ unit



39.



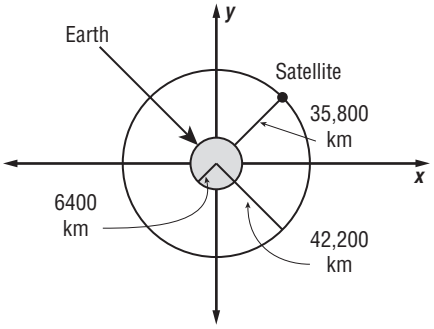
(123, −18),
$(122\frac{1}{4}, -18)$,
$y = -18$,
$x = 123\frac{3}{4}$, left,
3 units

41. Rewrite it as $y = (x - h)^2$, where $h > 0$.
43. When she added 9 to complete the square, she forgot to also subtract 9. The standard form is $y = (x + 3)^2 - 9 + 4$ or $y = (x + 3)^2 - 5$.
45. A parabolic reflector can be used to make a car headlight more effective. Answers should include the following.

- Reflected rays are focused at that point.
- The light from an unreflected bulb would shine in all directions. With a parabolic reflector, most of the light can be directed forward toward the road. **47.** J **49.** 10 units **51.** about 3.82 days **53.** 4 **55.** 9 **57.** $2\sqrt{3}$ **59.** $4\sqrt{3}$

Pages 577–579 ***Lesson 10–3***

1. $(x - 3)^2 + (y + 1)^2 = 9$ **3.**



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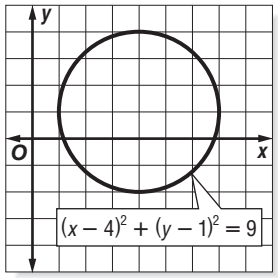
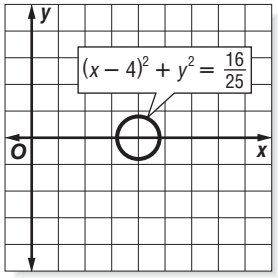
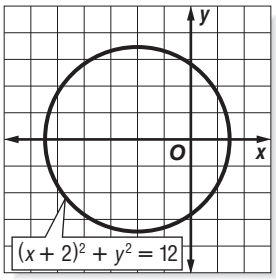
5. $x^2 + (y + 2)^2 = 25$ **7.** (4, 1), 3 units



9. (4, 0), $\frac{4}{5}$ unit

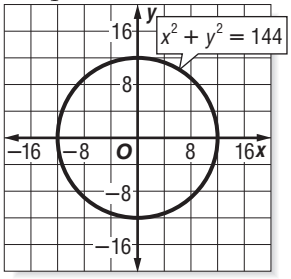


11. (−4, 3), 5 units

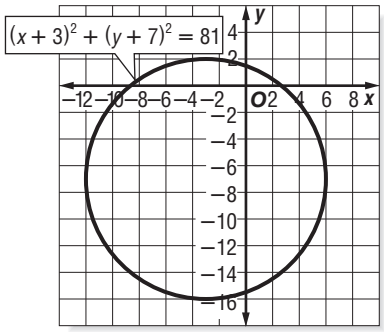


13. $(x + 1)^2 + (y - 1)^2 = 16$ **15.** $x^2 + y^2 = 18$

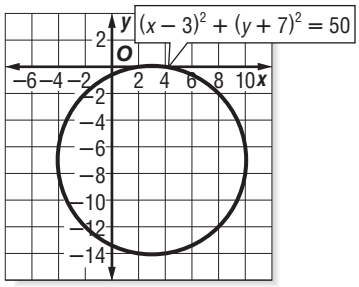
17. $(x + 8)^2 + (y - 7)^2 = \frac{1}{4}$ **19.** $(x + 1)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1945}{4}$ **21.** (0, 0), 12 units



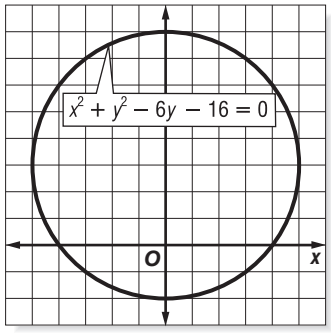
23. (−3, −7), 9 units



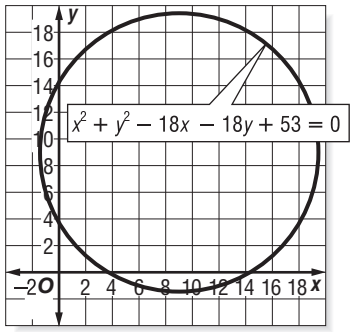
25. (3, −7), $5\sqrt{2}$ units



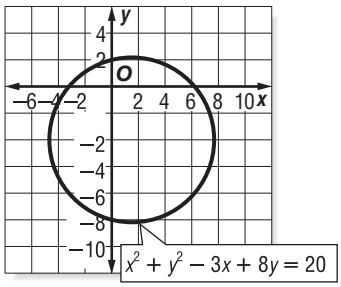
27. (0, 3), 5 units



29. (9, 9), $\sqrt{109}$ units



31. $\left(\frac{3}{2}, -4\right)$, $\frac{3\sqrt{17}}{2}$ units



33. $(x + \sqrt{13})^2 + (y - 42)^2 = 1777$ **35.** $(x - 4)^2 + (y - 2)^2 = 4$ **37.** $(x + 5)^2 + (y - 4)^2 = 25$ **39.** about 109 mi **41.** $y = \sqrt{16 - (x + 3)^2}$, $y = -\sqrt{16-(x + 3)^2}$ **43.** $x = -3 \pm \sqrt{16 - y^2}$; The equations $x = -3 - \sqrt{16 - y^2}$; and $x = -3 - \sqrt{16 - y^2}$ represent the right and left halves of the circle, respectively. **45.** $(x + 3)^2 + (y - 1)^2 = 64$; left 3 units, up 1 unit **47.** $(x + 1)^2 + (y + 2)^2 = 5$

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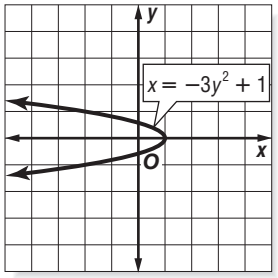
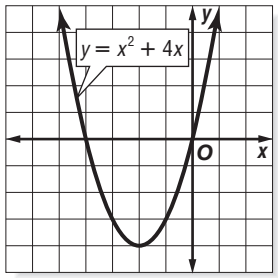
49. D **51.** (1, 0), $\left(\frac{11}{12}, 0\right)$, $y = 0$, $x = 1\frac{1}{12}$, left, $\frac{1}{3}$ unit



53. $(-2, -4)$, $\left(-2, -3\frac{3}{4}\right)$, $x = -2$, $y = -\frac{17}{4}$, upward 1 unit

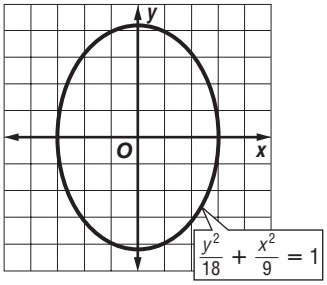


55. $(-1, -2)$ **57.** $-4, -2, 1$ **59.** 28 in. by 15 in.
61. 6 **63.** $2\sqrt{5}$

Pages 586–588 ***Lesson 10-4***

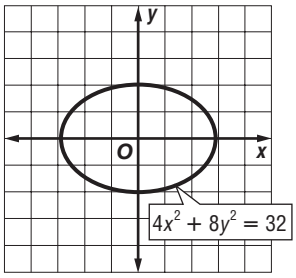
1. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ **3.** $\frac{y^2}{100} + \frac{x^2}{36} = 1$

5.



(0, 0); (0, ±3); $6\sqrt{2}$; 6

7.

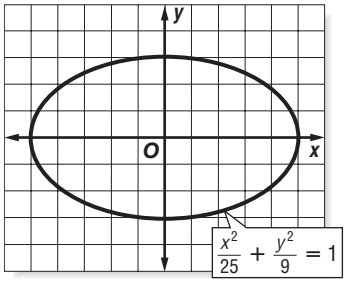


(0, 0); (±2, 0); $4\sqrt{2}$; 4

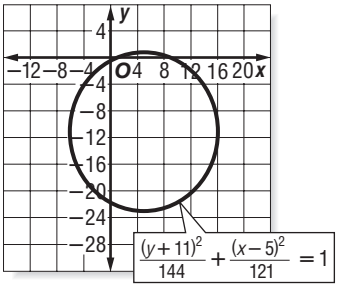
9. $\frac{y^2}{64} + \frac{x^2}{36} = 1$ **11.** $\frac{(x-5)^2}{64} + \frac{(y-4)^2}{9} = 1$ **13.** $\frac{(x+2)^2}{81} + \frac{(y-5)^2}{16} = 1$ **15.** $\frac{(y-2)^2}{100} + \frac{(x-4)^2}{9} = 1$

17. $\frac{x^2}{324} + \frac{y^2}{196} = 1$ **19.** (0, 0);

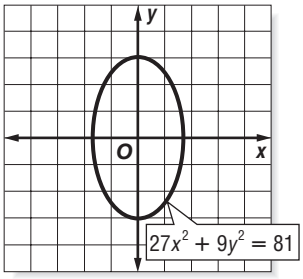
(±4, 0); 10; 6;



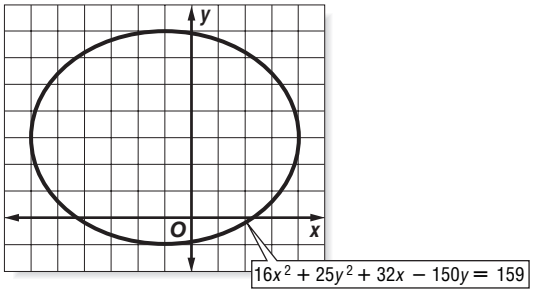
21. (5, −11); $(5, -11 \pm \sqrt{23})$; 24; 22;



23. (0, 0); $(0, \pm\sqrt{6})$; 6; $2\sqrt{3}$;



25. (−1, 3); (2, 3), (−4, 3); 10; 8;



27. $\frac{(x-1)^2}{81} + \frac{(y-2)^2}{56} = 1$ **29.** $\frac{y^2}{20} + \frac{x^2}{4} = 1$

31. $\frac{y^2}{279{,}312.25} + \frac{x^2}{193{,}600} = 1$ **33.** Let the equation of a circle be $(x-h)^2 + (y-k)^2 = r^2$. Divide each side by r^2 to get $\frac{(x-3)^2}{r^2} + \frac{(y+k)^2}{r^2} = 1$. This is the equation of an ellipse with a and b both equal to r. In other words, a circle is an ellipse whose major and minor are both diameters. **35.** $\frac{x^2}{12} + \frac{y^2}{9} = 1$ **37.** C

39. $(x-3)^2 + (y+2)^2 = 25$ **41.** $y = \frac{1}{2}(x-3)^2 + 1$

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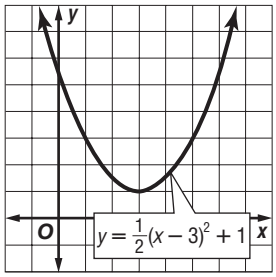
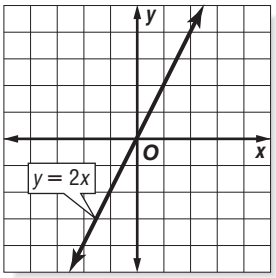


43. Sample answer using (0, 104.6) and (10, 112.6): $y = 0.8x + 104.6$

45.



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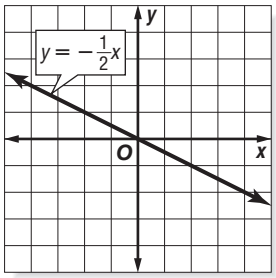

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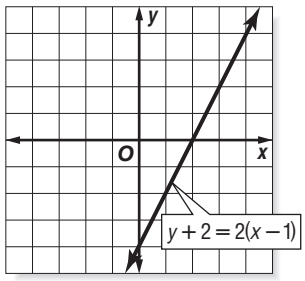
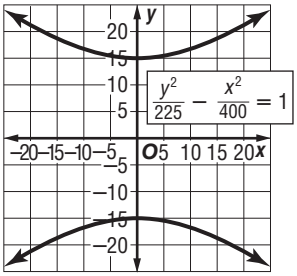
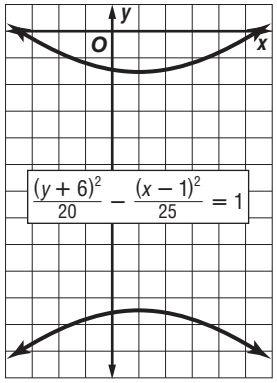


Pages 594–597 ***Lesson 10–5***

1. $\frac{y^2}{4} - \frac{x^2}{21} = 1$ **3.** $(0, \pm 15)$; $(0, \pm 25)$; $y = \pm\frac{3}{4}x$;

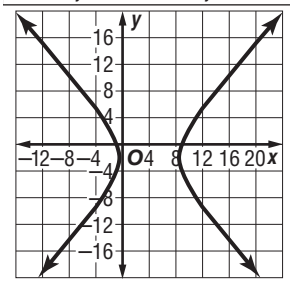


5.



$(1, -6 \pm 2\sqrt{5})$; $(1, -6 \pm 3\sqrt{5})$; $y + 6 = \pm\frac{2\sqrt{5}}{5}(x - 1)$

7. $5x^2 - 4y^2 - 40x - 16y - 36 = 0$

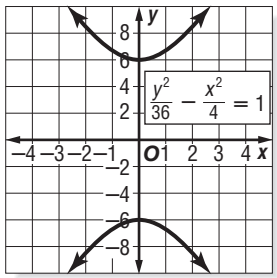


$(4 \pm 2\sqrt{5}, -2)$; $(4 \pm 3\sqrt{5}, -2)$; $y + 2 = \pm\frac{\sqrt{5}}{2}(x - 4)$

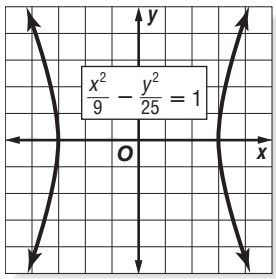
9. $\frac{(y-3)^2}{1} - \frac{(x+2)^2}{4} = 1$ **11.** $\frac{(x-3)^2}{4} - \frac{(y+5)^2}{9} = 1$

13. $\frac{y^2}{16} - \frac{x^2}{49} = 1$

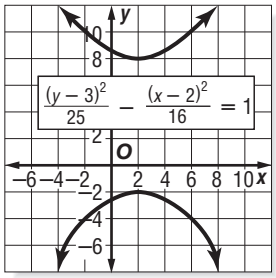
15. $\frac{(y-5)^2}{16} - \frac{(x+4)^2}{81} = 1$ **17.** $(0, \pm 6)$; $(0, \pm 2\sqrt{10})$; $y = \pm 3x$



19. $(\pm 3, 0)$; $(\pm\sqrt{34}, 0)$; $y = \pm\frac{5}{3}x$



21. $(2, -2)$, $(2, 8)$; $(2, 3 \pm \sqrt{41})$; $y - 3 = \pm\frac{5}{4}(x - 2)$



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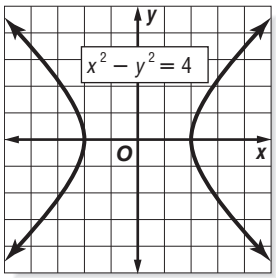
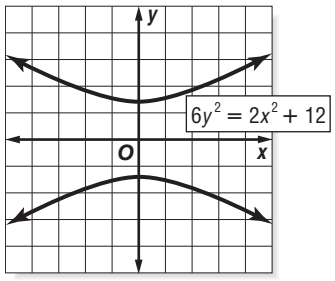
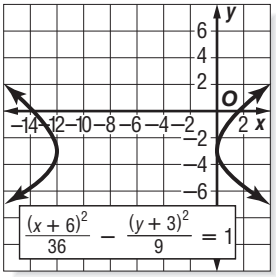
23. $(\pm 2, 0)$; $(\pm 2\sqrt{2}, 0)$;
$y = \pm x$



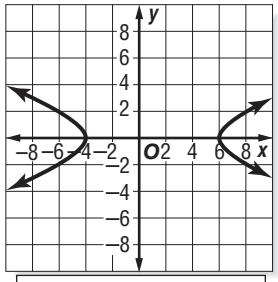
25. $(0, \pm\sqrt{2})$; $(0, \pm 2\sqrt{2})$;
$y = \pm\frac{\sqrt{3}}{3}x$



27. $(-12, -3)$, $(0, -3)$; $(-6 \pm 3\sqrt{5}, -3)$;
$y + 3 = \pm\frac{1}{2}(x + 6)$



29.



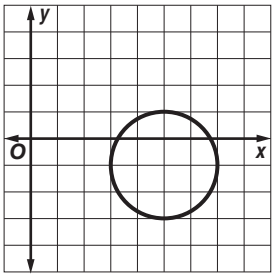
$(-4, 0)$, $(6, 0)$; $(1 \pm \sqrt{29}, 0)$; $y = \pm\frac{2}{5}(x - 1)$

31. $\frac{y^2}{36} - \frac{x^2}{4} = 1$ **33.** about 47.32 ft

35. $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ **37.** The graph of $xy = -2$ can be obtained by reflecting the graph of $xy = 2$ over the x-axis or over the y-axis. The graph of $xy = -2$ can also be obtained by rotationg the graph of $xy = 2$ by 90°. **39.** As k increases, the branches of the hyperbola become wider. **41.** Hyperbolas and parabolas have different graphs and different reflective properties. Answers should include the following.

- Hyperbolas have two branches, two foci and two vertices. Parabolas have only one branch, one focus, and one vertex. Hyperbolas have asymptotes, but parabolas do not.
- Hyperbolas reflect rays directed at one focus toward the other focus. Parabolas reflect parallel incoming rays toward the only focus.

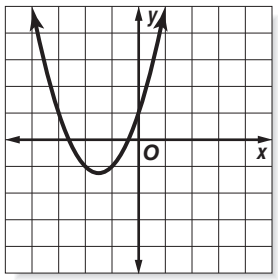
43. F **45.** $\frac{(x+3)^2}{9} + \frac{(y-1)^2}{16} = 1$ **47.** $(5, -1)$, 2 units



49. $-7, \frac{3}{2}$ **51.** $-5, 4$ **53.** $2, 3, -5$ **55.** $1, 0, 0$

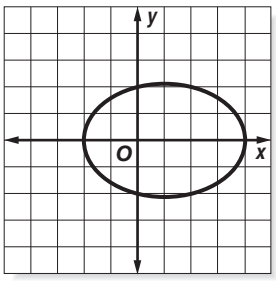
Pages 599–602 ***Lesson 10-6***

1. parabola



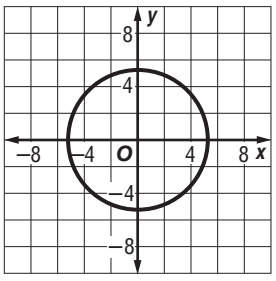
$y = \left(x + \frac{3}{2}\right)^2 - \frac{5}{4}$

3. circle $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{9}{4}$

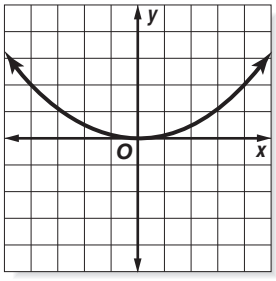


5. parabola **7.** hyperbola

9. circle $x^2 + y^2 = 27$



11. parabola $y = \frac{1}{8}x^2$



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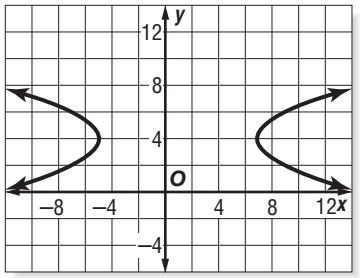
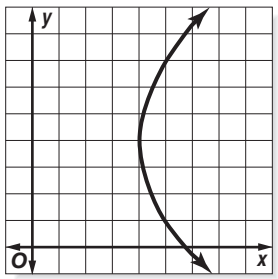
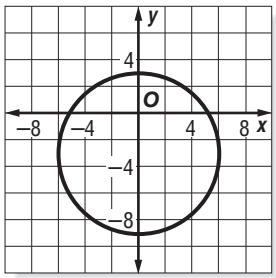
13. hyperbola $\frac{(x-1)^2}{36} - \frac{(y-4)^2}{4} = 1$



15. parabola $x = \frac{1}{9}(y-4)^2 + 4$

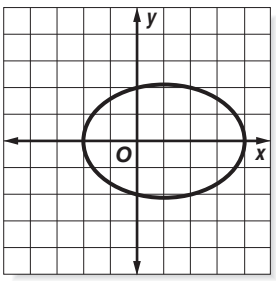


17. circle; $x^2 + (y+3)^2 = 36$

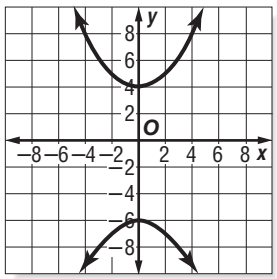


19. circle **21.** parabola **23.** b **25.** c **27.** about 1321 ft **29.** hyperbola **31.** hyperbola

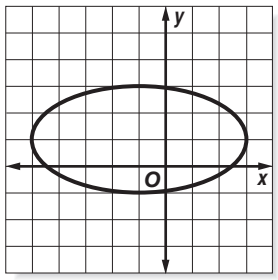
33. ellipse $\frac{(x-1)^2}{9} + \frac{y^2}{\frac{9}{2}} = 1$



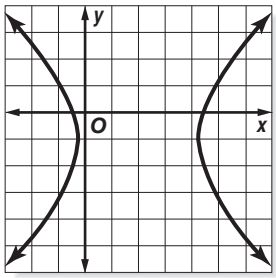
35. hyperbola $\frac{(y-1)^2}{25} - \frac{x^2}{9} = 1$



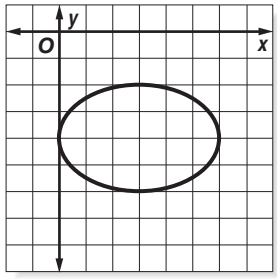
37. ellipse $\frac{(x+1)^2}{16} + \frac{(y-1)^2}{4} = 1$



39. hyperbola $\frac{(x-2)^2}{5} - \frac{(y+1)^2}{6} = 1$



41. ellipse **43.** parabola **45.** Sample answer: $2x^2 + 2y^2 - 1 = 0$ **47.** 2 intersecting lines **49.** $0 < e < 1$, $e > 1$ **51.** C **53.** $\frac{(y-4)^2}{36} - \frac{(x-5)^2}{16} = 1$ **55.** $(3, -4)$; $(3 \pm \sqrt{5}, -4)$, 6; 4;

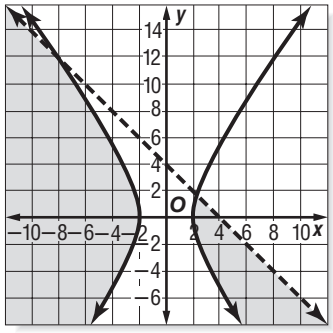


57. $m^{12}n$ **59.** 196 beats per min **61.** $y = -\frac{5}{3}x - \frac{4}{3}$ **63.** (3, 2)

Pages 606–608 ***Lesson 10–7***

1. $(\pm 4, 5)$ **3.** no solution **5.** (40, 30)

7.



9. $\left(\frac{3}{2}, \frac{9}{2}\right)$, $(-1, 2)$ **11.** no solution **13.** no solution **15.** $(0, 3)$, $\left(\pm\frac{\sqrt{23}}{2}, -\frac{11}{4}\right)$ **17.** $(0, \pm 5)$

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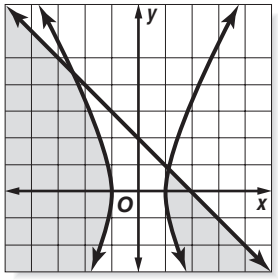
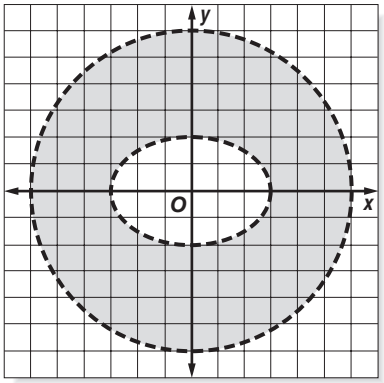
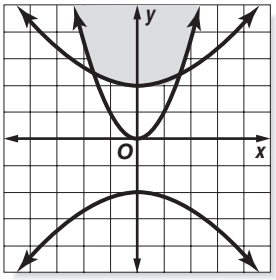
19. (4, ±3), (-4, ±3) **21.**



23.



25.

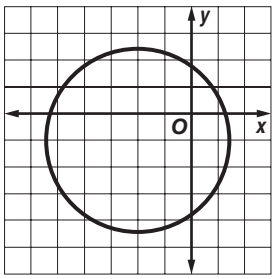


27. (39.2, ±4.4) **29.** $\left(-\frac{5}{3}, -\frac{7}{3}\right)$, (1, 3) **31.** 0.5 s

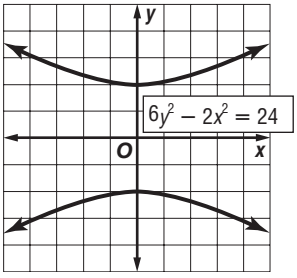
33. $y = \pm 900\sqrt{1 - \frac{x^2}{(300)^2}}$; $y = \pm 690\sqrt{1 - \frac{x^2}{(600)^2}}$

35. Sample answer: The orbit of the satellite modeled by the second equation is closer to a circle than the other orbit. The distance on the x-axis is twice as great for one satellite than the other. **37.** Sample answer: $x^2 + y^2 = 36$, $\frac{(x+2)^2}{16} - \frac{y^2}{4} = 1$ **39.** Sample answer: $x^2 + y^2 = 81$, $\frac{x^2}{4} + \frac{y^2}{100} = 1$ **41.** impossible **43.** Sample answer: $x^2 + y^2 = 40$, $y = x^2 + x$ **45.** none **47.** none **49.** $x^2 + y^2 = 20$, $\frac{x^2}{25} + \frac{y^2}{16} = 1$ This system has four solutions whereas the other three only have two solutions. **51.** C

53. $(x + 2)^2 + (y + 1)^2 = 11$, circle,

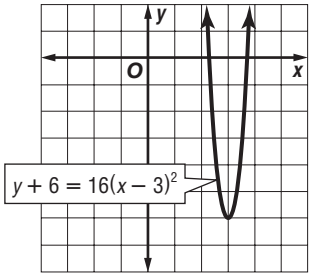


55. (0, ± 2), (0, ± 4), $y = \pm\frac{\sqrt{3}}{3}x$

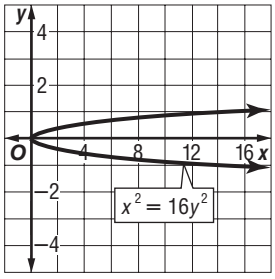


Pages 609–614 Chapter 10 Study Guide and Review

1. true **3.** False; a parabola is the set of all points that are the same distance from a given point called the focus and a given line called the directrix. **5.** False; the conjugate axis of a hyperbola is a line segment perpendicular to the transverse axis. **7.** true **9.** $\left(-5, \frac{3}{2}\right)$ **11.** (16, 26) **13.** $\sqrt{290}$ units **15.** $\sqrt{2}$ **17.** $\left(4, -\frac{5}{2}\right)$ **19.** (3, −6); $\left(3, -5\frac{63}{64}\right)$; $x = 3$; $y = -6\frac{1}{64}$; upward; $\frac{1}{16}$ unit



21. (0, 0); $(\frac{1}{64}, 0)$; $y = 0$; $x = -\frac{1}{64}$; right; $\frac{1}{16}$ unit



23. $y = -\frac{1}{400}x^2 + x$ **25.** $(x + 4)^2 + y^2 = \frac{9}{16}$

27. $(x + 1)^2 + (y - 2)^2 = 4$

29. (−5, 11), 7 units

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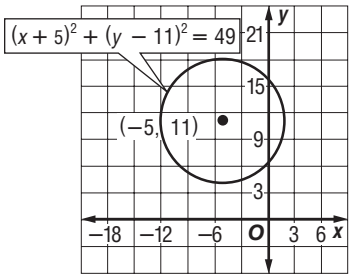
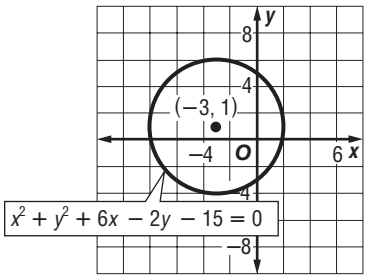
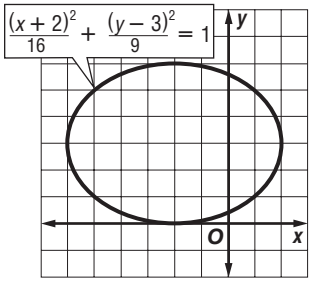


31. (−3, 1), 5 units



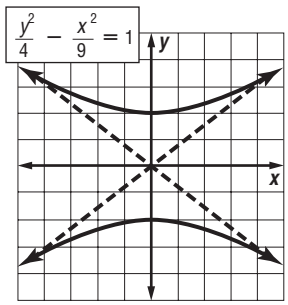
33. $\frac{(x+1)^2}{25}+\frac{(y-1)^2}{4}=1$

35. $(-2, 3)$; $(-2 \pm \sqrt{7}, 3)$; 8; 6

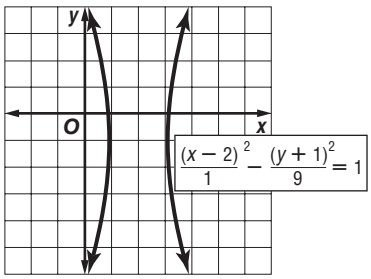


37. $\frac{x^2}{5.45^2}+\frac{y^2}{4.4^2}=1$

39. $(0, \pm 2)$; $(0 \pm \sqrt{13})$; $y = \pm \frac{2}{3}x$

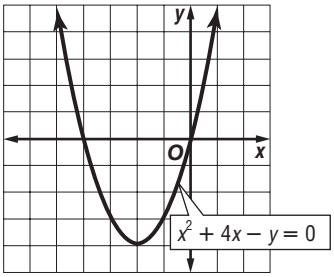


41. $(1, -1)$, $(3, -1)$; $(2 \pm \sqrt{10}, -1)$; $y + 1 = \pm 3(x - 2)$

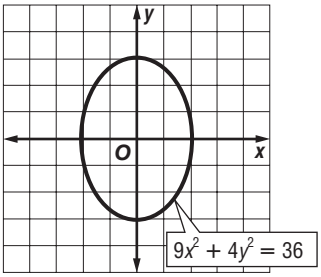


43. $\left(\frac{40-24\sqrt{5}}{5}, \frac{45-12\sqrt{5}}{5}\right)$

45. parabola, $y = (x + 2)^2 - 4$



47. ellipse $\frac{y^2}{9}+\frac{x^2}{4}=1$

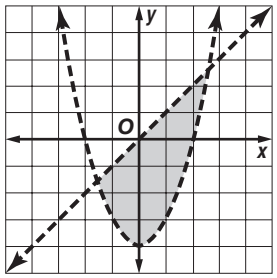


49. hyperbola

51. circle

53. (6, −8), (12, −16)

55.



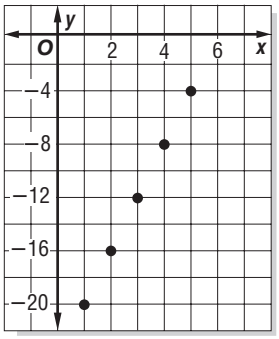
57. (0, 10) and (20, 10)

Chapter 11 Sequences and Series

Page 621 ***Chapter 11*** ***Get Ready***

1. −10 **3.** −5 **5.** 6

7.



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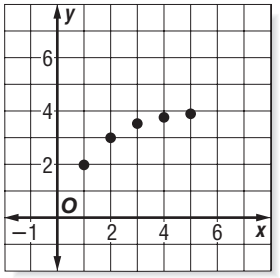
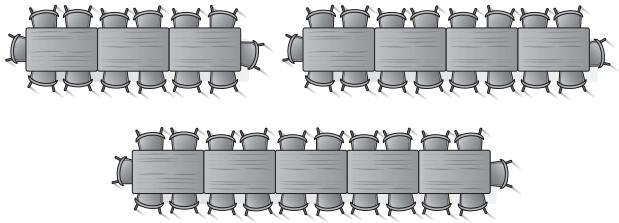
9.



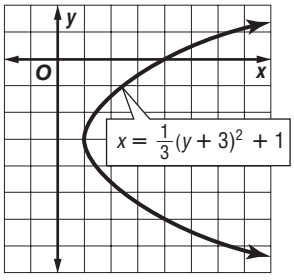
11. 17 **13.** $\frac{1}{32}$

Pages 625–628 Lesson 11-1

1. 24, 28, 32, 36 **3.** 5, 8, 11, 14, 17 **5.** $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}$
7. 43 **9.** 79 **11.** 39.15 **13.** $a_n = 11n - 37$ **15.** 56, 68, 80
17. 30, 37, 44, 51 **19.** 6, 10, 14, 18 **21.** 2, 15, 28, 41, 54
23. 6, 2, −2, −6, −10 **25.** 28 **27.** 94 **29.** 335 **31.** 27
33. 176 ft **35.** 30th **37.** $a_n = 9n - 2$ **39.** $a_n = -2n - 1$
41. 70, 85, 100 **43.** −5, −2, 1, 4 **45.** $\frac{7}{3}, 3, \frac{11}{3}, \frac{13}{3}$ **47.** 5.5, 5.1, 4.7, 4.3 **49.** $\frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0$ **51.** 29
53. 14, 18, 22



55. No; there is no whole number n for which $4n + 2 = 100$. **57.** $-\frac{25}{2}$ **59.** 173 **61.** $a_n = 7n - 600$
63. $a_n = -6n + 615$ **65.** Sample answer: Maya has $50 in her savings account. She withdraws $5 each week to pay for music downloads. **67.** $z = 2y - x$ **69.** B
71. (−1, ±4), (5, ±2) **73.** $x = \frac{1}{3}(y + 3)^2 + 1$; parabola

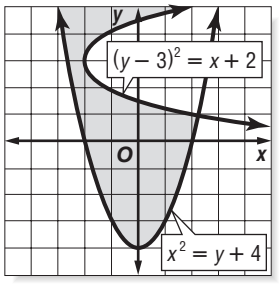


75. 15 **77.** 2 **79.** $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}$ **81.** $y = 3x + 57$ **83.** 5, 4, 3, 2
85. −2, −4, −6, −8, −10

Pages 632–635 Lesson 11-2

1. 800 **3.** 260 **5.** 135h **7.** 230 **9.** 552 **11.** 19 **13.** −6, 0, 6
15. 95 **17.** 663 **19.** −88 **21.** 182 **23.** 225 **25.** 8 days
27. 2 **29.** 18 **31.** −13, −8, −3 **33.** 13, 18, 23 **35.** 735
37. −204 **39.** −35 **41.** 510 **43.** 24,300 **45.** 2646 **47.** 119
49. $-\frac{245}{6}$ **51.** 166,833 **53.** $522,500 **55.** 3649 **57.** 6900.5
59. 600 **61.** True; for any series, $2a_1 + 2a_2 + 2a_3 + \cdots + 2a_n = 2(a_1 + a_2 + a_3 + \cdots + a_n)$.

63. Arithmetic series can be used to find the seating capacity of an amphitheater. The sequence represents the numbers of seats in the rows. The sum of the first n terms of the series is the seating capacity of the first n rows. One method is to write out the terms and add them: 18 + 22 + 26 + 30 + 34 + 38 + 42 + 46 + 50 + 54 = 360. Another method is to use the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$: $S_{10} = \frac{10}{2}[2(18) + (10 - 1)4]$ or 360.
65. G **67.** −135 **69.**



71. $\log_7 x = 3$ **73.** $1\frac{5}{7}$ days **77.** $-\frac{9}{2}$ **79.** $\frac{3 \pm \sqrt{89}}{2}$
81. −25.21 **83.** $a = -2, b = 2$ **85.** $c = 9, d = 4$ **87.** −54

Pages 639–641 Lesson 11-3

1. 67.5, 101.25 **3.** A **5.** 16 **7.** $\frac{1}{27}$ **9.** $a_n = 15\left(\frac{1}{3}\right)^{n-1}$
11. −4 **13.** 6, 18 **15.** 192, 256 **17.** 48, 32 **19.** 1, 4, 16, 64, 256 **21.** 2592 **23.** $46,794.34 **25.** 1024 **27.** 2 **29.** 192
31. $a_n = 64\left(\frac{1}{4}\right)^{n-1}$ **33.** $a_n = 4(-3)^{n-1}$ **35.** ±12, 36, ±108 **37.** 6, 12, 24, 48 **39.** $\frac{125}{24}, \frac{625}{48}$ **41.** −21.875, 54.6875
43. 576, −288, 144, −72, 36 **45.** 8 days **47.** 243
49. −8748 **51.** 800 **53.** Sample answer: $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \cdots$ **55.** The sequence 9, 16, 25, … does not belong with the other three. The other three sequences are geometric sequences, but 9, 16, 25, … is not. **57.** False; the sequence 1, 1, 1, 1, …, for example, is arithmetic ($d = 0$) and geometric ($r = 1$). **59.** C **61.** 632.5
63. 19, 23 **65.** $5\sqrt{2} + 3\sqrt{10}$ units **67.** $\frac{63}{32}$

Pages 646–649 Lesson 11-4

1. 81,915 **3.** $\frac{1330}{9}$ **5.** 93 in. or 7 ft 9 in. **7.** $\frac{1093}{9}$
9. 32,552 **11.** $\frac{4921}{27}$ **13.** 39,063 **15.** −504 **17.** 3
19. −2 **21.** 765 **23.** 1,328,600 **25.** 1441 **27.** 300
29. $10,737,418.23 **31.** 206,668 **33.** −364 **35.** 1024
37. 6 **39.** $\frac{215}{4}$ **41.** 7.96875 **43.** −118,096 **45.** 156.248
47. $-\frac{182}{9}$ **49.** 3,145,725 **51.** $\frac{387}{4}$ **53.** 8 **55.** −1,048,575
57. 6.24999936 **59.** Sample answer: $4 + 2 + 1 + \frac{1}{2}$
61. If the first term and common ratio of a geometric series are integers, then all the terms of the series are integers. Therefore, the sum of the series is an integer.
63. If the number of people that each person sends the joke to is constant, then the total number of people who have seen the joke is the sum of a geometric series. Increase the number of days that the joke circulates so that it is inconvenient to find and add all the terms of the series.

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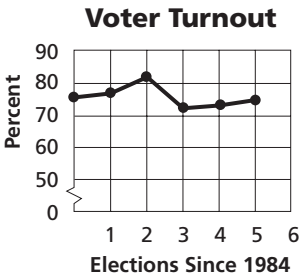
65. J **67.** $-3, -\frac{9}{2}, -\frac{27}{4}, -\frac{81}{8}$
69. 192 **71.** -14 **73.** even; 4 **75.** $(7p + 3)(6q - 5)$
77.



79. Sample answer: 70.4 **81.** 2 **83.** $\frac{2}{3}$ **85.** 0.6

Pages 653–655 Lesson 11-5

1. 108 **3.** does not exist **5.** 96 cm **7.** 100 **9.** $\frac{4}{5}$ **11.** $\frac{73}{99}$
13. 14 **15.** 7.5 **17.** 64 **19.** does not exist **21.** 3 **23.** 144
25. does not exist **27.** does not exist **29.** $\frac{1}{9}$ **31.** $\frac{82}{99}$
33. 78 cm **35.** 1 **37.** 7.5 **39.** 6 **41.** $\frac{82}{333}$ **43.** $\frac{41}{90}$ **45.** 6 ft
47. 75, 30, 12 **49.** $-8, -3\frac{1}{5}, -1\frac{7}{25}, -\frac{64}{125}$ **51.** 0.999999… can be written as the infinite geometric series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$. The first term of this series is $\frac{9}{10}$ and the common ratio is $\frac{1}{10}$, so the sum is $\frac{\frac{9}{10}}{1 - \frac{1}{10}}$ or 1.

53.
$$S = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots$$
$$(-)\ rS = a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \cdots$$
$$S - rS = a_1 + 0 + 0 + 0 + 0 + \cdots$$
$$S(1 - r) = a_1$$
$$S = \frac{a_1}{1 - r}$$

55. D **57.** -182 **59.** 32.768% **61.** $-\frac{3}{2}$ **63.** $\frac{-2a + 5b}{a^2b}$
65. $\frac{3x + 7}{(x + 4)(x + 2)}$ **67.** $x^2 + 9x + 14 = 0$
69. about $-46{,}037$ visitors per year **71.** 2 **73.** 2 **75.** 4

Pages 660–662 Lesson 11-6

1. 12, 9, 6, 3, 0 **3.** 0, -4, 4, -12, 20
5. $b_n = 1.03b_{n-1} - 10$ **7.** 5, 11, 29 **9.** 3, 11, 123 **11.** 13, 18, 23, 28, 33 **13.** 6, 10, 15, 21, 28 **15.** 4, 6, 12, 30, 84
17. -2.1 **19.** 5, 17, 65 **21.** $-4, -19, -94$ **23.** $a_n = a_{n-2} + a_{n-1}$ **25.** 1, 3, 6, 10, 15 **27.** 20,100 **29.** \$75.77
31. $-1, -1, -1$ **33.** $\frac{4}{3}, \frac{10}{3}, \frac{76}{3}$ **35.** Sometimes; if $f(x) = x^2$ and $x_1 = 2$, then $x_2 = 2^2$ or 4, so $x_2 \neq x_1$. But, if $x_1 = 1$, then $x_2 = 1$, so $x_2 = x_1$. **37.** Under certain conditions, the Fibonacci sequence can be used to model the number of shoots on a plant. The 13th term of the sequence is 233, so there are 233 shoots on the plant during the 13th month. **39.** F **41.** $\frac{1}{6}$
43. -5208 **45.** $3x + 7$ units **47.** 6

Pages 667–669 Lesson 11-7

1. $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$ **3.** $x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4$ **5.** 40,320 **7.** 17,160
9. $56a^5b^3$ **11.** $a^3 - 3a^2b + 3ab^2 - b^3$ **13.** $r^8 + 8r^7s + 28r^6s^2 + 56r^5s^3 + 70r^4s^4 + 56r^3s^5 + 28r^2s^6 + 8rs^7 + s^8$
15. $x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$
17. 362,880 **19.** 72 **21.** $-126x^4y^5$ **23.** $280x^4$ **25.** 10
27. $16b^4 - 32b^3x + 24b^2x^2 - 8bx^3 + x^4$ **29.** $243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$
31. $\frac{a^5}{32} + \frac{5a^4}{8} + 5a^3 + 20a^2 + 40a + 32$ **33.** 495
35. $1{,}088{,}640a^6b^4$ **37.** $\frac{35}{27}x^4$ **39.** 500 **41.** Sample answer: $(5x + y)^4$ **43.** The coefficients in a binomial expansion give the numbers of sequences of births resulting in given numbers of boys and girls. $(b + g)^5 = b^5 + 5b^4g + 10b^3g^2 + 10b^2g^3 + 5bg^4 + g^5$; There is one sequence of births with all five boys, five sequences with four boys and one girl, ten sequences with three boys and two girls, ten sequences with two boys and three girls, five sequences with one boy and four girls, and one sequence with all five girls. **45.** H **47.** 3, 5, 9, 17, 33
49. hyperbola **51.** $\frac{\log 5}{\log 2}$; 2.3219 **53.** $\frac{\log 8}{\log 5}$; 1.2920
55. asymptotes: $x = -4$, $x = 1$ **57.** true **59.** true **61.** true

Pages 672–673 Lesson 11-8

1. Step 1: When $n = 1$, the left side of the given equation is 1. The right side is $\frac{1(1 + 1)}{2}$ or 1, so the equation is true for $n = 1$. **Step 2:** Assume $1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2}$ for some positive integer k. **Step 3:** $1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1) = \frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$
The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$ for all positive integers n. **3. Step 1:** After the first guest has arrived, no handshakes have taken place. $\frac{1(1-1)}{2} = 0$, so the formula is correct for $n = 1$. **Step 2:** Assume that after k guests have arrived, a total of $\frac{k(k - 1)}{2}$ handshakes have taken place, for some positive integer k. **Step 3:** When the $(k + 1)$st guest arrives, he or she shakes hands with the k guests already there, so the total number of handshakes that have then taken place is $\frac{k(k - 1)}{2} + k$.
$$\frac{k(k - 1)}{2} + k = \frac{k(k - 1) + 2k}{2}$$
$$= \frac{k[(k - 1) + 2]}{2}$$
$$= \frac{k(k + 1)}{2} \text{ or } \frac{(k + 1)k}{2}$$
The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the formula is true for $n = k + 1$.

Therefore, the total number of handshakes is $\frac{n(n - 1)}{2}$ for all positive integers n. **5. Step 1:** $5^1 + 3 = 8$, which is divisible by 4. The statement is true for $n = 1$.

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Step 2: Assume that $5^k + 3$ is divisible by 4 for some positive integer k. This means that $5^k + 3 = 4r$ for some positive integer r.

Step 3: $5^k + 3 = 4r$

$5^k = 4r - 3$

$5^{k+1} = 20r - 15$

$5^{k+1} + 3 = 20r - 12$

$5^{k+1} + 3 = 4(5r - 3)$

Since r is a positive integer, $5r - 3$ is a positive integer. Thus, $5^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$. Therefore, $5^n + 3$ is divisible by 4 for all positive integers n. **7.** Sample answer: $n = 3$

9. Step 1: When $n = 1$, the left side of the given equation is 2. The right side is $\frac{1[3(1) + 1]}{2}$ or 2, so the equation is true for $n = 1$.**Step 2:** Assume $2 + 5 + 8 + \ldots + (3k - 1) = \frac{k(3k + 1)}{2}$ for some positive integer k.

Step 3: $2 + 5 + 8 + \ldots + (3k - 1) + [3(k + 1) - 1] = \frac{k(3k + 1)}{2} + [3(k + 1) - 1]$

$= \frac{k(3k + 1) + 2[3(k + 1) - 1]}{2}$

$= \frac{3k^2 + k + 6k + 6 - 2}{2}$

$= \frac{3k^2 + 7k + 4}{2}$

$= \frac{(k + 1)(3k + 4)}{2}$

$= \frac{(k + 1)[3(k + 1) + 1]}{2}$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}$ for all positive integers n. **11. Step 1:** When $n = 1$, the left side of the given equation is 1^2 or 1. The right side is $\frac{1[2(1) - 1][2(1) + 1]}{3}$ or 1, so the equation is true for $n = 1$. **Step 2:** Assume $1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}$ for some positive integer k.

Step 3: $1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 + [2(k + 1) - 1]^2$

$= \frac{k(2k - 1)(2k + 1)}{3} + [2(k + 1) - 1]^2$

$= \frac{k(2k - 1)(2k + 1) + 3(2k + 1)^2}{3}$

$= \frac{(2k + 1)[k(2k - 1) + 3(2k + 1)]}{3}$

$= \frac{(2k + 1)(2k^2 - k + 6k + 3)}{3}$

$= \frac{(2k + 1)(2k^2 + 5k + 3)}{3}$

$= \frac{(2k + 1)(k + 1)(2k + 3)}{3}$

$= \frac{(k + 1)[2(k + 1) - 1][2(k + 1) + 1]}{3}$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$ for all positive integers n.

13. Step 1: $9^1 - 1 = 8$, which is divisible by 8. The statement is true for $n = 1$. **Step 2:** Assume that $9^k - 1$ is divisible by 8 for some positive integer k. This means that $9^k - 1 = 8r$ for some whole number r.

Step 3: $9^k - 1 = 8r$

$9^k = 8r + 1$

$9^{k+1} = 72r + 9$

$9^{k+1} - 1 = 72r + 8$

$9^{k+1} - 1 = 8(9r + 1)$

Since r is a whole number, $9r + 1$ is a whole number. Thus, $9^{k+1} - 1$ is divisible by 8, so the statement is true for $n = k + 1$. Therefore, $9^n - 1$ is divisible by 8 for all positive integers n. **15. Step 1:** When $n = 1$, the left side of the given equation is f_1. The right side is $f_3 - 1$. Since $f_1 = 1$ and $f_3 = 2$ the equation becomes $1 = 2 - 1$ and is true for $n = 1$. **Step 2:** Assume $f_1 + f_2 + \ldots + f_k = f_{k+2} - 1$ for some positive integer k.

Step 3: $f_1 + f_2 + \ldots + f_k + f_{k+1}$ $= f_{k+2} - 1 + f_{k+1} = f_{k+1} + f_{k+2} - 1$ $= f_{k+3} - 1$, since Fibonacci numbers are produced by adding the two previous Fibonacci numbers. The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$ for all positive integers n.

17. Sample answer: $n = 4$ **19.** Sample answer: $n = 3$

21. Sample answer: $n = 41$

23. Step 1: When $n = 1$, the left side of the given equation is $\frac{1}{4}$. The right side is $\frac{1}{3}\left(1 - \frac{1}{4}\right)$ or $\frac{1}{4}$, so the equation is true for $n = 1$. **Step 2:** Assume $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^k} = \frac{1}{3}\left(1 - \frac{1}{4^k}\right)$ for some positive integer k.

Step 3: $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^k} + \frac{1}{4^{k+1}}$

$= \frac{1}{3}\left(1 - \frac{1}{4^k}\right) + \frac{1}{4^{k+1}}$

$= \frac{1}{3} - \frac{1}{3 \cdot 4^k} + \frac{1}{4^{k+1}}$

$= \frac{4^{k+1} - 4 + 3}{3 \cdot 4^{k+1}}$

$= \frac{4^{k+1} - 1}{3 \cdot 4^{k+1}}$

$= \frac{1}{3}\left(\frac{4^{k+1} - 1}{4^{k+1}}\right)$

$= \frac{1}{3}\left(1 - \frac{1}{4^{k+1}}\right)$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots + \frac{1}{4^n} = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$ for all positive integers n.

25. Step 1: $13^1 + 11 = 24$, which is divisible by 12. The statement is true for $n = 1$. **Step 2:** Assume that $13^k + 11$ is divisible by 12 for some positive integer k. This

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means that $13^k + 11 = 12r$ for some positive integer r.

Step 3:
$$\begin{aligned} 13^k + 11 &= 12r \\ 13^k &= 12r - 11 \\ 13^{k+1} &= 156r - 143 \\ 13^{k+1} + 11 &= 156r - 132 \\ 13^{k+1} + 11 &= 12(13r - 11) \end{aligned}$$

Since r is a positive integer, $13r - 11$ is a positive integer. Thus, $13^{k+1} + 11$ is divisible by 12, so the statement is true for $n = k + 1$. Therefore, $13^n + 11$ is divisible by 12 for all positive integers n.

27. Step 1: When $n = 1$, the left side of the given equation is a_1.

The right side is $\frac{a_1(1-r^1)}{1-r}$ or a_1, so the equation is true for $n = 1$. **Step 2:** Assume $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{k-1} = \frac{a_1(1-r^k)}{1-r}$ for some positive integer k.

Step 3: $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{k-1} + a_1r^k$

$$= \frac{a_1(1-r^k)}{1-r} + a_1r^k$$
$$= \frac{a_1(1-r^k) + (1-r)a_1r^k}{1-r}$$
$$= \frac{a_1 - a_1r^k + a_1r^k - a_1r^{k+1}}{1-r}$$
$$= \frac{a_1(1-r^{k+1})}{1-r}$$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $a_1 + a_1r + a_1r^2 + \ldots + a_1r^{n-1} = \frac{a_1(1-r^n)}{1-r}$ for all positive integers n. **29.** Sample answer: $3^n - 1$ **31.** An analogy can be made between mathematical induction and a ladder with the positive integers on the steps. Showing that the statement is true for $n = 1$ (Step 1) corresponds to stepping on the first step. Assuming that the statement is true for some positive integer k and showing that it is true for $k + 1$ (Steps 2 and 3) corresponds to knowing that you can climb from one step to the next. **33.** H **35.** $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$ **37.** 4, 10, 28 **39.** 12 h

Pages 674–678 Chapter 11 Study Guide and Review

1. partial sum **3.** sigma notation **5.** Binomial Theorem **7.** arithmetic series **9.** 38 **11.** −11 **13.** −3, 1, 5 **15.** 6, 3, 0, −3 **17.** 48 **19.** 990 **21.** 282 **23.** 32 **25.** 3 **27.** 6, 12 **29.** \$5796.37 **31.** $\frac{21}{8}$ **33.** $\frac{11}{16}$ **35.** 72 **37.** $-\frac{16}{13}$ **39.** −2, 3, 8, 13, 18 **41.** 1, 1, 1 **43.** \$13,301 **45.** $243r^5 + 405r^4s + 270r^3s^2 + 90r^2s^3 + 15rs^4 + s^5$ **47.** $-13{,}107{,}200x^9$

49. Step 1: When $n = 1$, the left side of the given equation is 1. The right side is $2^1 - 1$ or 1, so the equation is true for $n = 1$.

Step 2: Assume $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$ for some positive integer k. **Step 3:** $1 + 2 + 4 + \ldots + 2^{k-1} + 2^{(k+1)-1} = 2^k - 1 + 2^{(k+1)-1} = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$ for all positive integers n. **51. Step 1:** $3^1 - 1 = 2$ which is divisible by 2. The statement holds true for $n = 1$. **Step 2:** Assume that $3^k - 1$ is divisible by 2 for some positive integer k. This means that $3^k - 1 = 2r$ for some whole number r.

Step 3:
$$\begin{aligned} 3^k - 1 &= 2r \\ 3^k &= 2r + 1 \\ 3(3^k) &= 3(2r + 1) \\ 3^{k+1} &= 6r + 3 \\ 3^{k+1} - 1 &= 6r + 2 \\ 3^{k+1} - 1 &= 2(3r + 1) \end{aligned}$$

Since r is a whole number, $3r + 1$ is a whole number. Thus, $3^{k+1} - 1$ is divisible by 2, so the statement is true for $n = k + 1$. Therefore, $3^n - 1$ is divisible by 2 for all positive integers n. **53.** $n = 2$ **55.** $n = 2$

Chapter 12 Probability and Statistics

Page 683 Chapter 12 Get Ready

1. $\frac{1}{6}$ **3.** $\frac{1}{6}$ **5.** $\frac{1}{2}$ **7.** $\frac{5}{32}$ **9.** $a^3 + 3a^2b + 3ab^2 + b^3$
11. $m^5 - 5m^4n + 10m^3n^2 - 10m^2n^3 + 5mn^4 - n^5$
13. $(h + t)^5 = h^5 + 5h^4t + 10h^3t^2 + 10h^2t^3 + 5ht^4 + t^5$

Pages 687–689 Lesson 12-1

1. independent **3.** 30 **5.** 256 **7.** 20 **9.** independent **11.** dependent **13.** 16 **15.** 30 **17.** 1024 **19.** 6 **21.** 160 **23.** 240 **25.** 27,216. **27.** Sample answer: buying a shirt that comes in 3 sizes and 6 colors. **29.** 17 **31.** A

33. Step 1: When $n = 1$, the left side of the given equation is 4. The right side is $\frac{1[3(1) + 5]}{2}$ or 4, so the equation is true for $n = 1$.

Step 2: Assume $4 + 7 + 10 + \ldots + (3k + 1) = \frac{k(3k + 5)}{2}$ for some positive integer k.

Step 3: $4 + 7 + 10 + \ldots + (3k + 1) + [k(3 + 1) + 1] + [3(k \ldots$

$$= \frac{k(3k + 5)}{2} + [3(k + 1) + 1]$$
$$= \frac{k(3k + 5) + 2[3(k + 1) + 1]}{2}$$
$$= \frac{3k^2 + 5k + 6k + 6 + 2}{2}$$
$$= \frac{3k^2 + 11k + 8}{2}$$
$$= \frac{(k + 1)(3k + 8)}{2}$$
$$= \frac{(k + 1)[3(k + 1) + 5]}{2}$$

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The last expression is the right side of the equation to be proved, where $n = k + 1$. Thus, the equation is true for $n = k + 1$. Therefore, $4 + 7 + 10 + \ldots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all positive integers n.

35. $280a^3b^4$ **37.** $\{-4, 4\}$ **39.** $\left\{-3, \frac{1}{3}\right\}$ **41.** (1, 3) **43.** 30 **45.** 720 **47.** 15 **49.** 1

Pages 693–695 Lesson 12-2

1. 60 **3.** 6 **5.** permutation; 5040 **7.** permutation; 1260 **9.** 84 **11.** 56 **13.** 2520 **15.** 10 **17.** 792 **19.** 27,720 **21.** permutation; 5040 **23.** permutation; 2520 **25.** combination; 28 **27.** combination; 45 **29.** 111,540 **31.** 267,696 **33.** 60 **35.** 80,089,128 **37.** Sample answer: There are six people in a contest. How many ways can the first, second, and third prizes be awarded? **39.** Sometimes; the statement is true when $r = 1$. **41.** Permutations and combinations can be used to find the number of different lineups. There are 9! different 9-person lineups available: 9 choices for the first player, 8 choices for the second player, 7 for the third player, and so on. So, there are 362,880 different lineups. There are $C(16, 9)$ ways to choose 9 players from 16: $C(16, 9) = \frac{16!}{7!9!}$ or 11,440. **43.** J **45.** 80 **47.** Sample answer: $n = 2$ **49.** $x > 0.8047$ **51.** 20 days **53.** $\frac{(y-4)^2}{9} + \frac{(x-4)^2}{4} = 1$ **55.** $\frac{1}{2}$ **57.** $\frac{1}{3}$

Pages 700–702 Lesson 12-3

1. $\frac{1}{7}$ **3.** $\frac{4}{7}$ **5.** $\frac{1}{210}$ **7.** $\frac{3}{8}$ **9.** $\frac{11}{115}$ **11.** $\frac{6}{115}$ **13.** $\frac{24}{115}$ **15.** 0 **17.** $\frac{1}{56}$ **19.** $\frac{1}{70}$ **21.** $\frac{1}{70}$ **23.** $\frac{9}{20}$ **25.** $\frac{1}{20}$ **27.** $\frac{9}{20}$ **29.** 0.007 **31.** 0.109 **33.** $\frac{1}{120}$ **35.** theoretical; $\frac{1}{36}$ **37.** theoretical; $\frac{1}{17}$ **39.** C **41.** permutation; 120 **43.** combination; 35 **45.** direct variation **47.** $\frac{6}{35}$ **49.** $\frac{1}{4}$ **51.** $\frac{9}{20}$

Pages 706–709 Lesson 12-4

1. $\frac{1}{36}$ **3.** $\frac{1}{4}$ **5.** $\frac{1}{16}$ **7.** $\frac{11}{850}$ **9.** dependent; $\frac{5}{204}$ or about 0.025 **11.** $\frac{4}{663}$ **13.** $\frac{25}{36}$ **15.** $\frac{1}{6}$ **17.** $\frac{5}{6}$ **19.** $\frac{25}{49}$ **21.** $\frac{1}{42}$ **23.** 0 **25.** $\frac{10}{171}$ or about 0.058 **27.** $\frac{20}{171}$ or about 0.117 **29.** $\frac{1}{20}$ **31.** independent; $\frac{1}{9}$ or about 0.111 **33.** dependent; $\frac{1}{21}$ **35.** independent; $\frac{8}{27}$ or about 0.296 **37.** $\frac{31,213}{3,200,000}$ or about 0.0098

39.

		First Spin: blue ($\frac{1}{3}$)	First Spin: yellow ($\frac{1}{3}$)	First Spin: red ($\frac{1}{3}$)
Second Spin	blue ($\frac{1}{3}$)	BB ($\frac{1}{9}$)	BY ($\frac{1}{9}$)	BR ($\frac{1}{9}$)
	yellow ($\frac{1}{3}$)	YB ($\frac{1}{9}$)	YY ($\frac{1}{9}$)	YR ($\frac{1}{9}$)
	red ($\frac{1}{3}$)	RB ($\frac{1}{9}$)	RY ($\frac{1}{9}$)	RR ($\frac{1}{9}$)

41. $\frac{1}{3}$ **43.** $\frac{19}{1,160,054}$ **45.** $\frac{6327}{20,825}$ **47.** no **49.** Sample answer: putting on your socks, and then your shoes **51.** 21 **53.** D **55.** $\frac{1}{204}$ **57.** 1440 ways **59.** 36 **61.** $x, x - 4$ **63.** $\frac{5}{6}$ **65.** $\frac{11}{12}$ **67.** $1\frac{5}{12}$

Pages 713–715 Lesson 12-5

1. $\frac{1}{3}$ **3.** $\frac{1}{3}$ **5.** $\frac{1}{2}$ **7.** mutually exclusive; $\frac{2}{13}$ **9.** $\frac{13}{16}$ **11.** $\frac{11}{56}$ **13.** $\frac{55}{56}$ **15.** $\frac{128}{1001}$ **17.** $\frac{202}{429}$ **19.** $\frac{227}{429}$ **21.** inclusive; $\frac{1}{2}$ **23.** mutually exclusive; $\frac{4}{13}$ **25.** $\frac{2}{3}$ **27.** $\frac{11}{221}$ **29.** $\frac{63}{221}$ **31.** $\frac{19}{36}$ **33.** $\frac{4}{15}$ **35.** $\frac{1}{30}$ **37.** 0.42 **39.** $\frac{53}{108}$ **41.** $\frac{17}{162}$ **43.** Sample answer: mutually exclusive events: tossing a coin and rolling a die; inclusive events: drawing a 7 or a diamond from a standard deck of cards **45.** Probability can be used to estimate the percents of what teens do online. The events are inclusive because some people send/read email and buy things online. Also, you know that the events are inclusive because the sum of the percents is not 100%. **47.** G **49.** $\frac{125}{216}$ **51.** $\frac{1}{8}$ **53.** 254 **55.** $(x + 1)^2(x - 1)(x^2 + 1)$ **57.** direct variation **59.** 35.4, 34, no mode, 72 **61.** 63.75, 65, 50 and 65, 30 **63.** 12.98, 12.9, no mode, 4.7

Pages 720–723 Lesson 12-6

1. \$7912.50, \$6460.75 **3.** 40, 6.3 **5.** 424.3, 20.6 **7.** 1.6, 1.3 **9.** 4.8, 2.2 **11.** 569.4, 23.9 **13.** 43.6, 6.6 **15.** The median seems to represent the center of the data. **17.** Mode; it is the least expensive price. **19.** Mean; it is highest. **21.** 2,290,403; 2,150,000; 2,000,000 **23.** Mean; it is highest. **25.** 64% **27.** 19.3 **29.** 19.5 **31.** Different scales are used on the vertical axes. **33.** Sample answer: The second graph might be shown by the company owner to a prospective buyer of the company. It looks like there is a dramatic rise in sales. **35.** Sample answer: The variance of the set {0, 1} is 0.25, and the standard deviation is 0.5. **37.** The first histogram is lower in the middle and higher on the ends, so it represents data that are more spread out. Since set B has the greater standard

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deviation, set B corresponds to the first histogram and set A corresponds to the second. **39.** The statistic(s) that best represent a set of test scores depends on the distribution of the particular set of scores. Answers should include the following. The mean, median, and mode of the data set are 73.9, 76.5, 94. The mode is not representative at all because it is the highest score. The median is more representative than the mean because it is influenced less than the mean by the two very low scores of 34 and 19. Each measure is increased by 5.

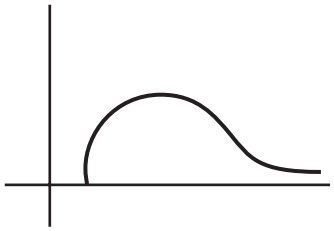
41. J **43.** mutually exclusive; $\frac{3}{7}$ **45.** $\frac{4}{663}$ **47.** $\frac{1}{16}$

49. 136 **51.** 380 **53.** 396

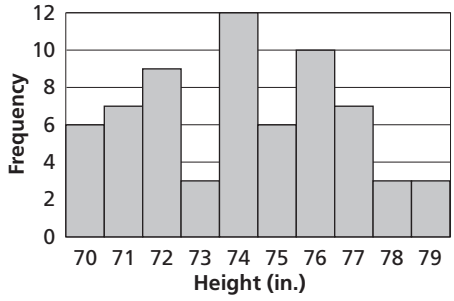
Pages 726–728 Lesson 12-7

1. normally distributed **3.** 13.5% **5.** 13,600 **7.** 3200 **9.** positively skewed **11.** Negatively skewed; the histogram is high at the right and has a tail to the left. **13.** 733 **15.** 50% **17.** 50% **19.** 25 **21.** 34% **23.** 16% **25.** 4.81

27.



Sample answer: the use of cassettes since CDs were introduced **29.** If a large enough group of athletes is studied, some of the characteristics may be normally distributed; others may have skewed distributions. Since the histogram goes up and down several times, the data may not be normally distributed. This may be due to players who play certain positions tending to be of similar large sizes while players who play the other positions tend to be of similar smaller sizes.



31. J **33.** 42.5, 6.5 **35.** $\frac{4}{13}$ **37.** 0.0183 **39.** 0.6065

Pages 731–733 Lesson 12-8

1. 0.22 **3.** 0.31 **5.** 0.105 **7.** 0.25 **9.** 0.05 **11.** 0.61 **13.** 0.49 **15.** 0.37 **17.** App. 10 **19.** 0.53 **21.** 0.98 **23.** 9 **25.** Never; The probability that x will be greater than the mean is always 36.8% for exponential distributions. **27.** The poll will give you a percent of people supporting the science wing addition. The percent of supporters represents the probability of success. You can use the formula for the expected number of success in a binomial distribution with the total number of students in the school to predict the number that will support the science wing addition.

29. G **31.** 97.5% **33.** $\frac{1}{3}$ **35.** $x^2 + 2x - 63$

37. -8 **39.** $56c^5d^3$

Pages 737–739 Lesson 12-9

1. $\frac{3}{8}$ **3.** $\frac{7}{8}$ **5.** $\frac{48}{28{,}561}$ **7.** about 0.075 **9.** $\frac{1}{32}$ **11.** $\frac{5}{16}$ **13.** $\frac{3}{16}$ **15.** $\frac{125}{324}$ **17.** $\frac{425}{432}$ **19.** $\frac{4096}{15{,}625}$ **21.** $\frac{48}{3125}$ **23.** $\frac{2816}{3125}$ **25.** $\frac{105}{512}$ **27.** $\frac{319}{512}$ **29.** $\frac{560}{2187}$ **31.** about 0.002 **33.** $\frac{1}{64}, \frac{3}{32}, \frac{15}{64}, \frac{5}{16}, \frac{15}{64}, \frac{3}{32}, \frac{1}{64}$ **35.** normal distribution **37a.** Each trial has more than two possible outcomes. **37b.** The number of trials is not fixed. **37c.** The trials are not independent. **39.** Getting a right answer and a wrong answer are the outcomes of a binomial experiment. The probability is far greater that guessing will result in a low grade than in a high grade. Use $(r + w)^5 = r^5 + 5r^4w + 10r^3w^2 + 10r^2w^3 + 5rw^4 + w^5$ and the chart on page 729 to determine the probabilities of each combination of right and wrong.

P(5 right): $r^5 = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$ or about 0.098%;

P(4 right, 1 wrong): $\frac{15}{1024}$ or about 1.5%;

P(3 right, 2 wrong): $10r^3w^2 = 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{45}{512}$ or about 8.8%;

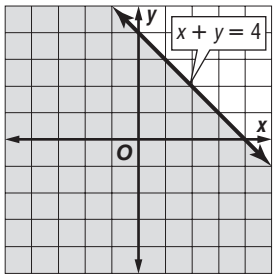
P(3 wrong, 2 right): $10r^2w^3 = 10\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = \frac{135}{512}$ or about 26.4%;

P(4 wrong, 1 right): $5rw^4 = 5\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^4 = \frac{405}{1024}$ or about 39.6%;

P(5 wrong): $w^5 = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$ or about 23.7%.

41. G **43.** 10 **45.** Mean; it is highest.

47.



49. 0.1 **51.** 0.039 **53.** 0.041

Pages 737–738 Lesson 12-10

1. Yes; the last digits of social security numbers are random. **3.** 9% **5.** 5% **7.** 1089 **9.** Yes; all seniors would have the same chance of being selected. **11.** No; basketball players are more likely to be taller than the average high school student, so a sample of basketball playetrs would not give representative heights for the whole school. **13.** 4% **15.** 3% **17.** 2% **19.** 4% **21.** 2% **23.** 36 or 64 **25.** 3% **27.** The margin of sampling error decreases when the size of the sample n increases. As n increases, $\frac{p\,(1-p)}{n}$ decreases. **29.** A **31.** $\frac{1}{32}$ **33.** $\frac{1}{2}$ **35.** 97.5%

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Pages 740–744 ***Chapter 12*** ***Study Guide and Review***

1. probability **3.** dependent events **5.** mutually exclusive events **7.** sample space **9.** 46,656 passwords **11.** 4 **13.** $\frac{4}{7}$ **15.** independent; $\frac{1}{36}$ **17.** dependent; $\frac{1}{7}$ **19.** inclusive; $\frac{1}{3}$ **21.** mutually exclusive; $\frac{1}{2}$ **23.** 8 **25.** 125 **27.** 34% **29.** 24 **31.** 0.12 **33.** $\frac{1}{2,176,782,336}$ **35.** $\frac{1}{64}, \frac{3}{32}, \frac{15}{64}, \frac{5}{16}, \frac{15}{64}, \frac{3}{32}, \frac{1}{64}$ **37.** about 4%

Chapter 13 Trigonometric Functions

Page 757 ***Chapter 13*** ***Getting Ready***

1. 10 **3.** 16.7 **5** 10.44 ft **7.** $x = 4\sqrt{3}, y = 8$

Pages 764–767 ***Lesson 13-1***

1. $\sin\theta = \frac{8}{17}; \cos\theta = \frac{15}{17}; \tan\theta = \frac{8}{15}; \csc\theta = \frac{17}{8}; \sec\theta = \frac{17}{15}; \cot\theta = \frac{15}{8}$ **3.** $\sin\theta = \frac{5}{6}; \cos\theta = \frac{\sqrt{11}}{6}; \tan\theta = \frac{5\sqrt{11}}{11}; \csc\theta = \frac{6}{5}; \sec\theta = \frac{6\sqrt{11}}{11}; \cot\theta = \frac{\sqrt{11}}{5}$ **5.** $\cos 23° = \frac{32}{x}; x \approx 34.8$ **7.** $B = 45°, a = 6, c \approx 8.5$ **9.** $a \approx 16.6, A \approx 67°, B \approx 23°$ **11.** 25.6 m

13. $\sin\theta = \frac{4}{11}; \cos\theta = \frac{\sqrt{105}}{11}; \tan\theta = \frac{4\sqrt{105}}{105}; \csc\theta = \frac{11}{4}; \sec\theta = \frac{11\sqrt{105}}{105}; \cot\theta = \frac{\sqrt{105}}{4}$

15. $\sin\theta = \frac{\sqrt{7}}{4}; \cos\theta = \frac{3}{4}; \tan\theta = \frac{\sqrt{7}}{3}; \csc\theta = \frac{4\sqrt{7}}{7}; \sec\theta = \frac{4}{3}; \cot\theta = \frac{3\sqrt{7}}{7}$ **17.** $\cos 60° = \frac{3}{x}, x = 6$

19. $\tan 17.5° = \frac{x}{23.7}; x \approx 7.5$ **21.** $\sin x° = \frac{16}{22}, x \approx 47$ **23.** $A = 63°, a \approx 13.7, c \approx 15.4$ **25.** $A = 75°, a \approx 24.1, b \approx 6.5$ **27.** $B = 45°, a = 7, b = 7$ **29.** about 142.8 ft **31.** $\sin\theta = \frac{\sqrt{5}}{5}; \cos\theta = \frac{2\sqrt{5}}{5}; \tan\theta = \frac{1}{2}; \csc\theta = \sqrt{5}; \sec\theta = \frac{\sqrt{5}}{2}; \cot\theta = 2$

33. $A = 72°, b \approx 1.3, c \approx 4.1$ **35.** $A \approx 63°, B \approx 27°, a \approx 11.5$ **37.** $A \approx 41°, B \approx 49°, b = 8, c \approx 10.6$

39a.

$\sin 30° = \frac{opp}{hyp}$	sine ratio
$\sin 30° = \frac{x}{2x}$	Replace *opp* with x and *hyp* with $2x$.
$\sin 30° = \frac{1}{2}$	Simplify.

39b.

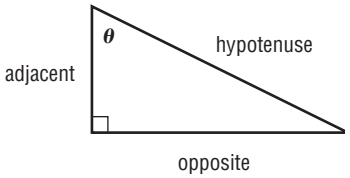
$\cos 30° = \frac{adj}{hyp}$	cosine ratio
$\cos 30° = \frac{\sqrt{3}x}{2x}$	Replace *adj* with $\sqrt{3}x$ and *hyp* with $2x$.
$\cos 30° = \frac{\sqrt{3}}{2}$	

39c.

$\sin 60° = \frac{opp}{hyp}$	sine ratio
$\sin 60° = \frac{\sqrt{3}x}{2x}$	Replace *opp* with $\sqrt{3}x$ and *hyp* with $2x$.
$\sin 60° = \frac{\sqrt{3}}{2}$	Simplify.

41. about 6° **43.** 93.53 units^2

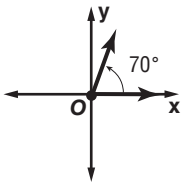
45.



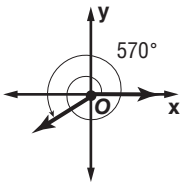
47. The sine and cosine ratios of acute angles of right triangles each have the longest measure of the triangle, the hypotenuse, as their denominator. A fraction whose denominator is greater than its numerator is less than 1. The tangent ratio of an acute angle of a right triangle does not involve the measure of the hypotenuse, $\frac{opp}{adj}$. If the measure of the opposite side is greater than the measure of the adjacent side, the tangent ratio is greater than 1. If the measure of the opposite side is less than the measure of the adjacent side, the tangent ratio is less than 1. **49.** C **51.** No; Band members may be more likely to like the same kinds of music. **53.** $\frac{3}{8}$ **55.** $\frac{15}{16}$ **57.** $\{-2, -1, 0, 1, 2\}$ **59.** 20 qt **61.** 12 m^2

Pages 772–774 ***Lesson 13-2***

1.

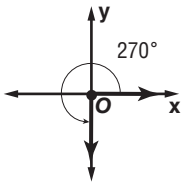


3.

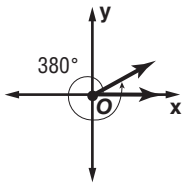


5. $\frac{13\pi}{18}$ **7.** $\frac{97\pi}{36}$ **9.** $-30°$ **11.** 21 h **13.** Sample answer: 420°, −300° **15.** Sample answer: $\frac{7\pi}{3}, -\frac{5\pi}{3}$

17.



19.



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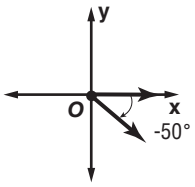
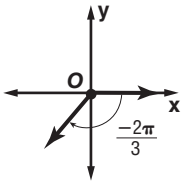
21. $\frac{\pi}{3}$ **23.** $-\frac{5\pi}{4}$ **25.** 495° **27.** −60° **29.** Sample answer: 390°, −330° **31.** Sample answer: $\frac{11\pi}{4}$, $-\frac{5\pi}{4}$ **33.** Sample answer: $\frac{3\pi}{4}$, $-\frac{13\pi}{4}$ **35.** about 188.5 m^2

37.

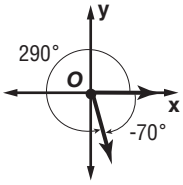


39.



41. $\frac{19\pi}{6}$ **43.** $\frac{13\pi}{9}$ **45.** 510° **47.** $\frac{540}{\pi} \approx 171.9°$ **49.** Sample answer: 8°, −352° **51.** Sample answer: $\frac{4\pi}{3}$, $-\frac{8\pi}{3}$ **53.** Sample answer: $\frac{25\pi}{4}$, $-\frac{7\pi}{4}$ **55.** number 17

57.



59a. $a^2 + (-b)^2 = a^2 + b^2 = 1$

59b. $b^2 + a^2 = a^2 + b^2 = 1$

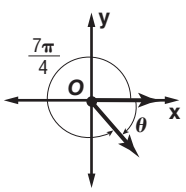
59c. $b^2 + (-a)^2 = a^2 + b^2 = 1$

61. An angle with a measure of more than 180° gives an indication of motion in a circular path that ended at a point more than halfway around the circle from where it started. Negative angles convey the same meaning as positive angles, but in an opposite direction. The standard convention is that negative angles represent rotations in a clockwise direction. Rates over 360° per minute indicate that an object is rotating or revolving more than one revolution per minute. **63.** J **65.** $A = 22°$, $a \approx 5.9$, $c \approx 15.9$ **67.** $c = 0.8$, $A = 30°$, $B = 60°$ **69.** about 7.07% **71.** combination, 35 **73.** $[g \circ h](x) = 4x^2 - 6x + 23$, $[h \circ g](x) = 8x^2 + 34x + 44$ **75.** $\frac{3\sqrt{5}}{5}$ **77.** $\frac{\sqrt{10}}{2}$ **79.** $\frac{\sqrt{10}}{4}$

Pages 781–783 ***Lesson 13-3***

1. $\sin\theta = \frac{8}{17}$, $\cos\theta = -\frac{15}{17}$, $\tan\theta = -\frac{8}{15}$, $\csc\theta = \frac{17}{8}$, $\sec\theta = -\frac{17}{15}$, $\cot\theta = -\frac{15}{8}$ **3.** $\sin\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = 1$, $\csc\theta = \sqrt{2}$, $\sec\theta = \sqrt{2}$, $\cot\theta = 1$ **5.** −1

7. $-\frac{2\sqrt{3}}{3}$ **9.** ; $\frac{\pi}{4}$

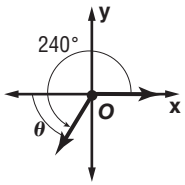


11. $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\csc\theta = \frac{2\sqrt{3}}{3}$, $\sec\theta = -2$, $\cot\theta = -\frac{\sqrt{3}}{3}$ **13.** about 12.4 ft

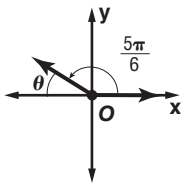
15. $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$, $\csc\theta = \sqrt{5}$, $\sec\theta = \frac{\sqrt{5}}{2}$, $\cot\theta = 2$ **17.** $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$

19. $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$, $\csc\theta$ = undefined, $\sec\theta = -1$, $\cot\theta$ = undefined **21.** $\sin\theta = -\frac{\sqrt{6}}{3}$, $\cos\theta = -\frac{\sqrt{3}}{3}$, $\tan\theta = \sqrt{2}$, $\csc\theta = -\frac{\sqrt{6}}{2}$, $\sec\theta = -\sqrt{3}$, $\cot\theta = \frac{\sqrt{2}}{2}$ **23.** −2 **25.** $-\sqrt{3}$ **27.** $\frac{1}{2}$ **29.** $\frac{\sqrt{2}}{2}$ **31.** 2

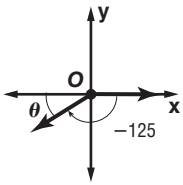
33. −1 **35.** ; 60°



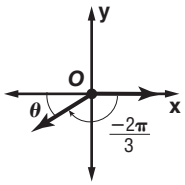
37. ; $\frac{\pi}{6}$



39. ; 55°



41. ; $\frac{\pi}{3}$



43. $\sin\theta = \frac{\sqrt{26}}{26}$, $\cos\theta = -\frac{5\sqrt{26}}{26}$, $\csc\theta = \sqrt{26}$, $\sec\theta = -\frac{\sqrt{26}}{5}$, $\cot\theta = -5$ **45.** $\sin\theta = -\frac{2\sqrt{5}}{5}$, $\cos\theta = -\frac{\sqrt{5}}{5}$, $\tan\theta = 2$, $\csc\theta = -\frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$ **47.** 45°; $2 \times 45°$ or 90° yields the greatest value for $\sin 2\theta$.

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49. 0.2, 0, −0.2, 0, 0.2, 0, and −0.2; or about 11.5°, 0°, −11.5°, 0°, 11.5°, 0°, and −11.5°

51. Sample answer: 200° **53.** Answers should include the following.

- The cosine of any angle is defined as $\frac{x}{r}$, where x is the x-coordinate of any point on the terminal ray of the angle and r is the distance from the origin to that point. This means that for angles with terminal sides to the left of the y-axis, the cosine is negative, and those with terminal sides to the right of the y-axis, the cosine is positive. Therefore the cosine function can be used to model real-world data that oscillate between being positive and negative.
- If we knew the length of the cable we could find the vertical distance from the top of the tower to the rider. Then if we knew the height of the tower we could subtract from it the vertical distance calculated previously. This will leave the height of the rider from the ground.

55. F **57.** 300° **59.** 635 **61.** (−4, 3) **63.** 4.7 **65.** 2.7

Pages 790–792 Lesson 13-4

1. 57.5 in^2 **3.** $C = 30°$, $a \approx 2.9$, $c \approx 1.5$ **5.** $B \approx 20°$, $A \approx 20°$, $a \approx 20.2$ **7.** two; $B \approx 42°$, $C \approx 108°$, $c \approx 5.7$; $B \approx 138°$, $C \approx 12°$, $c \approx 1.2$ **9.** one; $B \approx 19°$, $C \approx 16°$, $c \approx 8.9$ **11.** 43.1 m^2 **13.** 572.8 ft^2 **15.** 4.2 m^2 **17.** $B = 101$, $c \approx 3.0$, $b \approx 3.4$ **19.** $B \approx 21$, $C \approx 37$, $b \approx 13.1$ **21.** $C = 97°$, $a \approx 5.5$, $b \approx 14.4$ **23.** no **25.** two; $B \approx 72°$, $C \approx 75°$, $c \approx 3.5$; $B \approx 108°$, $C \approx 39°$, $c \approx 2.3$ **27.** one; $B = 90°$, $C = 60°$, $c \approx 24.2$ **29.** two; $B \approx 56°$, $C \approx 72°$, $c \approx 229.3$; $B \approx 124°$, $C \approx 4°$, $c \approx 16.8$ **31.** 4.6 and 8.5 mi **33.** $C \approx 67°$, $B \approx 63°$, $b \approx 2.9$ **35.** 690 ft **37.** Gabe; Dulce used the wrong angle. The Law of Sines must first be used to find $\angle B$. Then $m\angle C$ can be found. Once $m\angle C$ is found, $A = \frac{1}{2}ba \sin C$ will yield the area of the triangle. **39.** If the height of the triangle is not given, but the measure of two sides and their included angle are given, then the formula for the area of a triangle using the sine function should be used. You might use this formula to find the area of a triangular piece of land, since it might be easier to measure two sides and use surveying equipment to measure the included angle than to measure the perpendicular distance from one vertex to its opposite side.

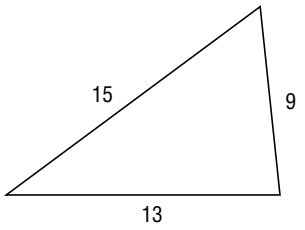
41. F **43.** $\sqrt{3}$ **45.** 660°, −60° **47.** $\frac{17\pi}{6}$, $-\frac{7\pi}{6}$

49. 5.6 **51.** 39.4°

Pages 796–798 Lesson 13-5

1. cosines; $A \approx 77°$, $B \approx 68°$, $c \approx 6.5$ **3.** sines; $C \approx 101°$, $B \approx 37°$, $c \approx 92.5$ **5.** 19.5 m **7.** sines; $A = 60°$, $b \approx 14.3$, $c \approx 11.2$ **9.** cosines; $A \approx 47°$, $B \approx 74°$, $C \approx 60°$ **11.** cosines; $A \approx 57°$, $B \approx 82°$, $c \approx 11.5$ **13.** cosines; $A \approx 55°$, $C \approx 78°$, $b \approx 17.9$ **15.** no **17.** cosines; $A \approx 103°$, $B \approx 49°$, $C \approx 28°$ **19.** 4.4 cm, 9.0 cm **21.** cosines; $A \approx 15°$, $B \approx 130°$, $C \approx 35°$ **23.** sines; $C = 102°$, $b \approx 5.5$, $c \approx 14.4$ **25.** cosines; $A \approx 107°$, $B \approx 35°$, $c \approx 13.8$ **27.** about 159.7° **29.** Since the step angle for the carnivore is closer to 180°, it appears as though the carnivore made more forward progress with each step than the herbivore did. **31. 1a.** Use the Law of Cosines to find the measure of one angle. Then use the Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of these two angles from 180° to find the measure of the third angle. **1b.** Use the Law of Cosines to find the measure of the third side. Then use the Law of Sines or the Law of Cosines to find the measure of a second angle. Finally, subtract the sum of these two angles from 180° to find the measure of the third angle.

33. Sample answer:



35. Given the latitude of a point on the surface of Earth, you can use the radius of the Earth and the orbiting height of a satellite in geosynchronous orbit to create a triangle. This triangle will have two known sides and the measure of the included angle. Find the third side using the Law of Cosines and then use the Law of Sines to determine the angles of the triangle. Subtract 90 degrees from the angle with its vertex on Earth's surface to find the angle at which to aim the receiver dish. **37.** F **39.** $\sin\theta = \frac{12}{13}$, $\cos\theta = \frac{15}{13}$, $\tan\theta = \frac{12}{5}$, $\csc\theta = \frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = \frac{5}{12}$ **41.** $\sin\theta = -\frac{\sqrt{6}}{4}$, $\cos\theta = \frac{\sqrt{10}}{4}$, $\tan\theta = \frac{\sqrt{15}}{5}$, $\csc\theta = \frac{2\sqrt{6}}{3}$, $\sec\theta = \frac{2\sqrt{10}}{5}$, $\cot\theta = \frac{\sqrt{15}}{3}$ **43.** $\{x \mid x > -0.6931\}$ **45.** 405, −315° **47.** 540°, −180° **49.** $\frac{19\pi}{6}$, $-\frac{5\pi}{6}$

Pages 803–805 Lesson 13-6

1. $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$ **3.** $\frac{\sqrt{3}}{2}$ **5.** 2 s **7.** $\sin\theta = \frac{4}{5}$; $\cos\theta = -\frac{3}{5}$ **9.** $\sin\theta = \frac{15}{17}$; $\cos\theta = \frac{8}{17}$ **11.** $\sin\theta = \frac{\sqrt{3}}{2}$; $\cos\theta = -\frac{1}{2}$ **13.** $-\frac{1}{2}$ **15.** −1 **17.** 1 **19.** 6 **21.** 2π **23.** $\frac{1}{440}$ s **25.** $\frac{1}{4}$ **27.** $\frac{1-\sqrt{3}}{2}$ **29.** $-3\sqrt{3}$ **31.** $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, (−1, 0), $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **33.** $\frac{y}{x}$ **35.** $-\frac{x}{y}$ **37.** $\sqrt{3}$

39. Sample answer: the motion of the minute hand on a clock; 60 s **41.** sine: D = {all reals}, R = $\{-1 \le y \le 1\}$; cosine: D = {all reals}, R = $\{-1 \le y \le 1\}$ **43.** B **45.** cosines: $c \approx 12.4$, $B \approx 59°$, $A \approx 76°$ **47.** 27.0 in^2 **49.** does not exist **51.** 8 **53.** 110°

Pages 809–811 Lesson 13-7

1. 45° **3.** 30° **5.** $\pi \approx 3.14$ **7.** 0.75 **9.** 0.58 **11.** 30° **13.** 30° **15.** 90° **17.** does not exist **19.** 0.52 **21.** 0.66 **23.** 0.5 **25.** 60° south of west **27.** 0.81 **29.** 3 **31.** 1.57 **33.** does not exist **35.** 0.87 **37.** No; with

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this point on the terminal side of the throwing angle θ, the measure of θ is found by solving the equation $\tan\theta = \frac{17}{18}$. Thus $\theta = \tan^{-1}\frac{17}{18}$ or about 43.3°, which is greater than the 40° requirement.

39. $\text{Sin}^{-1}x + \text{Cos}^{-1}x = \frac{\pi}{2}$ for all values of x

41. Sample answer: $\text{Cos } 45° = \frac{\sqrt{2}}{2}$; $\text{Cos}^{-1}\frac{\sqrt{2}}{2} = 45°$

43. 102° **45.** Trigonometry is used to determine proper banking angles. Answers should include the following.

- Knowing the velocity of the cars to be traveling on a road and the radius of the curve to be built, then the banking angle can be determined. First find the ratio of the square of the velocity to the product of the acceleration due to gravity and the radius of the curve. Then determine the angle that had this ratio as its tangent. This will be the banking angle for the turn.
- If the speed limit were increased and the banking angle remained the same, then in order to maintain a safe road the curvature would have to be decreased. That is, the radius of the curve would also have to increase, which would make the road less curved.

47. J **49.** −1 **51.** sines; $B \approx 69°$, $C \approx 81°$, $c \approx 6.1$ or $B \approx 111°$, $C \approx 39°$, $c \approx 3.9$ **53.** 46, 39 **55.** 11, 109

Pages 812–816 Chapter 13 Study Guide and Review

1. false; coterminal. **3.** true **5.** true **7.** $A \approx 26°$, $B \approx 64°$, $b \approx 14.4$ **9.** $A = 45°$, $a \approx 8.5$, $b \approx 8.5$ **11.** $A = 41°$, $b \approx 10.4$, $c \approx 13.7$ **13.** 587.6 ft **15.** $-\frac{7\pi}{6}$

17. −720° **19.** 320°, −400° **21.** $\frac{\pi}{4}$; $-\frac{15\pi}{4}$ **23.** $\sin\theta = \frac{5\sqrt{29}}{29}$, $\cos\theta = \frac{2\sqrt{29}}{29}$, $\tan\theta = \frac{5}{2}$, $\csc\theta = \frac{\sqrt{29}}{5}$, $\sec\theta = \frac{\sqrt{29}}{2}$, $\cot\theta = \frac{2}{5}$ **25.** −1 **27.** about 86.2 ft **29.** two; $B \approx 53°$, $C \approx 87°$, $c \approx 12.4$; $B \approx 127°$, $C \approx 13°$, $c \approx 3.0$ **31.** no **33.** 107 mph **35.** sines; $C = 105°$, $a \approx 28.3$, $c \approx 38.6$ **37.** sines; $B \approx 52°$, $C \approx 92°$, $c \approx 10.2$; $B \approx 128°$; $C \approx 16°$, $c \approx 2.7$ **39.** about 1148.5 ft **41.** $\frac{1}{2}$ **43.** $\frac{-\sqrt{2}}{2}$ **45.** $-\sqrt{3}$ **47.** −1.57 **49.** 0.75 **51.** 1125°

Chapter 14 Trigonometric Graphs and Identities

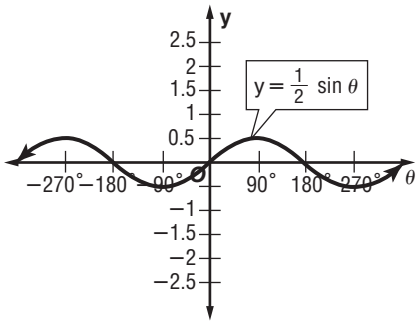
Page 821 Chapter 14 Get Ready

1. $\frac{\sqrt{2}}{2}$ **3.** 0 **5.** $-\frac{\sqrt{2}}{2}$ **7.** $-\frac{\sqrt{3}}{2}$ **9.** $\frac{2\sqrt{3}}{2}$

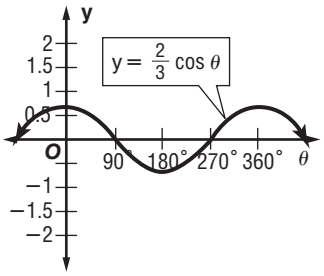
11. $-\sqrt{3}$ **13.** 60 ft **15.** $2x^2(x^2 - 2)$ **17.** $(2x + 1)(x - 2)$ **19.** 8, −3 **21.** 0, 12 **23.** −8, 5

Pages 826–828 Lesson 14-1

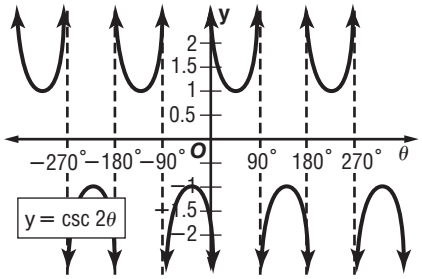
1. amplitude: $\frac{1}{2}$; period 360° or 2π



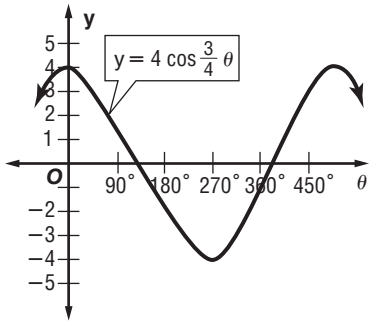
3. amplitude: $\frac{2}{3}$; period 360° or 2π



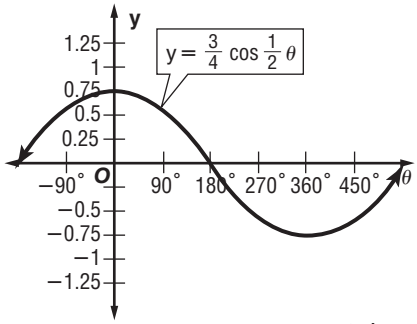
5. amplitude: does not exist; period: 180° or π



7. amplitude: 4; period: 480° or $\frac{8\pi}{3}$



9. amplitude: $\frac{3}{4}$; period: 720° or 4π



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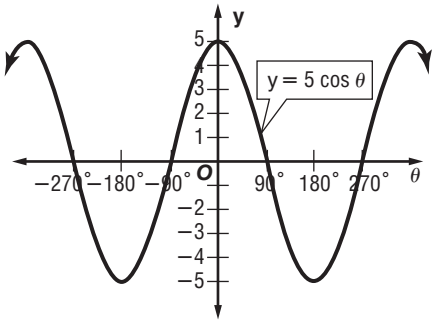
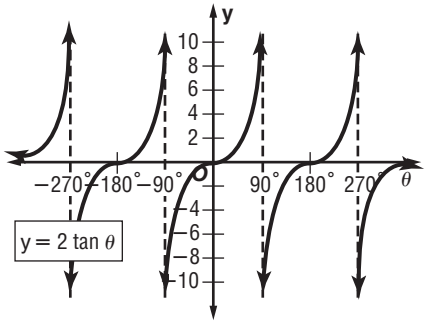
11. 4250; June 1

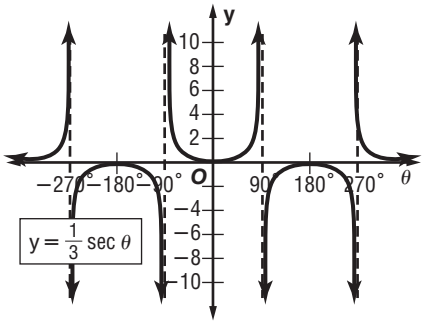
13. amplitude 5; period: 360° or 2π



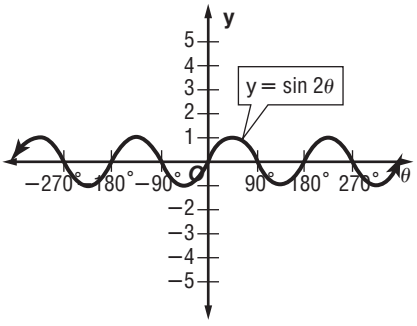
15. amplitude: does not exist; period: 180° or π



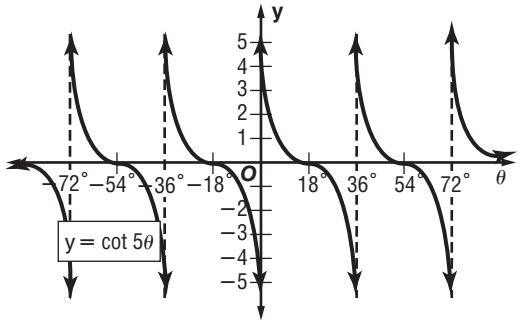
17. amplitude: does not exist; period: 360° or 2π



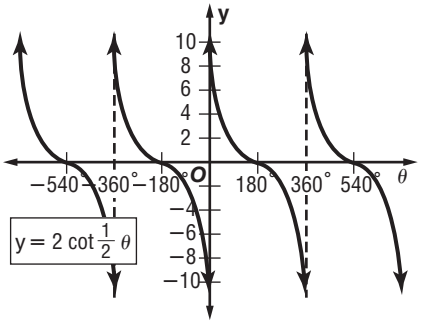
19. amplitude: 1; period: 180° or π



21. amplitude: does not exist; period: 36° or $\frac{\pi}{5}$

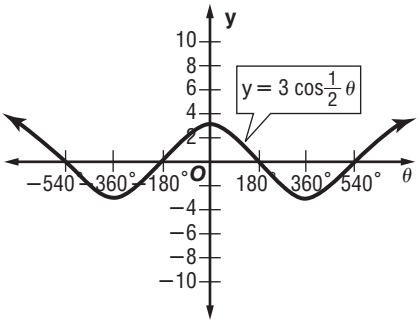


23. amplitude: does not exist; period: 360° or 2π

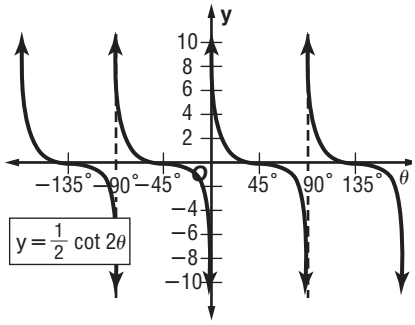


25. Sample answer: The amplitudes are the same. As the frequency increases, the period decreases.

27. amplitude: 3; period: 720° or 4π



29. amplitude: does not exist; period: 90° or $\frac{\pi}{2}$



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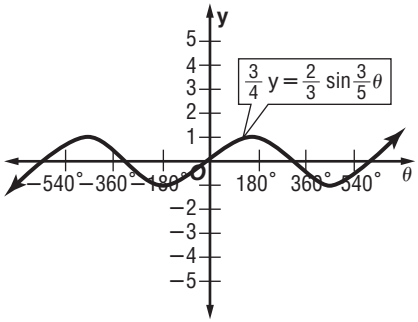
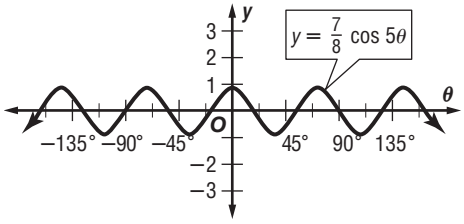
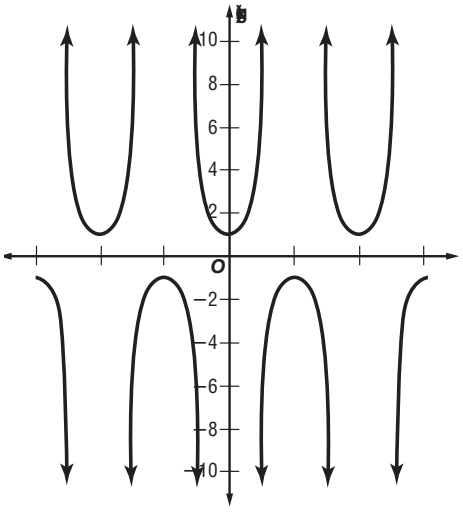
31. amplitude: $\frac{8}{9}$; period: 600° or $\frac{10\pi}{3}$



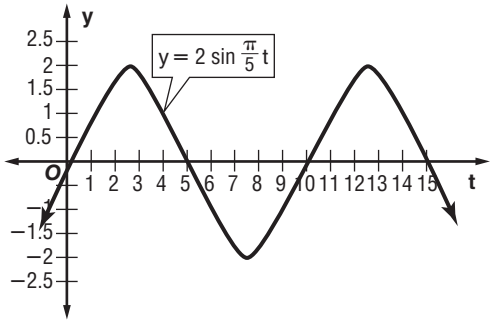
33. $y = \frac{7}{8}\cos 5\theta$



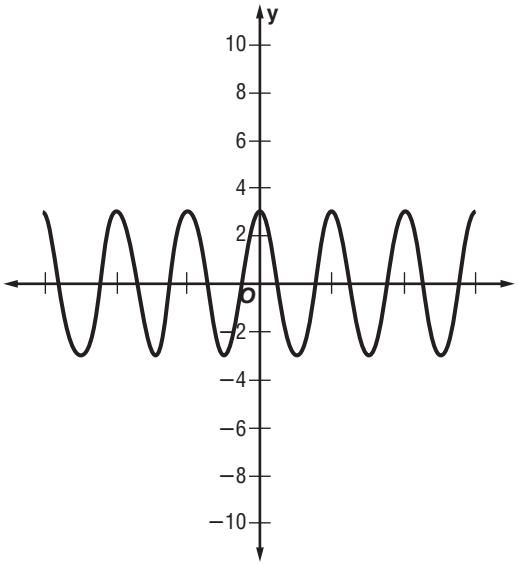
35. Vertical asymptotes located at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, etc. and $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, etc.



37.



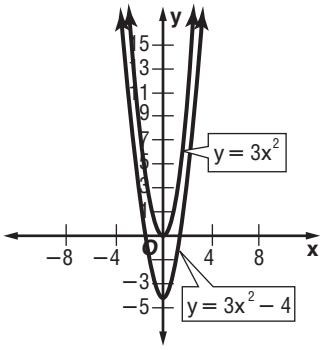
39. Sample answer: $y = 3\cos(2\theta)$



41. Jamile; the amplitude is 3, and the period is 3π. **43.** Sample answer: Tides display periodic behavior. This means that their pattern repeats at regular intervals. Tides rise and fall in a periodic manner, similar to the sine function.

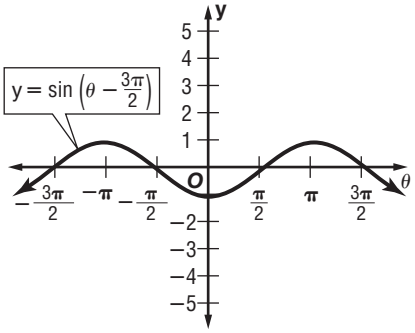
45. G **47.** −90° **49.** $\frac{1}{2}$ **51.** $\frac{\sqrt{2}}{2}$ **53.** 3, 11, 27, 59, 123

55.



Pages 834–836 ***Lesson 14-2***

1. 1; 2π; $\frac{\pi}{2}$



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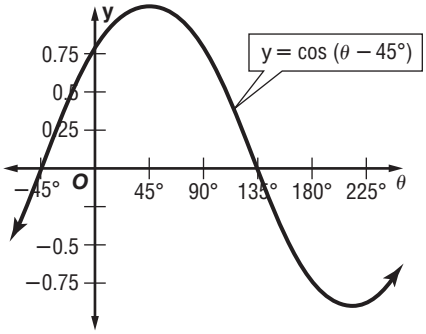
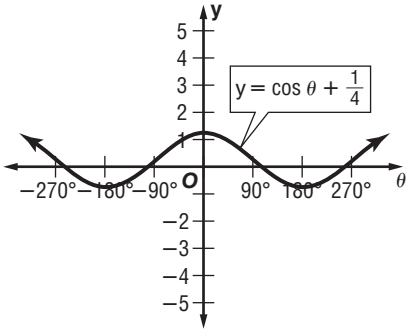
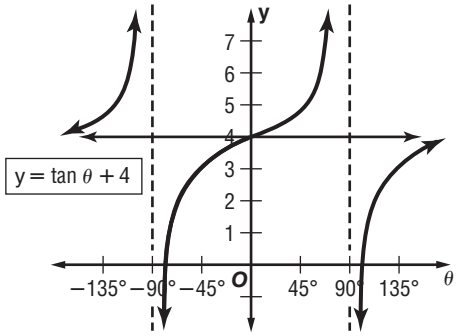
3. 1; 360°; 45°



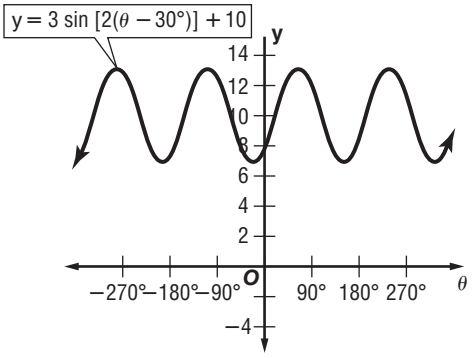
5. $\frac{1}{4}$; $y = \frac{1}{4}$; 1; 360°



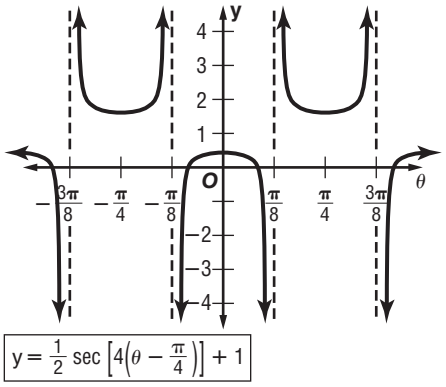
7. 4; $y = 4$; no amplitude; 180°



9. 10; 3; 180°; 30°

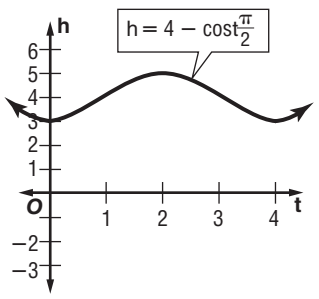


11. 1; no amplitude; $\frac{\pi}{2}$; $\frac{\pi}{4}$

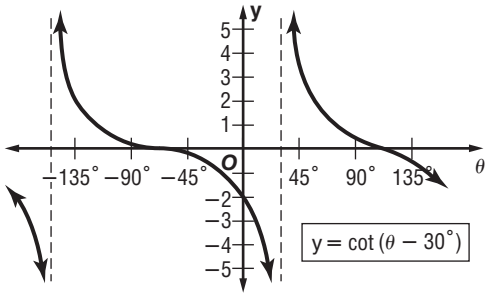


13. 4; 1; 4 s

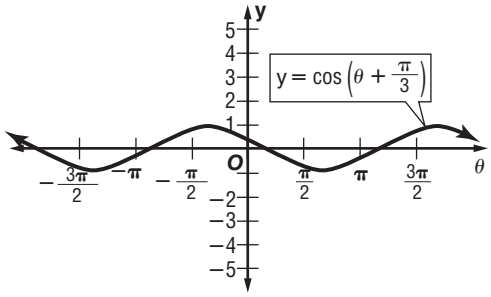
15.



17. no amplitude; 180°; 30°



19. 1; 2π; $-\frac{\pi}{3}$



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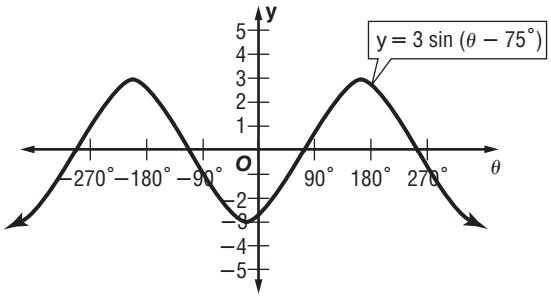
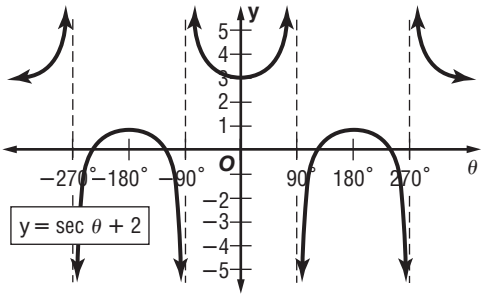
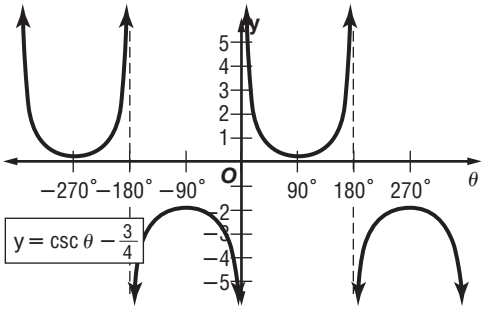
21. 3; 360°; 75°



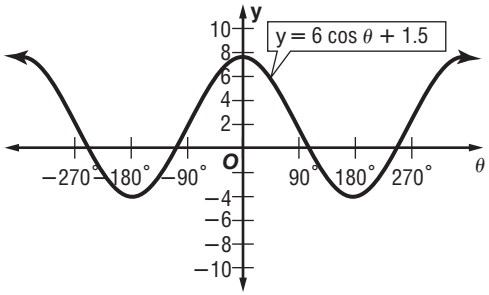
23. 2; $y = 2$; no amplitude; 360°



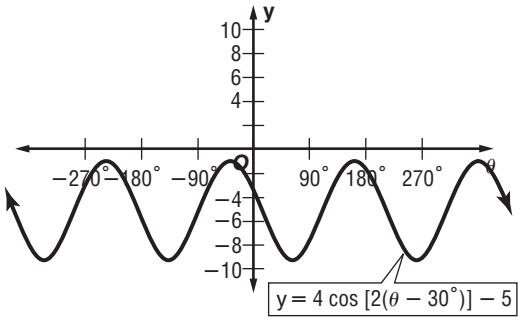
25. $-\frac{3}{4}$; $y = -\frac{3}{4}$; no amplitude; 360°



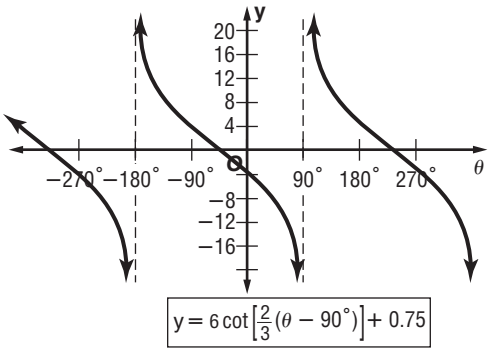
27. 1.5; $y = 1.5$; 6; 360°



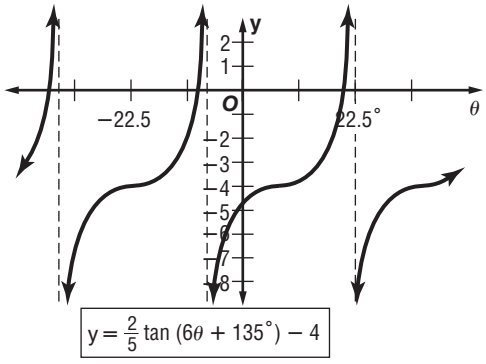
29. −5; 4; 180°; −30°



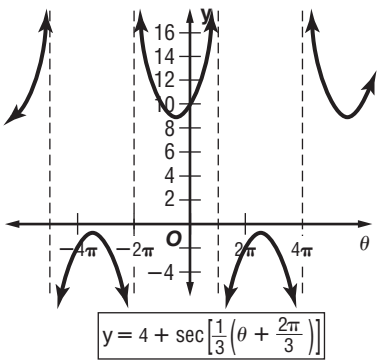
31. 0.75; does not exist; 270°; 90°



33. −4; does not exist; 30°; −22.5°



35. 4; does not exist; 6π; $-\frac{2\pi}{3}$



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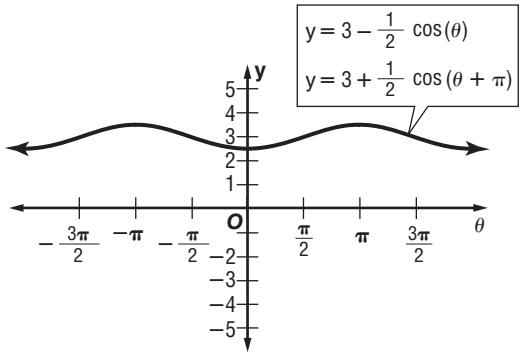
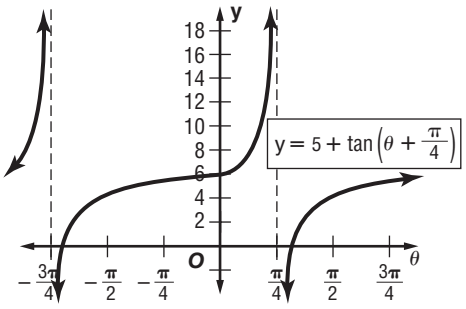
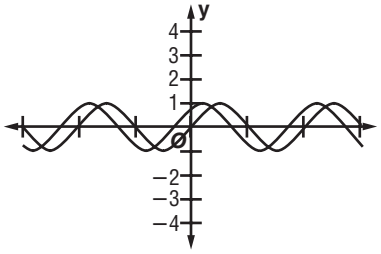
37. 300; 14.5 yr
39. The graphs are identical.



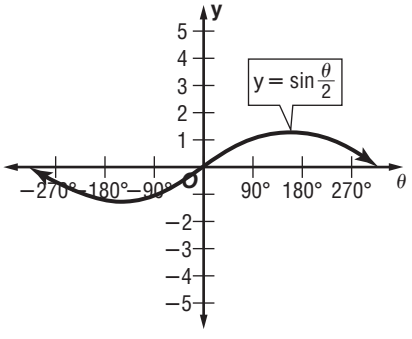
41. translation $\frac{\pi}{4}$ units left and 5 units up



43. c **45.** Sample answer: $y = \sin(\theta + 45°)$



47. Sample answer: You can use changes in amplitude and period along with vertical and horizontal shifts to show an animal population's starting point and display changes to that population over a period of time. The equation shows a rabbit population that begins at 1200, increases to a maximum of 1450, then decreases to a minimum of 950 over a period of 4 years. **49.** H **51.** amplitude: 1; period: 720° or 4π



53. 0.75 **55.** 0.83 **57.** 35 **59.** 0.66

61. $\frac{5a - 13}{(a - 2)(a - 3)}$ **63.** $\frac{3y^2 + 10y + 5}{2(y - 5)(y + 3)}$ **65.** −1

67. $\frac{1}{2}$ **69.** $\frac{\sqrt{3}}{3}$ **71.** 1

Pages 839–841 ***Lesson 14-3***

1. $-\frac{\sqrt{3}}{3}$ **3.** $\frac{3}{5}$ **5.** 1 **7.** $\sec\theta$ **9.** $\sin\theta = \cos\theta\frac{v^2}{gR}$

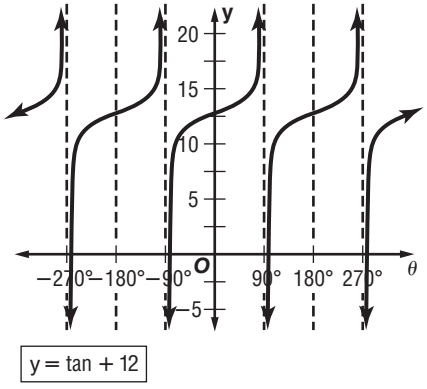
11. $\frac{\sqrt{5}}{3}$ **13.** $2\sqrt{2}$ **15.** $\frac{3}{5}$ **17.** $-\frac{3\sqrt{5}}{5}$ **19.** 1

21. $\sin\theta$ **23.** −3 **25.** $\tan\theta$ **27.** $P = I^2R\sin^2 2\pi ft$

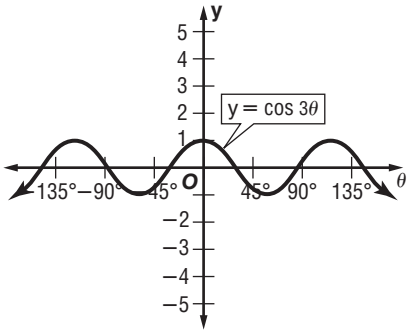
29. $\frac{3}{4}$ **31.** $-\frac{4\sqrt{7}}{7}$ **33.** $\cot^2\theta$ **35.** 1 **37.** about 4 m/s

39. $E = \frac{I\cos\theta}{R^2}$ **41.** Sample answer: The sine function is negative in the third and fourth quadrants. Therefore, the terminal side of the angle must lie in one of those two quadrants. **43.** $\frac{10\pi}{3}$

45. Sample answer: You can use equations to find the height and the horizontal distance of a baseball after it has been hit. The equations involve using the initial angle the ball makes with the ground with the sine function. Both equations are quadratic in nature with a leading negative coefficient. Thus, both are inverted parabolas which model the path of a baseball. **47.** F
49. 12; $y = 12$; no amplitude; 180°



51. amplitude: 1; period: 120° or $\frac{2\pi}{3}$



53. 93 **55.** 498 **57.** Subtraction (=)
59. Substitution (=)

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Pages 844–846 Lesson 14-4

1. $\tan\theta(\cot\theta + \tan\theta) \stackrel{?}{=} \sec^2\theta$

$1 + \tan^2\theta \stackrel{?}{=} \sec^2\theta$

$\sec^2\theta = \sec^2\theta$

3. $\frac{\cos^2\theta}{1-\sin\theta} \stackrel{?}{=} 1 + \sin\theta$

$\frac{1-\sin^2\theta}{1-\sin\theta} \stackrel{?}{=} 1 + \sin\theta$

$\frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta} \stackrel{?}{=} 1 + \sin\theta$

$1 + \sin\theta = 1 + \sin\theta$

5. $\frac{\sin\theta}{\sec\theta} \stackrel{?}{=} \frac{1}{\tan\theta + \cot\theta}$

$\frac{\sin\theta}{\sec\theta} \stackrel{?}{=} \frac{1}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$

$\frac{\sin\theta}{\sec\theta} \stackrel{?}{=} \frac{1}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$

$\frac{\sin\theta}{\sec\theta} \stackrel{?}{=} \frac{\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta}$

$\frac{\sin\theta}{\sec\theta} \stackrel{?}{=} \frac{\sin\theta\cos\theta}{1}$

$\frac{\sin\theta}{\sec\theta} = \frac{\sin\theta}{\sec\theta}$

7. D

9. $\cot\theta(\cot\theta + \tan\theta) \stackrel{?}{=} \csc^2\theta$

$\cot^2\theta + \cot\theta\tan\theta \stackrel{?}{=} \csc^2\theta$

$\cot^2\theta + \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \stackrel{?}{=} \csc^2\theta$

$\cot^2\theta + 1 \stackrel{?}{=} \csc^2\theta$

$\csc^2\theta = \csc^2\theta$

11. $\sin\theta\sec\theta\cot\theta \stackrel{?}{=} 1$

$\sin\theta \cdot \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} \stackrel{?}{=} 1$

$1 = 1$

13. $\frac{1-2\cos^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \tan\theta - \cot\theta$

$\frac{(1-\cos^2\theta) - \cos^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \tan\theta - \cot\theta$

$\frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \tan\theta - \cot\theta$

$\frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{\cos^2\theta}{\sin\theta\cos\theta} \stackrel{?}{=} \tan\theta - \cot\theta$

$\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta} \stackrel{?}{=} \tan\theta - \cot\theta$

$\tan\theta - \cot\theta = \tan\theta - \cot\theta$

15. $\sin\theta + \cos\theta \stackrel{?}{=} \frac{1+\tan\theta}{\sec\theta}$

$\sin\theta + \cos\theta \stackrel{?}{=} \frac{1 + \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$\sin\theta + \cos\theta \stackrel{?}{=} \frac{\frac{\sin\theta + \cos\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$\sin\theta + \cos\theta \stackrel{?}{=} \frac{\sin\theta + \cos\theta}{\cos\theta} \cdot \cos\theta$

$\sin\theta + \cos\theta = \sin\theta + \cos\theta$

17. $\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} \stackrel{?}{=} 2\csc\theta$

$\frac{\sin\theta}{\sin\theta} \cdot \frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1-\cos\theta} \cdot \frac{1-\cos\theta}{\sin\theta} \stackrel{?}{=} 2\csc\theta$

$\frac{\sin^2\theta}{\sin\theta(1-\cos\theta)} + \frac{1-2\cos\theta+\cos^2\theta}{\sin\theta(1-\cos\theta)} \stackrel{?}{=} 2\csc\theta$

$\frac{\sin^2\theta + \cos^2\theta + 1 - 2\cos\theta}{\sin\theta(1-\cos\theta)} \stackrel{?}{=} 2\csc\theta$

$\frac{2-2\cos\theta}{\sin\theta(1-\cos\theta)} \stackrel{?}{=} 2\csc\theta$

$\frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)} \stackrel{?}{=} 2\csc\theta$

$\frac{2}{\sin\theta} \stackrel{?}{=} 2\csc\theta$

$2\csc\theta = 2\csc\theta$

19. $\frac{\sin^2\theta}{1-\cos\theta} \stackrel{?}{=} 1 + \cos\theta$

$\frac{\sin^2\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \stackrel{?}{=} 1 + \cos\theta$

$\frac{\sin^2\theta(1+\cos\theta)}{1-\cos^2\theta} \stackrel{?}{=} 1 + \cos\theta$

$\frac{\sin^2\theta(1+\cos\theta)}{\sin^2\theta} \stackrel{?}{=} 1 + \cos\theta$

$1 + \cos\theta = 1 + \cos\theta$

21. 598.7 m

23. $\frac{1+\tan\theta}{1+\cot\theta} \stackrel{?}{=} \frac{\sin\theta}{\cos\theta}$

$\frac{1 + \frac{\sin\theta}{\cos\theta}}{1 + \frac{\cos\theta}{\sin\theta}} \stackrel{?}{=} \frac{\sin\theta}{\cos\theta}$

$\frac{\frac{\sin\theta + \cos\theta}{\cos\theta}}{\frac{\sin\theta + \cos\theta}{\sin\theta}} \stackrel{?}{=} \frac{\sin\theta}{\cos\theta}$

$\frac{\sin\theta + \cos\theta}{\cos\theta} \cdot \frac{\sin\theta}{\sin\theta + \cos\theta} \stackrel{?}{=} \frac{\sin\theta}{\cos\theta}$

$\frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta}$

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25. $1 + \frac{1}{\cos\theta} \stackrel{?}{=} \frac{\tan^2\theta}{\sec\theta - 1}$

$1 + \frac{1}{\cos\theta} \stackrel{?}{=} \frac{\tan^2\theta}{\sec\theta - 1} \cdot \frac{\sec\theta + 1}{\sec\theta + 1}$

$1 + \frac{1}{\cos\theta} \stackrel{?}{=} \frac{\tan^2\theta\,(\sec\theta + 1)}{\sec^2\theta - 1}$

$1 + \frac{1}{\cos\theta} \stackrel{?}{=} \frac{\tan^2\theta\,(\sec\theta + 1)}{\tan^2\theta - 1}$

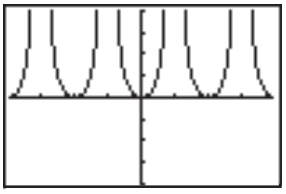
$1 + \frac{1}{\cos\theta} \stackrel{?}{=} \sec\theta + 1$

$1 + \frac{1}{\cos\theta} = 1 + \frac{1}{\cos\theta}$

27.
$$\cos^4\theta - \sin^4\theta \stackrel{?}{=} \cos^2\theta - \sin^2\theta$$
$$(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \stackrel{?}{=} \cos^2\theta - \sin^2\theta$$
$$(\cos^2\theta - \sin^2\theta) \cdot 1 \stackrel{?}{=} \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta$$

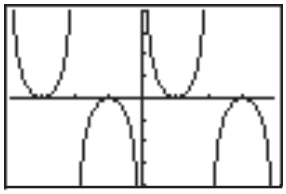
29.
$$\frac{\cos\theta}{1 + \sin\theta} + \frac{\cos\theta}{1 - \sin\theta} \stackrel{?}{=} 2\sec\theta$$
$$\frac{\cos\theta}{1 + \sin\theta} \cdot \frac{1 - \sin\theta}{1 - \sin\theta} + \frac{\cos\theta}{1 - \sin\theta} \cdot \frac{1 + \sin\theta}{1 + \sin\theta} \stackrel{?}{=} 2\sec\theta$$
$$\frac{\cos\theta\,(1 - \sin\theta) + \cos\theta\,(1 + \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)} \stackrel{?}{=} 2\sec\theta$$
$$\frac{\cos\theta - \sin\theta\cos\theta + \cos\theta + \sin\theta\cos\theta}{1 - \sin^2\theta} \stackrel{?}{=} 2\sec\theta$$
$$\frac{2\cos\theta}{\cos^2\theta} \stackrel{?}{=} 2\sec\theta$$
$$\frac{2}{\cos\theta} \stackrel{?}{=} 2\sec\theta$$
$$2\sec\theta = 2\sec\theta$$

31. [−360, 360] scl: 90 by [−5, 5] scl: 1; may be



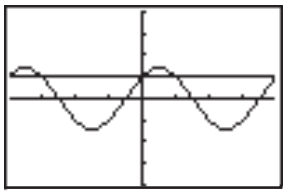
[−360, 360] scl: 90 by [−5, 5] scl: 1

33. [−360, 360] scl: 90 by [−5, 5] scl: 1; may be



[−360, 360] scl: 90 by [−5, 5] scl: 1

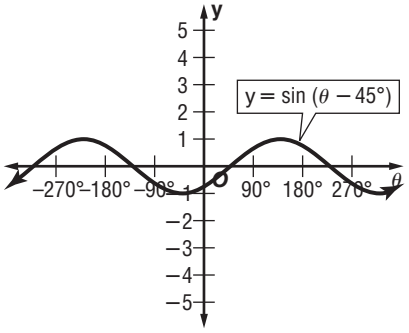
35. [−360, 360] scl: 90 by [−5, 5] scl: 1; is not



[−360, 360] scl: 90 by [−5, 5] scl: 1

37. $\sin^2\theta - \cos^2\theta = 2\sin^2\theta$ does not belong with the others. The other equations are identities, but $\sin^2\theta - \cos^2\theta = 2\sin^2\theta$ is not. $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1$ would be an identity. **39.** Sample answer: The expressions have not yet been shown to be equal, so you could not use the properties of equality on them. Graphing two expressions could result in identical graphs for a set interval, that are different elsewhere. **41.** G

43. $-\frac{\sqrt{5}}{3}$ **45.** $-\frac{\sqrt{7}}{4}$ **47.** 1; 360°; 45°



49. $\frac{1}{10^7}$ **51.** −5, −1 **53.** −2 **55.** $\frac{\sqrt{2}}{4}$ **57.** $\frac{2 - \sqrt{3}}{4}$

Pages 851–852 ***Lesson 14-5***

1. $\frac{\sqrt{6} + \sqrt{2}}{4}$ **3.** $\frac{\sqrt{2} - \sqrt{6}}{4}$ **5.** $\frac{\sqrt{3}}{2}$ **7.** $\frac{5 - \sqrt{3}}{1 + 5\sqrt{3}}$

9.
$$\sin\left(\theta + \frac{\pi}{2}\right) \stackrel{?}{=} \cos\theta$$
$$\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \stackrel{?}{=} \cos\theta$$
$$\sin\theta \cdot 0 + \cos\theta \cdot 1 \stackrel{?}{=} \cos\theta$$
$$\cos\theta = \cos\theta$$

11. $\frac{\sqrt{2}}{2}$ **13.** $\frac{-\sqrt{6} - \sqrt{2}}{4}$ **15.** $\frac{-\sqrt{6} - \sqrt{2}}{4}$

17. $-\frac{\sqrt{2}}{2}$ **19.** $\frac{\sqrt{2} - \sqrt{6}}{4}$ **21.** $-\frac{\sqrt{3}}{2}$

23. 0.3681 *E* **25.** 0.6157 *E*

27.
$$\sin(270° - \theta) \stackrel{?}{=} \sin 270°\cos\theta - \cos 270°$$
$$\stackrel{?}{=} -1\cos\theta - 0$$
$$= -\cos\theta$$

29.
$$\cos(90° - \theta) \stackrel{?}{=} \cos 90°\cos\theta + \sin 90°\sin\theta$$
$$\stackrel{?}{=} 0 \cdot \cos\theta + 1 \cdot \sin\theta$$
$$= \sin\theta$$

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31. $\sin\left(\theta + \frac{3\pi}{2}\right) \stackrel{?}{=} -\cos\theta$

$\sin\theta\cos\frac{3\pi}{2} + \cos\theta\sin\frac{3\pi}{2} \stackrel{?}{=} -\cos\theta$

$\sin\theta \cdot 0 + \cos\theta \cdot (-1) \stackrel{?}{=} -\cos\theta$

$0 + (-\cos\theta) \stackrel{?}{=} -\cos\theta$

$-\cos\theta = -\cos\theta$

33. $\cos(2\pi + \theta) \stackrel{?}{=} \cos\theta$

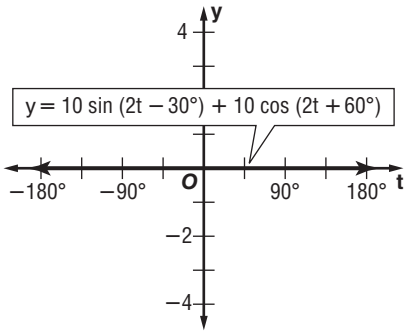
$\cos 2\pi\cos\theta - [\sin 2\pi\sin\theta] \stackrel{?}{=} \cos\theta$

$1 \cdot \cos\theta - [0 \cdot \sin\theta] \stackrel{?}{=} \cos\theta$

$1 \cdot \cos\theta - 0 \stackrel{?}{=} \cos\theta$

$\cos\theta = \cos\theta$

35.



37. $\sin(60° + \theta) + \sin(60° - \theta)$

$\stackrel{?}{=} \sin 60°\cos\theta + \cos 60°\sin\theta + \sin 60°\cos\theta - \cos 60°\sin\theta$

$\stackrel{?}{=} \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta$

$= \sqrt{3}\cos\theta$

39.

$\sin(\alpha + \beta)\sin(\alpha - \beta) \stackrel{?}{=} \sin^2\alpha - \sin^2\beta$

$\stackrel{?}{=} (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\stackrel{?}{=} \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$

$\stackrel{?}{=} \sin^2\alpha(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta$

$\stackrel{?}{=} \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta$

$= \sin^2\alpha - \sin^2\beta$

41. Sample answer: $\alpha = \frac{\pi}{4}; \beta = \frac{3\pi}{2}$

43. $\tan(\alpha + \beta) \stackrel{?}{=} \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$

$\stackrel{?}{=} \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$

$\stackrel{?}{=} \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(\alpha - \beta) \stackrel{?}{=} \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)}$

$\stackrel{?}{=} \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta}$

$\stackrel{?}{=} \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

45. A

47. $\cot\theta + \sec\theta \stackrel{?}{=} \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$

$\cot\theta + \sec\theta \stackrel{?}{=} \frac{\cos^2\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta}$

$\cot\theta + \sec\theta \stackrel{?}{=} \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta}$

$\cot\theta + \sec\theta = \cot\theta + \sec\theta$

49. $\sin\theta(\sin\theta + \csc\theta) \stackrel{?}{=} 2 - \cos^2\theta$

$\sin^2\theta + 1 \stackrel{?}{=} 2 - \cos^2\theta$

$1 - \cos^2\theta + 1 \stackrel{?}{=} 2 - \cos^2\theta$

$2 - \cos^2\theta + 1 = 2 - \cos^2\theta$

51. 1 **53.** $\sec\theta$ **55.** about 228 mi **57.** $\pm\frac{\sqrt{5}}{2}$

59. $\pm\frac{\sqrt{5}}{2}$

Pages 857–859 ***Lesson 14-6***

1. $\frac{24}{25}, -\frac{7}{25}, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}$ **3.** $\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt{2+\sqrt{3}}}{2}$

5. $-\frac{\sqrt{2-\sqrt{3}}}{2}$ **7.** 1.64

9. $\cos^2 2x + 4\sin^2 x\cos^2 x \stackrel{?}{=} 1$

$\cos^2 2x + \sin^2 2x \stackrel{?}{=} 1$

$1 = 1$

11. $-\frac{4\sqrt{6}}{25}, -\frac{23}{25}, \frac{\sqrt{10}}{5}, -\frac{\sqrt{15}}{5}$ **13.** $\frac{24}{25}, \frac{7}{25}, \frac{3\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}$

15. $-\frac{\sqrt{15}}{8}, -\frac{7}{8}, \frac{\sqrt{10}}{4}, \frac{\sqrt{6}}{4}$ **17.** $\frac{\sqrt{2-\sqrt{2}}}{2}$ **19.** $-\frac{\sqrt{2-\sqrt{3}}}{2}$

21. $-\frac{\sqrt{2-\sqrt{3}}}{2}$

23. $2\cos^2\frac{x}{2} \stackrel{?}{=} 1 + \cos x$

$2\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2 \stackrel{?}{=} 1 + \cos x$

$2\left(\frac{1+\cos x}{2}\right) \stackrel{?}{=} 1 + \cos x$

$1 + \cos x = 1 + \cos x$

25. $\sin^2 x \stackrel{?}{=} \frac{1}{2}(1 - \cos 2x)$

$\sin^2 x \stackrel{?}{=} \frac{1}{2}[1 - (1 - 2\sin^2 x)]$

$\sin^2 x \stackrel{?}{=} \frac{1}{2}(2\sin^2 x)$

$\sin^2 x = \sin^2 x$

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27. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} \stackrel{?}{=} \tan x$

$\frac{1 - \cos^2 x}{\sin x \cos x} \stackrel{?}{=} \tan x$

$\frac{\sin^2 x}{\sin x \cos x} \stackrel{?}{=} \tan x$

$\frac{\sin x}{\cos x} \stackrel{?}{=} \tan x$

$\tan x = \tan x$

29. $\frac{1}{4} \tan \theta$ **31.** $\frac{120}{169}, \frac{119}{169}, \frac{5\sqrt{26}}{26}, -\frac{\sqrt{26}}{26}$

33. $\frac{\sqrt{15}}{8}, \frac{7}{8}, \frac{\sqrt{8 + 2\sqrt{15}}}{4}, -\frac{\sqrt{8 - 2\sqrt{15}}}{4}$

35. $-\frac{4\sqrt{21}}{5}, \frac{17}{25}, \frac{\sqrt{5\sqrt{2} + 10\sqrt{21}}}{10}, \frac{\sqrt{5\sqrt{10} - 10\sqrt{21}}}{10}$

37. $\frac{1 \pm \sqrt{\frac{1 - \cos L}{1 + \cos L}}}{1 \pm \sqrt{\frac{1 - \cos L}{1 + \cos L}}}$

39. Sample answer: If x is in the third quadrant, then $\frac{x}{2}$ is between 90° and 135°. Use the half-angle formula for cosine knowing that the value is negative.

41. Sample answer: 45°; $\cos 2(45°) = \cos 90° \text{ or } 0$, $2 \cos 45° = 2 \cdot \frac{\sqrt{2}}{2} \text{ or } \sqrt{22}$

43. D **45.** $\frac{\sqrt{6} + \sqrt{2}}{4}$ **47.** $-\frac{\sqrt{2}}{2}$ **49.** $\frac{\sqrt{6} + \sqrt{2}}{4}$

51. $\cot^2 \theta - \sin^2 \theta \stackrel{?}{=} \frac{\cos^2 \theta \csc^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta}$

$\cot^2 \theta - \sin^2\theta \stackrel{?}{=} \frac{\cos^2 \theta \frac{1}{\sin^2 \theta} - \sin^2\theta}{\sin^2 \theta \frac{1}{\sin^2 \theta}}$

$\cot^2 \theta - \sin^2\theta \stackrel{?}{=} \frac{\cot^2 \theta - \sin^2\theta}{1}$

$\cot^2 \theta - \sin^2\theta = \cot^2 \theta - \sin^2\theta$

53. 10^1 or 10 **55.** $(a^4)2 - 7(a^4) + 13$ **57.** $4(d^3)^2 + 2(d^3) + 104$ **59.** 5 **61.** $n^2 - 7n + 5f$ **63.** 1, −1

65. $\frac{5}{2}, -2$ **67.** $0, -\frac{1}{2}$

Pages 864–866 Lesson 14-7

1. 60°, 120°, 240°, 300° **3.** $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **5.** $0 + \frac{2k\pi}{3}$

7. $90° + k \cdot 360°, 180° + k \cdot 360°$ **9.** $\frac{7\pi}{6} + 2k\pi$, $\frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°, 330° + k \cdot 360°$

11. 31.3° **13.** 240°, 300° **15.** 30°, 150°, 180°, 330°

17. $\pi + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi$ **19.** $0 + 2k\pi$

21. $0° + k \cdot 180°$ **23.** $30° + k \cdot 360°, 150° + k \cdot 360°$

25. $\frac{7\pi}{6} + 2k\pi, \frac{11\pi}{6} + 2k\pi$ or $210° + k \cdot 360°, 330° + k \cdot 360°$ **27.** $\frac{\pi}{2} + k\pi, \frac{2\pi}{3} + 2k\pi, \frac{4\pi}{3} + 2k\pi$ or $90° + k \cdot 180°, 120° + k \cdot 360°, 240° + k \cdot 360°$ **29.** 10

31. $\frac{\pi}{2}$ **33.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$ **35.** $0 + k\pi, \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi$ **37.** $\frac{5\pi}{4} + k \cdot 2\pi, \frac{7\pi}{4} + k \cdot 2\pi, \frac{\pi}{6} + k \cdot 2\pi, \frac{5\pi}{6} + k \cdot 2\pi$ **39.** $120° + k \cdot 360°, 240° + k \cdot 360°$

41. $120° + k \cdot 360°, 240° + k \cdot 360°$ **43.** $\frac{\pi}{2} + 4k\pi$ or $90° + k \cdot 720°$ **45.** $\frac{\pi}{3} + k \cdot 2\pi, \frac{4\pi}{3} + k \cdot 2\pi, \frac{\pi}{4} + k \cdot 2\pi, \frac{5\pi}{4} + k \cdot 2\pi$, or $60° + k \cdot 360°, 240° + k \cdot 360°, 45° + k \cdot 360°, 225° + k \cdot 360°$

47. about 32° **49.** Sample answer: If $\sec \theta = 0$ then $\frac{1}{\cos \theta} = 0$. Since no value of θ makes $\frac{1}{\cos \theta} = 0$, there are no solutions. **51.** Sample answer: The function is periodic with two solutions in each of its infinite number of periods. **53.** D

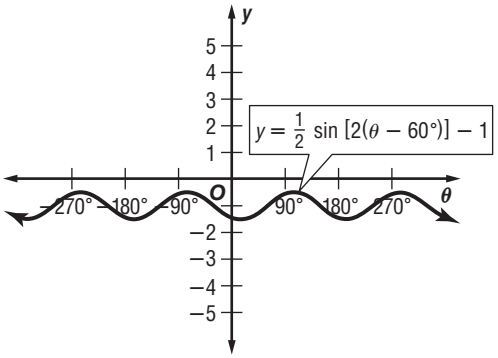
55. $\frac{24}{25}, \frac{7}{25}, \frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}$ **57.** $\frac{5\sqrt{11}}{18}, \frac{7}{18}, \frac{\sqrt{3}}{6}, \frac{\sqrt{33}}{6}$

59. $-\frac{\sqrt{3}}{2}$ **61.** $\frac{1}{2}$

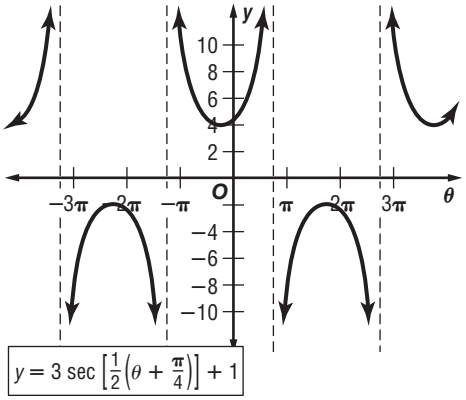
Pages 867–870 Chapter 14 Study Guide and Review

1. phase shift **3.** vertical shift **5.** double-angle formula **7.** trigonometric identity **9.** amplitude **11.** amplitude: 4; period: 180° or π **13.** amplitude: does not exist; period: 360° or 2π **15.** amplitude: does not exist; period: 45° or $\frac{\pi}{4}$

17. $-1, \frac{1}{2}, 180°, 60°$



19. 1, does not exist, $4\pi - \frac{\pi}{4}$



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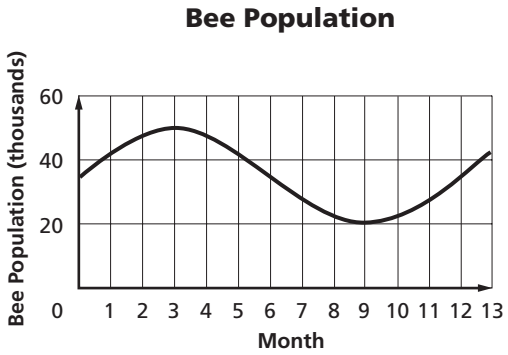
21. $p = 35{,}000 + 15{,}000 \sin\left(\frac{\pi}{6}t\right)$



23. $-\frac{4}{3}$ **25.** $\cot\theta$ **27.** $\csc\theta$

29.
$$\frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\frac{\sin\theta}{} + \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\frac{\sin\theta}{} \cdot \frac{\cos\theta}{\sin\theta} + \cos\theta \cdot \frac{\sin\theta}{\cos\theta} \stackrel{?}{=} \cos\theta + \sin\theta$$
$$\cos\theta + \sin\theta = \cos\theta + \sin\theta$$

31.
$$\cot^2\theta \sec^2\theta \stackrel{?}{=} 1 + \cot^2\theta$$
$$\frac{\cos^2\theta}{\sin^2\theta} \cdot \frac{1}{\cos^2\theta} \stackrel{?}{=} 1 + \cot^2\theta$$
$$\frac{1}{\sin^2\theta} \stackrel{?}{=} 1 + \cot^2\theta$$
$$\csc^2\theta \stackrel{?}{=} 1 + \cot^2\theta$$
$$1 + \cot^2\theta = 1 + \cot^2\theta$$

33.
$$I_m \cos^2\theta = I_m\left(1 - \frac{1}{\csc^2\theta}\right)$$
$$I_m \cos^2\theta = I_m(1 - \sin^2\theta)$$
$$I_m \cos^2\theta = I_m \cos^2\theta$$

35. $\frac{\sqrt{6} - \sqrt{2}}{4}$ **37.** $\frac{\sqrt{2} - \sqrt{6}}{4}$ **39.** $\frac{-\sqrt{6} - \sqrt{2}}{4}$

41.
$$\sin(30 - \theta) = \cos(60 + \theta)$$
$$\sin 30^\circ \cos\theta - \cos 30^\circ \sin\theta \stackrel{?}{=} \cos 60^\circ \cos\theta - \sin 60^\circ \sin\theta$$
$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \stackrel{?}{=} \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$$
$$\sin(\theta + \pi) \stackrel{?}{=} -\sin\theta$$
$$\sin\theta\cos\pi + \cos\theta\sin\pi \stackrel{?}{=} -\sin\theta$$
$$(\sin\theta)(-1) + (\cos\theta)(0) \stackrel{?}{=} -\sin\theta$$
$$-\sin\theta = -\sin\theta$$

43.
$$-\cos\theta \stackrel{?}{=} \cos(\pi + \theta)$$
$$-\cos\theta \stackrel{?}{=} \cos\pi\cos\theta - \sin\pi\sin\theta$$
$$-\cos\theta \stackrel{?}{=} -1 \cdot \cos\theta - 0 \cdot \sin\theta$$
$$-\cos\theta = -\cos\theta$$

45. $\frac{120}{169}, \frac{119}{169}, \frac{5\sqrt{26}}{26}, \frac{-\sqrt{26}}{26}$ **47.** $-\frac{120}{169}, \frac{119}{169}, \frac{\sqrt{26}}{26}, -\frac{5\sqrt{26}}{26}$ **49.** 0°

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